1

Examples

1.1 Cones in Vector Spaces

Vector optimization in partially ordered spaces requires, among other things, that one studies properties of cones; as arguments recall:

- Conic approximation of sets, e.g., in order to prove necessary optimality conditions; think of tangent cones, generated cones ....
- Cone-valued mappings; think of the subdifferential of indicator functions, which coincide with the normal cone, used in order to state optimality conditions for optimization problems with nonfunctional constraints.
- Order relations in linear spaces induced by cones; think of Pareto efficiency and the consideration of smaller or larger cones in order to vary the set of efficient points.
- Cones in order to represent inequalities; think of side restrictions or vector variational inequalities.
- Cones in connection with some theoretical procedures; think of Phelps's cone related to Ekeland's variational principle in normed spaces, minimal points with respect to cone orderings.
- Dual cones or polar cones; think of dual assertions and the bipolar theorem.

A cone in a vector space is an algebraic notion, but its use in theory and applications of optimization demands that one considers cones in topological vector spaces and studies, besides algebraic properties such as pointedness and convexity, also analytical ones such as those of being closed, normal, Daniell, nuclear, or having a base with special properties. A short overview of such qualities can be found at the end of this section; the corresponding proofs are given by hints to references or to later sections. The examples of cones given below show interesting results on cones in infinite-dimensional vector spaces, which are important in vector optimization and control theory.

Example 1.1.1. Let X be a normed vector space with $\dim X = \infty$, and let $x' : X \to \mathbb{R}$ be a linear but not continuous function. Then the cone

$$C = \{0\} \cup \{x \mid x'(x) > 0\}$$

has a base $B = \{x \in C \mid x'(x) = 1\}$, but $0 \in \operatorname{cl} B$, since B is dense in C. For bases see Section 2.2.

Example 1.1.2. Let X be a locally convex space and C a convex cone in X. Then

$$C^{\#} \neq \emptyset \Rightarrow \operatorname{cl} C \cap (-C) = \{0\} \Rightarrow C \text{ is pointed,}$$

where

$$C^{\#} := \left\{x^* \in X^* \mid x^*(x) > 0 \;\forall x \in C \setminus \{0\}\right\}.$$

The first implication becomes an equivalence if $\dim X < \infty$. The converse of the second implication is not true, even if $X = \mathbb{R}^2$; $C = \big((0,\infty) \times \mathbb{R}) \cup (\{0\} \times [0,\infty)\big)$ is a counterexample.

To prove the first implication consider $u^* \in C^{\#}$. Assume that there exists $\overline{c} \in \operatorname{cl} C \cap (-C)$, $\overline{c} \neq 0$. Then there exists a sequence $(c^n)_{n\in\mathbb{N}} \subset C$ such that $(c^n) \to \overline{c}$. Because $u^*(c^n) \geq 0$, we have that $u^*(\overline{c}) = \lim u^*(c^n) \geq 0$, which contradicts the fact that $u^* \in C^{\#}$ and $\overline{c} \in -C \setminus \{0\}$.

Assume $\dim X < \infty$; let us prove the converse of the first implication. Let $C \neq \{0\}$ (if $C = \{0\}$ then $C^{\#} = X^*$) and suppose that $\operatorname{cl} C \cap (-C) = \{0\}$. Then

