2

Asymptotic Cones and Functions

This chapter provides the fundamental tools used throughout this monograph. For a given subset of $\mathbb{R}^n$ we are interested in studying its behavior at infinity. This leads to the concepts of asymptotic cone and asymptotic function through its epigraph. Using elementary real analysis and geometrical concepts we develop a complete mathematical treatment to handle the asymptotic behavior of sets, functions, and other induced functional operations.

2.1 Definitions of Asymptotic Cones

The set of natural numbers is denoted by $\mathbb{N}$, so that $k \in \mathbb{N}$ means $k = 1, 2, \ldots$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ or simply $\{x_k\}$ in $\mathbb{R}^n$ is said to converge to x if $\|x_k - x\| \to 0$ as $k \to \infty$, and this will be indicated by the notation $x_k \to x$ or $x = \lim_{k\to\infty} x_k$. We say that x is a cluster point of $\{x_k\}$ if some subsequence converges to x. Recall that every bounded sequence in $\mathbb{R}^n$ has at least one cluster point. A sequence in $\mathbb{R}^n$ converges to x if and only if it is bounded and has x as its unique cluster point.

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. We are interested in knowing how to handle situations when the sequence $\{x_k\} \subset \mathbb{R}^n$ is unbounded. To derive some convergence properties, we are led to consider directions $d_k := x_k\|x_k\|^{-1}$ with $x_k \neq 0$, $k \in \mathbb{N}$. From classical analysis, the Bolzano–Weierstrass theorem implies that we can extract a convergent subsequence $d = \lim_{k\in K} d_k$, $K \subset \mathbb{N}$, with $d \neq 0$. Now suppose that the sequence $\{x_k\} \subset \mathbb{R}^n$ is such that

$\|x_k\| \to +\infty$. Then

$$\exists\ t_k := \|x_k\|,\ k \in K \subset \mathbb{N},\ \text{such that}\ \lim_{k\in K} t_k = +\infty\ \text{and}\ \lim_{k\in K} \frac{x_k}{t_k} = d.$$

This leads us to introduce the following concepts.

Definition 2.1.1 *A sequence $\{x_k\} \subset \mathbb{R}^n$ is said to converge to a direction $d \in \mathbb{R}^n$ if*

$$\exists \{t_k\},\ \textit{with}\ t_k \to +\infty\ \textit{such that}\ \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Definition 2.1.2 *Let C be a nonempty set in $\mathbb{R}^n$. Then the asymptotic cone of the set C, denoted by C_∞, is the set of vectors $d \in \mathbb{R}^n$ that are limits in direction of the sequences $\{x_k\} \subset C$, namely*

$$C_\infty = \left\{ d \in \mathbb{R}^n \mid \exists t_k \to +\infty, \exists x_k \in C\ \textit{with}\ \lim_{k\to\infty} \frac{x_k}{t_k} = d \right\}.$$

From the definition we immediately deduce the following elementary facts.

Proposition 2.1.1 *Let $C \subset \mathbb{R}^n$ be nonempty. Then:*
(a) C_∞ is a closed cone*.*
(b) $(\operatorname{cl} C)_\infty = C_\infty$*.*
(c) If C is a cone, then $C_\infty = \operatorname{cl} C$.

The importance of the asymptotic cone is revealed by the following key property, which is an immediate consequence of its definition.

Proposition 2.1.2 *A set $C \subset \mathbb{R}^n$ is bounded if and only if $C_\infty = \{0\}$.*

Proof. It is clear that C_∞ cannot contain any nonzero direction if C is bounded. Conversely, if C is unbounded, then there exists a sequence $\{x_k\} \subset C$ with $x_k \neq 0,\ \forall k \in \mathbb{N}$, such that $t_k := \|x_k\| \to \infty$ and thus the vectors $d_k = t_k^{-1} x_k \in \{d :\ \|d\| = 1\}$. Therefore, we can extract a subsequence of $\{d_k\}$ such that $\lim_{k\in K} d_k = d,\ \ K \subset \mathbb{N}$, and with $\|d\| = 1$. This nonzero vector d is an element of C_∞ by Definition 2.1.2, a contradiction. $\square$

Associated with the asymptotic cone C_∞ is the following related concept, which will help us in simplifying the definition of C_∞ in the particular case where $C \subset \mathbb{R}^n$ is assumed convex.

Definition 2.1.3 *Let $C \subset \mathbb{R}^n$ be nonempty and define*

$$C^1_\infty := \left\{ d \in \mathbb{R}^n \mid \forall t_k \to +\infty,\ \exists x_k \in C\ \textit{with}\ \lim_{k\to\infty} \frac{x_k}{t_k} = d \right\}.$$

We say that C is asymptotically regular if $C_\infty = C^1_\infty$.

Proposition 2.1.3 *Let C be a nonempty convex set in $\mathbb{R}^n$. Then C is asymptotically regular.*

Proof. The inclusion $C^1_\infty \subset C_\infty$ clearly holds from the definitions of C^1_∞ and C_∞, respectively. Let $d \in C_\infty$. Then $\exists \{x_k\} \in C$, $\exists s_k \to \infty$ such that $d = \lim_{k\to\infty} s_k^{-1} x_k$. Let $x \in C$ and define $d_k = s_k^{-1}(x_k - x)$. Then we have

$$d = \lim_{k\to\infty} d_k, \quad x + s_k d_k \in C.$$

Now let $\{t_k\}$ be an arbitrary sequence such that $\lim_{k\to\infty} t_k = +\infty$. For any fixed $m \in N$, there exists $k(m)$ with $\lim_{m\to\infty} k(m) = +\infty$ such that $t_m \le s_{k(m)}$, and since C is convex, we have $x'_m := x + t_m d_{k(m)} \in C$. Hence, $d = \lim_{m\to\infty} t_m^{-1} x'_m$, showing that $d \in C^1_\infty$. □

We note that a set can be nonconvex, yet asymptotically regular. Indeed, consider, for example, sets defined by $C := S + K$, with S compact and K a closed convex cone. Then clearly C is not necessarily convex, but it can be easily seen that $C_\infty = C^1_\infty$.

Remark 2.1.1 Note that the definitions of C_∞ and C^1_∞ are related to the theory of set convergence of Painlevé–Kuratowski. Indeed, for a family $\{C_t\}_{t>0}$ of susbsets of $\mathbb{R}^n$, the outer limit as $t \to +\infty$ is the set

$$\limsup_{t\to+\infty} C_t = \left\{ x \mid \liminf_{t\to+\infty} d(x, C_t) = 0 \right\},$$

while the inner limit as $t \to +\infty$ is the set

$$\liminf_{t\to+\infty} C_t = \left\{ x \mid \limsup_{t\to+\infty} d(x, C_t) = 0 \right\}.$$

It can then be verified that the corresponding asymptotic cones can be written as

$$C_\infty = \limsup_{t\to+\infty} t^{-1} C, \quad C^1_\infty = \liminf_{t\to+\infty} t^{-1} C.$$

Proposition 2.1.4 *Let $C \subset \mathbb{R}^n$ be nonempty and define the normalized set*

$$C_N := \left\{ d \in \mathbb{R}^n \mid \exists \{x_k\} \in C, \|x_k\| \to +\infty \text{ with } d = \lim_{k\to\infty} \frac{x_k}{\|x_k\|} \right\}.$$

Then $C_\infty = \operatorname{pos} C_N$, where for any set C, $\operatorname{pos} C = \{\lambda x \mid x \in C, \lambda \ge 0\}$.

Proof. Clearly, one always has $\operatorname{pos} C_N \subset C_\infty$. Conversely, let $0 \ne d \in C_\infty$. Then there exists $t_k \to \infty$, $x_k \in C$ such that

$$d = \lim_{k\to\infty} t_k^{-1} x_k = \lim_{k\to\infty} t_k^{-1} \|x_k\| \frac{x_k}{\|x_k\|}, \quad \text{with } \|x_k\| \to \infty.$$

Thus the sequence $\{t_k^{-1}\|x_k\|\}$ is a nonnegative bounded sequence, and by the Bolzano–Weierstrass theorem, there exists a subsequence $\{t_k^{-1}\|x_k\|\}_{k\in K}$ with $K \subset \mathbb{N}$ such that $\lim_{k\in K} t_k^{-1}\|x_k\| = \lambda \geq 0$, which means that $d = \lambda d_N$ with $d_N \in C_N$, namely $d \in \operatorname{pos} C_N$. $\square$

We now turn to some useful formulations of the asymptotic cone for convex sets.

Proposition 2.1.5 *Let C be a nonempty convex set in $\mathbb{R}^n$. Then the asymptotic cone C_∞ is a closed convex cone. Moreover, define the following sets:*

$$\begin{aligned} D(x) &:= \{d \in \mathbb{R}^n \mid x + td \in \operatorname{cl} C,\ \forall t > 0\}\ \forall x \in C, \\ E &:= \{d \in \mathbb{R}^n \mid \exists x \in C \textit{ such that } x + td \in \operatorname{cl} C,\ \forall t > 0\}, \\ F &:= \{d \in \mathbb{R}^n \mid d + \operatorname{cl} C \subset \operatorname{cl} C\}. \end{aligned}$$

Then $D(x)$ is in fact independent of x, which is thus now denoted by D, and $C_\infty = D = E = F$.

Proof. We already know that C_∞ is a closed cone. The convexity property simply follows from Proposition 2.1.3, which ensures that $C^1_\infty = C_\infty$. We now prove the three equivalent formulations. Let $d \in C_\infty,\ x \in C,\ t > 0$. From Definition 2.1.2,

$$\exists t_k \to +\infty\ \exists d_k \to d, \text{ with } x + t_k d_k \in C.$$

Take k sufficiently large such that $t \leq t_k$, then since C is convex, we have

$$x + td_k = (1 - t_k^{-1}t)x + t_k^{-1}t(x + t_k d_k) \in C.$$

Passing to the limit, we thus have $x + td \in \operatorname{cl} C,\ \forall t > 0$, thus proving the inclusion $C_\infty \subset D(x),\ \forall x \in C$. Clearly, we also have the inclusion $D(x) \subset E,\ \forall x \in C$. We now show that $E \subset C_\infty$. For that, let $d \in E$ and $x \in C$ be such that $x(t) := x + td \in \operatorname{cl} C,\ \forall t > 0$. Then, since $d = \lim_{t\to\infty} t^{-1}x(t)$ and $x(t) \in \operatorname{cl} C$, we have $d \in (\operatorname{cl} C)_\infty = C_\infty$ (by Proposition 2.1.1), which also proves that $D(x)$ is in fact independent of x, and we can write $D(x) = D = C_\infty = E$. Finally, let $d \in C_\infty$. Using the representation $C_\infty = D$ we thus have $x + d \in \operatorname{cl} C,\ \forall x \in C$, and hence $d + \operatorname{cl} C \subset \operatorname{cl} C$, proving the inclusion $C_\infty \subset F$. Now, if $d \in F$, then $d + \operatorname{cl} C \subset \operatorname{cl} C$, and thus

$$2d + \operatorname{cl} C = d + (d + \operatorname{cl} C) \subset d + \operatorname{cl} C \subset \operatorname{cl} C,$$

and by induction $\operatorname{cl} C + md \subset \operatorname{cl} C, \forall m \in N$. Therefore, $\forall x \in \operatorname{cl} C,\ d_m := x + md \in \operatorname{cl} C$ and $d = \lim_{m\to\infty} m^{-1} d_m$, namely $d \in C_\infty$, showing that $F \subset C_\infty$, and hence the proof of the three equivalent formulations for C_∞ is completed. $\square$

Remark 2.1.2 When C is a closed convex set the asymptotic cone is also called the recession cone. However, we will keep the terminology asymptotic cone throughout this book.

From Proposition 2.1.5, using the set D as a representation of C_∞, an alternative useful representation of the asymptotic cone of a closed convex set is

$$C_\infty = \bigcap_{t>0} t^{-1}(C - x), \quad \forall x \in C.$$

It is important to note that in the definition of the set D given above, the closure assumption on the convex set $C \subset \mathbb{R}^n$ is crucial and cannot be removed. Indeed, consider the convex set

$$C = \{x \in \mathbb{R}^2 : \ x_1 > 0, x_2 > 0\} \cup \{(0,0)\}.$$

Then from Definition 2.1.2 we have $C_\infty = \mathbb{R}^2_+$, while any of the three formulations given in Proposition 2.1.5 would lead to the wrong result $C_\infty = C$.

Proposition 2.1.6 *For any nonempty closed convex set $C \subset \mathbb{R}^n$, one has $C = C + C_\infty$.*

Proof. The inclusion $C \subset C + C_\infty$ is clear. Let $x \in C + C_\infty$. Then there exists $c \in C$, $d \in C_\infty$ with $x = c + d$. Thus, there exists $d_k \in C$, $t_k \to \infty$ such that $t_k^{-1} d_k \to d$, which implies that for k sufficiently large $(1 - t_k^{-1})c + t_k^{-1} d_k \in C$ and $(1 - t_k^{-1})c + t_k^{-1} d_k \to x \in C$. □

Proposition 2.1.7 *For any nonempty closed convex set $C \subset \mathbb{R}^n$ that contains no lines one has $C = \operatorname{conv}(\operatorname{ext} C) + C_\infty$.*

Proof. Thanks to Proposition 2.1.6, we have only to prove that

$$C \subset \operatorname{conv}(\operatorname{ext} C) + C_\infty.$$

Let $x \in C$. Then from Theorem 1.1.3, we can write

$$x = \sum_{i=1}^{k} \lambda_i x_i + \sum_{i=k+1}^{m} \lambda_i d_i, \quad \sum_{i=1}^{m} \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1, \ldots, m,$$

with $x_i \in \operatorname{ext} C$, $\ d_i \in \operatorname{extray} C$, $\ i = 1, \ldots, m$. Thus $d_i \in \operatorname{extray} C$ implies that $d_i = e_i + v_i$, where e_i, the endpoint of the ray, is an extreme point of C and $v_i \in C_\infty$, from which it follows that $x \in \operatorname{conv}(\operatorname{ext} C) + C_\infty$. □

We now prove the following useful result, which as we shall see is often used as a technical device in analyzing closedness properties involving operations on closed sets in $\mathbb{R}^n$.

Lemma 2.1.1 *Let C be a nonempty closed set of $\mathbb{R}^n$, and K the cone in $\mathbb{R}^{n+1}$ generated by $\{(1,x) \mid x \in C\}$, i.e., $K = \operatorname{pos}\{(1,x) \mid x \in C\}$. Define $D := \{(0,x) \mid x \in C_\infty\}$. Then $\operatorname{cl} K = K \cup D$.*

Proof. The cone $K \subset \mathbb{R}^{n+1}$ generated by $\{(1,x) \mid x \in C\}$ can be written as

$$K = \{\lambda(1,x) \mid \lambda \geq 0, x \in C\} = \{\lambda(1,x) \mid \lambda > 0, x \in C\} \cup \{0\}.$$

Let $y = (t,x) \in \operatorname{cl} K$. Then $\exists t_k \geq 0,\ t_k \to t,\ x^k \in C$ such that $t_k(1,x^k) \to y,\ t_k x^k \to x$. If $t = 0,\ x \in C_\infty$, and thus $y \in D$. Otherwise, $t^{-1}x \in C$ and $y = t(1,t^{-1}x) \in K$, showing that $\operatorname{cl} K \subset K \cup D$. Conversely, any $y \in D$ can be written as $y = \lim_{k\to\infty} t_k(1,x^k)$, with $t_k \to 0,\ x^k \in C$, so that $D \subset \operatorname{cl} K$, and hence since $K \subset \operatorname{cl} K$, that $D \cup K \subset \operatorname{cl} K$. □

Some further elementary operations on asymptotic cones of nonempty convex sets are collected below.

Proposition 2.1.8 *Let $C \subset \mathbb{R}^n$ be nonempty and convex. Then:*
(a) $(\operatorname{cl} C)_\infty = (\operatorname{ri} C)_\infty = C_\infty$.
(b) For any $x \in \operatorname{ri} C$, *one has* $d \in (\operatorname{cl} C)_\infty \iff x + td \in \operatorname{ri} C,\ \forall t > 0$.
(c) $C \subset C_\infty \implies \operatorname{ri} C_\infty = \operatorname{ri} \operatorname{pos} C$.
(d) C is closed with $0 \in C \implies C_\infty = \{d :\ t^{-1}d \in C, ,\ \forall t > 0\}$.

Proof. Since C is nonempty and convex, the relative interior of C is nonempty, and thus applying Proposition 2.1.1(b) to $\operatorname{ri} C$ we obtain $(\operatorname{ri} C)_\infty = (\operatorname{cl} \operatorname{ri} C)_\infty = (\operatorname{cl} C)_\infty = C_\infty$, where in the second equation we use Proposition 1.1.4, and (a) is proved, while (b) follows from (a) and the line segment principle, cf. Proposition 1.1.4. To prove (c), we first note that $\operatorname{pos} C$ is convex. Then under the assumption $C \subset C_\infty$ one has $\operatorname{pos} C \subset \operatorname{pos} C_\infty = C_\infty$ and hence $\operatorname{cl} \operatorname{pos} C \subset \operatorname{cl} C_\infty = C_\infty$, which together with the fact $C_\infty \subset \operatorname{cl} \operatorname{pos} C$ gives that $\operatorname{cl} \operatorname{pos} C = C_\infty = \operatorname{cl} C_\infty$, which is equivalent to $\operatorname{ri} \operatorname{pos} C = \operatorname{ri} C_\infty$. In part (d), since $0 \in C$, the representation of C_∞ follows from the representation of C_∞ via the set D given in Proposition 2.1.5.

□

Some useful operations with asymptotic cones are now given for arbitrary sets in $\mathbb{R}^n$.

Proposition 2.1.9 *Let $C_i \subset \mathbb{R}^n$, $i \in I$, be an arbitrary index set. Then:*
(a) $(\cap_{i\in I} C_i)_\infty \subset \cap_{i\in I} (C_i)_\infty$, *whenever* $\cap_{i\in I} C_i$ *is nonempty.*
(b) $(\cup_{i\in I} C_i)_\infty \supset \cup_{i\in I} (C_i)_\infty$.
The inclusion in (a) holds as an equation for closed convex sets C_i having a nonempty intersection. The inclusion (b) holds as an equation when I is a finite index set.

Proof. Let d be any point in the set $C := (\cap_{i \in I} C_i)_\infty$, which is closed by definition of the asymptotic cone and satisfies

$$\exists t_k \to \infty,\ \exists x_k \in \cap_{i \in I} C_i \text{ such that } \frac{x_k}{t_k} \to d.$$

Therefore, $\exists t_k \to \infty,\ \exists x_k \in C_i,\ \forall i \in I$ such that $t_k^{-1} x_k \to d$, implying that $d \in (C_i)_\infty$ for all $i \in I$ and proving (a). The relation (b) is proved in a similar way through the definition of the corresponding asymptotic cone. Finally, the special result for the convex case follows by applying the characterization given in Proposition 2.1.5. □

Proposition 2.1.10 *For any sets $C_i \subset \mathbb{R}^n$, $i = 1, \ldots, m$, one has*

$$(C_1 \times \cdots \times C_m)_\infty \subset (C_1)_\infty \times \cdots \times (C_m)_\infty.$$

The inclusion holds as an equation if every C_i is nonempty and convex.

Proof. The inclusion is proved by direct use of the definition of the asymptotic cone, and in the convex case using the characterization provided in Proposition 2.1.5. □

Proposition 2.1.11 *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping and C a closed convex set in $\mathbb{R}^n$ such that the inverse image of C is nonempty. Then $(A^{-1}(C))_\infty = A^{-1}(C_\infty)$.*

Proof. Since A is continuous and C is closed and convex, $A^{-1}(C)$ is closed and convex. Let $x \in A^{-1}(C)$. Then $d \in (A^{-1}(C))_\infty$ if and only if $A(x + td) = Ax + tAd \subset C$, $\forall t \geq 0$, but the latter means that $Ad \in C_\infty$, namely $d \in A^{-1}(C_\infty)$. □

We close the section by giving explicit formulas of asymptotic cones for some other important sets.

Example 2.1.1 (i) Let C be a cone in $\mathbb{R}^n$. Then $C_\infty = \operatorname{cl} C$.
(ii) Let C be an affine set. Then C_∞ is the linear subspace parallel to C.
(iii) Let C be a polyhedral convex set $C := \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. Then $C_\infty = \{d \in \mathbb{R}^n \mid Ad \leq 0\}$.

2.2 Dual Characterization of Asymptotic Cones

There exists a close connection between the support function of a set and its asymptotic cone. We show below that for a closed convex set $C \subset \mathbb{R}^n$, one can in fact give a *dual* characterization of the asymptotic cone, via the barrier cone of C.

Theorem 2.2.1 *Let $C \subset \mathbb{R}^n$ be nonempty and let C_∞^* denote the polar cone of C_∞. Then the following relations hold:*
(a) $\operatorname{dom}\sigma_C \subset C_\infty^*$.
(b) If $\operatorname{int} C_\infty^* \neq \emptyset$, *then* $\operatorname{int} C_\infty^* \subset \operatorname{dom}\sigma_C$.
(c) If C is convex, then $(\operatorname{dom}\sigma_C)^* = C_\infty$.

