2
Combinatorial Tools

2.1 Induction

It is time to learn one of the most important tools in discrete mathematics. We start with a question:

We add up the first n odd numbers. What do we get?

Perhaps the best way to try to find the answer is to experiment. If we try small values of n, this is what we find:

$$\begin{aligned}
1 &= 1\\
1+3 &= 4\\
1+3+5 &= 9\\
1+3+5+7 &= 16\\
1+3+5+7+9 &= 25\\
1+3+5+7+9+11 &= 36\\
1+3+5+7+9+11+13 &= 49\\
1+3+5+7+9+11+13+15 &= 64\\
1+3+5+7+9+11+13+15+17 &= 81\\
1+3+5+7+9+11+13+15+17+19 &= 100
\end{aligned}$$

It is easy to observe that we get squares; in fact, it seems from these examples that *the sum of the first n odd numbers is n^2*. We have observed this for the first 10 values of n; can we be sure that it is valid for all? Well, I'd say we can be reasonably sure, but not with mathematical certainty. How can we *prove* the assertion?

Consider the sum for a general n. The nth odd number is $2n-1$ (check!), so we want to prove that

$$1 + 3 + \cdots + (2n-3) + (2n-1) = n^2. \tag{2.1}$$

If we separate the last term in this sum, we are left with the sum of the first $(n-1)$ odd numbers:

$$1 + 3 + \cdots + (2n-3) + (2n-1) = \Big(1 + 3 + \cdots + (2n-3)\Big) + (2n-1).$$

Now, here the sum in the large parenthesis is $(n-1)^2$, since it is the sum of the first $n-1$ odd numbers. So the total is

$$(n-1)^2 + (2n-1) = (n^2 - 2n + 1) + (2n-1) = n^2, \tag{2.2}$$

just as we wanted to prove.

Wait a minute! Aren't we using in the proof the statement that we are trying to prove? Surely this is unfair! One could prove everything if this were allowed!

In fact, we are not quite using the assertion we are trying to prove. What we were using, was the assertion about the sum of the first $n-1$ odd numbers; and we argued (in (2.2)) that this proves the assertion about the sum of the first n odd numbers. In other words, what we have actually shown is that if the assertion is true for a certain value $(n-1)$, then it is also true for the next value (n).

This is enough to conclude that the assertion is true for every n. We have seen that it is true for $n = 1$; hence by the above, it is also true for $n = 2$ (we have seen this anyway by direct computation, but this shows that this was not even necessary: It follows from the case $n = 1$). In a similar way, the truth of the assertion for $n = 2$ implies that it is also true for $n = 3$, which in turn implies that it is true for $n = 4$, etc. If we repeat this sufficiently many times, we get the truth for any value of n. So the assertion is true for *every* value of n.

This proof technique is called *induction* (or sometimes *mathematical induction*, to distinguish it from a notion in philosophy). It can be summarized as follows.

Suppose that we want to prove a property of positive integers. Also suppose that we can prove two facts:

(a) 1 has the property, and

(b) whenever $n-1$ has the property, then n also has the property ($n > 1$).

The *Principle of Induction* says that if (a) and (b) are true, then every natural number has the property.

This is precisely what we did above. We showed that the "sum" of the first 1 odd numbers is 1^2, and then we showed that *if* the sum of the first $n-1$ odd numbers is $(n-1)^2$, *then* the sum of the first n odd numbers is n^2, for whichever integer $n > 1$ we consider. Therefore, by the Principle of Induction we can conclude that for every positive integer n, the sum of the first n odd numbers is n^2.

Often, the best way to try to carry out an induction proof is the following. First we prove the statement for $n = 1$. (This is sometimes called the *base case*.) We then try to prove the statement for a general value of n, and we are allowed to assume that the statement is true if n is replaced by $n-1$. (This is called the *induction hypothesis*.) If it helps, one may also use the validity of the statement for $n-2$, $n-3$, etc., and in general, for every k such that $k < n$.

Sometimes we say that if 1 has the property, and every integer n *inherits* the property from $n-1$, then every positive integer has the property. (Just as if the founding father of a family has a certain piece of property, and every new generation inherits this property from the previous generation, then the family will always have this property.)

Sometimes we start not with $n = 1$ but with $n = 0$ (if this makes sense) or perhaps with a larger value of n (if, say, $n = 1$ makes no sense for some reason, or the statement is not for $n = 1$). For example, we want to prove that $n!$ *is an even number if* ≥ 1. We check that this is true for $n = 2$ (indeed, $2! = 2$ is even), and also that it is inherited from $n-1$ to n (indeed, if $(n-1)!$ is even, then so is $n! = n \cdot (n-1)!$, since every multiple of an even number is even). This proves that $n!$ is even for all values of n *from the base case* $n = 2$ *on*. The assertion is false for $n = 1$, of course.