$$0 \notin \operatorname{raint} C^+ \subset C^{\#}. \tag{1.1}$$

Indeed, if $0 \in \operatorname{raint} C^+$, then C^+ is a linear subspace, and consequently $C^{++} = \operatorname{cl} C$ is a linear subspace. This implies that $\operatorname{cl} C \cap (-C) = -C \neq \{0\}$, a contradiction. Now let $\overline{x}^* \in \operatorname{raint} C^+$. Assume that $\overline{x}^* \notin C^{\#}$; then there exists $\overline{x} \in C \setminus \{0\}$ such that $\overline{x}^*(\overline{x}) = 0$. Let $x^* \in C^+$. Then there is $\lambda > 0$ such that $(1+\lambda)\overline{x}^* - \lambda x^* \in C^+$. So $((1+\lambda)\overline{x}^* - \lambda x^*)(\overline{x}) = -\lambda x^*(\overline{x}) \geq 0$, whence $x^*(\overline{x}) \leq 0$; it follows that $-\overline{x} \in C^{++} = \operatorname{cl} C$. It follows that $\operatorname{cl} C \cap (-C) \neq \{0\}$, a contradiction. Taking into account $\dim X < \infty$, (1.1) gives $C^{\#} \neq \emptyset$, since $\operatorname{raint} C^+$ is nonempty (recall that every nonempty finite-dimensional convex set has a nonempty relative algebraic interior; see [168, p. 9]).

Example 1.1.3. The convex cone C in a Banach space X has the **angle property** if for some $\varepsilon \in (0,1]$ and $x^* \in X^* \setminus \{0\}$ we have $C \subset \{x \in X \mid x^*(x) \geq \varepsilon \|x^*\| \cdot \|x\|\}$. It follows that $x^* \in C^{\#}$. Since for $X = \mathbb{R}^n = X^*$ the last inequality means $\cos(x^*, x) \geq \varepsilon$, it is clear where the name “angle property” comes from. The class of convex cones with the angle property is very large (for all $\varepsilon \in (0,1)$ and $x^* \in X^* \setminus \{0\}$ the set $\{x \in X \mid x^*(x) \geq \varepsilon \|x^*\| \cdot \|x\|\}$ is a closed convex cone with the angle property and nonempty interior). In fact, in normed spaces, a convex cone has the angle property iff it is well-based. So, the cone $\mathbb{R}^n_+$ in $\mathbb{R}^n$ has the angle property (with $x^* = (1,1,\ldots,1) \in \mathbb{R}^n$ and $\varepsilon = 2^{-1/2}$), but the ordinary order cone $\ell_2^+ \subset \ell_2$ does not have it.

Indeed, if for some $x^* \in \ell_2 \setminus \{0\}$ and some $\varepsilon > 0$, $\ell_2^+ \subset \{x \in \ell_2 \mid x^*(x) \geq \varepsilon\|x^*\| \cdot \|x\|\}$, then $x_n^* \geq \varepsilon\|x^*\|$ (because $e_n = (0,\ldots,1,\ldots) \in \ell_2^+$), whence the contradiction $\varepsilon\|x^*\| \leq 0$ (since $(x_n^*) \to 0$ for $n \to \infty$).

Overview of Several Properties of Cones

Let X be a Banach space, C and K proper (i.e., $\{0\} \neq C, K \neq X$) convex cones in X, K^+ the continuous dual cone of K, and

$$K^{\#} := \{x^* \in K^+ \mid x^*(x) > 0 \;\forall\, x \in K \setminus \{0\}\}$$

the quasi-interior of K^+. Then the relations below hold. For more relationships among different kinds of cones and spaces look at Section 3.2; for cones with base see Section 2.2.

$$\begin{array}{ccccc}
 & & K \text{ has compact base} & \Longrightarrow & K \text{ is Daniell} \\
 & & \Downarrow \;\; \Uparrow \begin{smallmatrix} K = \overline{K} \\ X = \mathbb{R}^n \end{smallmatrix} & & \\
K \text{ has angle property} & \Longleftrightarrow & K \text{ is well-based} & \Longleftrightarrow & \operatorname{int} K^+ \neq \emptyset \\
 & & \Downarrow & & \\
\exists\, C \;:\; K \setminus \{0\} \subset \operatorname{int} C & \Longleftrightarrow & K \text{ is based} & \Longleftrightarrow & K^{\#} \neq \emptyset \\
 & & \Downarrow \;\; \Uparrow \begin{smallmatrix} K = \overline{K} \\ X \text{ separable} \end{smallmatrix} & & \\
K \text{ well-based} & \overset{K=\overline{K},\; X=\mathbb{R}^n}{\Longleftarrow} & K \text{ pointed} & \Longleftarrow & K^+ - K^+ = X^* \\
\Downarrow & & \Uparrow & & \Updownarrow \\
\overline{K} \text{ is normal} & \Longleftrightarrow & K \text{ is normal} & \Longrightarrow & K \text{ is } w\text{-normal.}
\end{array}$$

