Preface

The subject of local dynamical systems is concerned with the following two questions:

1. Given an $n \times n$ matrix A, describe the behavior, in a neighborhood of the origin, of the solutions of all systems of differential equations having a rest point at the origin with linear part Ax, that is, all systems of the form

$$\dot{x} = Ax + \cdots ,$$

   where $x \in \mathbb{R}^n$ and the dots denote terms of quadratic and higher order.

2. Describe the behavior (near the origin) of all systems close to a system of the type just described.

To answer these questions, the following steps are employed:

1. A normal form is obtained for the general system with linear part Ax. The normal form is intended to be the simplest form into which any system of the intended type can be transformed by changing the coordinates in a prescribed manner.

2. An unfolding of the normal form is obtained. This is intended to be the simplest form into which all systems close to the original system can be transformed. It will contain parameters, called *unfolding parameters*, that are not present in the normal form found in step 1.

3. The normal form, or its unfolding, is truncated at some degree k, and the behavior of the truncated system is studied.
4. Finally, it is shown (if possible) that the full (untruncated) system has the same behavior established for the truncated system (or, in certain cases, the value of k for which this is the case is computed before determining the behavior of the truncated system). This is called *establishing k-determinacy* or *k-sufficiency*.

As the title of this book suggests, our focus will be primarily on the first two of these steps; a secondary focus will be the determinacy question. That is, we will study in detail the procedures available for obtaining normal forms and unfoldings (in the sense described later in this preface). To a smaller (but still substantial) degree, we will study procedures for deciding how many terms of a system are sufficient to establish its behavior. When it comes to *actually establishing* the behavior of systems, considerations of time and space prevent us from going beyond a limited number of examples, which it is hoped will serve to illustrate the techniques presented. The justification for this is that there are a number of books available (which will be cited at the appropriate time) that treat more complicated examples, often with the aim of obtaining conclusions in the quickest way possible. These books tend to treat each problem in an ad hoc manner, without laying a careful foundation of general principles useful for all problems. It is that need that is addressed here. Most of these general principles exist either in the journal literature (mostly of a "pure" mathematical character) or in the "folklore" of the subject, but have never been written down in book form. Armed with the ideas presented in this book, one should be able to understand much more clearly the books and papers that present specific conclusions for complicated examples not presented here.

As in my earlier book on perturbation theory, my goal in this book is to say the things that I found I needed to understand, but that other books did not say. Unlike that earlier book, this one is not an "introduction" to its subject. The book is self-contained, and does not require any specific knowledge other than the basic theories of advanced calculus, differential equations, and linear algebra. But it is unlikely that a student will find this book to be understandable without some previous exposure to the subject of dynamical systems, at least enough to motivate the kinds of questions that are asked and the kinds of solutions that are sought.

An Outline of the Chapters

Chapter 1 examines two easy two-dimensional examples of normal forms, one semisimple and the other nilpotent. For each example, the normal form and the unfolding are computed, leading to the Hopf and Takens–Bogdanov bifurcations (the study of which is continued in Sections 6.5 and

6.6). The treatment of these examples in Chapter 1 begins with the elementary methods used in such texts as [52] and [111] and goes on to suggest the advantages of more advanced methods to be developed in Chapter 4.

Chapter 2 is an interlude devoted to certain topics in linear algebra. It will have become clear in Chapter 1 that the study of normal forms is concerned with the complement to the image of a certain linear operator, the so-called *homological operator*, which we denote by the British monetary symbol $\pounds$ (to be read as "pounds"). Chapter 2 collects four methods of finding a complement to the image of a linear operator. The development includes self-contained treatments of the Fredholm alternative (for finite-dimensional spaces), the Jordan canonical form, and the representation theory of the Lie algebra sl(2). (This representation theory is simplified considerably by adopting an unnecessary assumption, true in the general case but difficult to prove. It happens that this assumption is self-evident in our applications, so its adoption is harmless.) Readers who are already familiar with these topics may skip the chapter, but are advised that we occasionally introduce unfamiliar terminology (such as *entry point, triad, chain-weight basis*, and *pressure*) when no convenient term seems to exist. It will be useful to skim the relevant sections of Chapter 2 when these terms are encountered later in the book. Readers for whom parts of Chapter 2 are new may omit the starred sections (dealing with sl(2) representation theory) on a first reading; mostly, these are needed only for the starred sections in Chapters 3 and 4.