2.1.1 Prove, using induction but also without it, that $n(n+1)$ is an even number for every nonnegative integer n.

2.1.2 Prove by induction that the sum of the first n positive integers is $n(n+1)/2$.

2.1.3 Observe that the number $n(n+1)/2$ is the number of handshakes among $n+1$ people. Suppose that everyone counts only handshakes with people older than him/her (pretty snobbish, isn't it?). Who will count the largest number of handshakes? How many people count 6 handshakes? (We assume that no two people have exactly the same age.)

Give a proof of the result of Exercise 2.1.2, based on your answer to these questions.

2.1.4 Give a proof of Exercise 2.1.2, based on Figure 2.1.

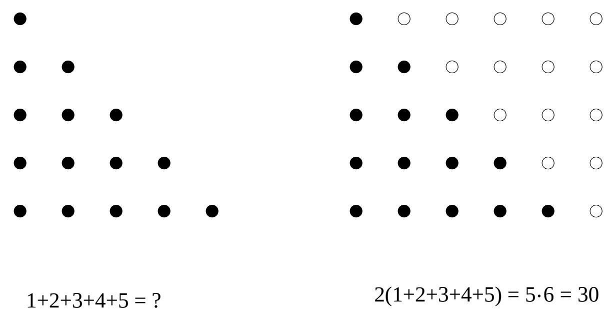

FIGURE 2.1. The sum of the first n integers

2.1.5 Prove the following identity:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \frac{(n-1) \cdot n \cdot (n+1)}{3}.$$

Exercise 2.1.2 relates to a well-known anecdote from the history of mathematics. Carl Friedrich Gauss (1777–1855), one of the greatest mathematicians of all times, was in elementary school when his teacher gave the class the task to add up the integers from 1 to 1000. He was hoping that he would get an hour or so to relax while his students were working. (The story is apocryphal, and it appears with various numbers to add: from 1 to 100, or 1900 to 2000.) To the teacher's great surprise, Gauss came up with the correct answer almost immediately. His solution was extremely simple: Combining the first term with the last, you get $1 + 1000 = 1001$; combining the second term with the last but one, you get $2 + 999 = 1001$; proceeding in a similar way, combining the first remaining term with the last one (and then discarding them) you get 1001. The last pair added this way is $500 + 501 = 1001$. So we obtained 500 times 1001, which makes 500500. We can check this answer against the formula given in exercise 2.1.2: $1000 \cdot 1001/2 = 500500$.

2.1.6 Use the method of the young Gauss to give a third proof of the formula in exercise 2.1.2

2.1.7 How would the young Gauss prove the formula (2.1) for the sum of the first n odd numbers?

2.1.8 Prove that the sum of the first n squares $\left(1 + 4 + 9 + \cdots + n^2\right)$ is $n(n + 1)(2n + 1)/6$.

2.1.9 Prove that the sum of the first n powers of 2 (starting with $1 = 2^0$) is $2^n - 1$.

In Chapter 1 we often relied on the convenience of saying "etc.": we described some argument that had to be repeated n times to give the result we wanted to get, but after giving the argument once or twice, we said "etc." instead of further repetition. There is nothing wrong with this, if the argument is sufficiently simple so that we can intuitively see where the repetition leads. But it would be nice to have some tool at hand that could be used instead of "etc." in cases where the outcome of the repetition is not so transparent.

The precise way of doing this is using induction, as we are going to illustrate by revisiting some of our results. First, let us give a proof of the formula for the number of subsets of an n-element set, given in Theorem 1.3.1 (recall that the answer is 2^n).

As the Principle of Induction tells us, we have to check that the assertion is true for $n = 0$. This is trivial, and we already did it. Next, we assume that $n > 0$, and that the assertion is true for sets with $n-1$ elements. Consider a set S with n elements, and fix any element $a \in S$. We want to count the subsets of S. Let us divide them into two classes: those containing a and those not containing a. We count them separately.

First, we deal with those subsets that don't contain a. If we delete a from S, we are left with a set S' with $n-1$ elements, and the subsets we are interested in are exactly the subsets of S'. By the induction hypothesis, the number of such subsets is 2^{n-1}.

Second, we consider subsets containing a. The key observation is that every such subset consists of a and a subset of S'. Conversely, if we take any subset of S', we can add a to it to get a subset of S containing a. Hence the number of subsets of S containing a is the same as the number of subsets of S', which is, by the induction hypothesis, 2^{n-1}. (We can exercise another bit of mathematical jargon introduced before: The last piece of the argument establishes a one-to-one correspondence between those subsets of S containing a and those not containing a.)