Proof. (a) Let $y \notin C_\infty^*$. Then $\exists\, 0 \neq d \in C_\infty$ such that $\langle d, y\rangle > 0$. Since $d \in C_\infty$, $\exists t_k \to \infty$, $\exists x_k \in C$ with $t_k^{-1}x_k \to d$, and with $\langle d, y\rangle > 0$, it follows that $\langle x_k, y\rangle \to +\infty$, proving that $y \notin \operatorname{dom}\sigma_C$.
(b) Let $y \notin \operatorname{dom}\sigma_C$. Then $\exists x_k \in C$ with $\langle x_k, y\rangle \to +\infty$. Passing to subsequences if necessary we can assume without loss of generality that $\|x_k\|^{-1}x_k \to d \neq 0$, with $d \in C_\infty$, and hence $\langle \|x_k\|^{-1}x_k, y\rangle \geq 0$. Therefore, $\forall \varepsilon > 0$ we have $\langle d, y + \varepsilon d\rangle \geq \varepsilon\|d\|^2 > 0$, showing that $y + \varepsilon d \notin C_\infty^*$, i.e., $y \notin \operatorname{int} C_\infty^*$.
(c) Since C is assumed convex, then C_∞ is a closed convex cone and $C_\infty^{**} = C_\infty$. Using (a) we thus have

$$\operatorname{dom}\sigma_C \subset C_\infty^* \Longrightarrow C_\infty = C_\infty^{**} \subset (\operatorname{dom}\sigma_C)^*.$$

We now prove the reverse inclusion. Let $d \in (\operatorname{dom}\sigma_C)^*$, $t > 0$, and let $\bar{x}$ be any point in C. Since $td \in (\operatorname{dom}\sigma_C)^*$, we have for any $y \in \operatorname{dom}\sigma_C$,

$$\begin{aligned} \langle \bar{x} + td, y\rangle &= \langle \bar{x}, y\rangle + \langle td, y\rangle \leq \langle \bar{x}, y\rangle \\ &\leq \sup_{x \in C}\langle x, y\rangle = \sigma_C(y). \end{aligned}$$

Since for any $y \notin \operatorname{dom}\sigma_C$ we have $\sigma_C(y) = +\infty$, the above inequality remains valid for any $y \in \mathbb{R}^n$. Therefore, invoking Theorem 1.3.1 we obtain for all $t > 0$, $\bar{x} + td \in \operatorname{cl} C$, which by Proposition 2.1.5 proves that $d \in C_\infty$. $\square$

2.3 Closedness Criteria

Given a nonempty closed set of $\mathbb{R}^n$, we are interested in answering the following question: Under what conditions does the image of a closed set under a linear mapping remain closed. This type of result is of fundamental importance and is at the root of several closedness criteria that are particularly useful for analyzing the existence of solutions of extremum problems. Asymptotic cones play a key role in the derivation of such results.
Let $C \subset \mathbb{R}^n$ be a nonempty closed set and let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Let $\ker A := \{x : Ax = 0\} := A^{-1}(0)$. As we shall see, a basic

sufficient condition that guarantees that $A(C)$ is closed is

$$\ker A \cap C_\infty = \{0\}. \tag{2.1}$$

Moreover, under the condition (2.1) we also have $A(C_\infty) = (A(C))_\infty$.
For a convex set, a weaker sufficient condition to guarantee closedness of $A(C)$ is

$$\ker A \cap C_\infty \quad \text{is a linear subspace.} \tag{2.2}$$

Unfortunately, even in the convex case, the above condition can fail, as shown in the following two examples.

Example 2.3.1 Let $C = \{x \in \mathbb{R}^2 \mid x_2 \geq x_1^2\}$ and let A be the linear mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $A(x_1, x_2) = (x_1, 0)$, i.e., the projection onto the x_1-axis. Then, one has

$$\begin{aligned} C_\infty &= \{d \in \mathbb{R}^2 \mid d_1 = 0, d_2 \geq 0\}, \\ A(C) &= \{x \in \mathbb{R}^2 \mid x_2 = 0\} = x_1\text{-axis}, \\ \ker A &= \{x \in \mathbb{R}^2 \mid x_1 = 0\} = x_2\text{-axis}, \end{aligned}$$

and condition (2.2) fails, yet $A(C)$ is a closed set.

Example 2.3.2 Let $D = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \geq 1\}$ and A be the linear mapping as in Example 2.3.1. One easily see that $\ker A \cap D_\infty = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\}$, and thus condition 2.2 fails; yet $A(D)$ is a closed set.

As we shall see, the sets C and D are quite different in nature. The set C belongs to a class of sets called *continuous sets*, while the set D is in the class of *asymptotically linear sets*. These two classes of sets are quite general and are very different one from the other and will be introduced later on. We thus need conditions that can handle the situations just described. In fact, the next general result will establish a necessary and sufficient condition for preserving closedness of $A(C)$.

Theorem 2.3.1 *Let C be any nonempty closed set of $\mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ a linear mapping. Let $\{y_k\}$ be any sequence in $A(C)$ converging to y and define the set*

$$S := \left\{ \{x_k\} \mid x_k \in C : \ \|x_k\| \to \infty, \ \|Ax_k - y\| \leq \|y - y_k\|, \ \frac{x_k}{\|x_k\|} \to \bar{x} \right\}. \tag{2.3}$$

Then a necessary and sufficient condition for the set $A(C)$ to be closed is that for each y and each sequence $\{y_k\} \in A(C)$ converging to y, (a) either S is empty or (b) for each sequence $\{x_k\} \in S$ there exists a sequence $\{z_k, \rho_k\} \subset \mathbb{R}^n \times \mathbb{R}_{++}$ such that for k sufficiently large,

$$x_k - \rho_k z_k \in C, \ \|A(x_k - \rho_k z_k) - y\| \leq \|y - y_k\|, \ \rho_k \in (0, \|x_k\|], \tag{2.4}$$

and $z_k \to z$ with $\|z - \bar{x}\| < 1$.

Proof. First, suppose that $y_k \in A(C)$, $y_k \to y$. Then there exists $x_k \in C$ such that $y_k = Ax_k$, which implies $\|Ax_k - y\| = \|y_k - y\|$. Thus, $S = \emptyset$ means that a subsequence of $\{x_k\}$ is bounded and has a cluster point, say $x^* \in C$. Passing to the limit in the latter relation implies that $\|Ax^* - y\| = 0$, showing that $A(C)$ is closed. We now prove the second case, described in (b). Suppose that $S \neq \emptyset$, and (2.4) holds, and let $y_k \to y$ with $y_k = Ax'_k$, $x'_k \in C$. Define the set $S_k := \{x \in C \mid \|Ax - y\| \leq \|y - y_k\|\}$ and consider the optimization problem $\inf\{\|x\| \mid x \in S_k\}$. By (2.3), we clearly have $x'_k \in S_k$, and hence S_k is nonempty. Furthermore, S_k is closed, and then the existence of $x_k \in \operatorname{argmin}\{\|x\| : x \in S_k\}$ is guaranteed. If we can show that the sequence $\{x_k\}$ is bounded, we are done. Indeed, in that case there would exist a subsequence $\{x_k\}_{k \in K}$, $K \subset \mathbb{N}$, such that $\lim_{k \in K} x_k = x \in C$ and with $Ax = y$, proving the closedness of $A(C)$. To prove this, suppose that $\{x_k\}$ is unbounded. Passing to subsequences if necessary, we can suppose without loss of generality that $\|x_k\|^{-1} x_k \to \bar{x} \neq 0$, and then using (2.3) there exists $\{z_k, \rho_k\}$ satisfying (2.4). As a consequence, $x_k - \rho_k z_k \in S_k$, and since by definition of x_k we have $\|x_k\| \leq \|x\|, \forall x \in S_k$, we obtain in particular that $\|x_k\| \leq \|x_k - \rho_k z_k\|$. Now,

$$\begin{aligned}
\|x_k - \rho_k z_k\| &= \|(1 - ||x_k||^{-1}\rho_k)x_k + \rho_k(||x_k||^{-1}x_k - z_k)\|, \\
&\leq (1 - ||x_k||^{-1}\rho_k)\|x_k\| + \rho_k\| \, ||x_k||^{-1}x_k - z_k\|, \\
&= \|x_k\| + \rho_k(\| \, ||x_k||^{-1}x_k - z_k\| - 1),
\end{aligned}$$

and hence $\| \, ||x_k||^{-1}x_k - z_k\| \geq 1$. Passing to the limit we get $\|\bar{x} - z\| \geq 1$, contradicting assumption (2.4). Conversely, suppose that $A(C)$ is closed and let $y_k \to y$ with $y_k \in A(C)$. Then, since $A(C)$ is closed, there exists $x \in C$ with $Ax = y$. Set $z_k = ||x_k||^{-1}(x_k - x)$, $\rho_k = \|x_k\|$, and then (2.4) is clearly satisfied. □

In general, it is easier and more convenient to choose in Theorem 2.3.1 $z_k = \bar{x}$ or more generally $z_k = \alpha\bar{x}$, $\alpha \in (0, 2)$, in assumption (2.4). In this case with $\alpha = 1$ we can say more about the image of C as shown in the theorem given below. As an application of that theorem we will also show how situations previously discussed, illustrating the failure of condition (2.2) to characterize closedness, can now be adequately handled.
For any nonempty closed set C of $\mathbb{R}^n$ and any linear map $A : \mathbb{R}^n \to \mathbb{R}^m$, it is immediate, using the definition of the asymptotic cone, to verify the inclusion $A(C_\infty) \subset A(C)_\infty$. However, the reverse inclusion requires a much finer analysis.

Theorem 2.3.2 *Let C be any nonempty closed set of $\mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ a linear mapping. Let $\{y_k\}$ be any sequence in $A(C)$ converging to y and let S be the set defined in (2.3). Suppose that for each y and any sequence $y_k \in A(C)$ converging to y, either S is empty or for any sequence $\{x_k\} \in S$ there exists a sequence $\{\rho_k\} \subset \mathbb{R}_{++}$ such that for k sufficiently* *large, $x_k - \rho_k \bar{x} \in C,\ \rho_k \le \|x_k\|$. Then $A(C)$ is closed, and we have*

$$A(C_\infty) = (A(C))_\infty. \tag{2.5}$$

Proof. The closedness of $A(C)$ follows as an immediate consequence of the previous theorem. Indeed, if we take $z_k = z = \bar{x}$, it follows from (2.3) that $A\bar{x} = 0$, and (2.4) holds. Now, to prove (2.5) we first note that the inclusion $A(C)_\infty \supset A(C_\infty)$ follows immediately from the definition of the asymptotic cone. Thus it remains to prove the reverse inclusion $(A(C))_\infty \subset A(C_\infty)$. Let $y \in (A(C))_\infty$ and let $u_k \in C,\ t_k \to +\infty$, and $y_k = Au_k$ with $t_k^{-1} y_k \to y$. Define $S_k = \{x \in C \mid Ax = y_k\}$. Then, S_k is a nonempty closed set, and the existence of $x_k \in \operatorname{argmin}\{\|x\| \mid x \in S_k\}$ is guaranteed. We prove now that we cannot have $\lim_{k\infty} t_k^{-1}\|x_k\| = +\infty$. Indeed, in the contrary case, we would have $\lim_{k\infty} \|x_k\| = +\infty$. Without loss of generality, we can suppose that $\|x_k\|^{-1} x_k \to \bar{x}$ with $\|\bar{x}\| = 1$. We thus obtain

$$A\bar{x} = \lim_{k\to\infty} A\frac{x_k}{\|x_k\|} = \lim_{k\to\infty} \frac{y_k}{\|x_k\|} = \lim_{k\to\infty} \frac{y_k}{t_k}\frac{t_k}{\|x_k\|} = 0.$$

Since (2.3) holds, there exists $\rho_k \in (0, \|x_k\|]$ such that $x_k - \rho_k\bar{x} \in C$ and since $A\bar{x} = 0$, it follows that $x_k - \rho_k\bar{x} \in S_k$. By definition of x_k we have $\|x_k\| \le \|x_k - \rho_k\bar{x}\|$, and as in the proof of Theorem 2.3.1 it follows that $\|\|x_k\|^{-1}x_k - \bar{x}\| \ge 1$ and passing to the limit, we get a contradiction. Finally, since the sequence $\{t_k^{-1}\|x_k\|\}$ is bounded, passing to subsequences if necessary, we can conclude that $\{t_k^{-1}x_k\}$ converges to some x_∞. Since $x_k \in S_k$, we have $x_\infty \in C_\infty$ and $t_k^{-1}Ax_k \to y = Ax_\infty$, proving the desired result. □

We consider now the situation in which Theorem 2.3.2 is satisfied and give further applications.

Corollary 2.3.1 *Let C be a nonempty closed set of $\mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ a linear mapping. Let $L(C) = C_\infty \cap -C_\infty$ and $L = L(C) \cap \ker A$ and suppose that the following two conditions hold:*
(a) For k sufficiently large, $C_k + L \subset C$, with $C_k = \{x \in C : \|x\| \ge k\}$.
(b) $z \in \ker A \cap C_\infty$ implies $z \in -C_\infty$.
Then $A(C)$ is closed and $(A(C))_\infty = A(C_\infty)$.

Proof. Let $\{y_k\} \subset A(C)$ be a sequence converging to y and let $\{x_k\} \in S$ be an unbounded sequence satisfying (2.3). Then clearly, $\bar{x} \in \ker A \cap C_\infty$, and from assumption (b) it follows that $\bar{x} \in -C_\infty$, which together with $\bar{x} \in C_\infty$ implies that $-\bar{x} \in L$. Let $\rho > 0$. Then by assumption (a), for k sufficiently large we have $x_k - \rho\bar{x} \in C$, and thus we can apply Theorem 2.3.2. □

Remark 2.3.1 It is possible to replace assumption (a) in Corollary 2.3.1 by the somewhat stronger assumption $C + L \subset C$.

Corollary 2.3.2 *Let C be a nonempty closed set of $\mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ a linear mapping. Then, $A(C)$ is closed and $A(C_\infty) = (A(C))_\infty$ under either of the following conditions:*
(a) $\ker A \cap C_\infty = \{0\}$,
(b) C is convex and $\ker A \cap C_\infty$ *is a linear subspace.*

Proof. Assume (a) holds, and in order to use Theorem 2.3.2 let $\{y_k\} \subset A(C)$ converge to y. Then there cannot exist any sequence $x_k \in C$ satisfying (2.3). Indeed, in the contrary case, we would have

$$\left\| A\frac{x_k}{\|x_k\|} - \frac{y_k}{\|x_k\|} \right\| \le \frac{\|y - y_k\|}{\|x_k\|},$$

and passing to the limit we obtain $A\bar{x} = 0$. Furthermore, since $x_k \in C$, it follows that $\bar{x} \in C_\infty$ with $\|\bar{x}\| = 1$, in contradiction to the assumption (a), asserting $\ker A \cap C_\infty = \{0\}$. Now assume (b). Then since C is convex, assumption (a) in Corollary 2.3.1 is satisfied by Proposition 2.1.5. Furthermore, since $\ker A \cap C_\infty$ is a linear subspace, the assumption $z \in \ker A \cap C_\infty$ implies that $z \in -C_\infty$, and assumption (b) in Corollary 2.3.1 holds. □

Corollary 2.3.3 *Let $S \subset \mathbb{R}^n$ be closed with $0 \notin S$. Then*

$$\operatorname{cl}\operatorname{pos} S = \operatorname{pos} S \cup S_\infty.$$

Furthermore, if S is bounded, then $\operatorname{pos} S$ *is closed.*

Proof. Let K be the cone in $\mathbb{R}^{n+1}$ generated by $\{(1,x) \mid x \in S\}$, and let $A : (\alpha, x) \to x$. Then $\operatorname{pos} S = A(K)$. By Lemma 2.1.1 one has $\operatorname{cl} K = K \cup \{(0,x) \mid x \in S_\infty\}$ and $A(\operatorname{cl} K) = \operatorname{pos} S \cup S_\infty$. Invoking Corollary 2.3.2 (a) with $C := \operatorname{cl} K$ and since $0 \notin S$ noting that in that case $\ker A \cap C_\infty = \{0\}$, it follows that $\operatorname{pos} S \cup S_\infty$ is closed. Now, since $\operatorname{pos} S \subset \operatorname{pos} S \cup S_\infty$ and $S_\infty \subset \operatorname{cl}\operatorname{pos} S$, then

$$\operatorname{cl}\operatorname{pos} S \subset \operatorname{cl}(\operatorname{pos} S \cup S_\infty) = \operatorname{pos} S \cup S_\infty \subset \operatorname{pos} S \cup \operatorname{cl}\operatorname{pos} S \subset \operatorname{cl}\operatorname{pos} S,$$

and hence $\operatorname{pos} S \cup S_\infty = \operatorname{cl}\operatorname{pos} S$. Finally, when S is bounded $S_\infty = \{0\}$, and this proves the last statement. □

Asymptotic Linear Sets

We introduce now the class of asymptotically linear sets. As we shall see, this notion is of great interest. We begin with the basic definition of this concept.

Definition 2.3.1 *Let C be a nonempty closed set of $\mathbb{R}^n$. Then C is said to be an asymptotically linear set if for each $\rho > 0$ and each sequence $\{x_k\}$ satisfying*

$$x_k \in C, \quad \|x_k\| \to +\infty, \quad x_k\|x_k\|^{-1} \to \bar{x}, \tag{2.6}$$

there exists $k_0 \in \mathbb{N}$ such that

$$x_k - \rho\bar{x} \in C \quad \forall k \geq k_0. \tag{2.7}$$

If C is a closed convex set, we also have $x_k + \rho\bar{x} \in C$. This justifies the terminology "asymptotically linear".
As a consequence of Definition 2.3.1 a set that is the intersection of a finite number of sets, each of them being the union of a finite number of asymptotically linear sets, is also an asymptotically linear set.

Definition 2.3.2 *A set $C \subset \mathbb{R}^n$ is said to be a simple asymptotically polyhedral set if there exists a nonnegative integer l for which the set $C_l := C \cap \{x : \|x\| \geq l\}$ admits the decomposition*

$$C_l = K + M,$$

with K compact and M a polyhedral cone.
The set C is said to be asymptotically polyhedral if it is the intersection of a finite number of sets, each of them being the union of a finite number of simple asymptotically polyhedral sets.

When $l = 0$, the simplest set of this type is clearly a polyhedral set.

Recall that when C is convex without lines, the Krein–Milman theorem (cf. Theorem 1.1.3), always permits the decomposition $C_l = K + M$, where K is the convex hull of the extreme points of C, $M = C_\infty$, and $l = 0$. In this case when C_∞ is polyhedral, and when the set of extreme points is bounded (which is the case if C is a polyhedral set), then C is asymptotically polyhedral. It is also easy to note that the extension to sets for which $l > 0$ gives a wider class of sets.

Proposition 2.3.1 *An asymptotically polyhedral set C of $\mathbb{R}^n$ is asymptotically linear.*

Proof. Obviously the proposition holds if it holds for simple asymptotically polyhedral sets. Thus, we suppose that C is a simple asymptotically polyhedral set and let $\{x_k\}$ be a sequence satisfying (2.6), and $\rho > 0$. If (2.7) is not satisfied, there exists a subsequence of $\{x_k - \rho\bar{x}\} \notin C$. Without loss of generality we can suppose that $x_k - \rho\bar{x} \notin C$, and that $\|x_k\| \geq l$ for each k.
Since M is a polyhedral cone, M is finitely generated (cf. Chapter 1) and there exist rays $d_i,\ i = 1, \ldots, r$, such that

$$M = \left\{ y \mid \exists \lambda_i \geq 0 \ i = 1 \ldots r \text{ such that } y = \sum_{i=1}^{r} \lambda_i d_i \right\}.$$

Then for each k, since $x_k \in C_l$, there exist $y_k \in K$, a subset $I_k \subset \{1, \dots, m\}$, and $\lambda_i^k \geq 0$ for $i \in I_k$ such that

$$x_k = y_k + z_k, \; z_k = \sum_{i \in I_k} \lambda_i^k d_i,$$

and we can assume that the vectors $\{d_i\}, i \in I_k$, are linearly independent. Indeed it can be easily seen that M is a finite union of sets of the same type but with the vectors d_i linearly independent.
Since $\{y_k\}$ is bounded, it follows that $\lim_{k \to \infty} \|z_k\| = +\infty$. As a consequence, there exist a subsequence z_{k_m} and a nonempty set of indices I such that the following hold:

$$\lim_{m \to \infty} \lambda_i^{k_m} = +\infty \quad , \quad \frac{\lambda_i^{k_m}}{\sum_{i \in I} \lambda_i^{k_m}} \to \mu_i \geq 0 \quad \forall i \in I, \quad \lambda_i^{k_m} \geq 0 \; \forall i \notin I,$$

the vectors $d_i, i \in I$, are linearly independent, and the sequence $\{\lambda_i^{k_m}\}$ is bounded for $i \notin I$. Now let $w_k = \sum_{i \in I} \lambda_i^k d_i$, $w = \sum_{i \in I} \mu_i d_i$. Then $w \neq 0$ and

$$\lim_{m \to \infty} \frac{w_{k_m}}{\sum_{i \in I} \lambda_i^{k_m}} = w, \quad \bar{x} = \frac{w}{\|w\|},$$

and we thus have

$$x_{k_m} - \rho \bar{x} = y_{k_m} + \sum_{i \in I} \left(\lambda_i^{k_m} - \rho \frac{\mu_i}{\|w\|} \right) d_i + \sum_{i \notin I} \lambda_i^{k_m} d_i.$$

Since for m sufficiently large $\lambda_i^{k_m} - \rho \frac{\mu_i}{\|w\|} \geq 0$ for $i \in I$, it follows that $x_{k_m} - \rho \bar{x} \in C$ for m sufficiently large, which is impossible. □

As we shall see in the next chapter, where we introduce the class of asymptotically stable functions, which contains in particular convex polynomial functions, we can enlarge considerably the class of asymptotically linear sets.