Chapter 3 is concerned with matrix perturbation theory, presented as a linear special case of normal form theory. This chapter should be taken seriously, even by readers mainly interested in the nonlinear case (Chapter 4). One reason is that every idea and notation developed in Chapter 4 is introduced first in Chapter 3. Many of these ideas, in particular those connected with Lie theory, are easier to understand in the linear case; the Lie groups and Lie algebras that arise in the linear case are finite-dimensional, and the matrix exponential suffices to explain them. (In Chapter 4, the one-parameter groups defined by matrix exponentials must be replaced by local flows.) A second reason for paying attention to linear normal forms is that they play an important role in the unfolding theory of Chapter 6. In addition to normal forms, Chapter 3 introduces the ideas of *hypernormal forms* (in Section 3.3) and *metanormal forms* (Section 3.7). Hypernormal forms exploit the arbitrariness remaining after ordinary normalization to achieve additional simplifications, ultimately leading to what are sometimes called *unique normal forms.* Metanormal forms introduce fractional powers of parameters (shearing or scaling, related to Newton diagrams) to obtain yet further simplifications.

Chapter 4 is the central chapter of the book, where the project outlined in Chapter 1 is carried out. This project is to explain the structure of normal forms using the language of a module of equivariants (of some one-parameter group) over a ring of invariants, and (as a secondary goal) to give algorithms suitable for use in symbolic computation systems. The treatment includes both semisimple and nonsemisimple cases, and includes both the inner product (or "Elphick–Iooss") and sl(2) (or "Cushman–Sanders") normal form styles, as well as my own "simplified" normal form style. The work of Richard Cushman and Jan Sanders is already at least 15 years old, but has not (until now) received a systematic treatment in monograph form. Section 4.7 contains very recent work extending the Cushman–Sanders theory to the inner product and simplified normal form styles, and providing an algorithmic approach to the determination of Stanley decompositions for normal form modules. Section 4.10 introduces the hypernormal form theory of Alberto Baider and others. (Metanormal forms are not considered in Chapter 4, but reappear in Section 6.6. See also the open problems list at the end of this preface.)

With regard to the algorithmic portions of Chapters 2, 3, and 4, it is unfortunate that no specific implementations of these algorithms are available. They have been successfully implemented in Maple by Jan Sanders (who originated most of them), but he informs me that his programs no longer work, because of changes in the Maple software since they were written.

Until this point in the book, no attention has been given to the possible application of normal forms (except briefly in Chapter 1). Chapters 5 and 6 are an attempt to remedy this omission, as far as time and space allow. Although Chapters 2, 3, and 4 are intended to be exhaustive treatments of their subjects, Chapters 5 and 6 (as already mentioned at the beginning of this preface) are not. Instead, I have presented selected topics in which I have a personal interest and feel that I have something to say. Some of these topics end with open problems and suggestions for further research.

Chapter 5 concerns the geometrical structures (invariant manifolds and preserved fibrations and foliations) that exist in truncated systems in normal form, and the estimates that relate these structures, and the solutions lying in them, to the corresponding structures and solutions in the full (not truncated) system (which is in normal form only up to the truncation point).