1.2 Equilibrium Problems

Let us consider a common scalar optimization problem

$$\varphi(x) \to \min \text{ s.t. } x \in \mathcal{B}, \tag{1.2}$$

where $\mathcal{B}$ is a given nonempty set in a space X and $\varphi : \mathcal{B} \to \mathbb{R}$ a given function. Let $\overline{x} \in \mathcal{B}$ be a solution of (1.2); that is,

$$\varphi(\overline{x}) \leq \varphi(y) \;\forall\, y \in \mathcal{B}. \tag{1.3}$$

Then, setting $f(x, y) := \varphi(x) - \varphi(y)$ for $x, y \in \mathcal{B}$, $\overline{x}$ solves also the problem

$$\text{find}\, \overline{x} \in \mathcal{B} \,\text{such that}\, f(\overline{x}, y) \leq 0 \;\forall\, y \in \mathcal{B}. \tag{1.4}$$

For given $\mathcal{B}$ and $f : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$, a problem of the kind (1.4) is called an **equilibrium problem** and $\mathcal{B}$ its feasible set. A large number of quite different problems can be subsumed under the class of equilibrium problems as, e.g., saddle point problems, Nash equilibria in noncooperative games, complementarity problems, variational inequalities, and fixed point problems. To sketch the last one, consider X a Hilbert space and $T : \mathcal{B} \to \mathcal{B}$ a given mapping. With $f(x, y) := (x - Tx \,|\, x - y)$ we have that $\overline{x} \in \mathcal{B}$ is a fixed point of T (i.e., $T\overline{x} = \overline{x}$) if and only if $\overline{x}$ satisfies $f(\overline{x}, y) \leq 0 \; \forall\, y \in \mathcal{B}$. Indeed, if $\overline{x}$ is an equilibrium point, then taking $\overline{y} := T\overline{x}$, we have

$$0 \geq (\overline{x} - T\overline{x} \,|\, \overline{x} - T\overline{x}) = \|\overline{x} - T\overline{x}\|^2,$$

and so $\overline{x} = T\overline{x}$ as claimed. The other direction of the assertion is obvious.

There are powerful results that ensure the existence of a solution of the equilibrium problem (1.4); one of the most famous is **Fan's theorem**; see [111]:

Theorem 1.2.1. *Let $\mathcal{B}$ be a compact convex subset of the Hausdorff locally convex space (H.l.c.s.) X and let $f : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ be a function satisfying*

$\forall\, y \in \mathcal{B} : x \to f(x, y)$ *is lower semicontinuous,*
$\forall\, x \in \mathcal{B} : y \to f(x, y)$ *is concave,*
$\forall\, y \in \mathcal{B} : f(y, y) \leq 0.$

Then there is $\overline{x} \in \mathcal{B}$ with $f(\overline{x}, y) \leq 0$ for every $y \in \mathcal{B}$.

Since we mainly deal with multicriteria problems, we now take $\mathcal{B} \subset X$ as above and $\varphi : \mathcal{B} \to Y$, where Y is the Euclidean space $\mathbb{R}^n$ $(n > 1)$ or any locally convex space (l.c.s.). In Y we consider a convex pointed cone C, for instance $C = \mathbb{R}^n_+$ if $Y = \mathbb{R}^n$. Then, as usual and in accordance with (1.3), $\overline{x} \in \mathcal{B}$ solves the multicriteria problem $\varphi(x) \to \min, x \in \mathcal{B}$, if

$$\forall\, y \in \mathcal{B} : \varphi(y) \notin \varphi(\overline{x}) - (C \setminus \{0\}). \tag{1.5}$$