Theorem 2.3.3 *Let C be asymptotically linear. Then $A(C)$ is closed and $A(C_\infty) = (A(C))_\infty$.*

Proof. Let $\{y_k\} \subset A(C)$ converge to y and let $\{x_k\}$ be any sequence satisfying (2.3). Then, since C is asymptotically linear, by definition there exists $\rho_k > 0$ such that for k sufficiently large $x_k - \rho_k \bar{x} \in C, \; \rho_k \leq \|x_k\|$, and we can thus invoke Theorem 2.3.2, which implies the results. □

Closure of the Sum of Closed Convex Sets

The sum of closed convex sets might not be closed even when the sets are themselves closed. It is thus important to find conditions under which closedness is preserved. Obviously, if C_1, C_2 are closed and one of the two sets is also bounded, then $C_1 + C_2$ is also closed. As another application of Theorem 2.3.2, we now derive weaker and general conditions preserving closedness under appropriate assumptions on the sets involved.

Definition 2.3.3 *Let $C_i \subset \mathbb{R}^n,\ i = 1, \dots, m$, be a collection of nonempty closed sets such that for any $z_i \in (C_i)_\infty, i = 1, \dots, m,\ \sum_{i=1}^m z_i = 0$. Then:*
(a) If one has $z_i = 0$ for all $i = 1, \dots, m$, the collection C_i is said to be in general position.
(b) If for any $i = 1, \dots, m$ one has

(i) $z_i \in -(C_i)_\infty$,

(ii) $z_i + C_i \subset C_i$,

the collection C_i is said to be in relative general position.

Remark 2.3.2 Inclusion (ii) is in fact equivalent to $\mathbb{R}_+ z_i + C_i \subset C_i$.

The condition (a) imposed on the set C_i in the definition is stronger than the second one, asking for relative general position of the sets C_i, but is often easier to use for proving closedness of the sum of sets; see Corollary 2.3.4 below.

Theorem 2.3.4 *Let $C_i \subset \mathbb{R}^n, i = 1, \dots, m$, be nonempty closed sets, which are supposed to be in relative general position. Then $\sum_{i=1}^m C_i$ is a closed set, and we have*

$$(C_1 + \cdots + C_m)_\infty \subset (C_1)_\infty + \cdots + (C_m)_\infty,$$

where the inclusion holds as an equation if in addition the C_i are all convex.

Proof. Let $C := C_1 \times \cdots \times C_m$ and define the linear map $A : (x_1, \dots, x_m) \to x_1 + \cdots + x_m,\ x_i \in \mathbb{R}^n$. Then $A(C) = C_1 + \ldots + C_m$. Since by Proposition 2.1.10 one has $C_\infty \subset (C_1)_\infty \times \ldots \times (C_m)_\infty$, then $\forall z \in \ker A \cap C_\infty$ it follows from hypothesis (i) and (ii) of Definition 2.3.3 that $-z + C \subset C$. Therefore, $-2z + C = -z + (-z + C) \subset C$, and by recurrence, $-kz + C \subset C,\ \forall k \in \mathbb{N}_*$. Thus, $\forall x \in C,\ x_k = x - kz \in C$ and $\lim_{k\to\infty} \frac{x_k}{k} = -z$, implying $-z \in C_\infty$. Invoking Corollary 2.3.1 (with hypothesis (a) of that theorem replaced by (ii) of Definition 2.3.3, cf. Remark 2.3.2, the result concerning the inclusion is proved. If the C_i are in addition convex, the reverse inclusion is an immediate consequence of Proposition 2.1.5. □

As another application of Theorem 2.3.4 we obtain the following simple test to check the closedness of the sum of nonempty closed sets.

Corollary 2.3.4 *Let $C_i \subset \mathbb{R}^n, i = 1, \dots, m$, be nonempty closed sets. Suppose that $z_i \in (C_i)_\infty$ with $\sum_{i=1}^m z_i = 0$ implies that $z_i = 0 \ \forall i = 1, \dots, m$. Then $\sum_{i=1}^m C_i$ is a closed set*

The next application of Theorem 2.3.2 concerns a closedness criterion of the set of convex combinations of a finite number of nonempty closed sets $C_i,\ i = 1, \dots, m$, of $\mathbb{R}^n$. Define the following two sets:

$$S := \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^m \lambda_i x_i, \lambda \in \Delta_m,\ x_i \in C_i,\ i = 1, \dots, m \right\} \quad (2.8)$$

$$T := \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^m \lambda_i * x_i, \lambda \in \Delta_m \right\} \quad (2.9)$$

where

$$\lambda_i * x_i := \begin{cases} \lambda_i x_i, & \text{with } x_i \in C_i \text{ if } \lambda_i > 0, \\ x_i, & \text{with } x_i \in (C_i)_\infty \text{ if } \lambda_i = 0, \end{cases}$$

and Δ_m denotes the simplex in $\mathbb{R}^m$.

Theorem 2.3.5 *For a finite collection of nonempty closed sets $C_i \subset \mathbb{R}^n,\ i = 1, \dots, m$, that are in relative general position, one has $\operatorname{cl} S = T$.*

Proof. Let K_i be a cone in $\mathbb{R}^{n+1}$ generated by $\{(1, x_i) \mid x_i \in C_i\}$ and let $D_i := \{(0, x_i) \mid x_i \in (C_i)_\infty\}$. Applying Lemma 2.1.1 for each $i = 1, \dots, m$ we have

$$\operatorname{cl} K_i = K_i \cup D_i. \quad (2.10)$$

Furthermore, since K_i is a cone, we also have

$$\operatorname{cl} K_i = (K_i)_\infty = (\operatorname{cl} K_i)_\infty = K_i \cup D_i. \quad (2.11)$$

We now want to apply Theorem 2.3.4 to the sets $\operatorname{cl} K_i$. Let $z_1, \dots, z_m$ be vectors such that

$$z_i := (t_i, d_i) \in (\operatorname{cl} K_i)_\infty,\ \forall i \in [1, m],\ \sum_{i=1}^m z_i = 0.$$

By (2.11), $z_i = (t_i, d_i) \in (\operatorname{cl} K_i)_\infty$ implies that $z_i = (t_i, d_i) \in K_i \cup D_i$, which in turns implies $t_i = 0,\ d_i \in (C_i)_\infty$, and $\sum_{i=1}^m d_i = 0$, i.e., $z_i = (0, d_i)$ with $d_i \in (C_i)_\infty$, and therefore, since hypothesis (i) of Definition 2.3.3(b) is satisfied for the sets C_i, it is also satisfied for the sets $\operatorname{cl} K_i$, namely $-z_i \in \operatorname{cl} K_i,\ \forall i = 1, \dots, m$. Now, we need to verify that hypothesis (ii) of Definition 2.3.3(b) holds for the sets $\operatorname{cl} K_i$, given that it holds for C_i; i.e., we have to prove that

$$z_i + \operatorname{cl} K_i \subset \operatorname{cl} K_i,\ \forall i = 1, \dots, m. \quad (2.12)$$

Let $z_i = (0, d_i),\ d_i \in (C_i)_\infty$ and take any $0 \neq y_i \in \operatorname{cl} K_i$. Then by (2.10), $y_i \in K_i$ or $y_i \in D_i$. If $y_i \in K_i$, one has $y_i = \lambda_i(1, \bar{y}_i), \lambda_i > 0$, with $\bar{y}_i \in C_i$ and

$$y_i + z_i = (\lambda_i, \lambda_i \bar{y}_i) + (0, d_i) = \lambda_i \left(1, \bar{y}_i + \frac{d_i}{\lambda_i}\right).$$

Since $\bar{y}_i \in C_i,\ \lambda_i^{-1} d_i \in (C_i)_\infty$, and the sets C_i satisfy hypothesis (ii) of Definition 2.3.3(b), then $u_i := \bar{y}_i + \lambda_i^{-1} d_i \in C_i$, and hence $y_i + z_i = \lambda_i(1, u_i) \in K_i$. Otherwise, if $y_i \in D_i$, one has $y_i = (0, \bar{d}_i),\ \bar{d}_i \in (C_i)_\infty$, and $y_i + z_i = (0, \bar{d}_i + d_i)$. But since $\bar{d}_i \in (C_i)_\infty$, then for all $i = 1, \ldots, m$,

$$\exists t_i^k > 0, t_i^k \to 0^+,\ \exists x_i^k \in C_i \ \text{ such that } \ \bar{d}_i = \lim_{k \to \infty} t_i^k x_i^k,$$

so that

$$\bar{d}_i + d_i = \lim_{k \to \infty} t_i^k \left(x_i^k + \frac{d_i}{t_i^k}\right), \ \text{ with } \ x_i^k + \frac{d_i}{t_i^k} \in C_i,$$

by hypothesis (b)(ii) of Definition 2.3.3 on the sets C_i. This implies that $\bar{d}_i + d_i \in (C_i)_\infty$ and therefore $y_i + z_i \in D_i$, proving (2.12). We can thus apply Theorem 2.3.4 to the sets $\operatorname{cl} K_i$ to conclude that $\operatorname{cl} K_1 + \cdots + \operatorname{cl} K_m$ is closed. But since one always has

$$\operatorname{cl}(K_1 + \cdots + K_m) = \operatorname{cl}(\operatorname{cl} K_1 + \cdots + \operatorname{cl} K_m),$$

it follows that

$$\operatorname{cl}(K_1 + \cdots + K_m) = \operatorname{cl} K_1 + \cdots + \operatorname{cl} K_m. \tag{2.13}$$

Now consider the hyperplane $H := \{(1, x) \,|x \in \mathbb{R}^n\}$ and let

$$\begin{aligned} E &:= (\operatorname{cl} K_1 + \cdots + \operatorname{cl} K_m) \cap H, \\ F &:= \operatorname{cl}(K_1 + \cdots + K_m) \cap H. \end{aligned}$$

Since by (2.11) one has $\operatorname{cl} K_i = K_i \cup D_i$, one can verify that in fact $E = (1, T)$, with T defined in (2.9). Now $y := (t, x) \in F$ if and only if $y = \lim_{k \to \infty} y^k$ with

$$y^k = \sum \lambda_i^k (1, x_i^k),\ \lambda_i^k \geq 0, x_i^k \in C_i \ \text{ and } \ \lim_{k \to \infty} \sum_{i=1}^m \lambda_i^k = 1.$$

Define $\mu_i^k := \lambda_i^k (\sum_{i=1}^m \lambda_i^k)^{-1}$. Then $y \in F$ if and only if $y = (1, x)$ with $x = \lim_{k \to \infty} \sum_{i=1}^m \mu_i^k x_i^k$ with $x_i^k \in C_i$. In other words, one has $F = (1, \operatorname{cl} S)$. Finally, by the definition of E and F and from (2.13) it follows that $T = \operatorname{cl} S$. $\square$

The convex case is much simpler and follows as an immediate consequence of Theorem 2.3.5.

Corollary 2.3.5 *Let $C_i \subset \mathbb{R}^n, i = 1, \dots, m$, be nonempty closed convex sets such that for any $z_i \in (C_i)_\infty$ $\forall i = 1, \dots, m$, and $\sum_{i=1}^m z_i = 0$ one has $z_i \in -(C_i)_\infty$ for each $i = 1, \dots, m$. Then*

$$T = \operatorname{cl} S = \operatorname{cl}(\operatorname{conv} \cup_{i=1}^m C_i),$$

where S, T are as defined in (2.8)–(2.9).

Proof. Note that when C_i is convex for each i, then one has $S = \operatorname{conv}(\cup_{i=1}^m C_i)$, and hypothesis (ii) of Definition 2.3.3(b) holds. Applying Theorem 2.3.5, the result follows. $\square$

As another consequence of Theorem 2.3.5 we obtain a decomposition formula for the closed convex hull of an arbitrary closed set in $\mathbb{R}^n$, in terms of its convex hull and the associated asymptotic cone. This leads to the following notions.

Definition 2.3.4 *Let C be a nonempty set of $\mathbb{R}^n$ and suppose that for any vectors $c_1, \dots, c_{n+1}$ such that $\sum_{i=1}^{n+1} c_i = 0$, $c_i \in C_\infty$, $i = 1, \dots, n+1$, one has*

$$c_i \in -C_\infty, \quad c_i + C \subset C, \ \forall i = 1, \dots, n+1.$$

Then C is said to be weakly semibounded. If instead of the above inclusion we have the stronger assumption

$$c_i = 0 \quad \forall i = 1 \dots m,$$

C is said to be semibounded.

Another way to say that a set C is semibounded is to suppose that C_∞ is a pointed cone; cf. Definition 1.1.3.

Corollary 2.3.6 *Let C be a closed weakly semibounded set. Then*

$$\operatorname{cl} \operatorname{conv} C = \operatorname{conv} C + \operatorname{conv} C_\infty.$$

Proof. Apply Theorem 2.3.5 with $m = n+1$ and $C_i = C$ $\forall i$. $\square$

Lemma 2.3.1 *Let K be a closed pointed cone in $\mathbb{R}^n$. Then:*
(a) $\exists \theta > 0$ such that $\|x_i\| \le \theta \| \sum_{i=1}^{n+1} x_i \|$, $\forall x_i \in K, i = 1, \dots, n+1$.
(b) $\operatorname{conv} K$ is closed pointed cone.

Proof. (a) The proof is by contradiction. Then for all $j \in \mathbb{N}$, there exist for $i = 1, \dots, n+1$, $u_i^j \in K$ such that $\|u_i^j\| > j \| \sum_{i=1}^{n+1} u_i^j \|$. Since K is pointed, it follows that $\sum_{i=1}^{n+1} u_i^j \neq 0$, and then if we set

$$x_i^j = \left\| \frac{u_i^j}{\sum_{i=1}^{n+1} u_i^j} \right\|, \quad y^j = \sum_{i=1}^{n+1} x_i^j,$$

we have

$$\forall i \quad x_i^j \in K,\ \|y^j\| = 1,\ \lambda^j := \left(\max_{1\le i\le n+1} \|x_i^j)\|\right)^{-1} \to 0.$$

Since $\|\lambda^j x_i^j\| \le 1$, there exists a subsequence $\{\lambda^{j_l}, x_i^{j_l},\ i = 1,\dots,n+1\}$ such that for each i, $\lambda^{j_l} x_i^{j_l} \to z_i$. Now since K is a closed cone, $z_i \in K$ and since $\|\sum_{i=1}^{n+1} \lambda^{j_l} x_i^{j_l}\| = \lambda^{j_l}$, it follows that $\sum_{i=1}^{n+1} z_i = 0$. Furthermore, since one has $1 \le \sum_{i=1}^{n+1} \lambda^j \|x_i^j\| \le n+1$, then at least one z_i is nonzero, which contradicts the hypothesis that K is pointed.
(b) The fact that $\operatorname{conv} K$ is pointed is obvious. Now let us prove that it is closed, and let $x^j \in \operatorname{conv} K$ with $x^j \to x$. Then by Theorem 1.1.1 there exist $x_i^j \in K$ with $i = 1,\dots,n+1$ and such that $x^j = \sum_{i=1}^{n+1} x_i^j$. By part (a) the sequences $\{x_i^j\}_{j\in\mathbb{N}}$ are bounded. Passing to subsequences if necessary, it follows that there exist $\{x_i^j\}$ with $x_i^j \to y_i$. Then, $x = \sum_{i=1}^{n+1} y_i \in \operatorname{conv} K$, and we can conclude that $\operatorname{conv} K$ is closed. □

Lemma 2.3.2 *Let C be a nonempty set of $\mathbb{R}^n$. Then:*
(a) $(\operatorname{conv} C)_\infty = \operatorname{conv} C_\infty$.
(b) If in addition C is semibounded then $\operatorname{conv} C$ *is semibounded.*

Proof. (a) Clearly, one has that $(\operatorname{conv} C)_\infty$ is a convex set. Therefore, the inclusion $C_\infty \subset (\operatorname{conv} C)_\infty$ implies that $\operatorname{conv} C_\infty \subset (\operatorname{conv} C)_\infty$. To prove the reverse inclusion $(\operatorname{conv} C)_\infty \subset \operatorname{conv} C_\infty$, let $d \in (\operatorname{conv} C)_\infty$. Then there exist $x_k \in \operatorname{conv} C,\ t_k \to +\infty$ such that $t_k^{-1} x_k \to d$. Every $x_k \in \operatorname{conv} C$ can be written as

$$x_k = \sum_{i=0}^{n} \lambda_k^i x_k^i,\ \text{with } x_k^i \in C,\ \lambda_k^i \ge 0, \sum_{i=0}^{n} \lambda_k^i = 1.$$

Since $\{t_k^{-1} \lambda_k^i x_k^i\}$ is a bounded sequence, passing to subsequences if necessary, it follows that $t_k^{-1} \lambda_k^i x_k^i \to d^i \in C_\infty$ and therefore $d = \sum_{i=0}^{n} d^i$, so that $d \in \operatorname{conv} C_\infty$. To prove (b), note that $C \subset \mathbb{R}^n$ being semi-bounded means that C_∞ is pointed. Invoking Lemma 2.3.1(b), the latter implies that $\operatorname{conv} C_\infty$ is pointed, which in turn implies that $(\operatorname{conv} C)_\infty$ is pointed and hence $\operatorname{conv} C$ is semibounded. □

Lemma 2.3.3 *Let C be a nonempty convex and semibounded set of $\mathbb{R}^n$. Then*

$$\operatorname{int} C_\infty \ne \emptyset \Longleftrightarrow \operatorname{cl} \operatorname{dom} \sigma_C \ \textit{ is pointed.}$$

Proof. Applying Theorem 2.2.1 to the closed convex set $\operatorname{cl} C$ one has $C_\infty = (\operatorname{cl} C)_\infty = (\operatorname{cl} \operatorname{dom} \sigma_C)^*$, and the desired result follows from Proposition 1.1.15. □

2.4 Continuous Convex Sets

Convex compact sets in $\mathbb{R}^n$ enjoy many important properties that are not shared by closed convex sets in general. For example, if C and D are compact and convex, then the sets $C + D$ and $\operatorname{conv}(C \cup D)$ enjoy the same property. Moreover, if $C \cap D = \emptyset$, then C and D can be strongly separated by a hyperplane and $\delta(C, D) = \inf\{\|c - d\| \mid c \in C, d \in D\} > 0$. In general, none of these properties remain true for arbitrary closed convex sets in $\mathbb{R}^n$. However, there is a natural class of closed convex sets in $\mathbb{R}^n$ that are not necessarily compact, yet which enjoy these properties. This is the class of *continuous* sets.

Definition 2.4.1 *A nonempty closed convex set of $\mathbb{R}^n$ is called continuous if its support function $x \to \sigma_C(x)$ is continuous for any $x \neq 0$.*

We recall that if $\sigma_C(x) = +\infty$, the continuity of σ_C at x means that there exists a neighborhood V of x such that $\sigma_C(y) = +\infty$ for all $y \in V$.
Continuous convex sets can be characterized in a more geometric sense using the concepts of boundary rays and asymptotes.

Definition 2.4.2 *Let C be a nonempty closed set of $\mathbb{R}^n$. A boundary ray of the set C is a half-line that is contained in the boundary of C. An asymptote of C is a half-line ρ contained in $\mathbb{R}^n \setminus C$ such that*

$$\delta(\rho, C) := \inf\{\|x - y\| \mid x \in \rho, y \in C\} = 0.$$

Proposition 2.4.1 *Let C be a nonempty closed convex set of $\mathbb{R}^n$, $x \in C$, and suppose that the half-line $\{y + t\rho : \ t \geq 0\}$ with $\rho \in C_\infty$ and $y \in \mathbb{R}^n$ has an empty intersection with C. For $\lambda \in \mathbb{R}$, let $x(\lambda) = (1 - \lambda)x + \lambda y$. Then there exist $\lambda \in [0, 1], s \geq 0$ such that the half-line $D(\lambda, s) = \{x(\lambda) + t\rho \mid t \geq s\}$ is a boundary ray or an asymptote of C.*

Proof. For any $\lambda \in \mathbb{R}$ and any $t \geq 0$, define $G(\lambda, t) = \operatorname{dist}(x(\lambda) + t\rho, C)$ and $g(\lambda) = \inf\{G(\lambda, t) \mid t \geq 0\}$. Then, since C is a closed convex set, the function g is finite and convex on $\mathbb{R}$, and as a consequence, is continuous. Let $E := \{\lambda \in [0, 1] \mid g(\lambda) = 0\}$. Since $x \in C$ and $\rho \in C_\infty$, one has $0 \in E$ and since g is continuous, it follows that E is a nonempty compact set. Therefore, there exists $\lambda^* \in [0, 1]$ that maximizes λ on E. If $\lambda^* = 1$, then by definition $D(1, 0)$ is an asymptote of C; otherwise $\lambda^* < 1$. Now suppose that $D(\lambda^*, 0)$ is not an asymptote. Then $D(\lambda^*, 0) \cap C \neq \emptyset$. Take $u \in D(\lambda^*, 0) \cap C$. Then $D := \{u + t\rho \mid t \geq 0\} \subset C$. Without loss of generality we can suppose that $\operatorname{int} C \neq \emptyset$ (otherwise, D would be a boundary ray), and if D were not a boundary ray, there would exist $s \geq 0$ such that $z := u + s\rho \in \operatorname{int} C$. As a consequence, there would exist $\Delta\lambda > 0$ such that $\{x(\lambda^* + \Delta\lambda) + t\rho \mid t \geq 0\} \cap C \neq \emptyset$, a contradiction with the fact that λ^* is the maximum of λ on E. $\square$

The next result gives an important characterization of continuous convex sets.