Chapter 6 is a collection of results related to bifurcation theory. Sections 6.1 and 6.2 contain two developments of the basic theory of bifurcation from a single zero eigenvalue, one using Newton diagrams and fractional power series, and the other using singularity theory; the developments are parallel, so that one can see the relative strengths and weaknesses of the two methods. In both sections, the theme is sufficiency of jets. Sections 6.3 and 6.4 contain a treatment of my theory of "asymptotic unfoldings," somewhat improved from the original paper (because of results from Sections 3.4 and 4.6). Sections 6.5 through 6.7 compute the unfoldings of certain specific bifurcation problems by the methods of Section 6.4, and either analyze the bifurcation in detail (Section 6.5) or give annotated references to the literature for this analysis (Sections 6.6 and 6.7).

Appendices A and B contain background material on rings and modules useful in Chapters 4 and 6. Topics include smooth germs, flat functions, formal power series, relations and syzygies, an introduction to Gröbner bases, and the formal and Malgrange preparation theorems. Appendices C and D contain additional sections that originally belonged to Chapters 3 and 4, but were moved to the appendices because of their optional character. (References to these appendices are given at the appropriate points in Chapters 3 and 4.)

Formats, Styles, Description, and Computation

Two distinctions are maintained throughout this book, one between the "description problem" and the "computation problem" for normal forms, and the other between "formats" and "styles."

1. If a normal form is thought of as the "simplest" form into which a given system can be placed, there might be disagreement as to what form is considered simplest. A systematic policy for deciding what counts as simplest is called a *normal form style*. The important normal form styles are the semisimple, inner product, simplified, and sl(2), or triad, styles.

2. In order to put a system into normal form, one or more near-identity transformations are employed. A *format* is a scheme for handling these transformations, and includes the decisions as to whether a single transformation (with coefficients determined recursively) or a sequence of transformations (applied iteratively) is used, and whether the transformations are handled directly or via generators (in the sense of Lie theory).

3. The *description problem* for a given normal form style is to describe what that normal form style "looks like" in a given situation (usually given the leading term or linear part, depending on the context). For linear problems (Chapter 3), what a style "looks like" is usually answered by giving a diagonal structure, or block structure, or stripe structure. For nonlinear problems (Chapter 4), it is best answered by describing the vector fields in normal form as a module over a ring of scalar functions, using such algebraic devices as Stanley decompositions and Gröbner bases.

4. The *computation problem* is the problem of placing a specific system (having either numerical or symbolic coefficients) into normal form of a particular style, using a particular format. This entails determining the coefficients of the normalized system, either numerically or as functions of the symbolic coefficients in the original system.

It is very helpful to keep styles separate from formats and the description problem separate from the computation problem. Normal forms of any style can be worked out using any format. (There is one exception to this: A generated, or Lie-theoretic, format must be used if the original system and its normalization are both required to belong to a specific Lie subalgebra.) It follows that one can think about styles without worrying which format is to be used, and about formats without regard to the style. The description and computation problems are not quite as distinct as this, but are more so than might be expected. They share a good deal of machinery, such as the homological equation. But it is possible to write a computer program (for a symbolic processing system) that puts a system into normal form (that is, solves the computation problem) without having a theoretical description in advance of what the normal form will look like. It is equally possible to describe the normal form without giving algorithms that generate the coefficients. In fact, the mathematical techniques required for the two problems are rather different, at least superficially: The computation problem involves algorithms and questions of algorithmic efficiency, while the description problem brings in abstract algebra (theory of rings and modules). At a deeper level, these subjects do interact with one another, as the recent burst of activity in the application of Gröbner bases to computational algorithms attests.

