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## Preface

Population dynamics is an important subject in mathematical biology. A central problem is to study the long-term behavior of modeling systems. Most of these systems are governed by various evolutionary equations such as difference, ordinary, functional, and partial differential equations (see, e.g., [165, 142, 218, 119, 55]). As we know, interactive populations often live in a fluctuating environment. For example, physical environmental conditions such as temperature and humidity and the availability of food, water, and other resources usually vary in time with seasonal or daily variations. Therefore, more realistic models should be nonautonomous systems. In particular, if the data in a model are periodic functions of time with commensurate period, a periodic system arises; if these periodic functions have different (minimal) periods, we get an almost periodic system. The existing reference books, from the dynamical systems point of view, mainly focus on autonomous biological systems. The book of Hess [106] is an excellent reference for periodic parabolic boundary value problems with applications to population dynamics. Since the publication of this book there have been extensive investigations on periodic, asymptotically periodic, almost periodic, and even general nonautonomous biological systems, which in turn have motivated further development of the theory of dynamical systems.

In order to explain the dynamical systems approach to periodic population problems, let us consider, as an illustration, two species periodic competitive systems

$$\begin{aligned}\frac{du_1}{dt} &= f_1(t, u_1, u_2), \\ \frac{du_2}{dt} &= f_2(t, u_1, u_2),\end{aligned}\tag{0.1}$$

where  $f_1$  and  $f_2$  are continuously differentiable and  $\omega$ -periodic in  $t$ , and  $\partial f_i / \partial u_j \leq 0$ ,  $i \neq j$ . We assume that for each  $v \in \mathbb{R}^2$ , the unique solution  $u(t, v)$  of system (0.1) satisfying  $u(0) = v$  exists globally on  $[0, \infty)$ .

Let  $X = \mathbb{R}^2$ , and define a family of mappings  $T(t) : X \rightarrow X, t \geq 0$ , by  $T(t)x = u(t, x), \forall x \in X$ . It is easy to see that  $T(t)$  satisfies the following properties:

- (1)  $T(0) = I$ , where  $I$  is the identity map on  $X$ ;
- (2)  $T(t + \omega) = T(t) \circ T(\omega), \forall t \geq 0$ ;
- (3)  $T(t)x$  is continuous in  $(t, x) \in [0, \infty) \times X$ .

$T(t)$  is called the periodic semiflow generated by periodic system (0.1), and  $P := T(\omega)$  is called its associated Poincaré map (or period map). Clearly,  $P^n(v) = u(n\omega, v), \forall n \geq 1, v \in \mathbb{R}^2$ . It then follows that the study of the dynamics of (0.1) reduces to that of the discrete dynamical system  $\{P^n\}$  on  $\mathbb{R}^2$ .

If  $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$ , then we write  $u \leq v$  whenever  $u_i \leq v_i$  holds for  $i = 1, 2$ . We write  $u \leq_K v$  whenever  $u_1 \leq v_1$  and  $u_2 \geq v_2$ . By the well-known Kamke comparison theorem, it follows that the following key properties hold for competitive system (0.1) (see, e.g., [218, Lemma 7.4.1]):

- (P1) If  $u \leq_K v$ , then  $Pu \leq_K Pv$ ;
- (P2) If  $Pu \leq Pv$ , then  $u \leq v$ .

Then the Poincaré map  $P$ , and hence the discrete dynamical system  $\{P^n\}$ , is monotone with respect to the order  $\leq_K$  on  $\mathbb{R}^2$ . Consequently, system (0.1) admits convergent dynamics (see [218, Theorem 7.4.2]).

**Theorem** *Every bounded solution of a competitive planar periodic system asymptotically approaches a periodic solution.*

We use the proof provided in [218, Theorem 7.4.2]. Indeed, it suffices to prove that every bounded orbit of  $\{P^n\}$  converges to a fixed point of  $P$ . Given two points  $u, v \in \mathbb{R}^2$ , one or more of the four relations  $u \leq v, v \leq u, u \leq_K v, v \leq_K u$  must hold. Now, if  $P^{n_0}u_0 \leq_K P^{n_0+1}u_0$  (or the reverse inequality) holds for some  $n_0 \geq 0$ , then (P1) implies that  $P^n u_0 \leq_K P^{n+1}u_0$  (or the reverse inequality) holds for all  $n \geq n_0$ . Therefore,  $\{P^n u_0\}$  converges to some fixed point  $\bar{u}$ , since the sequence is bounded and eventually monotone. The proof is complete in this case, so we assume that there does not exist such an  $n_0$  as just described. In particular, it follows that  $u_0$  is not a fixed point of  $P$ . Then it follows that for each  $n$  we must have either  $P^{n+1}u_0 \leq P^n u_0$  or the reverse inequality. Suppose for definiteness that  $u_0 \leq Pu_0$ , the other case being similar. We claim that  $P^n u_0 \leq P^{n+1}u_0$  for all  $n$ . If not, there exists  $n_0$  such that

$$u_0 \leq Pu_0 \leq P^2u_0 \leq \dots \leq P^{n_0-1}u_0 \leq P^{n_0}u_0$$

but  $P^{n_0}u_0 \geq P^{n_0+1}u_0$ . Clearly,  $n_0 \geq 1$  since  $u_0 \leq Pu_0$ . Applying (P2) to the displayed inequality yields  $P^{n_0-1}u_0 \geq P^{n_0}u_0$  and therefore  $P^{n_0-1}u_0 = P^{n_0}u_0$ . Since  $P$  is one-to-one,  $u_0$  must be a fixed point, in contradiction to our assumption. This proves the claim and implies that the sequence  $\{P^n u_0\}$  converges to some fixed point  $\bar{u}$ .