Theorem 2.4.1 *A nonempty closed convex subset C of $\mathbb{R}^n$ is continuous if and only if C has no boundary ray or asymptote.*

Proof. Assume first that C has no boundary ray or asymptote and suppose that C is not continuous at some point $u \neq 0$. Then there exists a sequence $\{u_k\}$ converging to u such that $\sigma_C(u_k)$ does not converge to $\sigma_C(u)$. Since σ_C is lsc, it follows that $\alpha := \limsup_{k\to\infty} \sigma_C(u_k) > \sigma_C(u)$. Furthermore, there exist $\beta \in \mathbb{R}$ with $\sigma_C(u) < \beta \leq \alpha$ and a sequence $\{x_{k_j}\} \subset C$ such that

$$\lim_{j\to\infty} \langle u_{k_j}, x_{k_j}\rangle \geq \beta. \tag{2.14}$$

When $\alpha \in \mathbb{R}$, taking $\beta = \alpha$, the latter inequality follows from the definition of the support functional, and when $\alpha = +\infty$, it follows for the same reason. Let us prove now that the sequence $\{x_{k_j}\}$ cannot be bounded. Assume the contrary. Then with $\{x_{k_j}\}$ bounded, without loss of generality we can suppose that $x_{k_j} \to x \in C$, and taking the limit in (2.14) it follows that $\sigma_C(u) < \beta \leq \langle u, x\rangle$, which is impossible. Thus, without loss of generality we can suppose that $\|x_{k_j}\| \to +\infty$ and $x_{k_j}\|x_{k_j}\|^{-1} \to \bar{x}$. Dividing both members of (2.14) by $\|x_{k_j}\|$ and passing to the limit when $j \to +\infty$, it follows that $\langle u, \bar{x}\rangle \geq 0$. Now let $y \notin C$, with $\langle y, u\rangle > \sigma_C(u)$; as a consequence it follows that $\langle y + \lambda\bar{x}, u\rangle > \sigma_C(u), \ \ \forall\lambda > 0$, and then invoking Theorem 1.3.2, this implies that $y + \lambda\bar{x} \notin C, \ \ \forall\lambda > 0$. Therefore, by Proposition 2.4.1, there exists some ray of direction $\bar{x}$, which is a boundary ray or an asymptote of C, in contradiction to the hypothesis of the theorem. We now prove the reverse statement. Suppose that σ_C is continuous, and there exist some ray y and a direction $\bar{x}$ such that the set $D = \{\bar{x} + \lambda y \mid \lambda \geq 0\}$ is a boundary ray or an asymptote. In both cases, invoking Proposition 1.1.11, it can be separated from C by a hyperplane; i.e., there exist $0 \neq a \in \mathbb{R}^n, \ \ b \in \mathbb{R}$ such that

$$\inf_{z\in D} \langle a, z\rangle \geq b, \ \ \sup_{x\in C} \langle a, x\rangle \leq b. \tag{2.15}$$

Let us prove that σ_C is not continuous at a. Since for $z \in D$ one has $\langle a, z\rangle = \langle a, \bar{x}\rangle + \lambda\langle a, y\rangle$, letting $\lambda \to +\infty$, it follows from the first inequality in (2.15) that $\langle a, y\rangle \geq 0$. Suppose that D is a boundary ray. Then from the second inequality in (2.15) it follows that $\lambda\langle a, y\rangle \leq b - \langle a, \bar{x}\rangle$, and letting $\lambda \to +\infty$, this implies $\langle a, y\rangle \leq 0$, so that we have proved that $\langle a, y\rangle = 0$. The same holds if D is an asymptote. Indeed, by definition there exist

$z_k = \bar{x} + \lambda_k y$, with $\lambda_k \to +\infty$, $x_k \in C$ such that $\|x_k - z_k\| \to 0$. Then using the second inequality in (2.15) we obtain

$$\langle a, x_k \rangle = \langle a, \bar{x} \rangle + \lambda_k \langle a, y \rangle + \langle a, x_k - z_k \rangle \le b,$$

and passing to the limit we obtain $\langle a, y \rangle \le 0$, and hence $\langle a, y \rangle = 0$. Now from the second inequality in (2.15) one has that $\sigma_C(a)$ is finite. Let $\varepsilon > 0$ and let us prove that $\sigma_C(a + \varepsilon y) = +\infty$. This will prove that σ_C is not continuous at a. We consider two cases. If D is a boundary ray, then for all $\lambda > 0$,

$$\sigma_C(a + \varepsilon y) \ge \langle a + \varepsilon y, \bar{x} + \lambda y \rangle = \langle a, \bar{x} \rangle + \varepsilon \langle y, \bar{x} \rangle + \varepsilon \lambda \|y\|^2 + \lambda \langle a, y \rangle,$$

and letting $\lambda \to +\infty$, it follows that $\sigma_C(a+\varepsilon y) = +\infty$. If D is an asymptote,

$$\sigma_C(a + \varepsilon y) \ge \langle a + \varepsilon y, x_k \rangle = \langle a + \varepsilon y, \bar{x} \rangle + \langle a + \varepsilon y, x_k - z_k \rangle + \lambda_k \langle a + \varepsilon y, y \rangle.$$

Since $\langle a + \varepsilon y, x_k - z_k \rangle \to 0$ and since $\lambda_k \langle a + \varepsilon y, y \rangle = \varepsilon \lambda_k \|y\|^2$, it follows that $\sigma_C(a + \varepsilon y) = +\infty$. □

As previously discussed in the beginning of Section 2.3, a key question in convex analysis is to know when the image of a closed convex set under a linear map remains closed. As demonstrated Example 2.3.1, where the set is continuous, the standard sufficient condition (2.2) fails, and in Theorem 2.3.2 we gave a necessary and sufficient condition for preserving closedness of the image of a closed convex set. However, thanks to this theorem, the next result shows that for continuous convex sets, the closedness property is guaranteed.

Proposition 2.4.2 *Let $C \subset \mathbb{R}^n$ be a nonempty closed convex and continuous set and let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then $A(C)$ is closed. Furthermore, if A is a surjective map, then $A(C)$ is a continuous set.*

Proof. In order to prove that $A(C)$ is closed we can suppose that $\operatorname{aff} A(C) \neq A(C)$. Otherwise, there is nothing to prove. Then there exists $u \in \operatorname{bd}(\operatorname{cl} AC)$ with $u \notin \operatorname{ri} A(C)$, and invoking the separation Proposition 1.1.11 there exists $0 \neq a \in \mathbb{R}^n$ such that the hyperplane $H = \{x \in \mathbb{R}^n \mid \langle a, x \rangle = \beta\}$ separates properly $\{u\}$ and $\operatorname{cl} A(C)$, namely, one has

$$\langle a, u \rangle = \beta,\ \langle a, z \rangle \le \beta,\ \forall z \in \operatorname{cl} A(C),\ \inf_{z \in \operatorname{cl} A(C)} \langle a, z \rangle < \beta,$$

so that

$$\langle A^T a, x \rangle \le \beta,\ \forall x \in C.$$

Now take d and x satisfying

$$\langle A^T a, d \rangle = \beta,\ \langle A^T a, x \rangle < \beta,$$

and set $z = 2d - x$. In order to prove that $A(C)$ is closed, let $\{y_k\}$ be a sequence converging to y. Then invoking Theorem 2.3.2 it is sufficient to prove only that there cannot exist any sequence $\{x_k\}$ satisfying (2.3). Suppose the contrary, and let $\{x_k\}$ be such a sequence. Then since $A\bar{x} = 0$, for each $\lambda \geq 0$ one obtains

$$\begin{aligned}\langle A^T a, z + \lambda\bar{x}\rangle &= 2\langle A^T a, d\rangle - \langle A^T a, x\rangle + \lambda\langle A^T a, \bar{x}\rangle \\ &= 2\beta - \langle A^T a, x\rangle > \beta,\end{aligned}$$

showing that the ray $\Delta = \{z + \lambda\bar{x} : \lambda \geq 0\}$ does not meet C. Therefore, since $\bar{x} \in C_\infty$, it follows from Proposition 2.4.1 that C is not a continuous set, leading to the desired contradiction. To prove the last part of the proposition, assume now that the linear map is surjective. Since $(\ker(A^T))^\perp = A(\mathbb{R}^n) = \mathbb{R}^m$, it follows that $\ker(A^T) = \{0\}$. Furthermore, from Proposition 1.3.3(e) we have $\sigma_{A(C)}(z) = \sigma_C(A^T z)$. Since $A^T z \neq 0,\ \forall z \neq 0$, and since C is continuous, this implies from Definition 2.4.1 that $\sigma_{A(C)}$ is continuous on $\mathbb{R}^m \setminus \{0\}$, and hence $A(C)$ is continuous. $\square$

Proposition 2.4.3 *For any nonempty closed convex and unbounded set $C \subset \mathbb{R}^n$, $C \not\equiv \mathbb{R}^n$, the following statements are equivalent:*
(a) C is continuous.
(b) $\operatorname{dom}\sigma_C \setminus \{0\} = \operatorname{int}(\operatorname{dom}\sigma_C) \neq \emptyset$.
(c) $\arg\sup\sigma_C(d)$ *is a nonempty compact set for all* $d \in \operatorname{dom}\sigma_C \setminus \{0\}$.

Proof. We first show that $(a) \Longrightarrow (b)$. If C is continuous, since C is assumed unbounded then $\operatorname{dom}\sigma_C \neq \mathbb{R}^n$ and $\operatorname{int}\operatorname{dom}\sigma_C \subset \operatorname{dom}\sigma_C \setminus \{0\}$. Furthermore, since $C \not\equiv \mathbb{R}^n$, one has $\operatorname{dom}\sigma_C \setminus \{0\} \neq \emptyset$. Now let $0 \neq d \in \operatorname{dom}\sigma_C$. Then $d \in \operatorname{int}\operatorname{dom}\sigma_C$. Indeed, in the contrary case there would exist a sequence $\{d_k\}$ converging to d and such that $\sigma_C(d_k) = +\infty$, which is impossible, since $\sigma_C(d) = \lim_{k\to\infty}\sigma_C(d_k)$. To prove the reverse implication $(b) \Longrightarrow (a)$, note that by Theorem 1.2.3, σ_C is continuous on $\operatorname{dom}\sigma_C \setminus \{0\}$, hence everywhere on $\mathbb{R}^n \setminus \{0\}$, implying that the set C is continuous. Finally, by Proposition 1.2.18 one has $\arg\sup\sigma_C(d) = \partial\sigma_C(d)$. But, by Proposition 1.2.16 the subdifferential of σ_C at d is nonempty and compact if and only if $d \in \operatorname{int}\operatorname{dom}\sigma_C$, and this proves the equivalence between (b) and (c). $\square$

2.5 Asymptotic Functions

Building on the concept of asymptotic cone, we are now interested in understanding the behavior in the large of functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

The epigraph plays a key role. Let F be a nonempty closed set in $\mathbb{R}^{n+1}$ satisfying the property

$$(x, \mu) \in F \Longrightarrow (x, \mu') \in F,\ \forall \mu' > \mu. \tag{2.16}$$

Then clearly, there exists one and only one function g such that $\operatorname{epi} g = F$, defined by

$$g(x) := \inf\{\mu \mid (x, \mu) \in F\}.$$

Now let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper. Then $\operatorname{epi} f$ is nonempty, and therefore $(\operatorname{epi} f)_\infty$ is a nonempty closed cone in $\mathbb{R}^{n+1}$ satisfying (2.16); i.e.,

$$(x, \mu) \in (\operatorname{epi} f)_\infty \Longrightarrow (x, \mu') \in (\operatorname{epi} f)_\infty,\ \forall \mu' > \mu.$$

Indeed, from the definition of the asymptotic cone given in Definition 2.1.2 and applied to the set $\operatorname{epi} f$ we have

$$\exists (x_k, \mu_k) \in \operatorname{epi} f,\ \exists t_k \to \infty :\ t_k^{-1}(x_k, \mu_k) \to (x, \mu).$$

Set $\mu_k^{'} := \mu_k + (\mu' - \mu)t_k$. Since $\mu' > \mu$, we have $\mu'_k > \mu_k \geq f(x_k)$ and $t_k^{-1}(x_k, \mu'_k) \to (x, \mu')$, showing that $(x, \mu') \in (\operatorname{epi} f)_\infty,\ \forall \mu' > \mu$.
The above discussion leads us to introduce the following concept.

Definition 2.5.1 *For any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, there exists a unique function $f_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ associated with f, called the asymptotic function, such that* $\operatorname{epi} f_\infty = (\operatorname{epi} f)_\infty$.

This definition indicates that the epigraph of an asymptotic function is in fact a closed cone. This can be further elaborated through the use of lower semicontinuity and positively homogeneous functions; cf. Definition 1.3.2. The asymptotic function f_∞ enjoys the following basic properties.

Proposition 2.5.1 *For any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we have:*
(a) f_∞ is lsc and positively homogeneous.
(b) $f_\infty(0) = 0$ or $f_\infty(0) = -\infty$.
(c) If $f_\infty(0) = 0$, then f_∞ is proper.

Proof. (a) Since by Definition (2.5.1) $\operatorname{epi} f_\infty = (\operatorname{epi} f)_\infty$ and $(\operatorname{epi} f)_\infty$ is a closed set by definition, it follows that f_∞ is lsc. First, note that $0 \in \operatorname{dom} f_\infty$. So, let $x \in \operatorname{dom} f_\infty$. Since $\operatorname{epi} f_\infty$ is a cone, we have

$$(x, f_\infty(x)) \in \operatorname{epi} f_\infty \implies (\lambda x, \lambda f_\infty(x)) \in \operatorname{epi} f_\infty,\ \forall \lambda > 0,$$

i.e., $f_\infty(\lambda x) \leq \lambda f_\infty(x)$. Likewise, one has $(\lambda x, f_\infty(\lambda x)) \in \operatorname{epi} f_\infty,\ \forall x \in \operatorname{dom} f_\infty,\ \forall \lambda > 0$, and hence $(x, \lambda^{-1} f_\infty(\lambda x)) \in \operatorname{epi} f_\infty$ by the cone property. Therefore, $\lambda f_\infty(x) \leq f_\infty(\lambda x)$, and (a) is proved whenever $x \in \operatorname{dom} f$. Finally, if $x \notin \operatorname{dom} f_\infty$, then $\lambda x \notin \operatorname{dom} f_\infty,\ \forall \lambda > 0$, and hence $f_\infty(\lambda x) = \lambda f_\infty(x) = +\infty$.

(b) Since f is proper, then $\operatorname{epi} f$ is nonempty, and hence either $f_\infty(0)$ is finite or $f_\infty(0) = -\infty$. If $f_\infty(0)$ is finite, then by (a), $f_\infty(0) = \lambda f_\infty(0)$, $\forall \lambda > 0$, and hence $f_\infty(0) = 0$.
(c) Suppose f_∞ is not a proper function. Then there exists x such that $f_\infty(x) = -\infty$. Now let $\{\lambda_k\} \subset \mathbb{R}_{++}$ be a positive sequence converging to 0. Then $\lambda_k x \to 0$, and under our assumption $f_\infty(0) = 0$, the lower semicontinuity of f_∞, and property (a), we obtain

$$0 = f_\infty(0) \le \liminf_{k\to\infty} f_\infty(\lambda_k x) = \liminf_{k\to\infty} \lambda_k f_\infty(x) = -\infty,$$

leading to a contradiction. □

We can now give a fundamental analytic representation of the asymptotic function f_∞.

Theorem 2.5.1 *For any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ the asymptotic function f_∞ is given by*

$$f_\infty(d) = \liminf_{\substack{d' \to d \\ t \to +\infty}} \frac{f(td')}{t}, \tag{2.17}$$

or equivalently

$$f_\infty(d) = \inf \left\{ \liminf_{k\to\infty} \frac{f(t_k d_k)}{t_k} \mid t_k \to +\infty, d_k \to d \right\}, \tag{2.18}$$

where $\{t_k\}$ and $\{d_k\}$ are sequences in $\mathbb{R}$ and $\mathbb{R}^n$, respectively.

Proof. The equivalence of the two formulas above is clear. Let $g(d)$ denote the right-hand side of (2.18). We will first show that $(\operatorname{epi} f)_\infty \subset \operatorname{epi} g$. Let $(d, \mu) \in (\operatorname{epi} f)_\infty$. Then by Definition 2.1.2, $\exists t_k \to \infty$, $(d_k, \mu_k) \in \operatorname{epi} f$ such that $t_k^{-1}(d_k, \mu_k) \to (d, \mu)$. As a consequence, since $f(d_k) \le \mu_k$ can be written as $t_k^{-1} f(t_k^{-1} d_k \cdot t_k) \le t_k^{-1}\mu_k$, passing to the limit, it follows that $g(d) \le \mu$ and hence $(d, \mu) \in \operatorname{epi} g$. Conversely, let $(d, \mu) \in \operatorname{epi} g$. By definition of g, there exist sequences $\{d_k\} \subset \mathbb{R}^n$ and $\{t_k\} \in \mathbb{R}$ such that

$$g(d) = \lim_{k\to\infty} \frac{f(t_k d_k)}{t_k},\ t_k \to \infty,\ d_k \to d,$$

and since $(d, \mu) \in \operatorname{epi} g$, it follows from the definition of the limit that $\forall \varepsilon > 0$ and all $k \in \mathbb{N}$ sufficiently large, we have $f(t_k d_k) \le (\mu + \varepsilon) t_k$ and hence $z_k := t_k(d_k, \mu + \varepsilon) \in \operatorname{epi} f$. Since $t_k^{-1} z_k \to (d, \mu + \varepsilon)$, it follows that $(d, \mu + \varepsilon) \in (\operatorname{epi} f)_\infty$, and therefore since $(\operatorname{epi} f)_\infty$ is a closed set and $\varepsilon > 0$ was arbitrary, we also have $(d, \mu) \in (\operatorname{epi} f)_\infty$. □

Corollary 2.5.1 *For a nonempty set $C \subset \mathbb{R}^n$ one has $(\delta_C)_\infty = \delta_{C_\infty}$.*

Proof. This is an immediate consequence of the definition of the asymptotic cone and the previous theorem.

Corollary 2.5.2 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper function such that* $\operatorname{dom} f^*$ *is nonempty. Then* $f_\infty(0) = 0$.

Proof. Indeed, in the contrary case $f_\infty(0) = -\infty$, but by hypothesis there exist $v \in \operatorname{dom} f^*,\ \alpha \in \mathbb{R}$ such that

$$f(x) \geq \langle v, x\rangle - \alpha \quad \forall x.$$

Then from the above inequality we get $t_k^{-1} f(t_k d_k) \geq \langle v, d_k\rangle - \alpha t_k^{-1}$. Passing to the limit with $d_k \to 0,\ t_k \to \infty$, it follows from Theorem 2.5.1 that $f_\infty(0) \geq 0$, which is impossible.
When f is a convex function, there are simpler ways to express the asymptotic function, which is also called in this case the recession function. We will keep the terminology asymptotic function in both the convex and nonconvex cases.

Proposition 2.5.2 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper, lsc, convex function. The asymptotic function is a positively homogeneous, lsc, proper convex function, and for any* $d \in \mathbb{R}^n$*, one has*

$$f_\infty(d) = \sup\{f(x+d) - f(x) \mid x \in \operatorname{dom} f\} \tag{2.19}$$

and

$$f_\infty(d) = \lim_{t \to +\infty} \frac{f(x+td) - f(x)}{t} = \sup_{t>0} \frac{f(x+td) - f(x)}{t}, \ \forall x \in \operatorname{dom} f. \tag{2.20}$$

Proof. By Proposition 2.5.1(a), the asymptotic function f_∞ is lsc and positively homogeneous, while the convexity of f_∞ follows from the convexity of f. Since f_∞ is lsc, that f_∞ is proper follows from (2.19), which we now prove. By definition, the asymptotic function f_∞ associated with f is uniquely determined via $\operatorname{epi} f_\infty = (\operatorname{epi} f)_\infty$. Using Proposition 2.1.5 via the representation F of the asymptotic cone, one has $(d, \mu) \in (\operatorname{epi} f)_\infty$ if and only if for all $(x, \alpha) \in \operatorname{epi} f$ it holds that $(x, \alpha) + (d, \mu) \in \operatorname{epi} f$, namely if and only if $f(x+d) \leq \alpha + \mu$. The latter inequality is clearly equivalent to $f(x+d) - f(x) \leq \mu,\ \forall x \in \operatorname{dom} f$, proving formula (2.19). To verify (2.20), let $x \in \operatorname{dom} f$. Then again using Proposition 2.1.5 but via the representation D of the asymptotic cone, one has for any $x \in \operatorname{dom} f$,

$$(\operatorname{epi} f)_\infty = \{(d, \mu) \in \mathbb{R}^n \times \mathbb{R} \mid (x, f(x)) + t(d, \mu) \in \operatorname{epi} f,\ \forall t > 0\},$$

and hence $(d, \mu) \in (\operatorname{epi} f)_\infty$ if and only if for any $x \in \operatorname{dom} f$ we have

$$f(x+td) \leq f(x) + t\mu,\ \forall t > 0,$$

which means exactly that

$$g(d) := \sup_{t>0} \frac{f(x+td) - f(x)}{t} \le \mu,$$

and hence $(\operatorname{epi} f)_\infty = \operatorname{epi} g$, $\forall x \in \operatorname{dom} f$, proving the formula (2.20). That the limit in t in the first part of formula (2.20) coincides with the supremum in $t > 0$ simply follows by recalling that the convexity of f implies that for fixed $x, d \in \mathbb{R}^n$, for any $\tau > 0$, the function $\tau \to (\tau)^{-1}(f(x+\tau d) - f(x))$ is nondecreasing. $\square$

The next result gives a useful simplification of the formula for f_∞.