To conclude: The total number of subsets of S is $2^{n-1}+2^{n-1} = 2\cdot 2^{n-1} = 2^n$. This proves Theorem 1.3.1 (again).

2.1.10 Use induction to prove Theorem 1.5.1 (the number of strings of length n composed of k given elements is k^n) and Theorem 1.6.1 (the number of permutations of a set with n elements is $n!$).

2.1.11 Use induction on n to prove the "handshake theorem" (the number of handshakes between n people is $n(n-1)/2$).

2.1.12 Read carefully the following induction proof:

> ASSERTION: $n(n+1)$ *is an odd number for every* n.
>
> PROOF: Suppose that this is true for $n-1$ in place of n; we prove it for n, using the induction hypothesis. We have
>
> $$n(n+1) = (n-1)n + 2n.$$

> Now here $(n-1)n$ is odd by the induction hypothesis, and $2n$ is even. Hence $n(n+1)$ is the sum of an odd number and an even number, which is odd.

The assertion that we proved is obviously wrong for $n = 10$: $10 \cdot 11 = 110$ is even. What is wrong with the proof?

2.1.13 Read carefully the following induction proof:

> Assertion: *If we have n lines in the plane, no two of which are parallel, then they all go through one point.*
>
> Proof: The assertion is true for one line (and also for 2, since we have assumed that no two lines are parallel). Suppose that it is true for any set of $n-1$ lines. We are going to prove that it is also true for n lines, using this induction hypothesis.
>
> So consider a set $S = \{a, b, c, d, \dots\}$ of n lines in the plane, no two of which are parallel. Delete the line c; then we are left with a set S' of $n-1$ lines, and obviously, no two of these are parallel. So we can apply the induction hypothesis and conclude that there is a point P such that all the lines in S' go through P. In particular, a and b go through P, and so P must be the point of intersection of a and b.
>
> Now put c back and delete d, to get a set S'' of $n-1$ lines. Just as above, we can use the induction hypothesis to conclude that these lines go through the same point P'; but just as above, P' must be the point of intersection of a and b. Thus $P' = P$. But then we see that c goes through P. The other lines also go through P (by the choice of P), and so all the n lines go through P.

But the assertion we proved is clearly wrong; where is the error?

2.2 Comparing and Estimating Numbers

It is nice to have formulas for certain numbers (for example, for the number $n!$ of permutations of n elements), but it is often more important to have a rough idea about how large these numbers are. For example, how many digits does 100! have?

Let us start with simpler questions. Which is larger, n or $\binom{n}{2}$? For $n = 2, 3, 4$ the value of $\binom{n}{2}$ is $1, 3, 6$, so it is less than n for $n = 2$, equal for $n = 3$, but larger for $n = 4$. In fact, $n = \binom{n}{1} < \binom{n}{2}$ if $n \geq 4$.

More can be said: The quotient

$$\frac{\binom{n}{2}}{n} = \frac{n-1}{2}$$

becomes arbitrarily large as n becomes large; for example, if we want this quotient to be larger than 1000, it suffices to choose $n > 2001$. In the language of calculus, we have

$$\frac{\binom{n}{2}}{n} \to \infty \qquad (n \to \infty)\,.$$

Here is another simple question: Which is larger, n^2 or 2^n? For small values of n, this can go either way: $1^2 < 2^1$, $2^2 = 2^2$, $3^2 > 2^3$, $4^2 = 2^4$, $5^2 < 2^5$. But from here on, 2^n takes off and grows much faster than n^2. For example, $2^{10} = 1024$ is much larger than $10^2 = 100$. In fact, $2^n/n^2$ becomes arbitrarily large, as n becomes large.

2.2.1 (a) Prove that $2^n > \binom{n}{3}$ if $n \ge 3$.
(b) Use (a) to prove that $2^n/n^2$ becomes arbitrarily large as n becomes large.

Now we tackle the problem of estimating 100! or, more generally, $n! = 1 \cdot 2 \cdots n$. The first factor 1 does not matter, but all the others are at least 2, so $n! \ge 2^{n-1}$. Similarly, $n! \le n^{n-1}$, since (ignoring the factor 1 again) $n!$ is the product of $n-1$ factors, each of which is at most n. (Since all but one of them are smaller than n, the product is in fact much smaller.) Thus we know that

$$2^{n-1} \le n! \le n^{n-1}. \tag{2.3}$$

These bounds are very far apart; for $n = 10$, the lower bound is $2^9 = 512$, while the upper bound is 10^9 (one billion).