It is hoped that the reader will appreciate the elegance and simplicity of the arguments supporting the above theorem, which are motivated by a now classical paper of deMottoni and Schiaffino [68] for the special case of periodic Lotka–Volterra systems. This example also illustrates the roles that Poincaré maps and monotone discrete dynamical systems may play in the study of periodic systems. For certain nonautonomous perturbations of a periodic system (e.g., an asymptotically periodic system), one may expect that the Poincaré map associated with the unperturbed periodic system (e.g., the limiting periodic system) should be very helpful in understanding the dynamics of the original system. For an nonperiodic nonautonomous system (e.g., almost periodic system), we are not able to define a continuous or discrete-time dynamical system on its state space. The skew-product semiflow approach has proved to be very powerful in obtaining dynamics for certain types of nonautonomous systems (see, e.g., [193, 190, 200]).

The main purpose of this book is to provide an introduction to the theory of periodic semiflows on metric spaces and its applications to population dynamics. Naturally, the selection of the material is highly subjective and largely influenced by my personal interests. In fact, the contents of this book are predominantly from my own and my collaborators' recent works. Also, the list of references is by no means exhaustive, and I apologize for the exclusion of many other related works.

Chapter 1 is devoted to abstract discrete dynamical systems on metric spaces. We study chain transitivity, strong repellers, and perturbations. In particular, we will show that a dissipative, uniformly persistent, and asymptotically compact system must admit a coexistence state. This result is very useful in proving the existence of (all or partial componentwise) positive periodic solutions of periodic evolutionary systems.

The focus of Chapter 2 is on global dynamics in certain types of monotone discrete dynamical systems on ordered Banach spaces. Here we are interested in the abstract results on attracting order intervals, global attractivity, and global convergence, which may be easily applied to various population models.

In Chapter 3 we introduce the concept of periodic semiflows and prove a theorem on the reduction of uniform persistence to that of the associated Poincaré map. The asymptotically periodic semiflows, nonautonomous semiflows, skew-product semiflows, and continuous processes are also discussed.

In Chapter 4, as a first application of the previous abstract results, we analyze in detail a discrete-time, size-structured chemostat model that is described by a system of difference equations, although in this book our main concern is with global dynamics in periodic and almost periodic systems. The reason for this choice is that we want to show how the theory of discrete dynamical systems can be applied to discrete-time models governed by difference equations (or maps).

In the rest of the book we apply the results of Chapters 1–3 to continuous-time periodic population models: In Chapter 5 to the  $N$ -species competition in a periodic chemostat; in Chapter 6 to almost periodic competitive systems;

in Chapter 7 to competitor–competitor–mutualist parabolic systems; and in Chapter 8 to a periodically pulsed bioreactor model. Of course, for each chapter we need to use different qualitative methods and even to develop certain ad hoc techniques.

Chapter 9 is devoted to the global dynamics in an autonomous, nonlocal, and delayed predator–prey model. Clearly, the continuous-time analogues of the results in Chapters 1 and 2 can find applications in autonomous models. Note that an autonomous semiflow can be viewed as a periodic one with the period being any fixed positive real number, and hence it is possible to get some global results by using the theory of periodic semiflows. However, we should point out that there do exist some special theory and methods that are applicable only to autonomous systems. The fluctuation method in this chapter provides such an example.

The existence, attractivity, uniqueness, and exponential stability of periodic traveling waves in periodic reaction–diffusion equations with bistable nonlinearities are discussed in Chapter 10, which is essentially independent of the previous chapters. We appeal only to a convergence theorem from Chapter 2 to prove the attractivity and uniqueness of periodic waves. Here the Poincaré-type map associated with the system plays an important role once again.

Over the years I have benefited greatly from the communications, discussions, and collaborations with many colleagues and friends in the fields of differential equations, dynamical systems, and mathematical biology, and I would like to take this opportunity to express my gratitude to all of them. I am particularly indebted to Herb Freedman, Morris Hirsch, Hal Smith, Horst Thieme, Gail Wolkowicz and Jianhong Wu, with whom I wrote research articles that are incorporated in the present book.

Finally, I gratefully appreciate financial support for my research from the National Science Foundation of China, the Royal Society of London, and the Natural Sciences and Engineering Research Council of Canada.



<http://www.springer.com/978-0-387-00308-5>

Dynamical Systems in Population Biology

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2003, XIII, 276 p., Hardcover

ISBN: 978-0-387-00308-5