Sometimes, if the cone C has a nonempty interior (e.g., $C = \mathbb{R}^n_+$ when $X = \mathbb{R}^n$), one looks for so-called weak solutions of multicriteria problems replacing (1.5) by

$$\forall\, y \in \mathcal{B} : \varphi(y) \notin \varphi(\overline{x}) - \operatorname{int} C. \tag{1.6}$$

As above, we introduce $f(x, y) := \varphi(x) - \varphi(y)$ for $x, y \in \mathcal{B}$, and so in accordance with (1.4) we get the following **vector equilibrium problem**: find $\overline{x} \in \mathcal{B}$ (sometimes called vector equilibrium) with the property

$$\forall\, y \in \mathcal{B} : f(\overline{x}, y) \notin C \setminus \{0\}, \tag{1.7}$$

or, considering weak solutions of multicriteria problems, find $\overline{x} \in \mathcal{B}$ (sometimes called **weak vector equilibrium**) with the property

$$\forall\, y \in \mathcal{B} : f(\overline{x}, y) \notin \operatorname{int} C. \tag{1.8}$$

Looking for existence theorems for solutions of (1.7) we mention Sections 3.8, 3.9, 4.2.4, and 4.2.5, which are devoted to vector-valued equilibrium problems.

Vector equilibria play an important role in mathematical economics, e.g., if one deals with **traffic control**. As an example we describe an extension of **Wardrop's principle** for weak traffic equilibria to the case in which the route **flows** have to satisfy the travel demands between origin–destination pairs and route capacity restrictions, and the travel cost function is a mapping.

Let us consider a **traffic** (or **transportation) network** (N, L, P), where N denotes a finite set of nodes, L a finite set of links, $P \subseteq N \times N$ a set of origin–destination pairs, and $|P|$ the cardinality of P. We denote by

- $d \in \mathbb{R}^{|P|}$ the travel demand vector with $d_p > 0$ for all $p \in P$;
- $\mathbb{R}$ a finite set of routes (paths) between the origin–destination pairs such that for each $p \in P$, the set $R(p)$ of routes in $\mathbb{R}$ connecting p is nonempty;
- Q the pair–route incidence matrix ($Q_{pr} = 1$ if $r \in R(p), Q_{pr} = 0$ otherwise); i.e., Q is a matrix of type $(|P|, |R|)$.

We suppose that the demand vector is given on $\mathbb{R}^{|P|}$. Then we introduce the set K of traffic flow vectors (path flow vectors), where $K := \{v \in \mathbb{R}^{|R|} \mid 0 \leq v_r \; \forall\, r \in R,\; Qv = d\}$. Note that, in more detailed form, the condition $Qv = d$ states that for all $p \in P$, $\sum_{r \in R(p)} v_r = d_p$. The set K is the feasible set for the desired vector equilibrium problem.

In order to formulate the vector equilibrium problem, a valuation of the traffic flow vectors is necessary. Therefore, let Y be a topological vector space partially ordered by a suitable convex, pointed cone C with $\operatorname{int} C \neq \emptyset$ and F a travel cost function, which assigns to each route flow vector $v \in \mathbb{R}^{|R|}$ a vector of marginal travel costs $F(v) \in L(\mathbb{R}^{|R|}, Y)$, where $L(\mathbb{R}^{|R|}, Y)$ is the set of all linear operators from $\mathbb{R}^{|R|}$ into Y.

The **weak traffic equilibrium problem** we describe consists in finding $u \in K$ such that (in accordance to (1.4))

$$\phi(u, v) := -F(u)(v - u) \notin \operatorname{int} C \;\forall\, v \in K, \tag{1.9}$$

recalling that in the scalar case $Y = \mathbb{R}^1$, the product $F(u)v$ is a price for v.