Here is a question that is not answered by the simple bounds in (2.3). Which is larger, $n!$ or 2^n? In other words, does a set with n elements have more permutations or more subsets? For small values of n, subsets are winning: $2^1 = 2 > 1! = 1$, $2^2 = 4 > 2! = 2$, $2^3 = 8 > 3! = 6$. But then the picture changes: $2^4 = 16 < 4! = 24$, $2^5 = 32 < 5! = 120$. It is easy to see that as n increases, $n!$ grows much faster than 2^n: If we go from n to $n+1$, then 2^n grows by a factor of 2, while $n!$ grows by a factor of $n+1$.

2.2.2 Use induction to make the previous argument precise, and prove that $n! > 2^n$ if $n \ge 4$.

There is a formula that gives a very good approximation of $n!$. We state it without proof, since the proof (although not terribly difficult) needs calculus.

Theorem 2.2.1 [Stirling's formula]

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Here $\pi = 3.14\ldots$ is the area of the circle with unit radius, $e = 2.718\ldots$ is the base of the natural logarithm, and $\sim$ means approximate equality in the precise sense that

$$\frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \to 1 \qquad (n \to \infty).$$

Both of the funny irrational numbers e and π occur in the same formula!

Let us return to the question: How many digits does 100! have? We know by Stirling's formula that

$$100! \approx (100/e)^{100} \cdot \sqrt{200\pi}.$$

The number of digits of this number is its logarithm, in base 10, rounded up. Thus we get

$$\lg(100!) \approx 100 \lg(100/e) + 1 + \lg \sqrt{2\pi} = 157.969\ldots.$$

So the number of digits in 100! is about 158 (this is, in fact, the right value).

2.3 Inclusion-Exclusion

In a class of 40, many students are collecting the pictures of their favorite rock stars. Eighteen students have a picture of the Beatles, 16 students have a picture of the Rolling Stones and 12 students have a picture of Elvis Presley (this happened a long time ago, when we were young). There are 7 students who have pictures of both the Beatles and the Rolling Stones, 5 students who have pictures of both the Beatles and Elvis Presley, and 3 students who have pictures of both the Rolling Stones and Elvis Presley. Finally, there are 2 students who possess pictures of all three groups. Question: How many students in the class have no picture of any of the rock groups?

First, we may try to argue like this: There are 40 students altogether in the class; take away from this the number of those having Beatles pictures (18), those having Rolling Stones picture (16), and those having Elvis pictures (12); so we take away $18 + 16 + 12$. We get -6; this negative number warns us that there must be some error in our calculation; but what was not correct? We made a mistake when we subtracted the number of those students who collected the pictures of two groups twice! For example, a student having the Beatles and Elvis Presley was subtracted with the Beatles collectors as well as with the Elvis Presley collectors. To correct our calculations, we have to add back the number of those students who have pictures of two groups. This way we get $40 - (18 + 16 + 12) + (7 + 5 + 3)$. But we must be careful; we shouldn't make the same mistake again! What happened to the 2 students who have the pictures of all three groups? We subtracted these 3 times at the beginning, and then we added them back 3 times, so we must subtract them once more! With this correction, our final result is:

$$40 - (18 + 16 + 12) + (7 + 5 + 3) - 2 = 7. \tag{2.4}$$

We can not find any error in this formula, looking at it from any direction. But learning from our previous experience, we must be much more careful: We have to give an exact proof!

So suppose that somebody records picture collecting data of the class in a table like Table 2.1 below. Each row corresponds to a student; we did not put down all the 40 rows, just a few typical ones.

Name	Bonus	Beatles	Stones	Elvis	BS	BE	SE	BSE
Al	1	0	0	0	0	0	0	0
Bel	1	-1	0	0	0	0	0	0
Cy	1	-1	-1	0	1	0	0	0
Di	1	-1	0	-1	0	1	0	0
Ed	1	-1	-1	-1	1	1	1	-1
$\vdots$								

TABLE 2.1. Strange record of who's collecting whose pictures.

The table is a bit silly (but with reason). First, we give a bonus of 1 to every student. Second, we record in a separate column whether the student is collecting (say) both the Beatles and Elvis Presley (the column labeled BE), even though this could be read off from the previous columns. Third, we put a -1 in columns recording the collecting of an odd number of pictures, and a 1 in columns recording the collecting of an even number of pictures.