Corollary 2.5.3 *For any lsc, proper, and convex function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$*, one has*

$$f_\infty(d) = \lim_{t\to 0^+} t f(t^{-1}d),\ \forall d \in \operatorname{dom} f.$$

If $0 \in \operatorname{dom} f$*, the formula holds for every* $d \in \mathbb{R}^n$*.*

Proof. When $0 \in \operatorname{dom} f$ one has $f(0) < \infty$, and the formula above is immediate from (2.20). Consider now the case where $0 \notin \operatorname{dom} f$. Using (2.20) with $x := d \in \operatorname{dom} f$, one has for any $d \in \operatorname{dom} f$,

$$f_\infty(d) = \lim_{t\to+\infty} t^{-1}(f((1+t)d) - f(d)),$$

from which the desired formula for f_∞ follows. $\square$

Example 2.5.1 *(Some useful asymptotic functions.)* (a) Let Q be a symmetric $n \times n$ positive semidefinite matrix and $f(x) := (1 + \langle x, Qx\rangle)^{\frac{1}{2}}$. Then $f_\infty(d) = \langle d, Qd\rangle^{\frac{1}{2}}$.
(b) For the quadratic function associated with Q positive semidefinite and given by $f(x) := \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle + s,\ c \in \mathbb{R}^n, s \in \mathbb{R}$, one has

$$f_\infty(d) = \begin{cases} \langle c, d\rangle & \text{if } Qd = 0, \\ +\infty & \text{if } Qd \neq 0. \end{cases}$$

For a nonconvex quadratic function, that is, with Q not positive semidefinite, one needs to apply (2.17), and in that case one obtains

$$f_\infty(d) = \begin{cases} -\infty & \text{for } d \text{ with } d^TQd \le 0, \\ +\infty & \text{for } d \text{ with } d^TQd > 0. \end{cases}$$

Thus, the asymptotic function of a proper function, and in that case even of a finite function, can be improper.
(c) Let $f(x) = \sum_{j=1}^n e^{x_j}$. Then $f_\infty(d) = \delta_{\mathbb{R}^n_-}$.
(d) Let $f(x) = \log \sum_{j=1}^n e^{x_j},\ n > 1$. Then $f_\infty(d) = \max_{1\le j\le n} d_j$.

Further examples of asymptotic functions will be given in Sections 2.6–2.7.

Proposition 2.5.3 *For any proper function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *and any* $\alpha \in \mathbb{R}$ *such that* $\operatorname{lev}(f, \alpha) \neq \emptyset$ *one has* $(\operatorname{lev}(f, \alpha))_\infty \subset \operatorname{lev}(f_\infty, \alpha)$*, i.e.,*

$$\{x \mid f(x) \leq \alpha\}_\infty \subset \{d \mid f_\infty(d) \leq 0\}.$$

Equality holds in the inclusion when f *is lsc, proper, and convex.*

Proof. Since $\operatorname{lev}(f, \alpha) \neq \emptyset$, take $d \in (\operatorname{lev}(f, \alpha))_\infty$. We have to show that $f_\infty(d) \leq 0$, $\forall d$. By definition, $d \in (\operatorname{lev}(f, \alpha))_\infty$ means that

$$\exists x_k \in \operatorname{lev}(f, \alpha),\ \exists t_k \to +\infty : \lim_{k \to \infty} t_k^{-1} x_k = d.$$

Set $d_k = t_k^{-1} x_k$. Then $d_k \to d$, and since $x_k \in \operatorname{lev}(f, \alpha)$, we have $t_k^{-1} f(t_k d_k) = t_k^{-1} f(x_k) \leq t_k^{-1} \alpha \to 0$, which implies by the fundamental formula (2.18) of an asymptotic function that $f_\infty(d) \leq 0$, $\forall d$. In the case that f is assumed proper lsc convex, to prove the reverse inclusion, let d be such that $f_\infty(d) \leq 0$. Then for each $x \in \operatorname{lev}(f, \alpha)$ and $\lambda > 0$ one has $f(x+\lambda d) - f(x) \leq \lambda f_\infty(d) \leq 0$, so that $x + \lambda d \in \operatorname{lev}(f, \alpha)$, which by Proposition 2.1.5 means that $d \in (\operatorname{lev}(f, \alpha))_\infty$.

□

Corollary 2.5.4 *Let* $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in I$ *a collection of proper functions and, let* $S \subset \mathbb{R}^n$ *with* $S \neq \emptyset$*. Define* $C := \{x \in S \mid f_i(x) \leq 0,\ \forall i \in I\}$*. Then*

$$C_\infty \subset \{d \in S_\infty \mid (f_i)_\infty(d) \leq 0,\ \forall i \in I\}.$$

The inclusion holds as an equation when $C \neq \emptyset$, S *is closed convex in* $\mathbb{R}^n$*, and each* f_i *is lsc convex.*

Proof. Let $C_i := \{x \in \mathbb{R}^n : f_i(x) \leq 0\}$, so that $C = S \cap \cap_{i \in I} C_i$. Invoking Proposition 2.1.9, one has $C_\infty \subset S_\infty \cap_{i \in I} (C_i)_\infty$, and thus using Proposition 2.5.3, the inclusion statement of the theorem is proved. Furthermore, under the additional assumptions given here, by the same Proposition 2.1.9, the previous inclusion holds as an equation.

□

Example 2.5.2 (a) Let Q be an $n \times n$ symmetric positive semidefinite matrix, $a \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$. Let C be the set defined by

$$C = \{x \in \mathbb{R}^n \mid (x - a)^T Q (x - a) \leq \beta\}.$$

Then using the examples given in (2.5.1) one has

$$C_\infty = \{d \in \mathbb{R}^n \mid d^T Q a \leq 0,\ Qd = 0\}.$$

(b) Let $f : \mathbb{R}_{++} \to \mathbb{R}$ defined by $f(t) = t \sin t^{-1}$ and consider the set

$$\mathrm{Gr}(f) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = f(x_1),\ x_1 > 0\}.$$

Then $(\mathrm{Gr}(f))_\infty = \{(d_1, d_2) \in \mathbb{R}^2 \mid 0 \le d_2 \le d_1\}$.

The next result gives a useful relation between the asymptotic cone of the subdifferential of a convex function and the normal cone of its domain.

Proposition 2.5.4 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and convex. Then for any $z \in \operatorname{dom} f$ such that $\partial f(z) \neq \emptyset$ one has*

$$(\partial f(z))_\infty = N_{\operatorname{dom} f}(z).$$

Proof. Let $z \in \operatorname{dom} f$ be such that $\partial f(z) \neq \emptyset$. Then the subdifferential of f is a nonempty closed convex set that can be described as the infinite intersection of closed half-spaces. More precisely, one has

$$\partial f(z) = \bigcap_{y \in \operatorname{dom} f} \{u \in \mathbb{R}^n \mid \langle u, y - z\rangle \le f(y) - f(z)\} \equiv \bigcap_{y \in \operatorname{dom} f} C_y.$$

Applying Proposition 2.1.9(a) to the closed convex sets C_y one obtains

$$(\partial f(z))_\infty = \bigcap_{y \in \operatorname{dom} f} (C_y)_\infty.$$

But by Corollary 2.5.4 one obtains immediately that $(C_y)_\infty = \{u \mid \langle u, y - z\rangle \le 0\}$, and thus it follows that

$$(\partial f(z))_\infty = \bigcap_{y \in \operatorname{dom} f} \{u \mid \langle u, y - z\rangle \le 0\} = N_{\operatorname{dom} f}(z),$$

where the last equality follows from the definition of the normal cone of the nonempty convex set $\operatorname{dom} f$ at z. $\square$

Proposition 2.5.5 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc, proper, and convex. Then f is Lipschitz on $\mathbb{R}^n$, i.e.,*

$$|f(x) - f(y)| \le L\|x - y\|,\ \forall x, y \in \mathbb{R}^n,$$

if and only if f_∞ is finite everywhere on $\mathbb{R}^n$. The Lipschitz constant L is given by

$$L = \sup\{f_\infty(d) \mid \|d\| = 1\}.$$

Proof. The asymptotic function f_∞, being finite everywhere on $\mathbb{R}^n$, is continuous, and therefore $\sup\{f_\infty(d) \mid \|d\| = 1\} = L < \infty$. Since f_∞ is positively homogeneous, $f_\infty(d) \le L\|d\|, \ \forall d$. Invoking the formula (2.19) one thus has

$$f(x+d) - f(x) \le L\|d\|, \ \forall x \in \operatorname{dom} f, \ d \in \mathbb{R}^n,$$

and since $f(x+d) < \infty$, then $\operatorname{dom} f = \mathbb{R}^n$ and L is the Lipschitz constant for f. Conversely, let f be Lipschitz on $\mathbb{R}^n$ and such that $x \in \mathbb{R}^n$. Then

$$f(x+td) - f(x) \le tL\|d\|, \ \forall t > 0, \ d \in \mathbb{R}^n,$$

which implies by (2.20) that $f_\infty(d) \le L\|d\|, \ \forall d$, so that $\operatorname{dom} f_\infty = \mathbb{R}^n$ and $\sup\{f_\infty(d) \mid \|d\| = 1\} = L$. □

To prove the next result, we need the following obvious technical fact on one-dimensional convex functions, the proof of which is left to the reader.

Lemma 2.5.1 *Let $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc, convex function and let S denote the set of optimal solutions of the problem $\inf\{\psi(t) : t \in \mathbb{R}\}$. Then only one of the following two statements holds:*
(a) S is nonempty and compact.
(b) The function ψ is a monotone function on $\mathbb{R}$.

Theorem 2.5.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc, convex function, and for any $x \in \operatorname{dom} f$ and $0 \ne d \in \mathbb{R}^n$ let $\psi(t) := f(x+td)$.*
(a) If $f_\infty(d) \le 0$, then $\limsup_{t\to+\infty} \psi(t) < +\infty$.
(b) If there exists some $z \in \operatorname{dom} f$ such that $\liminf_{t\to+\infty} f(z+td) < +\infty$, then ψ is decreasing on $\mathbb{R}$, which is equivalent to saying that $f_\infty(d) \le 0$.

Proof. From (2.20) we have $t^{-1}(f(x+td) - f(x)) \le f_\infty(d) \le 0$, which implies that $f(x+td) \le f(x), \ \forall t > 0$, proving (a). To prove (b), let $z \in \operatorname{dom} f$ satisfy the hypothesis of the theorem. Then one has $f_\infty(d) \le 0$. Indeed, suppose the contrary. Then again using (2.20) there would exist $t_0 > 0$ and $\alpha > 0$ such that $f(z+td) \ge f(z) + t\alpha, \ \forall t \ge t_0$. Taking the limit as $t \to +\infty$ in the latter inequality implies that $\lim_{t\to+\infty} f(z+td) = +\infty$, which contradicts the assumption made in (b). Since for each $x \in \operatorname{dom} f$, one has $\psi_\infty(1) = f_\infty(d) \le 0$, invoking Lemma 2.5.1, it follows that the optimal solution set of the problem minimizing ψ is noncompact, and thus ψ is monotone. We can suppose that ψ is nonconstant. Then ψ is decreasing. Indeed, in the opposite case there would exist $t_0 \in \operatorname{dom} \psi$ such that $\psi(t) \le \psi(t_0), \ \forall t \le t_0$, and then it would follow that $\psi_\infty(-1) \le 0$. Since $\psi_\infty(0) = 0$, it follows from the convexity of ψ_∞ that $\psi_\infty(-1) = \psi_\infty(1) = 0$ and then that ψ is constant. Finally, if ψ is decreasing on $\mathbb{R}$, then $f_\infty(d) \le 0$ follows from (2.20). □

Definition 2.5.2 *Given $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ a proper convex function, we define:*
(a) The asymptotic cone of f by

$$\mathcal{K}_f = \{d \in \mathbb{R}^n \mid f_\infty(d) \le 0\} = (\operatorname{epi} f)_\infty \cap \{(d, 0) \mid d \in \mathbb{R}^n\}.$$

This is a closed convex cone containing zero. A vector $d \in \mathcal{K}_f$ is called an asymptotic direction of f.
(b) The constancy space of f by

$$\mathcal{C}_f = \{d \in \mathbb{R}^n \mid f_\infty(d) = f_\infty(-d) = 0\} = (-\mathcal{K}_f) \cap \mathcal{K}_f.$$

(c) The lineality space of f by $L_f = \{d \in \mathbb{R}^n \mid f_\infty(-d) = -f_\infty(d)\}$.

Theorem 2.5.3 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc, convex function. The following conditions on $d \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ are equivalent:*
(a) $f(x + td) = f(x) + t\mu$, $\forall x \in \operatorname{dom} f, t \in \mathbb{R}$.
(b) $(d, \mu) \in -(\operatorname{epi} f)_\infty \cap (\operatorname{epi} f)_\infty$.
(c) $-f_\infty(-d) = f_\infty(d) = \mu$.
(d) Furthermore, the vector d satisfies (a)–(c) with $\mu = f_\infty(d)$ if there exists one $x \in \operatorname{dom} f$ such that $f(x + td)$ is an affine function of t.
(e) $d \in \mathcal{C}_f$ if and only if f is constant along the direction d.

Proof. Suppose (a) holds, i.e., $f(x+td) = f(x)+t\mu$, $\forall x \in \operatorname{dom} f$, $\forall t \in \mathbb{R}$. Thus in particular, using formula (2.19) one has $\mu = f_\infty(d)$ and $-\mu = f_\infty(-d)$, proving the implication $(a) \Longrightarrow (c)$. Now if (c) holds, this means that both (d, μ) and $(-d, -\mu)$ are in $\operatorname{epi} f_\infty$, and hence (b) holds. We end the proof by showing that (b) $\Longrightarrow$ (a). If (b) holds, then for any $t \in \mathbb{R}$ one has $\operatorname{epi} f = \operatorname{epi} f - t(\mu, d)$. Take $x \in \operatorname{dom} f$. Then $(x, f(x)) \in \operatorname{epi} f$, and if we define $h(x) = f(x + td) - t\mu$, then $\operatorname{epi} h = \operatorname{epi} f - t(\mu, d)$, and hence (a) holds. Furthermore, (d) follows from the formula (2.20), and finally, since $f_\infty(0) = 0 \le f_\infty(d) + f_\infty(-d)$, it follows that $f_\infty(d) = f_\infty(-d) = 0$ if and only if $d \in \mathcal{C}_f$, i.e., if and only if f is constant on the direction d by using the equivalence between (a) and (c). $\square$

There exists an interesting interplay between a proper, lsc, convex function and the convex sets associated with it, together with its conjugate and the corresponding asymptotic function.

Theorem 2.5.4 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function, and f^* its conjugate. The following relations hold:*
(a) $(f^*)_\infty = \sigma_{\operatorname{dom} f}$.
(b) If f is also assumed lsc, then

$$f_\infty = \sigma_{\operatorname{dom} f^*}, \quad (f_\infty)^* = \delta_{\operatorname{cl} \operatorname{dom} f^*}.$$

Proof. (a) Using the definition of f_∞ given by the formula (2.19) and that of the conjugate function, one has

$$\begin{aligned}(f^*)_\infty(d) &= \sup\{f^*(x+d)-f^*(x):\ x\in\operatorname{dom} f^*\}\\ &= \sup_{x\in\operatorname{dom} f^*}\{\sup_{u\in\operatorname{dom} f}\{\langle u, x+d\rangle - f(u)\} - f^*(x)\}\}\\ &= \sup_{u\in\operatorname{dom} f}\{\sup_{x\in\operatorname{dom} f^*}\{\langle u, x\rangle - f^*(x)\} + \langle u,d\rangle - f(u)\}\}\\ &= \sup_{u\in\operatorname{dom} f}\{\langle u,d\rangle + f^{**}(u) - f(u)\},\\ &\leq \sigma_{\operatorname{dom} f}(d),\ \text{ since }\ f^{**}\leq f.\end{aligned}$$

We now prove the reverse inequality $(f^*)_\infty(d)\geq\sigma_{\operatorname{dom} f}(d)$. Clearly, for any $d\notin(\operatorname{dom} f^*)_\infty$ there is nothing to prove. Thus, let any $d\in(\operatorname{dom} f^*)_\infty$ and take $\mu=(f^*)_\infty(d)$. Then one has

$$f^*(z+td)-f^*(z)\leq \mu t,\ \forall z\in\operatorname{dom} f^*,\ \forall t>0.$$

On the other hand, by the definition of the conjugate, one has

$$f(x)+f^*(z+td)\geq\langle x,z\rangle + t\langle x,d\rangle,\ \forall t>0,$$

which combined with the inequality above implies

$$f(x)\geq\langle x,z\rangle - f^*(z)+t(\langle x,d\rangle-\mu),\ \forall t>0.$$

Therefore, as $t\to+\infty$ it follows that $\langle x,d\rangle\leq\mu$, $\forall x\in\operatorname{dom} f$, and hence

$$\sup_{x\in\operatorname{dom} f}\langle x,d\rangle\leq\mu \text{ and thus } \sigma_{\operatorname{dom} f}(d)\leq(f^*)_\infty(d).$$

(b) We now assume that f is lsc, which implies that $f^{**}=f$. Then replacing f by f^* in the formula proven in (a) yields $(f^{**})_\infty=\sigma_{\operatorname{dom} f^*}=f_\infty$, proving the first formula of (b). Taking conjugates in that formula, and recalling that for any set $C\subset\mathbb{R}^n$, $\sigma_C=\sigma_{\operatorname{cl} C}$ and $\sigma_C=\delta_C^*$ we obtain

$$(f_\infty)^*=(\sigma_{\operatorname{cl}\operatorname{dom} f^*})^*=\delta^{**}_{\operatorname{cl}\operatorname{dom} f^*}=\delta_{\operatorname{cl}\operatorname{dom} f^*},$$

proving the second formula in (b). □

Corollary 2.5.5 *Let $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be a proper convex function and suppose that $0\in E:=\operatorname{aff}\operatorname{dom} f$, i.e., E is a subspace. Then $\mathcal{C}_{f^*}=L_{f^*}=E^\perp$.*

Proof. From the definition of the lineality space and from Theorem 2.5.3 one has $d\in L_{f^*}$ if and only if the function $\langle\cdot,d\rangle$ is constant on $\operatorname{dom} f$, or equivalently on $E=\operatorname{aff}\operatorname{dom} f$, and since $0\in E$, this constant is 0, and one has $L_{f^*}=\mathcal{C}_{f^*}$ and $d\in E^\perp$ if and only if $d\in\mathcal{C}_{f^*}$. □

The characterization of the asymptotic function of a convex function given in terms of the support function of its conjugate in the previous theorem is of fundamental importance, and as we shall see, will be used very often in the following chapters. An interesting immediate application of this formula is the following dual characterization of the constancy subspace $\mathcal{C}_f = \{d \in \mathbb{R}^n \mid f_\infty(d) = f_\infty(-d) = 0\}$, which by replacing f_∞ by $\sigma_{\operatorname{dom} f^*}$ (when f is lsc) can thus be simply written as

$$\mathcal{C}_f = \{d \in \mathbb{R}^n \mid \langle d, v\rangle = 0, \ \ \forall v \in \operatorname{dom} f^*\}. \tag{2.21}$$

Lemma 2.5.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then for all $y \in \mathbb{R}^n$,*

$$f^*(y) = \sigma_{\operatorname{epi} f}(y, -1) = \sup\{\langle y, x\rangle - r \mid (x, r) \in \operatorname{epi} f\}.$$

As a consequence the following formula holds for the epigraph of f:

$$\sigma_{\operatorname{epi} f}(y, -t) = \begin{cases} tf^*(\frac{y}{t}) & \text{if } t > 0, \\ (f^*)_\infty(y) & \text{if } t = 0, \\ +\infty & \text{if } t < 0. \end{cases}$$

Proof. The first formula in the theorem follows from the definition of the epigraph and conjugate of f. In fact, one has

$$\begin{aligned} \sigma_{\operatorname{epi} f}(y, -1) = \sup_{f(x) \le r} \{\langle x, y\rangle - r\} &= \sup_x \sup_{r \ge f(x)} \{\langle x, y\rangle - r\} \\ &= \sup_x \{\langle x, y\rangle - f(x)\} = f^*(y). \end{aligned}$$

We now prove the second formula. First, if $t < 0$, one has $\sigma_{\operatorname{epi} f}(y, -t) = +\infty$. Suppose that $t > 0$. Then using the first formula just proved, we have

$$\sigma_{\operatorname{epi} f}(y, -t) = t \sup_{f(x) \le r} \{\langle x, y/t\rangle - r\} = t\sigma_{\operatorname{epi} f}(y/t, -1) = tf^*(y/t).$$

Finally, if $t = 0$, then

$$\begin{aligned} \sigma_{\operatorname{epi} f}(y, 0) &= \sup_{f(x) \le r} \{\langle x, y\rangle - r \times 0\}, \\ &= \sup\{\langle x, y\rangle \mid (x, r) \in \operatorname{epi} f \text{ for some } r\}, \\ &= \sigma_{\operatorname{dom} f}(y) = (f^*)_\infty(y), \end{aligned}$$

where the last equation follows from Theorem 2.5.4(a). □

For each function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ that is lsc and proper with $0 < f(0) < +\infty$, the positive hull, denoted by $\operatorname{pos} f$, is defined through its epigraph via

$$\operatorname{epi}(\operatorname{pos} f) = \operatorname{pos}(\operatorname{epi} f),$$

which is equivalent to (since $f(0) > 0$, i.e., $(0,0) \notin \operatorname{epi} f$)

$$(\operatorname{pos} f)(x) = \inf_{\lambda > 0} \lambda f(\lambda^{-1} x). \tag{2.22}$$

Proposition 2.5.6 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be lsc, and proper, with* $0 < f(0) < +\infty$*. Then:*
(a) $\operatorname{cl}(\operatorname{pos} f)(x) = \min\{(\operatorname{pos} f)(x), f_\infty(x)\} = \min\{\inf_{\lambda>0} \lambda f(\lambda^{-1}x), f_\infty(x)\}$.
(b) If in addition f is assumed convex, then $\operatorname{pos} f$ *is lsc, convex, proper, and positively homogeneous.*

Proof. (a) By the definition of the closure and the positive hull of a function, one has

$$\operatorname{epi}(\operatorname{cl}(\operatorname{pos} f)) = \operatorname{cl}(\operatorname{epi}(\operatorname{pos} f)) = \operatorname{cl}(\operatorname{pos}(\operatorname{epi} f)).$$

Using Corollary 2.3.3 one thus obtains

$$\begin{aligned} \operatorname{cl}(\operatorname{pos}(\operatorname{epi} f)) &= \operatorname{pos}(\operatorname{epi} f) \cup (\operatorname{epi} f)_\infty \\ &= \operatorname{epi}(\operatorname{pos} f) \cup \operatorname{epi} f_\infty \\ &= \operatorname{epi}(\min\{\operatorname{pos} f, f_\infty\}) \text{ (using Proposition 1.2.1)}, \end{aligned}$$

which establishes the first formula in (a), while the second formula follows from (2.22).
(b) Since f is convex and $0 \in \operatorname{dom} f$, one has $f_\infty(x) = \lim_{\lambda \to 0^+} \lambda f(\lambda^{-1}x)$. Furthermore, since $\lambda f(\lambda^{-1}x) \le f_\infty(x) - \lambda f(0)$ and $f(0) > 0$, it follows that $\operatorname{pos} f\ (x) = \inf_{\lambda>0} \lambda f(\lambda^{-1}x) \le f_\infty(x)$, and by part (a) we obtain $\operatorname{cl} \operatorname{pos} f = \operatorname{pos} f$, which proves that $\operatorname{pos} f$ is lsc. Since $\operatorname{epi} f$ is convex, it follows that $\operatorname{pos}(\operatorname{epi} f)$ is convex, and then $\operatorname{pos} f$ is convex. Furthermore, since $(\operatorname{pos} f)(0)=\inf_{\lambda>0} \lambda f(0) = 0$, one thus has that $\operatorname{pos} f$, which is lsc, is also proper. Finally, since $\operatorname{epi}(\operatorname{pos} f)$ is a cone, one thus has that $\operatorname{pos} f$ is positively homogeneous. $\square$

Remark 2.5.1 In the convex case, from part (b) of Proposition 2.5.6 we always have $\inf_{\lambda>0} \lambda f(\lambda^{-1}d) \le f_\infty(d)$. This inequality can be strict. For example, with $f(x) := x^2 + 1$ we obtain for $d = 1$, $\inf_{\lambda>0} \lambda f(\lambda^{-1}) < +\infty$ and $f_\infty(1) = +\infty$.