Any route flow $u \in K$ that satisfies (1.9) is called a **weak equilibrium flow**. It has been shown that u is a weak equilibrium flow if and only if u satisfies a generalized vector form of **Wardrop's principle**, namely, for all origin–destination $p \in P$ and all routes $s, t \in \mathbb{R}$ connecting p, i.e., $s, t \in R(p)$,

$$F(u)_s - F(u)_t \in -\operatorname{int} C \Rightarrow u_s = 0. \tag{1.10}$$

To explain $F(u)_s$ consider $u \in \mathbb{R}^{|R|}$; then $u = (u_1, \ldots, u_s, \ldots, u_{|R|})$, and for $v = (0, \ldots, 0, u_s, 0, \ldots 0)$ it is $F(u)_s = F(u)v$.

Condition (1.10), due to its decomposed (user-oriented) form, is often more practical than the original definition (1.9), since it deals with pairs p and paths $s, t \in R(p)$ directly: When the traffic flow is in vector equilibrium, users choose only Pareto-optimal or vector minimum paths to travel on. In the case

$Y = \mathbb{R}^m, C = \mathbb{R}^m_+$ it is easily seen that (1.10) follows from (1.9). Let $u \in K$ satisfy (1.9); take $p \in P$ and $s, t \in R(p)$. Choose a flow $\overline{v}$ such that

$$\overline{v}_\lambda = \begin{cases} u_\lambda & \text{if } \lambda \neq t, \lambda \neq s, \\ 0 & \text{if } \lambda = s, \\ u_t + u_s & \text{if } \lambda = t, \end{cases} \qquad (\lambda = 1, 2, \ldots, |R|).$$

Then $\overline{v}_\lambda \in K$, since $\sum_{1 \leq \lambda \leq |R|} \overline{v}_\lambda = \sum_{1 \leq \lambda \leq |R|} u_\lambda = d_p$. Consequently, from (1.9), writing $F(u)$ as a matrix $(F_{\mu\lambda}) \in \mathbb{R}^{m \times |R|}$, we have $\forall\, \mu = 1, \ldots, m$,

$$\sum_{1 \leq \lambda \leq |R|} F_{\mu\lambda}(\overline{v}_\lambda - u_\lambda) = F_{\mu t}(u_s) + F_{\mu s}(-u_s) = (F_{\mu t} - F_{\mu s})(u_s) \not< 0.$$

But from the first condition in (1.10) it is $F_{\mu s} - F_{\mu t} < 0 \;\forall\, \mu$, so $u_s = 0$. For further references see [86, 88, 134, 226, 254, 370, 382]. For other applications of multicriteria decision-making to economic problems see [15].

1.3 Location Problems in Town Planning

Urban development is connected with conflicting requirements of areas for dwelling, traffic, disposal of waste, recovery, trade, and others. Using methods of location theory may be one way of supporting urban planning to determine the **best location** for a special new layout or for arrangements. For references see e.g. [102, 101, 152, 156, 171, 173, 174, 170, 219, 305, 368, 369, 371].

In our investigations we consider the special situation in East German towns. One of the actual main problems of town planning is the traffic problem, due to the extremely high increase of motorized individual traffic in recent years. The lack of parking space is a part of the traffic problem. This is typical for many newly built residential areas in East Germany. Such a residential area is Halle-Silberhöhe, which was built at the beginning of the 1980s. In this district 5, 6, and 11-story blocks dominate. In our example we consider two residential sections, which count about 9300 inhabitants. This area has a size of 800 m × 1000 m. There exist 1750 parking facilities, representing a shortage of 1950. The impact is that many inhabitants park their cars on green areas.

One way to solve this problem of **inadequate parking facilities** is to build multistory parking garages. Now, the problem is to find the **best location** for such a multistory garages.

It would be possible to formulate our problem as a **real-valued location problem (Fermat–Weber problem)**, which is the problem to determine a location x of a new facility such that the weighted sum of distances between n given facilities a^i $(i = 1, \ldots, n)$ and x is minimal. Using this approach it is very difficult to say how the weights λ_i $(i = 1, \ldots, n)$ are to be chosen. Another difficulty may arise if the solution of the corresponding optimal location is not practically useful. Then we need new weights, and again we don't know how to choose the weights.