Unfoldings

This book adopts and advocates what might be called a "pragmatic" approach to unfoldings, rather than an "ideological" approach. The "ideological" approach is to choose an equivalence relation, and then to define a *universal unfolding* of a system to be (roughly) a system with parameters that (a) reduces to the given system when the parameters are zero, and (b) exhibits all possible behavior, up to the given equivalence relation, that can occur in systems close (in some topology) to the given system. The equivalence relations in common use fall into two classes that might be called *static* and *dynamic* equivalence relations; the former means that the sets of equilibrium states for two equivalent systems are topologically similar, while the latter requires that the full dynamics of the two systems be topologically similar. Neither equivalence relation takes into account any asymptotic properties of the systems, such as rates of approach to a stable rest point; only the stability itself matters. In addition, there may not exist a universal unfolding having finitely many parameters (finite codimension); for the dynamic equivalence relations, there almost never is. This approach is "ideological" in the sense that it begins from an a priori decision as to what behavior is interesting and what is expected from an unfolding.

The "pragmatic" approach, in contrast, begins from the fact that normal form calculations must always be terminated at some degree k (that is, with a normalized jet). Once k is chosen, it is always possible to consider an arbitrary perturbation of the jet (within its jet space). This perturbation already contains only finitely many parameters, and the number can be reduced through normal form (and hypernormal form) calculations. The resulting simplified perturbed jet is what we take as the unfolding. The next step in this approach is to ask what properties of this jet (or its associated flow) are k-determined; these are the properties that are so solidly established by the k-jet that they cannot be changed by the addition of higher-order terms. The unfolding that has been calculated is then a true (universal) unfolding *with regard to these properties*, although not necessarily with regard to all properties. (It can be shown that the collection of all properties that are k-determined does define an equivalence relation, and the unfolding is universal with respect to this relation. See the Notes and References to Section 6.4.) Some of the properties that are k-determined by this jet will be topological in character, and others will be asymptotic. Frequently, the unfolding computed by this method coincides with the universal unfolding in one of the original senses, when such an unfolding exists, or can be reduced to that unfolding by applying one or two additional coordinate changes, such as time scalings, that do not fit within the framework of normal forms. It can be the case that some interesting properties are fully unfolded at one value of k and others require a higher value. The case of "infinite codimension" is exactly the case in which no choice of k is sufficient to unfold the system fully with respect to one of the classical equivalence relations.

As already pointed out at the beginning of this preface, the treatment of bifurcation problems in this book is not intended to be at all complete. Especially for the more complicated bifurcations, such as mode interactions, the dynamical behavior is quite complicated; often it becomes understandable only under the assumption of various symmetry conditions. Symmetry is a subject that is quite important in itself, but is not addressed at all in this book (except with regard to the symmetries introduced by normalization, manifested as equivariance of vector fields). It does not fall within the scope of this book to discuss complicated bifurcation diagrams. In those complicated problems that we treat, we compute the normal form and unfolding, discuss to some extent the question of jet sufficiency, and refer to the literature for details.

Is the Nilpotent Normal Form Useful?

It is sometimes said that only the semisimple normal form is useful. (This includes what we call the "extended semisimple normal form," that is, the result of normalizing a nonsemisimple system with respect to the semisimple part of its linear term.) Since a fair portion of this book is concerned with the nilpotent (or, more generally, nonsemisimple) normal form, it is worthwhile to address the question of its usefulness. The answer has two parts: a list of existing results that use the nonsemisimple normal form, and a list of promising open problems. First is the list of the existing results (which may not be complete):

1. The nonsemisimple normal form for linear problems (matrix perturbations) is the starting point for the transplanting, or shearing, method described in Section 3.7.

2. The "simplified normal form," one version of the nonsemisimple normal form, is the starting point for the computation of unfoldings described in Section 6.4.

3. The nilpotent normal form in two dimensions is the basis for the various shearing, scaling, and blowup methods used to study the Takens–Bogdanov problem. These methods are touched upon in Sections 5.4 and 6.6. It is to be emphasized that in these approaches one does not study the system directly in its nilpotent normal form. Instead, one first creates the nilpotent normal form and then performs a transplanting operation leading to a new organizing center that is not nilpotent (and does not even have a rest point at the origin). The resulting problem is studied by an entirely different set of methods, which are global in character. So the difficulty of working with the nilpotent normal form itself does not even arise. Instead, the simplifications achieved by the nilpotent normal form are passed on to the new system in a different guise.