We compute the total sum of entries in this table in two different ways. First, what are the row sums? We get 1 for Al and 0 for everybody else. This is not a coincidence. If we consider a student like Al, who does not have any picture, then this student contributes to the bonus column, but nowhere else, which means that the sum in the row of this student is 1. Next, consider Ed, who has all 3 pictures. He has a 1 in the bonus column; in the next 3 columns he has 3 terms that are -1. In each of the next 3 columns he has a 1, one for each pair of pictures; it is better to think of this 3 as $\binom{3}{2}$. His row ends with $\binom{3}{3}$ -1's ($\binom{3}{3}$ equals 1, but in writing it this way the general idea can be seen better). So the sum of the row is

$$1-\binom{3}{1}+\binom{3}{2}-\binom{3}{3}=0.$$

Looking at the rows of Bel, Cy, and Di, we see that their sums are

$$1-\binom{1}{1}=0 \qquad \text{for Bel (1 picture),}$$

$$1-\binom{2}{1}+\binom{2}{2}=0 \qquad \text{for Cy and Di (2 pictures).}$$

If we move the negative terms to the other side of these equations, we get an equation with a combinatorial meaning: *The number of subsets of an* *n-set with an even number of elements is the same as the number of subsets with an odd number of elements.* For example,

$$\binom{3}{0} + \binom{3}{2} = \binom{3}{1} + \binom{3}{3}.$$

Recall that Exercise 1.3.3 asserts that this is indeed so for every $n \geq 1$.

Since the row sum is 0 for all those students who have any picture of any music group, and it is 1 for those having no picture at all, the sum of all 40 row sums gives exactly the number of those students having no picture at all.

On the other hand, what are the column sums? In the "bonus" column, we have 40 times $+1$; in the "Beatles" column, we have 18 times -1; then we have 16 and 12 times -1. Furthermore, we get 7 times $+1$ in the BS column, then 5- and 3 times $+1$ in the BE and SE columns, respectively. Finally, we get 2 -1's in the last column. So this is indeed the expression in (2.4).

This formula is called the *Inclusion–Exclusion Formula* or *Sieve Formula*. The origin of the first name is obvious; the second refers to the image that we start with a large set of objects and then "sieve out" those objects we don't want to count.

We could extend this method if students were collecting pictures of 4, or 5, or any number of rock groups instead of 3. Rather than stating a general theorem (which would be lengthy), we give a number of exercises and examples.

2.3.1 In a class of all boys, 18 boys like to play chess, 23 like to play soccer, 21 like biking and 17 like hiking. The number of those who like to play both chess and soccer is 9. We also know that 7 boys like chess and biking, 6 boys like chess and hiking, 12 like soccer and biking, 9 boys like soccer and hiking, and finally 12 boys like biking and hiking. There are 4 boys who like chess, soccer, and biking, 3 who like chess, soccer, and hiking, 5 who like chess, biking, and hiking, and 7 who like soccer, biking, and hiking. Finally there are 3 boys who like all four activities. In addition we know that everybody likes at least one of these activities. How many boys are there in the class?

2.4 Pigeonholes

Can we find in New York two persons having the same number of strands of hair? One would think that it is impossible to answer this question, since one does not even know how many strands of hair there are on one's own head, let alone about the number of strands of hair on every person living in New York (whose exact number is in itself quite difficult to determine). But there are some facts that we know for sure: Nobody has more than 500,000 strands of hair (a scientific observation), and there are more than 10 million inhabitants of New York. Can we now answer our original question? Yes. If there were no two people with the same number of strands of hair, then there would be at most one person having 0 strands, at most one person having exactly 1 strand, and so on. Finally, there would be at most one person having exactly 500,000 strands. But then this means that there are no more than 500,001 inhabitants of New York. Since this contradicts what we know about New York, it follows that there must be two people having the same number of strands of hair.[1]

We can formulate our solution as follows. Imagine 500,001 enormous boxes (or pigeon holes). The first one is labeled "New Yorkers having 0 strands of hair," the next is labeled "New Yorkers having 1 strand of hair," and so on. The last box is labeled "New Yorkers having 500,000 strands of hair". Now if everybody goes to the proper box, then about 10 million New Yorkers are properly assigned to some box (or hole). Since we have only 500,001 boxes, there certainly will be a box containing more than one New Yorker. This statement is obvious, but it is very often a powerful tool, so we formulate it in full generality:

> *If we have n boxes and we place more than n objects into them, then there will be at least one box that contains more than one object.*

Very often, the above statement is formulated using pigeons and their holes, and is referred to as the *Pigeonhole Principle*. The Pigeonhole Principle is simple indeed: Everybody understands it immediately. Nevertheless, it deserves a name, since we use it very often as the basic tool of many proofs. We will see many examples for the use of the Pigeonhole Principle, but to show you its power, we discuss one of them right away. This is not a theorem of any significance; rather, and exercise whose solution is given in detail.