So the following approach is of interest: We formulate the problem as a **vector-valued (or synonymously vector or multicriteria) location problem**

$$(P) \qquad \begin{pmatrix} \|x - a^1\|_{\max} \\ \|x - a^2\|_{\max} \\ \cdots \\ \|x - a^n\|_{\max} \end{pmatrix} \longrightarrow v - \min_{x \in \mathbb{R}^2},$$

where $x, a^i \in \mathbb{R}^2$ $(i = 1, \dots, n)$,

$$\|x\|_{\max} = \max\{| \, x_1 \, |, | \, x_2 \, |\},$$

and "$v - \min_{x \in \mathbb{R}^2}$" means that we study the problem of determining the set of efficient points of an objective function $f : X \longrightarrow \mathbb{R}^n$ with respect to a cone $C \subset \mathbb{R}^n$:

$$\mathrm{Eff}(f[X], C) := \{f(x) \mid x \in X, \quad f[X] \cap (f(x) - (C \setminus \{0\})) = \emptyset\}.$$

Remark 1.3.1. For applications in town planning it is important that we can choose different norms in the formulation of (P). The decision which of the norms will be used depends on the course of the roads in the city or in the district or on other influences coming from the practical background of the planning problem.

In the following example we study the problem (P) with $C = \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ denotes the usual ordering cone in n-dimensional Euclidean space.

In Section 4.3 we consider a location problem in town planning, formulate a multicriteria location problem, derive optimality conditions, and present several algorithms for solving multicriteria location problems and the corresponding computer programs. It is well known that the set of solutions in vector optimization (set of efficient elements) may be large, and so we will carry out a comparison of alternatives by using a graphical representation.

For the problem in town planning mentioned above we fix the given points $a^1 = (-1.5, 3.5)$, $a^2 = (1, 3)$, $a^3 = (1, 0)$, $a^4 = (-3, -2)$, $a^5 = (3.5, -1.5)$, $a^6 = (2, 2)$, $a^7 = (-2, 2)$, $a^8 = (4, 1)$, $a^9 = (-3, 2)$.

If the decision-maker prefers the maximum norm, we get the **solution set of the multicriteria location problem** (P) as shown in Figure 1.3.1.

But if the decision-maker prefers the dual norm to the maximum norm (this norm is called the Lebesgue norm), the **solution set** has the form shown in Figure 1.3.2

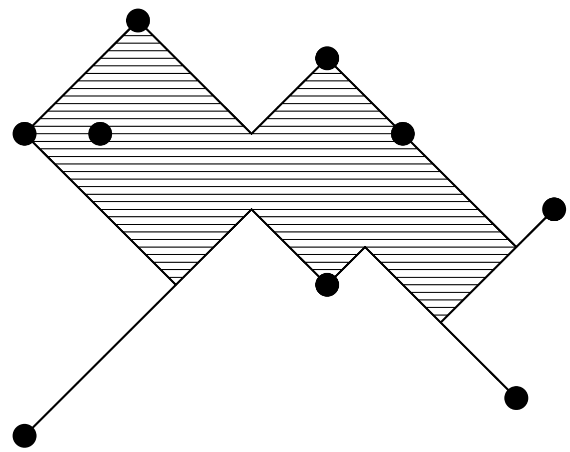

Figure 1.3.1. The set of efficient elements of the multicriteria location problem (P) with the maximum norm

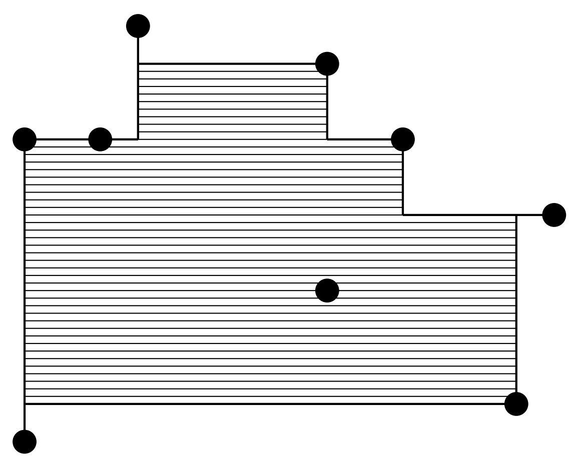

Figure 1.3.2. The set of efficient points of the multicriteria location problem (P) with the Lebesgue norm instead of the maximum norm

1.4 Multicriteria Control Problems

In control theory often one has the problem to minimize more than one objective function, for instance, a cost functional as well as the distance between the final state and a given point.