When f is lsc, proper, and convex, the corresponding asymptotic function can be viewed as the closure of another function associated with f.

Definition 2.5.3 *For any proper function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, *we associate the function* p_f *defined by*

$$p(t,x) = \begin{cases} tf(t^{-1}x) & \textit{if } t > 0, \\ 0 & \textit{if } t = 0, x = 0, \\ +\infty & \textit{otherwise,} \end{cases} \tag{2.23}$$

with $\operatorname{dom} p = \{(0,0\} \cup \{t(1,x) \mid t > 0,\ f(x) < \infty\} = \{(0,0)\} \cup \mathbb{R}_+(\{1\} \times \operatorname{dom} f)$.

Proposition 2.5.7 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be lsc, proper, and convex. Then the function* $p(t,x)$ *given in (2.23) is proper and jointly convex in* (t,x), *and for any* $d \in \mathbb{R}^n$ *one has*

$$(clp)(t,d) = \sigma_{\operatorname{epi} f^*}(y,-t) = \begin{cases} tf(t^{-1}x) & \textit{if } t > 0, \\ f_\infty(d) & \textit{if } t = 0, \\ +\infty & \textit{if } t < 0. \end{cases}$$

Proof. The joint convexity of $p(t,x)$ follows directly from applying the definition of the convexity of f by observing that for any $\alpha \in [0,1]$, $\beta = 1-\alpha$, and any $t_1, t_2 > 0$, $x_1, x_2 \in \mathbb{R}^n$ one has

$$\begin{aligned} p(\alpha t_1 + \beta t_2, \alpha x_1 + \beta x_2) &= (\alpha t_1 + \beta t_2) f\left(\frac{\alpha t_1 x_1}{(\alpha t_1 + \beta t_2)t_1} + \frac{\beta t_2 x_2}{(\alpha t_1 + \beta t_2)t_2}\right) \\ &\le \alpha t_1 f\left(\frac{x_1}{t_1}\right) + \beta t_2 f\left(\frac{x_2}{t_2}\right). \end{aligned}$$

Now define the function

$$g(s,x) = \begin{cases} f(x) & \text{if } s = 1, \\ +\infty & \text{if } s \neq 1. \end{cases}$$

Clearly, g is a proper lsc convex function, and since for any x, $g(0,x) = +\infty$, using (2.22) one has

$$(\operatorname{pos} g)(s,x) = \inf_{\lambda > 0} \lambda g(\lambda^{-1}s, \lambda^{-1}x) = \begin{cases} sf(s^{-1}x) & \text{if } s > 0, \\ +\infty & \text{if } s = 0, x = 0, \\ +\infty & \text{if } s < 0. \end{cases}$$

The point $(0,0)$ is not in the relative interior of p and $\operatorname{pos} g$, respectively, and therefore the functions p and $\operatorname{pos} g$ coincide on their relative interiors, and hence their closures coincide. Applying Proposition 2.5.6(a) one thus obtains

$$\operatorname{cl} p(t,x) = \operatorname{cl}(\operatorname{pos} g)(t,x) = \min\{(\operatorname{pos} g)(t,x), g_\infty(t,x)\},$$

and since here $g_\infty(t,d) = f_\infty(d)$ for $t = 0$ and is $+\infty$ whenever $t \neq 0$, the desired formula for $\operatorname{cl} p(t,x)$ follows using Lemma 2.5.2. $\square$

Corollary 2.5.6 *Let $C \subset \mathbb{R}^n$ be a nonempty closed convex set with $0 \in C$. Then the gauge function of C, denoted by γ_C and defined by*

$$\gamma_C = \inf\{\lambda \geq 0 \mid x \in \lambda C\}$$

is nonnegative, lsc, and positively homogeneous. Furthermore, one has:
(a) $C = \{x \mid \gamma_C(x) \leq 1\}$.
(b) $C_\infty = \{x \mid \gamma_C(x) = 0\}$.
(c) $\operatorname{pos} C = \operatorname{dom} \gamma_C$.

Proof. Let $h(x) := \delta(x|C) + 1$. Then $1 = h(0) > 0$ and $\lambda h(\lambda^{-1}x) = \delta(x|\lambda C) + \lambda, \quad \forall \lambda > 0$. Therefore, by definition one has $\gamma_C(x) = \operatorname{pos} h(x)$ and $\gamma_C \geq 0$. Hence, by Proposition 2.5.6(b) it follows that γ_C is lsc, convex, proper, and positively homogeneous. Moreover, (a) is satisfied, and then $C_\infty = \{d \mid (\gamma_C)_\infty(d) \leq 0\}$. Since γ_C is positively homogeneous, lsc convex, we have $(\gamma_C)_\infty = \gamma_C$, and since $\gamma_C \geq 0$, we obtain formula (b). Finally, using (a) and once more the fact that γ_C is positively homogeneous, we obtain $\operatorname{pos} C = \operatorname{dom} \gamma_C$. □

Domains of proper, lsc, convex functions can be usefully characterized via the use of asymptotic functions.

Proposition 2.5.8 *Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc, proper, and convex. Let z be a fixed vector and let $f(x) = h(x) - \langle x, z \rangle$. Then:*
(a) $z \in \operatorname{aff}(\operatorname{dom} h^*) \iff f_\infty(d) = 0, \ \forall d$ *such that* $-f_\infty(-d) = f_\infty(d)$.
(b) $z \in \operatorname{ri}(\operatorname{dom} h^*) \iff f_\infty(d) > 0, \ \forall d$ *except those satisfying* $-f_\infty(-d) = f_\infty(d) = 0$.
(c) $z \in \operatorname{int}(\operatorname{dom} h^*) \iff f_\infty(d) > 0, \ \forall d \neq 0$.
(d) $z \in \operatorname{cl}(\operatorname{dom} h^*) \iff f_\infty(d) \geq 0, \ \forall d$.

Proof. We have $f^*(y) = \sup_x\{\langle x, y + z\rangle - h(x)\} = h^*(y + z)$, so that $\operatorname{dom} f^* = \operatorname{dom} h^* - z$. Thus, $z \in \operatorname{cl} \operatorname{dom} h^*$ is equivalent to $0 \in \operatorname{cl} \operatorname{dom} f^*$, and similarly for each of the other statements in the theorem. By Theorem 2.5.4(b), one has $\sigma_{\operatorname{dom} f^*} = f_\infty$. Therefore, statements (a)–(d) correspond exactly to those of Theorem 1.3.2 stated for the support function of $\operatorname{dom} f^*$. □

2.6 Differential Calculus at Infinity

We develop here some useful formulas for computing asymptotic functions.

Proposition 2.6.1 *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \ i = 1, \ldots, p$, be a collection of proper functions, $f := \sum_{i=1}^p f_i$, and suppose that f is proper; i.e.,*

$\operatorname{dom} f = \cap_{i=1}^{p} \operatorname{dom} f_i \neq \emptyset$. *Then:*
(a) If the functions are all lsc, f is also lsc.
(b) $f_\infty(d) \geq \sum_{i=1}^{p} (f_i)_\infty(d)$ for all d satisfying the following condition:
if $(f_i)_\infty(d) = +\infty$ (respectively $-\infty$) for some i, then $(f_j)_\infty(d) > -\infty$ respectively ($< +\infty$) for $j \neq i$.
If in addition all the functions are lsc and convex, then equality holds in the inequality.

Proof. For arbitrary extended real valued functions g_i, one has for $y \in \mathbb{R}^n$,

$$\liminf_{x \to y} (g_1(x) + \cdots + g_p(x)) \geq \liminf_{x \to y} g_1(x) + \cdots + \liminf_{x \to y} g_p(x),$$

as long as the right side of the inequality has a meaning, and hence (a) follows immediately by using the definition of lower semicontinuity. The inequality of assertion (b) follows also from the above inequality by using the definition of the asymptotic function (2.18), while equality in the case of lsc convex functions follows from formula (2.20). □

Proposition 2.6.2 *For a collection $\{f_i\}_{i \in I}$ of proper functions $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ one has*

$$\left(\sup_{i \in I} f_i\right)_\infty \geq \sup_{i \in I} (f_i)_\infty \quad \textit{and} \quad \left(\inf_{i \in I} f_i\right)_\infty \leq \inf_{i \in I} (f_i)_\infty.$$

The first relation is an equality when f_i are lsc, proper, and convex. The second relation is an equality for a finite index set I.

Proof. Apply Proposition 2.1.9 to the corresponding epigraphs, recalling the epigraph algebraic properties of Proposition 1.2.1. □

Proposition 2.6.3 *Let $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a proper function, let A be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $A(\mathbb{R}^n) \cap \operatorname{dom} g \neq \emptyset$, and let $f(x) = g(Ax)$ be the proper composite function. The following properties hold for the function f:*
(a) If g is convex and there exists some y such that $Ay \in \operatorname{ri} \operatorname{dom} g$, then for any x, $\operatorname{cl} f(x) = \operatorname{cl} g(Ax)$.
(b) If g is lsc, then f is lsc and one has

$$f_\infty(d) \geq g_\infty(Ad) \quad \forall d.$$

If in addition g is convex (so is f), this holds also as an equality.

Proof. (a) Since $\operatorname{dom} f = A^{-1} \operatorname{dom} g$, using Proposition 1.1.6(c) we get $\operatorname{ri} \operatorname{dom} f = A^{-1}(\operatorname{ri} \operatorname{dom} g)$, which is equivalent to saying that $y \in \operatorname{ri} \operatorname{dom} f$ if and only if $Ay \in \operatorname{ri}\operatorname{dom} g$. Let y be such that $Ay \in \operatorname{ri}\operatorname{dom} g$ (which exists by hypothesis). Then, since $y \in \operatorname{ri}\operatorname{dom} f$, by Proposition 1.2.6 we have

$$\operatorname{cl} f(x) = \lim_{t\to 0^+} f(x + t(y - x)) = \lim_{t\to 0^+} g(Ax + t(Ay - Ax)).$$

By the same proposition, we also have

$$\operatorname{cl} g(Ax) = \lim_{t\to 0^+} g(Ax + t(Ay - Ax)),$$

so that $\operatorname{cl} f(x) = \operatorname{cl} g(Ax)$.
(b) If g is lsc, obviously f is also lsc. Furthermore, using the representation of the asymptotic function we obtain for each d

$$f_\infty(d) = \inf_{\substack{d_k\to d\\ t_k\to+\infty}} \liminf_{k\to\infty} t_k^{-1} g(t_k A d_k) \geq \liminf_{\substack{d'\to Ad\\ t\to\infty}} t^{-1} g(td') = g_\infty(Ad).$$

If in addition g is convex, then obviously f is also convex. Let $x_0 \in \operatorname{dom} f$. Then since f is lsc, proper, and convex, we have

$$f_\infty(d) = \lim_{\lambda\to\infty} \frac{g(Ax_0 + \lambda Ad) - g(Ax_o)}{\lambda} = g_\infty(Ad).$$

□

It is interesting and often useful to have a formula for the asymptotic function defined as a composition of a nondecreasing function with a convex function.

Proposition 2.6.4 *(Composition rule) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc, proper, and convex, and let $\psi : (-\infty, b) \to \mathbb{R}$ with $0 \leq b \leq +\infty$ be a convex nondecreasing function with $\psi_\infty(1) > 0$, and with $\operatorname{dom}\psi \cap f(\mathbb{R}^n) \neq \emptyset$. Consider the composite function*

$$g(x) = \begin{cases} \psi(f(x)) & \text{if } x \in \operatorname{dom} f,\\ +\infty & \text{otherwise.}\end{cases}$$

Then g is a proper, lsc, convex function, and one has

$$g_\infty(d) = \begin{cases} \psi_\infty(f_\infty(d)) & \text{if } d \in \operatorname{dom} f_\infty,\\ +\infty & \text{otherwise.}\end{cases}$$

Proof. Clearly, the composite function g is an lsc convex function. Let $x \in \operatorname{dom} f$ be such that $f(x) \in \operatorname{dom}\psi$. For every $s < f_\infty(d)$ there exists τ such that $f(x + td) \geq f(x) + ts$, for $t \geq \tau$, and since ψ is nondecreasing, we obtain

$$\frac{g(x + td) - g(x)}{t} \geq \frac{\psi(f(x) + ts) - \psi(f(x))}{t},$$

and passing to the limit with $t \to +\infty$ we deduce $g_\infty(d) \geq \psi_\infty(s)$. If $f_\infty(d) = +\infty$, using $\psi(1) > 0$ and letting $s \to +\infty$ we get $g_\infty(d) = +\infty$. In the other case, $f_\infty(d) < +\infty$, letting $s \to f_\infty(d)$ and since ψ_∞ is lsc, we deduce $g_\infty(d) \geq \psi_\infty(f_\infty(d))$. On the other hand, since $f(x + td) \leq f(x) + tf_\infty(d)$, using the monotonicity of ψ we also get

$$\begin{aligned} g_\infty(d) &= \lim_{t\to+\infty} \frac{g(x+td) - g(x)}{t} \\ &\leq \lim_{t\to+\infty} \frac{\psi(f(x) + tf_\infty(d)) - \psi(f(x))}{t} = \psi_\infty(f_\infty(d)). \end{aligned}$$

□

The next set of results concerns arbitrary proper functions.

Proposition 2.6.5 *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc proper functions, $i = 1, \ldots, m$, such that closed sets $\{\operatorname{epi} f_i \mid i = 1, \ldots, m\}$ are supposed to be in relative general position (see Definition 2.3.3) and let $h := \operatorname{conv}\{f_i \mid i = 1, \ldots, m\}$. Then for all $x \in \operatorname{dom} \operatorname{cl} h$,*

$$\exists x_i \in \operatorname{dom} f_i,\ y_j \in \operatorname{dom}(f_j)_\infty,\ \lambda \in \operatorname{int} \Delta_p,\ i \in [1, p],\ j \in [1, q],$$

with $p + q \leq n + 1,\ p \geq 1,$ such that

$$\operatorname{cl} h(x) = \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{j=1}^{q} (f_j)_\infty(y_j),$$

with $x = \sum_{i=1}^{p} \lambda_i x_i + \sum_{j=1}^{q} y_j$.

Proof. Apply Theorem 2.3.5 to the corresponding epigraphs. □

As in the case for sets where from Theorem 2.3.5 we have obtained an interesting corollary concerning the convex hull of a set and its closure we will now show how the convex hull of a function and its closure are related. For that purpose we need the following definition.

Definition 2.6.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then f is said to be semibounded if its epigraph is semibounded, i.e., if $\operatorname{epi} f_\infty$ is pointed.*

Proposition 2.6.6 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc, semibounded function. Then for all $x \in \operatorname{dom}(\operatorname{cl} \operatorname{conv} f)$,*

$$\exists x_i \in \operatorname{dom} f,\ y_j \in \operatorname{dom}(f)_\infty,\ \lambda \in \operatorname{int} \Delta_p,\ i \in [1, p],\ j \in [1, q],$$

with $p + q \leq n + 1,\ p \geq 1$ *such that*

$$\operatorname{cl}\operatorname{conv} f(x) = \sum_{i=1}^{p} \lambda_i f(x_i) + \sum_{j=1}^{q} f_\infty(y_j),$$

with $x = \sum_{i=1}^{p} \lambda_i x_i + \sum_{j=1}^{q} y_j$.

Proof. Use Proposition 2.6.5 with $m = n + 1,\ f = f_i\ \forall i = 1, \ldots, p$. $\square$

An example of a function where Proposition 2.6.6 becomes simpler concerns cofinite functions.

Definition 2.6.2 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper lsc function. Then* f *is said to be cofinite if*

$$f_\infty(0) = 0, \quad \textit{and} \quad f_\infty(y) = +\infty \quad \forall y \neq 0.$$

Corollary 2.6.1 *Let* f *be a cofinite function. Then* $\operatorname{cl}\operatorname{conv} f = \operatorname{conv} f$ *and*

$$\forall x \in \operatorname{dom}(\operatorname{conv} f),\ \exists x_i \in \operatorname{dom} f,\ \lambda \in \Delta_{n+1},\ i \in [1, n+1],$$

such that

$$\operatorname{conv} f(x) = \sum_{i=1}^{n+1} \lambda_i f(x_i),$$

with $x = \sum_{i=1}^{n+1} \lambda_i x_i$.

Proof. As an immediate consequence of Corollary 2.3.6, since $\operatorname{epi} f_\infty = \{0\}$, we obtain $\operatorname{cl}\operatorname{conv} f = \operatorname{conv} f$. From the definition of a cofinite function it follows that f is semibounded, and the claimed formula is obtained by using Proposition 2.6.6. $\square$

Proposition 2.6.7 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper lsc function. Then* f *is cofinite if and only if* $\operatorname{dom} f^* = \mathbb{R}^n$.

Proof. Suppose that f is cofinite and there exists some $v \notin \operatorname{dom} f^*$, i.e., $f^*(v) = +\infty$. Then we have

$$\inf\{f(x) - \langle v, x\rangle \mid x \in \mathbb{R}^n\} = -\infty,$$

and there exists a minimizing sequence $\{x_k\}$ such that $f(x_k) - \langle v, x_k\rangle \to -\infty$. Let j be an arbitrary integer. Then for k sufficiently large we have

$$f(x_k) - \langle v, x_k\rangle \leq -j. \tag{2.24}$$

Thus there are two cases:
(i) The sequence $\{x_k\}$ is bounded. Therefore, it has at least one cluster

point, and for each cluster point x, since f is lsc, passing to the limit in the last inequality it follows that

$$f(x) - \langle v, x \rangle \le -j,$$

and taking $j \to \infty$ we get a contradiction.
(ii) Suppose now that $\{x_k\}$ is unbounded and without loss of generality suppose that

$$\|x_k\| \to \infty, \quad \frac{x_k}{\|x_k\|} \to y \neq 0.$$

Then dividing both members of formula (2.24) by $\|x_k\|$ we obtain

$$\|x_k\|^{-1} f\left(\frac{x_k}{\|x_k\|}\|x_k\|\right) - \left\langle v, x_k \|x_k\|^{-1}\right\rangle \le -j\|x_k\|^{-1},$$

and passing to the limit we obtain

$$f_\infty(y) - \langle v, y \rangle \le 0,$$

a contradiction to the fact that f is cofinite. We prove now the reverse statement. Suppose that $\operatorname{dom} f^* = \mathbb{R}^n$. Then from Corollary 2.5.2 $f_\infty(0) = 0$, and f_∞ is proper. Suppose that f is not cofinite. Then there exists some $y \neq 0$ with $f_\infty(y) := \alpha \in \mathbb{R}$ such that we can find two sequences $\{y_k\} \to y$ and $\{t_k\} \to \infty$ for which

$$\lim_{k\to\infty} \frac{f(t_k y_k)}{t_k} \to \alpha.$$

Let $s \in \mathbb{R}^n$. Then we have $t_k^{-1}(\langle s, t_k y_k \rangle - f(t_k y_k)) \le t_k^{-1} f^*(s)$, and passing to the limit, since $f^*(s) \in \mathbb{R}$, we get $\langle s, y \rangle - \alpha \le 0 \quad \forall s$, which is obviously impossible.