Exercise. *We shoot 50 shots at a square target, the side of which is 70 cm long. We are quite a good shot, because all of our shots hit the target. Prove that there are two bulletholes that are closer than 15 cm.*

Solution: Imagine that our target is an old chessboard. One row and one column of it has fallen off, so it has 49 squares. The board received 50 shots, so there must be a square that received at least two shots (putting bulletholes in pigeonholes). We claim that these two shots are closer to each other than 15 cm.

[1]There is an interesting feature of this argument: we end up knowing that two such persons exist, without having the slightest hint about how to find these people. (Even if we suspect that two people have the same number of strands of hair, it is essentially impossible to verify that this is indeed so!) Such proofs in mathematics are called *pure existence proofs.*

The side of the square is obviously 10 cm, the diagonals of it are equal, and (from the Pythagorean Theorem) their length is $\sqrt{200} \approx 14.1$ cm. We show that

(∗) *the two shots cannot be at a larger distance than the diagonal.*

It is intuitively clear that two points in the square at largest distance are the endpoints of one of the diagonals, but intuition can be misleading; let us prove this. Suppose that two points P and Q are farther away then the length of the diagonal. Let A, B, C, and D be the vertices of the square. Connect P and Q by a line, and let P' and Q' be the two points where this line intersects the boundary of the square (Figure 2.2). Then the distance of P' and Q' is even larger, so it is also larger than the diagonal.

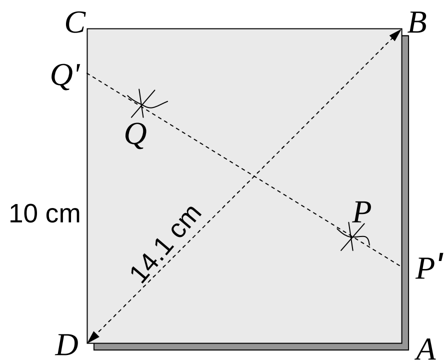

FIGURE 2.2. Two shots in the same square

We may assume without loss of generality that P' lies on the side AB (if this is not the case, we change the names of the vertices). One of the angles $Q'P'A$ and $Q'P'B$ is at least 90°; we may assume (again without loss of generality) that $Q'P'A$ is this angle. Then the segment AQ' is the edge of the triangle $Q'P'A$ opposite the largest angle, and so it is even longer than $P'Q'$, and so it is longer than the diagonal.

We repeat this argument to show that if we replace Q' by one of the endpoints of the side it lies on, we get a segment that is longer than the diagonal. But now we have a segment both of whose endpoints are vertices of the square. So this segment is either a side or a diagonal of the square, and in neither case is it longer than the diagonal! This contradiction shows that the assertion (∗) above must be true.

So we got not only that there will be two shots that are closer than 15 cm, but even closer than 14.2 cm. This concludes the solution of the exercise.

If this is the first time you have seen this type of proof, you may be surprised: we did not argue directly to prove what we wanted, but instead assumed that the assertion was not true, and then using this additional assumptions, we argued until we got a contradiction. This form of proof is called *indirect*, and it is quite often used in mathematical reasoning, as we will see throughout this book. (Mathematicians are strange creatures, one may observe: They go into long arguments based on assumptions they know are false, and their happiest moments are when they find a contradiction between statements they have proved.)

2.4.1 Prove that we can select 20 New Yorkers who all have the same number of strands of hair.

2.5 The Twin Paradox and the Good Old Logarithm

Having taught the Pigeonhole Principle to his class, the professor decides to play a little game: "I bet that there are two of you who have the same birthday! What do you think?" Several students reply immediately: "There are 366 possible birthdays, so you could only conclude this if there were at least 367 of us in the class! But there are only 50 of us, and so you'd lose the bet." Nevertheless, the professor insists on betting, and he wins.

How can we explain this? The first thing to realize is that the Pigeonhole Principle tells us that with 367 students in the class, the professor *always* wins the bet. But this is uninteresting as bets go; it is enough for him that he has a good chance of winning. With 366 students, he may already lose; could it be that with only 50 students he still has a good chance of winning?

The surprising answer is that even with as few as 23 students, his chance of winning is slightly larger than 50%. We can view this fact as a "Probabilistic Pigeonhole Principle", but the usual name for it is the *Twin Paradox*.

Let us try to determine the professor's chances. Suppose that on the class list, he writes down everybody's birthday. So he has a list of 50 birthdays. We know from Section 1.5 that there are 366^{50} different lists of this type.