To realize this task usually one takes as objective function a weighted sum of the different objectives. However, the more natural way would be to study the set of efficient points of a vector optimization problem with the given objective functions. It is well known that the weighted sum is only a special surrogate problem to find efficient points, which has the disadvantage that in the nonconvex case one cannot find all efficient elements in this way.

In order to formulate a multicriteria control problem we introduce a system of differential equations that describe the motion of the controlled system. Let $x(t)$ be an n-dimensional phase vector that characterizes the state of the controlled system at time t. Furthermore, let $u(t)$ be an m-dimensional vector that characterizes the controlling action realized at time t. In particular, consider in a time interval $0 \leq t \leq T$ the system of ordinary differential equations with initial condition

$$\left.\begin{aligned} \frac{dx}{dt}(t) &= \varphi(t, x(t), u(t)), \\ x(0) &= x_0 \in \mathbb{R}^n. \end{aligned}\right\} \tag{1.11}$$

It is assumed that the controlling actions $u(t)$ satisfy the restrictions

$$u(t) \quad \in \quad U \subset \mathbb{R}^m.$$

The vector x is often called the **state**, and u is called the **control** of (1.11); a pair (x, u) satisfying (1.11) and the control restriction is called an **admissible solution** (or **process**) of (1.11).

Under additional assumptions it is possible to ensure the existence of a solution (x, u) of (1.11) on the whole time interval $[0, T]$ or at least almost everywhere on $[0, T]$ (compare Section 4.5).

Introducing the criteria or performance space, the objective function $f : (X \times U) \longrightarrow (Y, C_Y)$, where Y is a linear space, and $C_Y \subset Y$ is a proper convex cone, we have as **multicriteria optimal control problem** (also called multiobjective or vector-valued or vector optimal control problem)

(P): Find some control $\bar{u}$ such that the corresponding trajectory $\bar{x}$ satisfies

$$f(x, u) \notin f(\bar{x}, \bar{u}) - (C_Y \setminus \{0\})$$

for all solution pairs (x, u) of (1.11). The pair $(\bar{x}, \bar{u})$ is then called an optimal process of (P).

If we study the problem to minimize the distance f_1 of the final state $x(T)$ of the system (1.11) and a given point as well as a cost functional $\int_0^T \Phi(t, x(t), u(t))dt$ by a control u, we have a special case of (P) with

$$f(x, u) \;=\; \begin{pmatrix} f_1(x(T)) \\ \int_0^T \Phi(t, x(t), u(t))dt \end{pmatrix}.$$

In the following we explain that **cooperative differential games** are special cases of the multicriteria control problem (P). We consider a game with $n \geq 2$ players and define

$$Y := Y_1 \times Y_2 \times \cdots \times Y_n,$$

the product of the criteria spaces of each of the n players,

$$C_Y := C_{Y_1} \times C_{Y_2} \times \cdots \times C_{Y_n},$$

the product of n proper convex cones,

$$f(x,u) := (f_1(x,u), \ldots, f_n(x,u)),$$

the vector of the loss functions of the n players,

$$U := U_1 \times U_2 \times \cdots \times U_n,$$

the product of the control sets of the n players, and define

$$u := (u_1, \ldots, u_n).$$

The player j tries to minimize his cost function f_j (the utility or profit function is $-f_j$) with respect to the partial order induced by the cone C_{Y_j} influencing the system (1.11) by means of the function $u_j \in U_j$. And "cooperative game" means that a state is considered optimal if no player can reduce his costs without increasing the costs of another player. Such a situation is possible because each cost function f_j depends on the same control tuple u. The optimal process gives the **Pareto minimum** of the cost function $f(x,u)$.