$\square$

We end this section with some interesting facts and properties of *one-dimensional* convex functions and their relations with the corresponding asymptotic function.

Lemma 2.6.1 *Let $\psi : (-\infty, b) \to \mathbb{R}$ with $0 \le b \le +\infty$ be a convex nondecreasing function with $\psi_\infty(1) > 0$. Then ψ^{-1}, the inverse of ψ exists and is a concave function, and the following relations hold:*
(a)

$$(-\psi^{-1})^*(-s) = \begin{cases} s\psi^*(s^{-1}) & \textit{if } s > 0, \\ +\infty & \textit{otherwise}. \end{cases}$$

(b) $s \in \operatorname{dom}\psi^* \iff -s^{-1} \in \operatorname{dom}(-\psi^{-1})^*$ *and* $\operatorname{dom}\psi^* \subset \mathbb{R}_+$.
(c) $\psi^{-1}(t) = \inf_{s>0} \{st + s\psi(s^{-1})\}$.
(d) $(-\psi^{-1})_\infty(t) = -\inf\{ts^{-1} : s \in \mathbb{R}_{++} \cap \operatorname{dom}\psi^*\}$.

Proof. The strict monotonicity and continuity of ψ implies that ψ^{-1} exists, and thus from the definition of the convexity of ψ it follows that ψ^{-1} is concave. By definition of the conjugate of the convex function $-\psi^{-1}$ one has

$$\begin{aligned}
(-\psi^{-1})^*(-s) &= \sup\{-ts + \psi^{-1}(t) : \ t \in \operatorname{dom} \psi^{-1}\} = \sup_u\{-s\psi(u) + u\} \\
&= \sup_u\{s(us^{-1} - \psi(u)\} \\
&= \begin{cases} s \sup_u\{us^{-1} - \psi(u)\} & \text{if } s > 0, \\ +\infty & \text{otherwise,} \end{cases}
\end{aligned}$$

proving the desired formula in (a). The relation between the domains in (b) follows as a direct consequence. Furthermore, since $\operatorname{ri} \operatorname{dom} \psi^* \subset \operatorname{rge} \partial\psi \subset \operatorname{dom} \psi^*$, and we assumed that ψ is nondecreasing, it follows that rge $\subset \partial\psi\mathbb{R}_{++}$, so that $\operatorname{ri} \operatorname{dom} \psi^* \subset \mathbb{R}_{++}$, and hence $\operatorname{dom} \psi^* \subset \mathbb{R}_+$, as stated. To prove the infimal representation of the inverse function ψ in (c) we use the fact that $-\psi^{-1}$ is lsc proper convex, so that $-\psi^{-1}(t) = (-\psi^{-1})^{**}(t)$, and with the help of the conjugate formula derived in (a) we then have

$$\begin{aligned}
(-\psi^{-1})^{**}(t) = -\psi^{-1}(t) &= \sup_{s<0}\{st - (-\psi^{-1})^*(s)\} \\
&= \sup_{s<0}\{st + s\psi^*(-s^{-1})\} = -\inf_{s>0}\{st + s\psi^*(s^{-1})\}.
\end{aligned}$$

Finally, using the first formula in (b) of the same theorem, and (b) proven above, we have

$$\begin{aligned}
(-\psi^{-1})_\infty(t) &= \sup\{st : \ s \in \operatorname{dom}(-\psi^{-1})^*\} \\
&= -\inf\{st : \ -s \in \operatorname{dom}(-\psi^{-1})^*\} \\
&= -\inf\{ts^{-1} : s \in \mathbb{R}_{++} \cap \operatorname{dom} \psi^*\},
\end{aligned}$$

proving (d). $\square$

Note that if in the above lemma we suppose in addition that $\psi_\infty(1) = +\infty, \psi_\infty(-1) = 0$, then the function $t \to -\psi^{-1}(-t)$ is convex and nondecreasing with domain the interval $(-\infty, b)$ and such that $-\psi^{-1}(1) = 0, -\psi^{-1}(-1) = +\infty$.

2.7 Application I: Semidefinite Optimization

For an arbitrary closed convex cone $K \subset \mathbb{R}^n$ we define a partial ordering $\succeq$ for vectors $x, y \in \mathbb{R}^n$ via

$$x \succeq y \iff x - y \in K.$$

The partial ordering $\succeq$ satisfies the following conditions:
(i) $x \succeq x, \ \forall x$.
$(ii)\, x \succeq y \Longrightarrow -y \succeq -x$.
$(iii)\, x \succeq y \Longrightarrow \lambda x \succeq \lambda y, \ \forall \lambda \geq 0$.
$(iv)\, x \succeq y$ and $x' \succeq y' \Longrightarrow x + x' \geq y + y'$.
$(v)\, x_k \succeq y_k, \ \ x_k \to x, \ \ y_k \to y \ \Longrightarrow x \succeq y$.
The additional antisymmetric property (vi) $x \succeq y$ and $y \succeq x \Longrightarrow x = y$ holds if and only if K is in addition a *pointed* cone.
The standard case, already mentioned in Chapter 1, is for vector inequalities when $K = \mathbb{R}^n_+$, the nonnegative orthant, which is a closed convex pointed cone. In that case $\succeq$ reduces simply to the usual coordinatewise inequalities

$$x \succeq y \iff x_j - y_j \geq 0, \ \forall j = 1, \ldots, n.$$

Another interesting and important partial ordering that fit the conditions $(i) - (v)$ is for the space of square real symmetric matrices. Let M_n be the space of real square matrices of order n equipped with the inner product

$$\langle A, B \rangle := \sum_{i,j=1}^{n} a_{ij} b_{ij} = \operatorname{tr} AB,$$

where tr stands for the trace operator of a matrix $C \in M_n$, which is the sum of the diagonal elements of C.
Of particular importance is the space of symmetric real matrices of order n, denotes by S_n. This in fact can be viewed as a linear subspace of M_n with dimension $n(n+1)/2$ and can be treated as a Euclidean vector space with the trace inner product.
Let $K \subset S_n$ be a given cone. Then as in the vector case we may define the dual cone

$$K^\circ = \{Y \in S_n \mid \langle Y, X \rangle \geq 0, \ \forall X \in K\}.$$

Among all the cones in S_n, one of particular interest is the cone of positive semidefinite matrices, denoted by S_n^+ and defined by

$$S_n^+ := \{A \in S_n \mid x^T A x \geq 0, \ \forall x \in \mathbb{R}^n\}.$$

It is easy to verify that S_n^+ is a closed convex pointed cone. The partial ordering associated with S_n^+ is quantified by

$$A \succeq B \iff A - B \ \text{ positive semidefinite}$$

and obeys the rules $(i) - (vi)$.
The cone S_n^+ is self dual, i.e., $(S_n^+)^\circ = S_n^+$. The interior of S_n^+ consists of positive definite matrices, namely

$$S_n^{++} := \{A \in S_n \mid x^T A x > 0, \ \forall x \neq 0\}.$$

Another particularly interesting partial ordering is the Lorentz cone (also called the second-order or ice-cream cone), defined by

$$L := \{x \in \mathbb{R}^{n+1} \mid x_{n+1} \geq \|x\|\}.$$

Spectrally Defined Matrix Functions

Spectrally defined functions arise in various areas of mathematical sciences and have recently gained strong interest within the area of matrix optimization problems, called semidefinite programming.
Recall that a function f on $\mathbb{R}^n$ is called symmetric if

$$f(Px) = f(x)$$

for all $n \times n$ permutation matrices P.

Definition 2.7.1 *The function $\Phi : S_n \to \mathbb{R} \cup \{+\infty\}$ is said to be spectrally defined if there exists a symmetric function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that*

$$\Phi(X) = \Phi_f(X) := f(\lambda(X)), \quad \forall X \in S_n,$$

where $\lambda(X) := (\lambda_1(X), \dots, \lambda_n(X))^T$ is the vector of eigenvalues of X in nondecreasing order.

The function Φ is spectrally defined if and only if Φ is orthonormally invariant, that is,

$$\Phi(U^T AU) = \Phi(A), \quad \forall U \in \mathcal{U}_n,$$

with $\mathcal{U}_n :=$ the set of $n \times n$ orthogonal matrices. The symmetric function f in the definition is then given by

$$f(x) = \Phi(\operatorname{diag} x), \quad \forall x \in \mathbb{R}^n,$$

where $\operatorname{diag} x$ is the diagonal matrix with diagonal elements $x_1, \dots x_n$.
Of particular interest is the class of spectrally defined functions associated with symmetric functions f that are proper convex and lsc, leading to convex matrix functions Φ_f. Most of these are in fact represented via $\operatorname{tr} f_s(A)$, for a particular choice of f. We recall that for a smooth scalar real-valued function f and a symmetric matrix A with real eigenvalues $\lambda_i(A)$, $i = 1, \dots, n$, one defines $f_s(A) := V^T \operatorname{diag}(f(\lambda_1(A)), \dots, f(\lambda_n(A)))V$, with $\lambda_i(A) \in \operatorname{dom} f$, $i = 1, \dots, n$, arranged in nondecreasing order, and the trace of $f_s(A)$ is thus given by

$$\operatorname{tr} f_s(A) = \sum_{i=1}^{n} f(\lambda_i(A)). \tag{2.25}$$

Example 2.7.1 *(i) Let*

$$f(\lambda) = \begin{cases} -\sum_{i=1}^n \log \lambda_i & \text{if } \lambda > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then for $X \in S_n$ one has

$$\Phi_f(X) = \begin{cases} -\log \det X & \text{if } X \succ 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\det X$ *is the determinant of the matrix X.*
(ii) Let $f(\lambda) = \max_{1 \le i \le n} \lambda_i$. Then $\Phi_f(X) = \lambda_{\max}(X)$, the largest eigenvalue of X.
(iii) Let

$$f(\lambda) = \begin{cases} -\sum_{i=1}^n \frac{1}{\lambda_i} & \text{if } \lambda > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then for $X \in S_n$ one has

$$\Phi_f(X) = \begin{cases} -\operatorname{tr}(X^{-1}) & \text{if } X \succ 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

(iv) Let

$$f(\lambda) = \begin{cases} -(\Pi_{i=1}^n \lambda_i)^{1/n} & \text{if } \lambda > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then for $X \in S_n$ one has

$$\Phi_f(X) = \begin{cases} -(\det X)^{1/n} & \text{if } X \succeq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Convex analysis tools in $\mathbb{R}^n$ can be translated to S_n. Most of the interesting properties of Φ_f can be deduced directly from those of f, and therefore convex analysis of spectrally defined convex functions can be easily developed.
The effective domain of Φ is defined by

$$\operatorname{dom} \Phi := \{X \in S_n \mid \Phi(X) < \infty\}.$$

The conjugate Φ^* of Φ is defined by

$$\Phi^*(Y) := \sup\{\langle X, Y\rangle - \Phi(X) \mid X \in S_n\}, \ \forall Y \in S_n,$$

and the subdifferential of Φ at $X \in \operatorname{dom} \Phi$ is

$$\begin{aligned} \partial\Phi(X) &= \{Y \in S_n \mid \Phi(Z) \ge \Phi(X) + \langle Z - X, Y\rangle, \ \forall Z \in S_n\} \\ &= \{Y \in S_n \mid \Phi^*(Y) + \Phi(X) = \langle X, Y\rangle\}. \end{aligned}$$

An important and useful inequality is the *trace inequality*

$$\langle X, Y\rangle \le \langle \lambda(X), \lambda(Y)\rangle, \ \ \forall X, Y \in S_n, \tag{2.26}$$

and equality holds if the matrices X, Y are simultaneously diagonalizable, i.e., if and only if there exists a matrix $U \in \mathcal{U}_n$ such that

$$U^T XU = \operatorname{diag}\lambda(X) \ \text{ and } \ U^T YU = \operatorname{diag}\lambda(Y).$$

Theorem 2.7.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a symmetric proper convex and lsc function and $\Phi_f : S_n \to \mathbb{R} \cup \{+\infty\}$ the induced convex spectral function. Then f^* is symmetric, and the following relations hold:*
(a) $\Phi_f^* = \Phi_{f^*}$.
(b) Φ_f is a proper, lsc, convex function.
(c) $\Phi_f^*(Y) + \Phi_f(X) \ge \langle \lambda(X), \lambda(Y)\rangle, \ \forall X, Y \in S_n$.
(d) $Y \in \partial\Phi_f(X) \iff \langle \lambda(X), \lambda(Y)\rangle = \langle X, Y\rangle$ and $\lambda(Y) \in \partial f(\lambda(X))$. Furthermore, X and Y are simultaneously diagonalizable.

Proof. The symmetry of f^* follows immediately from the fact that for any invertible matrix A, with $h(x) = f(Ax)$ one obtains $h^*(y) = f^*((A^T)^{-1}y)$, and thus if A is an orthogonal matrix (in particular a permutation matrix) one has $A^{-1} = A^T$ and hence $f^*(y) = f^*(Ay)$. To prove (a), clearly, we have $\Phi_f^*(Y) = \Phi^*(\operatorname{diag}\lambda(Y))$. Using the definition of the conjugate function one thus has

$$\begin{aligned}\Phi_f^*(Y) = \Phi^*(\operatorname{diag}\lambda(Y)) &= \sup_{X\in S_n}\{\langle X, \operatorname{diag}\lambda(Y)\rangle - \Phi_f(X)\},\\ &\ge \sup_{x\in\mathbb{R}^n}\{\langle \operatorname{diag} x, \operatorname{diag}\lambda(Y)\rangle - \Phi_f(\operatorname{diag} x))\},\\ &= \sup\{\langle x, \lambda(Y)\rangle - f(x)\} = f^*(\lambda(Y)) = \Phi_{f^*}(Y).\end{aligned}$$

The proof of the reverse inequality $\Phi_f^*(Y) \le \Phi_{f^*}(Y)$ is simply obtained by combining the Fenchel–Young inequality $f^*(\lambda(Y)) + f(\lambda(X)) \ge \langle \lambda(X), \lambda(Y)\rangle$ with the trace inequality (2.26). The statements (b), (c), (d) follow as easy consequences of the formula proven in (a), noting that in (d) the last statement follows from the trace equality condition given after (2.26). □

As a consequence of Theorem 2.7.1, the optimization problems $\min\{\Phi_f(X) \mid X \in S_n\}$ and $\min\{f(x) \mid x \in \mathbb{R}^n\}$ are equivalent. In fact, one has

$$\inf_{X\in S_n} \Phi_f(X) = -\Phi_f^*(0) = -\Phi_{f^*}(0) = -f^*(0) = \inf f.$$

Moreover,

$$X \in \operatorname{argmin}\Phi_f \iff \lambda(X) \in \operatorname{argmin} f,$$

with $\operatorname{argmin}\Phi := \{X \in S_n \mid \Phi(X) = \inf\Phi\}$.

Following Proposition 2.5.2, the asymptotic function of the proper convex lsc function $\Phi : S_n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\Phi_\infty(D) = \sup_{t>0} \frac{\Phi(A+tD) - \Phi(A)}{t} \quad \forall D \in S_n,$$

where $A \in \operatorname{dom} \Phi$. Using Theorem 2.5.4 one has, in fact,

$$\Phi_\infty(D) = \sup\{\langle B, D, \rangle : \ B \in \operatorname{dom} \Phi^*\} \ \ \forall D \in S_n.$$

We remark that if f is symmetric, lsc, convex, and proper, by using Proposition 2.5.2 so is f_∞.

Theorem 2.7.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be symmetric, lsc, proper, and convex with induced spectral function Φ_f. Then,*

$$(\Phi_f)_\infty(D) = \Phi_{f_\infty}(D) = f_\infty(\lambda(D)).$$

Proof. For any $D \in S_n$, one has

$$\begin{aligned} (\Phi_f)_\infty(D) &= \sup\{\langle Z, D\rangle \mid Z \in \operatorname{dom} \Phi_f^*\} \\ &= \sup\{\langle Z, D\rangle \mid Z \in \operatorname{dom} \Phi_{f^*}\} \\ &= \sup\{\langle Z, D\rangle \mid Z \in \operatorname{cl}\operatorname{dom} \Phi_{f^*}\}, \end{aligned}$$

where in the second equality we invoke Theorem 2.7.1(a). On the other hand, we also have

$$\begin{aligned} \Phi_{f_\infty}(D) &= \sup\{\langle Z, D\rangle \mid Z \in \operatorname{dom} \Phi_{f_\infty}^*\} \\ &= \sup\{\langle Z, D\rangle \mid Z \in \operatorname{dom} \Phi_{(f_\infty)^*}\} \\ &= \sup\{\langle Z, D\rangle \mid Z \in \operatorname{cl}\operatorname{dom} \Phi_{(f_\infty)^*}\}. \end{aligned}$$

To prove the desired statement, it thus suffices to verify that $\operatorname{cl}\operatorname{dom} \Phi_f^* = \operatorname{cl}\operatorname{dom} \Phi_{(f_\infty)^*}$. Using again Theorem 2.7.1(a) one has

$$\operatorname{dom} \Phi_{f^*} = \{U \operatorname{diag} x\, U^T \mid U \in \mathcal{U}_n, \ x \in \operatorname{dom} f^*\},$$

and therefore

$$\begin{aligned} \operatorname{cl}\operatorname{dom} \Phi_{f^*} &= \{U \operatorname{diag} x\, U^T \mid U \in \mathcal{U}_n, \ x \in \operatorname{cl}\operatorname{dom} f^*\} \\ &= \{U \operatorname{diag} x\, U^T \mid U \in \mathcal{U}_n, \ x \in \operatorname{dom}(f_\infty)^*\} = \operatorname{dom} \Phi_{(f_\infty)^*}, \end{aligned}$$

where in the second equality we use the fact $\operatorname{dom}(f_\infty)^* = \operatorname{cl}\operatorname{dom} f^*$, which follows from Theorem 2.5.4(b). $\square$

2.8 Application II: Modeling and Smoothing Optimization Problems

Asymptotic functions are essentially *built-in* within most optimization problems. This can be realized by considering the following general composite optimization model,

$$\inf\{f_0(x) + p(F(x)) \mid F(x) \in \operatorname{dom} p\},$$

where $F(x) = (f_1(x), \ldots, f_m(x))^T$ with all f_i real-valued, and $p : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is some given function. The composite model is rich enough to encompass most of the interesting formulations of nonlinear optimization problems and allows for recovering a generic class of both smooth and nonsmooth problems. In fact, as illustrated below, a viable choice for the function p is simply to pick the support function of some given subset $Y \subset \mathbb{R}^m$, i.e., to set $p(u) := \sup_{y \in Y} \langle y, u \rangle$.

Example 2.8.1 (i) l_1-norm approximation/optimization problems

$$p(u) = \sum_{i=1}^{p} |u_i|, \;\; Y = \{y : \|y\|_\infty \le 1\}.$$

(ii) Discrete minimax problems

$$p(u) = \max_{i=1,\ldots,m} u_i, \;\; Y = \Delta_m = \left\{ y \in \mathbb{R}^m \mid \sum_{i=1}^{m} y_i = 1, y \ge 0 \right\}.$$

(iii) l_2-norm nonsmooth problems

$$p(u) = \|u\|, \;\; Y = \{y \mid \|y\| \le 1\}.$$

Now suppose that there exists an lsc proper convex function H with conjugate H^* and such that $Y = \operatorname{dom} H^*$. Then by Theorem 2.5.4 one has $\sigma_Y = \sigma_{\operatorname{dom} H^*} = H_\infty$, and the composite model can be represented in terms of asymptotic functions and takes the form

$$\text{(CM)} \quad v = \inf\{\phi(x) \mid x \in \mathbb{R}^n\}$$

with

$$\phi(x) = \begin{cases} f_0(x) + H_\infty(f_1(x), \ldots, f_m(x)) & \text{if } x \in \bigcap_{i=1}^m \operatorname{dom} f_i, \\ +\infty & \text{otherwise.} \end{cases}$$

Now there are two key observations that can be made:
(i) The model above is in fact fairly generic, since it contains all usual constrained optimization problems. Indeed, for $p(u) = \delta_{\mathbb{R}^m_-}(u)$, problem (CM) is equivalent to

$$\inf\{f_0(x) \mid f_i(x) \le 0, \; i = 1, \ldots, m\}.$$

(ii) By Corollary 2.5.3 the asymptotic function H_∞ of a given lsc proper convex function H can be *approximated* via

$$H_\infty(y) = \lim_{r \to 0^+} \{H_r(y) := rH(r^{-1}y)\}, \quad \forall y \in \operatorname{dom} H,$$

and thus leads one naturally to consider as an approximate problem for (CM) the problem

$$(\mathrm{CM})_r \quad v_r = \inf\{\phi_r(x) \mid x \in \mathbb{R}^n\}$$

with

$$\phi_r(x) = \begin{cases} f_0(x) + H_r(f_1(x), \ldots, f_m(x)) & \text{if } x \in \bigcap_{i=1}^m \operatorname{dom}\ f_i, \\ +\infty & \text{otherwise.} \end{cases}$$

Thus, with H smooth enough, the above mechanism might lead to generic smooth approximation–type algorithms, such as penalty–barrier methods, which are often used as basic algorithms in the numerical treatment of optimization problems. Thus these methods can be analyzed within a unified framework based on the properties of asymptotic functions and where here $r > 0$ plays the role of *smoothing* parameter. The natural questions which thus arise are: Does there exist a solution x_r of problem $(\mathrm{CM})_r$? If it exists, do we have $x_r \to x$ as a solution of (CM) and $v_r \to v$? This type of questions can be efficiently handled via calculus rules of asymptotic functions, as illustrated below and in Chapter 5 later on.
Before formalizing the above approach, let us go back first to some of the examples above. For the corresponding function $p = H_\infty$ in the three examples, we can easily verify that the corresponding functions H are respectively given by the smooth functions

$$H(y) = \sum_{i=1}^m \sqrt{1 + y_i^2}, \quad H(y) = \log \sum_{i=1}^m e_i^y, \quad H(y) = \sqrt{\|y\|^2 + 1}.$$

In the general case of constrained optimization problems it turns out that there exists a wide class of functions H such that $H_\infty = \delta_{\mathbb{R}^m_-}$.