For how many of these does he lose? Again, we already know the answer from Section 1.7: $366 \cdot 365 \cdots 317$. So the probability that he loses the bet is[2]

$$\frac{366 \cdot 365 \cdots 317}{366^{50}}.$$

With some effort, we could calculate this value "by brute force", using a computer (or just a programmable calculator), but it will be much more

[2]Here we made the implicit assumption that all the 366^{50} birthday lists are equally likely. This is certainly not true; for example, lists containing February 29 are clearly much less likely. There are also (much smaller) variations between the other days of the year. It can be shown, however, that these variations only help the professor, making collisions of birthdays more likely.

useful to get upper and lower bounds by a method that will work in a more general case, when we have n possible birthdays and k students. In other words, how large is the quotient

$$\frac{n(n-1)\cdots(n-k+1)}{n^k}\ ?$$

It will be more convenient to take the reciprocal (which is then larger than 1):

$$\frac{n^k}{n(n-1)\cdots(n-k+1)}. \tag{2.5}$$

We can simplify this fraction by cancelling an n, but then there is no obvious way to continue. A little clue may be that the number of factors is the same in the numerator and denominator, so let us try to write this fraction as a product:

$$\frac{n^k}{n(n-1)\dots(n-k+1)} = \frac{n}{n-1}\cdot\frac{n}{n-2}\cdots\frac{n}{n-k+1}.$$

These factors are quite simple, but it is still difficult to see how large their product is. The individual factors are larger than 1, but (at least at the beginning) quite close to 1. But there are many of them, and their product may be large.

The following idea helps: *Take the logarithm!*[3] We get

$$\ln\left(\frac{n^k}{n(n-1)\cdots(n-k+1)}\right) = \ln\left(\frac{n}{n-1}\right) + \ln\left(\frac{n}{n-2}\right) + \cdots + \ln\left(\frac{n}{n-k+1}\right). \tag{2.6}$$

(Naturally, we took the natural logarithm, base $e = 2.71828\dots$.) This way we can deal with addition instead of multiplication, which is nice; but the terms we have to add up became much uglier! What do we know about these logarithms?

Let's look at the graph of the logarithm function (Figure 2.3). We have also drawn the line $y = x - 1$. We see that the function is below the line, and touches it at the point $x = 1$ (these facts can be proved by really elementary calculus). So we have

$$\ln x \le x - 1. \tag{2.7}$$

Can we say something about how good this upper bound is? From the figure we see that at least for values of x close to 1, the two graphs are

[3]After all, the logarithm was invented in the seventeenth century by Buergi and Napier to make multiplication easier, by turning it into addition.

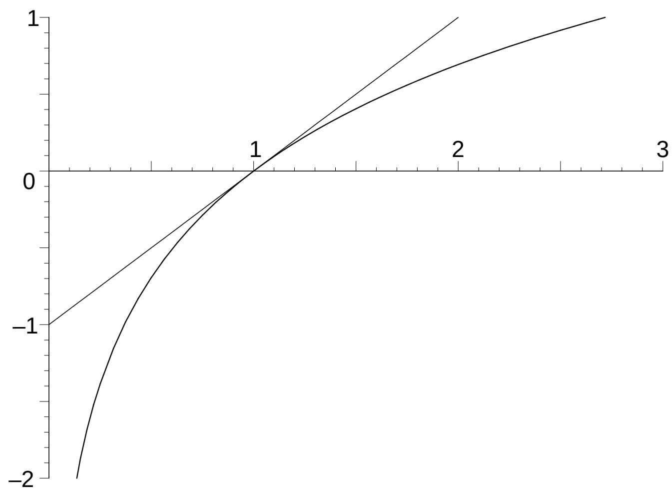

FIGURE 2.3. The graph of the natural logarithm function. Note that near 1, it is very close to the line $x - 1$.

quite close. Indeed, we can do the following little computation:

$$\ln x = -\ln \frac{1}{x} \geq -\left(\frac{1}{x} - 1\right) = \frac{x-1}{x}. \tag{2.8}$$

If x is a little larger than 1 (as are the values we have in (2.6)), then $\frac{x-1}{x}$ is only a little smaller than $x - 1$, and so the upper bound in (2.7) and lower bound in (2.8) are quite close.

These bounds on the logarithm function are very useful in many applications in which we have to do approximate computations with logarithms, and it is worthwhile to state them in a separate lemma. (A lemma is a precise mathematical statement, just like a theorem, except that it is not the goal itself, but some auxiliary result used along the way to the proof of a theorem. Of course, some lemmas are more interesting than some theorems!)