It is well known that it is difficult to show the existence of optimal (or efficient) controls of (P), whereas **suboptimal controls** exist under very weak conditions. So it is important to derive some assertions for suboptimal controls. This is done in Section 4.5. There an application of a variational principle for vector optimization problems yields an ε-minimum principle for (P), which is (for $\varepsilon = 0$) closely related to Pontryagin's minimum principle (cf. Section 4.5).

1.5 Multicriteria Fractional Programming Problems

Many problems in economics can be formulated as fractional programming problems. In the papers of Hirche ([166]), Schaible ([315]), Schaible and Ibaraki ([316]) the following **real-valued fractional programming problem** is considered:

$$\varphi(x) := a_1^T x + \frac{a_2^T x}{b_2^T x} \longrightarrow \max_{x \in \mathcal{A}},$$

where $x \in \mathbb{R}^n$, $a_1, a_2, b_2 \in \mathbb{R}^n_+$, $a_1^T x, a_2^T x, \ldots$ are scalar products, $b_2^T x \neq 0$, and $\mathcal{A} = \{x \in \mathbb{R}^n \mid Bx - b \in -K\}$. The cone K is a convex cone in $\mathbb{R}^m$. The objective function $a_1^T x$ describes the production scope, and $\frac{a_2^T x}{b_2^T x}$ describes the viability.

From the practical as well as from the mathematical point of view it is more useful to formulate the problem as a **multicriteria fractional programming problem**:

$$f(x) = \left(\frac{a_1^T x}{1}, \ \frac{a_2^T x}{b_2^T x} \right) \longrightarrow v - \max_{x \in \mathcal{A}} .$$

Applying the **Dinkelbach transformation**, one gets a useful surrogate parametric optimization problem and the corresponding algorithms:

$$f_\lambda(x) = \left(a_1^T x - \lambda_1, \ a_2^T x - \lambda_2 b_2^T x \right) \longrightarrow v - \max_{x \in \mathcal{A}},$$

where $\lambda \in \mathbb{R}^2$.

In Section 4.4 we discuss possibilities to handle this transformed vector optimization problem by means of parametric optimization. We then derive a three-level dialogue algorithm in order to solve the transformed problem.

1.6 Stochastic Efficiency in a Set

Uncertainty is the key ingredient in many decision problems. Financial planning, cancer screening, and airline scheduling are just examples of areas in which ignoring uncertainty may lead to inferior or simply wrong decisions. There are many ways to model uncertainty; one that has proved particularly fruitful is to use probabilistic models.

Two methods are frequently used for modeling choice among uncertainty prospects: stochastic dominance (Ogryczak and Ruszczynski [282], [283], Whitmore and Findlay ([377]); Levy ([236]) and mean-risk analysis (Markowitz ([251]).

The **stochastic dominance** is based on an axiomatic model of risk-averse preferences: It leads to conclusions that are consistent with the axioms. Unfortunately, the stochastic dominance approach does not provide us with a simple computational recipe; it is, in fact, a multiple criteria model with a continuum of criteria.

The **mean-risk approach** quantifies the problem in a lucid form of only two criteria:

- The mean, representing the expected outcome;
- The risk, a scalar measure of the variability of outcomes.

The mean-risk model is appealing to decision-makers and allows a simple trade-off analysis, analytical or geometrical. On the other hand, mean-risk approaches are not capable of modeling the gamut of risk-averse preferences. Moreover, for typical dispersion statistics used as risk measures, the mean-risk approach may lead to inferior conclusions (compare Section 4.6).

The seminal portfolio optimization model of Markowitz ([249]) uses the variance as the risk measure in the mean-risk analysis, which results in a formulation of a quadratic programming model. Since then, many authors have pointed out that the mean-variance model is, in general, not consistent with stochastic dominance rules. The use of the semivariance rather than variance as the risk measure was already suggested by Markowitz ([250]) himself.

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