4. The nonsemisimple normal form for Hamiltonian systems in two degrees of freedom is the starting point for the analysis of the Hamiltonian Hopf bifurcation. See Notes and References to Section 4.9.

Next is the list of open problems. Solution of any or all of these problems would increase the usefulness of the nonsemisimple normal form.

1. What is the correct general nonlinear transplanting (shearing) theory? Such a theory should generalize the linear shearing theory of Section 3.7 and include the scaling for the Takens–Bogdanov problem given in Section 6.6.

2. What is the complete theory of the following methods, best understood in two dimensions (see Notes and References to Sections 5.4 and 6.6):

   a. the blowup method (probably best explained in the writings of Freddy Dumortier)?

b. the power transformation method (introduced in the writings of Alexander Bruno)?

3. How do these methods (shearing, blowup, and power transform) relate to one another?
4. What is the correct general error estimate for nilpotent problems, and how do its time scales relate to the correct shearing scales? (In Chapter 5, error estimates are derived only for cases in which there is no nilpotent part on the center manifold.)
5. Is there a way to extract useful geometrical information from Lemma 5.4.2? This lemma describes the near-preservation of a foliation in the sl(2) normal form for nilpotent problems.
6. How can the unfolding theory of Section 6.4 be carried out using the sl(2) normal form? This question is significant in view of the previous one, which suggests that the sl(2) normal form contains significant dynamical information. The problem comes down to the following: to find a construction such that the analogue of Lemma 6.4.3 holds for the sl(2) normal form and preferably to state this construction in a coordinate-free manner. (Lemma 6.4.3 itself holds only for the simplified normal form and cannot be stated in a coordinate-free manner, since the definition of that normal form is itself not coordinate-free.)

What Is New?

The expert will want to know which results in this book, if any, are actually new (apart from matters of exposition and terminology). Here is a list of the items that appear to me to be new or possibly new:

1. The development of the simplified normal form given here is more complete than the other available presentations. This discussion occurs in Sections 3.4, 4.6, and 6.4.
2. The treatment of linear shearing (or transplanting) given in Section 3.7 is new. It extends results obtained by Bogaevsky and Povsner so that they hold for the inner product normal form style as well as the sl(2) style; the original arguments were valid for the sl(2) style only.
3. Section 3.7 gives an algorithmic procedure for deciding k-determined hyperbolicity. My previous work on this topic (some of it joint with Clark Robinson) was not algorithmic in character.
4. Section 4.7 is entirely new, although the paper (which contains additional material) may appear shortly before the book. This provides an algorithm to produce a Stanley decomposition for the normal form module in the nilpotent case, either in the inner product, simplified,

or sl(2) normal form styles, given the Stanley decomposition for the corresponding ring of invariants.

5. Chapter 5 (especially Sections 5.1 and 5.3) contains a number of general results that have not (to my knowledge) appeared in the literature except in particular examples. Most of the results that cannot be referenced should probably be classed as folklore rather than as new results, although it is not clear how widely the folklore is known. I have shown these results to several researchers and received responses ranging from surprise to "yes, I thought that was true, but I don't know any references." On the other hand, some ideas leading to open problems stated in this chapter may be new.

6. Although it is a rather minor point, I don't think the role of seminormality (Definition 3.4.10) in the theory of the inner product normal form has been noticed; this is because the condition is automatically satisfied when the leading term is in Jordan form, as is usually assumed. The sl(2) normal form never requires an assumption of seminormality, either in the linear or nonlinear setting.

7. The unfolding theory developed in Sections 6.3 and 6.4 is a few years old and has appeared previously in only one paper.