Lemma 2.5.1 *For every $x > 0$,*

$$\frac{x-1}{x} \leq \ln x \leq x - 1.$$

First we use the lower bound in this lemma to estimate (2.6) from below. For a typical term in the sum in (2.6) we get

$$\ln\left(\frac{n}{n-j}\right) \geq \frac{\frac{n}{n-j} - 1}{\frac{n}{n-j}} = \frac{j}{n},$$

and hence

$$\ln\left(\frac{n^k}{n(n-1)\cdots(n-k+1)}\right) \geq \frac{1}{n}+\frac{2}{n}+\cdots+\frac{k-1}{n}$$

$$= \frac{1}{n}(1+2+\cdots+(k-1)) = \frac{k(k-1)}{2n}$$

(remember the young Gauss's problem!). Thus we have a simple lower bound on (2.6). To get an upper bound, we can use the other inequality in Lemma 2.5.1; for a typical term, we get

$$\ln\left(\frac{n}{n-j}\right) \leq \frac{n}{n-j} - 1 = \frac{j}{n-j}.$$

We have to sum these for $j = 1, \dots, k-1$ to get an upper bound on (2.6). This is not as easy as in young Gauss's case, since the denominator is changing. But we only want an upper bound, so we could replace the denominator by the smallest value it can have for various values of j, namely $n-k+1$. We have $j/(n-j) \leq j/(n-k+1)$, and hence

$$\begin{aligned}\ln\left(\frac{n^k}{n(n-1)\cdots(n-k+1)}\right) &\leq \frac{1}{n-k+1}+\frac{2}{n-k+1}+\cdots+\frac{k-1}{n-k+1}\\ &= \frac{1}{n-k+1}(1+2+\cdots+(k-1))\\ &= \frac{k(k-1)}{2(n-k+1)}.\end{aligned}$$

Thus we have these similar upper and lower bounds on the logarithm of the ratio (2.5), and applying the exponential function to both sides, we get the following:

$$e^{\frac{k(k-1)}{2n}} \leq \frac{n^k}{n(n-1)\cdots(n-k+1)} \leq e^{\frac{k(k-1)}{2(n-k+1)}}. \tag{2.9}$$

Does this help to understand the professor's trick in the classroom? Let's apply (2.9) with $n = 366$ and $k = 50$; using our calculators, we get that

$$28.4 \leq \frac{366^{50}}{366\cdot 364\cdots 317} \leq 47.7.$$

(Using more computation, we can determine that the exact value is $33.414\ldots$.) So the probability that all students in the class have different birthdays (which is the reciprocal of this number) is less than 1/28. This means that if the professor performs this trick every year, he will likely fail only once or twice in his career!

Review Exercises

2.5.1 What is the following sum?

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1) \cdot n}.$$

Experiment, conjecture the value, and then prove it by induction.

2.5.2 What is the following sum?

$$0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + (n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n}.$$

Experiment, conjecture the value, and then prove it. (Try to prove the result by induction and also by combinatorial arguments.)

2.5.3 Prove the following identities:

$$1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots + n \cdot 2^{n-1} = (n-1)2^n + 1.$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

$$1 + 3 + 9 + 27 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}.$$

2.5.4 Prove by induction on n that

(a) $n^2 - 1$ is a multiple of 4 if n is odd,

(b) $n^3 - n$ is a multiple of 6 for every n.

2.5.5 There is a class of 40 girls. There are 18 girls who like to play chess, and 23 who like to play soccer. Several of them like biking. The number of those who like to play both chess and soccer is 9. There are 7 girls who like chess and biking, and 12 who like soccer and biking. There are 4 girls who like all three activities. In addition we know that everybody likes at least one of these activities. How many girls like biking?

2.5.6 There is a class of all boys. We know that there are a boys who like to play chess, b who like to play soccer, c who like biking and d who like hiking. The number of those who like to play both chess and soccer is x. There are y boys who like chess and biking, z boys who like chess and hiking, u who like soccer and biking, v boys who like soccer and hiking, and finally w boys who like biking and hiking. We don't know how many boys like, e.g.,chess, soccer and hiking, but we know that everybody likes at least one of these activities. We would like to know how many boys are in the class.

(a) Show by an example that this is not determined by what we know.

(b) Prove that we can at least conclude that the number of boys in the class is at most $a + b + c + d$, and at least $a + b + c + d - x - y - z - u - v - w$.

2.5.7 We select 38 even positive integers, all less than 1000. Prove that there will be two of them whose difference is at most 26.

2.5.8 A drawer contains 6 pairs of black, 5 pairs of white, 5 pairs of red, and 4 pairs of green socks.

(a) How many single socks do we have to take out to make sure that we take out two socks with the same color?

(b) How many single socks do we have to take out to make sure that we take out two socks with different colors?

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