Acknowledgments and Apologies

This book would never have come into existence in its present form without extensive discussions with Jan Sanders and Richard Cushman, and input from the following mathematicians over a number of years (listed in alphabetical order): Alberto Baider, Alexander Bruno, Kenneth Driessel, Karin Gatermann, Marty Golubitsky, George Haller, Irwin Hentzel, Alexander Kopanskii, Bill Langford, Victor Leblanc, A.H.M. Levelt, John Little, Ian Melbourne, Sri Namachchivaya, Peter Olver, Charlie Pugh, and Jonathan Smith. Special thanks are due to my graduate student David Malonza for his careful proofreading of much of this book, and for his many questions, which forced me to improve the exposition in several places. Thanks are also due to David Kramer, who copyedited the manuscript and also laboriously transformed my files from the dialect of TeX in which I had written them to another preferred by the publisher, in the process making a number of suggestions that improved the clarity of the exposition as well as the layout of the pages.

All of the shortcomings of this book, and any outright errors, are to be attributed to my own vast ignorance. Any errata that come to my attention after the book appears in print will be posted to my Web site,

http:\\www.math.iastate.edu \jmurdock\.

Any contributions to the open problems mentioned in this book will also be referenced there (with the authors' permission). If you discover errors or publish contributions that should be cited, please notify me at jmurdock@iastate.edu.

A Note About Notations

The following remarks are given to aid the reader who is already familiar with the ideas to recognize quickly certain notations used in this book, without having to find the passages in which these notations are introduced.

There are many places in normal form theory where a decision must be made that affects the signs that appear in many equations. Decisions that are convenient in one part of the theory are often inconvenient elsewhere. The following three principles have been adopted, because of their simplicity and because, together, they settle all of these decisions:

1. The Lie bracket of matrices (and of linear operators in general) should agree with the usual commutator bracket, so that
$$[A, B] = AB - BA$$
rather than the negative of this.

2. The Lie bracket of vector fields should reduce to the Lie bracket of matrices when the vector fields are linear. Thus,
$$[u, v] = u'v - v'u$$
rather than the negative of this. (Here, u and v are column vectors, and u' and v' are matrices of partial derivatives.)

3. The Lie operator L_u should agree with the Lie derivative as it is usually defined, so that $\mathsf{L}_u v$ is the rate of change (in the Lie sense) of v in the forward direction along the flow of u.

Unfortunately, the last two of these principles, taken together, imply that
$$\mathsf{L}_u v = [v, u] \quad \text{rather than} \quad [u, v].$$
Taking the first principle into account as well (to fix the meaning of the bracket of Lie operators), it follows that
$$\mathsf{L}_{[u,v]} = [\mathsf{L}_v, \mathsf{L}_u].$$
That is, L turns out to be a Lie algebra antihomomorphism, which is occasionally awkward.

We regard vector fields as column vectors and write these as

$$v = (v_1, v_2, \ldots, v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

(Thus, any vector written with parentheses and commas is automatically a column vector; if we want a row vector, it must be written $\begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$). We do not identify vectors with differential operators; instead, the differential operator associated with a vector field v is written

$$\mathcal{D}_v = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \cdots + v_n \frac{\partial}{\partial x_n}.$$

By the first principle above, the bracket of two such operators is their commutator. This implies that

$$\mathcal{D}_{[u,v]} = [\mathcal{D}_v, \mathcal{D}_u].$$

Therefore, $\mathcal{D}$ is also an antihomomorphism.

The usual way to avoid these problems (in advanced work) is to identify v and $\mathcal{D}_v$ and to eliminate the bracket that we call $[u, v]$; the Lie operator then reduces to the "Lie algebra adjoint" operator, defined by $\mathrm{ad}_u v = [u, v]$; this is a Lie algebra homomorphism. This works well for abstract considerations, but clashes with matrix notation (our principle 1 above) as soon as systems of differential equations are written in their familiar form.

The transpose of a matrix is denoted by $A^\dagger$, and the conjugate transpose by A^*. More generally, L^* denotes the adjoint of a linear operator with respect to some specified inner product.

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