Proposition 2.8.1 *Let $\theta : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be an lsc, proper convex and nondecreasing function with $\operatorname{dom}\theta = (-\infty, b)$ and $b \in [0, \infty)$, $\theta_\infty(-1) = 0, \theta_\infty(1) = +\infty$ and set $H(y) = \sum_{i=1}^m \theta(y_i)$. Then*

$$H_\infty(d) = \delta_{\mathbb{R}^m_-}(d), \quad \forall d \in \mathbb{R}^m.$$

Proof. By Proposition 2.6.1 one has $H_\infty(d) = \sum_{i=1}^m \theta_\infty(d_i), \quad \forall d \in \mathbb{R}^m$. Under the hypothesis on the function θ we have

$$\theta_\infty(s) = \begin{cases} s\theta_\infty(1) & \text{if } s > 0, \\ -s\theta_\infty(-1) & \text{if } s \le 0, \end{cases}$$

which proves the desired formula for H_∞. $\square$

Particularly interesting choices for the functions θ are

$$\begin{aligned}
\theta_1(u) &= \exp\,(u),\ \operatorname{dom}\theta = \mathbb{R},\\
\theta_2(u) &= -\log(1-u),\ \operatorname{dom}\theta = (-\infty, 1),\\
\theta_3(u) &= \frac{u}{1-u},\ \operatorname{dom}\theta = (-\infty, 1),\\
\theta_4(u) &= -\log(-u),\ \operatorname{dom}\theta = (-\infty, 0),\\
\theta_5(u) &= -u^{-1},\ \operatorname{dom}\theta = (-\infty, 0).
\end{aligned}$$

Another interesting example is provided by the class of semidefinite optimization problems introduced in the previous section.

Example 2.8.2 Semidefinite programming
Consider the following semidefinite optimization problem

$$\text{(SDP)}\quad \inf\ c^T x:\ \text{subject to } B(x) \preceq 0,$$

with $B(x) = B_0 + \sum_{i=1}^m x_i B_i$. The problem data are the vector $c \in \mathbb{R}^m$ and the $(m+1)$ symmetric matrices $B_0, B_1, \ldots, B_m$ of order $n \times n$. This special class of convex problems can also be handled through the general composite model by a simple adjustment of the involved finite-dimensional setting. Let us replace the set $Y \subset \mathbb{R}^m$ with $Y \subset S_n$, the space of $n \times n$ symmetric matrices. More precisely, recalling that the indicator of the negative cone in $\mathbb{R}^n$ is a symmetric function, (cf. Section 2.7), we can associate to it the spectral function $p : S_n \to \mathbb{R} \cup \{+\infty\}$ defined by $p(U) = \delta_{\mathbb{R}^n_-}(\lambda(U))$, where $\lambda(U)$ is the vector of eigenvalues of the symmetric matrix U in nondecreasing order. For any symmetric matrix U one has $\delta_{S_n^-}(U) = \delta_{\mathbb{R}^n_-}(\lambda(U))$, where S_n^- denotes the space of symmetric negative definite matrices, and thus one can write

$$p(U) = \delta_{S_n^-}(U) = \sup\{\langle Z, U\rangle \mid Z \in S_n^+\}.$$

Therefore, problem (SDP) can be written as the composite optimization model

$$\inf\{c^T x + p(B(x)) \mid B(x) \in \operatorname{dom} p\}.$$

From here, we can thus proceed as in Example 2.8.1. Let $h(y) = \sum_{i=1}^n \theta(y_i)$ with $\theta : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ a function satisfying Proposition 2.8.1. Then one has $h_\infty = \delta_{\mathbb{R}^n_-}$. Invoking Theorem 2.7.2 (once again recall that the indicator of the negative cone in $\mathbb{R}^n$ is a symmetric function), it follows that for any $D \in S_n$ one has $H_\infty(D) = \delta_{S_n^-}(D)$ with $H(D) = h(\lambda(D))$. Thus problem (SDP) can be written in the form of the composite model (CM)

$$\inf_{x \in \mathbb{R}^m} \{c^T x + H_\infty(B(x))\},$$

with corresponding approximate model given by

$$(SDP_r) \quad \inf_{x\in\mathbb{R}^m} \{c^T x + H_r(B(x))\},$$

where $H_r(D) = r^{-1}H(rD) = r^{-1}h(r\lambda(D))$.
Using for example the functions θ given above we thus obtain for any symmetric matrix D the following corresponding functions H

$H_1(D) = \mathrm{tr}(\exp D)$,
$H_2(D) = -\log(\det(I - D))$ for $D \prec I$, $+\infty$ otherwise,
$H_3(D) = \mathrm{tr}((I - D)^{-1}D)$ for $D \prec I$, $+\infty$ otherwise,
$H_4(D) = -\log(\det(-D))$ for $D \prec 0$, $+\infty$ otherwise,
$H_5(D) = \mathrm{tr}(-D^{-1})$ for $D \prec 0$, $+\infty$ otherwise.

We note that for these examples each H is C^∞ on the interior of its domain.

To formalize the above approach we need to make some minimal hypothesis on functionals involved in the structural representation of the problem (CM) $\inf\{\phi(x) \mid x \in \mathbb{R}^n\}$ and its corresponding approximation defined in $(\mathrm{CM})_r$ for every $r > 0$.

Definition 2.8.1 *A function $H : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ that is a proper, lsc, convex function with asymptotic function H_∞ is called a generating asymptotic approximation kernel if the following conditions hold:*
(a) H_∞ is isotone, i.e.,

$$y_i \le z_i \quad \forall i = 1, 2 \ldots, m \Longrightarrow H_\infty(y) \le H_\infty(z).$$

(b) $\lim_{r\to 0+} H_r(y) = H_\infty(y), \ \forall y \in \ \mathrm{ri}\,\mathrm{dom}\, H_\infty$.
(c) $\mathrm{ri}\,\mathrm{dom}\ H_\infty \subset \ \mathrm{ri}\,\mathrm{dom}\, H_r, \ \forall r > 0$.
(d) The constancy space of H_∞ satisfies $\mathcal{C}_{H_\infty} = \{0\}$.

It can be verified that the functions H defined through Proposition 2.8.1, and based on the choice of the functions θ outlined above, are generating asymptotic approximation kernels. In the case of semidefinite programming given in Example 2.8.2, a similar analysis can be developed to verify that claim. This can be done by either identifying the space S_n with $\mathbb{R}^{(n+1)n/2}$ or in a direct way. We refer the reader to the notes and references for details.

Lemma 2.8.1 *For any generating asymptotic approximation kernel H one has*

$$\lim_{\lambda\to+\infty} H_\infty(a_1, \ldots, a_{i-1}, a_i + \lambda, a_{i+1}, \ldots, a_m) = +\infty, \ \forall a \in \mathbb{R}^m.$$

Proof. By definition, H_∞ is convex and isotone. Suppose the result does not hold. Then one has

$$\liminf_{\lambda\to+\infty} \psi(\lambda) := H_\infty(a_1, \ldots, a_{i-1}, a_i + \lambda, a_{i+1}, \ldots, a_m) < +\infty,$$

and by Theorem 2.5.2, the function $\psi(\lambda)$ is decreasing, and since H_∞ is assumed isotone, it follows that H_∞ is constant along each direction $d = e_i$, where $e_i,\ i = 1, \ldots, m$, denotes the canonical basis of $\mathbb{R}^m$. This, however, contradicts the condition $C_{H_\infty} = \{0\}$ (cf. Definition 2.8.1) imposed on a generating asymptotic kernel. □

In the remainder of this section, for the problem's data (CM) we now make the following minimal assumptions, which are needed to guarantee that (CM) is well-defined.
(i) The functions $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 0, 1, \ldots, m$, are closed, proper and satisfy

$$(f_i)_\infty(d) > -\infty \quad \forall d.$$

(ii) There exists $x_0 \in \operatorname{dom} f_0$ such that $F(x_0) \in \operatorname{dom} H_\infty$, where $F(x) := (f_1(x), f_2(x), \ldots, f_m(x))$. When $0 \notin \operatorname{dom} H$ we suppose in addition that $F(x_0) \in \operatorname{ri}\operatorname{dom} H_\infty$.
Note that the assumption (i) on f_i is always satisfied when f_i is convex. Note also that whenever H is isotone, then the isotonicity of H_∞ follows at once, and thus if f_0 is in addition convex, then ϕ and ϕ_r are convex, and (CM) and $(\mathrm{CM})_r$ are convex problems.

Lemma 2.8.2 *Consider problems (CM) and $(CM)_r$ with a generating asymptotic approximation kernel H. Then:*
(a) The function ϕ is lsc, proper, and the same holds for any ϕ_r with $r > 0$ if H is isotone.
(b) If $0 \in \operatorname{dom} H$, one has $x \in \operatorname{dom} \phi \implies x \in \operatorname{dom} \phi_r$.

Proof. (a) We prove only that ϕ is lsc, since with the same arguments we can prove that ϕ_r is also lsc. Let x be arbitrary and let $\{x_n\}$ be a sequence converging to x. We have to prove that $\liminf_{n\to\infty} \phi(x_n) \geq \phi(x)$. As a consequence we have only to consider the case where $x_n \in \operatorname{dom} \phi$. Let $\epsilon > 0$ be arbitrary. Since the functions f_i are lsc, then for n sufficiently large we have

$$f_i(x_n) \geq f_i(x) - \epsilon, \quad \forall i = 0, 1 \ldots, m.$$

Now, since H_∞ is isotone, it follows for n sufficiently large that

$$\phi(x_n) \geq f_0(x) + H_\infty(f_1(x) - \epsilon, \ldots, f_m(x) - \epsilon) - \epsilon.$$

Passing to the limit as $n \to \infty$ we obtain

$$\liminf_{n\to\infty} \phi(x_n) \geq f_0(x) + H_\infty(f_1(x) - \epsilon, \ldots, f_m(x) - \epsilon) - \epsilon.$$

Now let $\epsilon \to 0^+$. Since H_∞ is lsc, we get

$$\phi(x) \leq \liminf_{n\to\infty} \phi(x_n),$$

and it follows that ϕ is lsc.
Since f_0, H_∞, H are proper, it follows that ϕ and ϕ_r never take the value $-\infty$. The hypothesis on the problem's data that there exists $x_0 \in \operatorname{dom} f_0$ with $F(x_0) \in \operatorname{dom} H_\infty$ implies that $\phi(x_0)$ is finite. If $0 \in \operatorname{dom} H$, then from Proposition 2.5.2, it follows that $\phi_r(x_0)$ is also finite. In the other case, since $\operatorname{ri}\operatorname{dom} H_\infty$ is a cone, it follows that $r^{-1}F(x_0) \in \operatorname{ri}\operatorname{dom} H_\infty$, and then from Definition 2.8.1(c) it follows that $\phi_r(x_0)$ is finite, proving that ϕ and ϕ_r are proper. To prove (b) we suppose now that $0 \in \operatorname{dom} H$ and let $x \in \operatorname{dom}\phi$. Then $F(x) \in \operatorname{dom} H_\infty$, $f_0(x)$ is finite, and from Corollary 2.5.2 it follows that $\phi_r(x)$ is finite. □

We arrive now with a key asymptotic formula that can be used in the analysis of the approximation problem $(\mathrm{CM})_r$.

Proposition 2.8.2 *Consider the problem (CM) with a given generating asymptotic approximation kernel H. Let*

$$\widetilde{\phi}_\infty(d) = (f_0)_\infty(d) + H_\infty(F_\infty(d)) \ \ \textit{if}\ d \in \bigcap_{i=1}^m \operatorname{dom}(f_i)_\infty, +\infty \ \textit{otherwise},$$

where $F_\infty(d) = ((f_1)_\infty(d), \dots, (f_m)_\infty(d))$. Then

$$\phi_\infty(d) \geq \widetilde{\phi}_\infty(d) \quad \forall d \in \mathbb{R}^n,$$

with equality when the functions f_i are convex.

Proof. Let $a_i < (f_i)_\infty(d)$ for $i = 0, 1, \dots, m$ and $d_n \to d, t_n \to +\infty$ with $\phi_\infty(d) = \liminf_{n\to+\infty} t_n^{-1}\phi(t_n d_n)$. Then using the fundamental analytical formula of an asymptotic function (cf. Theorem 2.5.1), for n sufficiently large we have for each i

$$f_i(t_n d_n) \geq a_i t_n \iff F(t_n d_n) \geq a t_n \text{ with } a = (a_1, \dots, a_m).$$

Furthermore, since asymptotic functions are positively homogeneous, it follows that

$$\frac{\phi(t_n d_n)}{t_n} = \frac{f_0(t_n d_n)}{t_n} + H_\infty\left(\frac{F(t_n d_n)}{t_n}\right).$$

Since H_∞ is assumed isotone, using the inequalities above we get

$$\frac{\phi(t_n d_n)}{t_n} \geq a_0 + H_\infty(a).$$

Now let $a_i \to (f_i)_\infty(d)$. Then since H_∞ is lsc, using the definition of the asymptotic function and passing to the limit in the above formula, we obtain

$$\phi_\infty(d) \geq (f_0)_\infty(d) + H_\infty(F_\infty(d)) \text{ if } d \in \bigcap_{i=1}^m \operatorname{dom}(f_i)_\infty,$$

proving the result for such a direction. Otherwise, the result remains valid by invoking Lemma 2.8.1. Now, if we suppose that for each $i = 0, 1, \ldots, m$ the functions f_i are convex, since H_∞ is sublinear and assumed isotone, it follows that

$$\frac{\phi(x + \lambda d) - \phi(x)}{\lambda} \leq (f_0)_\infty(d) + H_\infty(F_\infty(d)) \text{ if } d \in \bigcap_{i=1}^{m} \operatorname{dom}(f_i)_\infty.$$

Passing to the limit as $\lambda \to +\infty$ we get $\phi_\infty(d) \leq \widetilde{\phi}_\infty(d)$, and thus the desired equality in the convex case follows. □

We end this section by indicating that other approximating schemes can be naturally generated via the analytical asymptotic function formula. Indeed, consider a scalar function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\alpha(r) > 0,\ \forall r > 0,\ \lim_{r \to 0_+} \alpha(r) = 0,\ \lim_{r \to 0_+} r^{-1}\alpha(r) = +\infty.$$

Consider now the class of functions $\theta : \mathbb{R} \to \mathbb{R}_+$, that are nondecreasing, convex, and such that

$$\lim_{u \to -\infty} \theta(u) = 0,\ \ 0 < \theta_\infty(1) < +\infty.$$

Since here $\theta_\infty(1) \neq +\infty$, we no longer have the representation $\delta_{\mathbb{R}^m_-}(u) = \lim_{r \to 0_+} r\theta(r^{-1}u)$. However, it is possible to write now that $\delta_{\mathbb{R}^m_-}(u) = \lim_{r \to 0_+} \alpha(r)\theta(r^{-1}u)$. Thus, with the functions α and θ as above, we can now construct the approximation kernel $H_r(y) = \alpha(r)H(r^{-1}y)$, where $H(y) = \sum_{i=1}^m \theta(y_i)$, and we have $\lim_{r \to 0_+} H_r(y) = \delta_{\mathbb{R}^m_-}(y)$. Within this class we then construct the approximate models $(\mathrm{CM})_r$ and $(\mathrm{SDP})_r$ as before. Two particular examples of functions θ that can be used to generate the approximate models are

$$\theta_6(u) = \log(1 + e^u),\ \ \theta_7(u) = 2^{-1}(u + \sqrt{u^2 + 4}).$$

For any symmetric matrix D, the corresponding functions H used for approximating the semidefinite program (SDP) then take the forms

$$H_6(D) = \log\det(I + \exp(D)),\ \ H_7(D) = 2^{-1} \operatorname{tr}\left(D + \sqrt{D^2 + 4I}\right).$$

2.9 Notes and References

The concept of asymptotic cone and many of its properties seem to have appeared first in the literature in the work of Steinitz in 1913 [126]. Properties of unbounded convex point sets have been also studied in 1940 by

Stoker [127]. These concepts can also be found in Bourbaki [39] and Choquet [47]. For convex sets, the importance of asymptotic cones were already realized in the work of Dieudonné in 1966 [60] and for closed convex sets, in Rockafellar [119], who uses the terminology of *recession* cones. The generalization of the concept of asymptotic cone of convex sets to closed sets in an arbitrary topological vector space was considered in the works of Dedieu [58], [59]. Section 2.1 includes the fundamental results on asymptotic cones in both the convex and nonconvex cases, most of which can be found in the book of Rockafellar [119], while in the nonconvex case several results outlined here appear in the references cited above, but in greater details in the recent book of Rockafellar–Wets [123]. The dual characterization given in Section 2.2 can be found in the book of Aubin–Ekeland [4]. Many results on closedness criteria have been scattered in the literature, and our intent here was to present some of the most useful criteria. Most results were originally concerned with sufficient conditions that ensure the closedness of the image of a closed set. The conditions are expressed in terms of some properties shared by the map and the asymptotic cone of the set in question. In the convex case, such results can be found in Fenchel [72], Choquet [47], and Rockafellar [117]. The weaky coercive case described by Corollary 2.3.2(b) can be found in [119]. It seems that Dedieu [57] was the first to give the corresponding result to Corollary 2.3.2(a) in the nonconvex coercive case. The necessary and sufficient condition that ensures that the image of a closed set under a linear map remains closed and given in Theorem 2.3.1 is due to Auslender [13]. The concept of asymptotic linear sets and their properties were introduced by Auslender [18]. Theorem 2.3.2, stating that the image of the asymptotic cone of a set coincides with the asymptotic cone of its image, is new. However, such a result was proven earlier in the coercive case by Zalinescu [135] and can be found for the convex case in Rockafellar's book [119]. Specific results ensuring the closedness of the sum of closed sets and the set of convex combinations of a finite collection of closed sets have been given for the convex case in [119] and extended to the nonconvex case in [13]. Results concerning semibounded sets can be found in an article of Fadeev and Fadeva published in the collected works [100]. Continuous convex sets originated in the work of Gale and Klee [75], and the results given here follow this work. Propositions 2.4.2 and 2.4.3 are due to Auslender–Coutat [11]. Building on the concept of asymptotic cone, one can analyze the behavior in the large of real-valued functions through their epigraphs. This has been done by Rockafellar [119] for convex functions, who gave the representation formula of Proposition 2.5.2. For the nonconvex case, the notion of asymptotic function and the representation formula given in Theorem 2.5.1 were first given by Dedieu [58] and later on by Biaocchi, Butazzo, Gastaldi, and Tomeralli [22]. Most of the results presented in Section 2.5 and Section 2.6 are classical in the convex case and can be found in [119] as well as in the more recent book of Rockafellar and Wets [123], which also includes results for nonconvex functions. Some other less classical or not so well known results include Proposition 2.6.6, due to Benoist and Hirriart-Urruty [26]; Corollary 2.6.1, due to Valadier [128]; Proposition 2.6.5 from Auslender [13]. Proposition 2.6.7, (well known is convex analysis) is new, and the formula on the composite of a convex function in Proposition 2.6.4 given in Auslender, Cominetti, and Haddou [14], while the results given in Lemma 2.6.1 on one-dimensional convex functions is from Ben-Tal, Ben-Israel, and Teboulle [27]. Systems of matrix inequalities and related optimization problems, today called *Semidefinite Optimization* originated in 1963 with the work of Bellman and Fan [24]. This class of problems today forms an active research area; see for example, the recent handbook of semidefinite programming [134], which includes about one thousand references. We gave in Section 2.7 some basic notions on convex functions of matrices first characterized via their eigenvalues by Davis [55] and further studied and extended by Lewis in [87] and Seeger [124], on which our presentation is based. The formula for the associated asymptotic function given in Theorem 2.7.2 can be found in Seeger [124], with a slightly different proof given here. Section 2.8 considers smoothing and approximation of optimization problems. Initial studies on that topic can be found in Bertsekas in [33] and in [34], who proposed and studied in particular the logarithmic sum of exponentials. Related numerical algorithms based on this approach such as penalty–barrier and Lagrangians methods can be found in [34] and the more recent book of Berstekas [36], which also includes recent developments in the field together with the relevant references. The idea of smoothing optimization via asymptotic functions was originally proposed by Ben-Tal and Teboulle [28] for finite convex functions, with some applications that can be found in Ben-Tal, Teboulle, and Hang [29]. More recently, Auslender [17] proposed a unified framework within asymptotic functional representation of optimization problems to handle and analyze the more important case of extended real-valued functions, thus allowing the modeling of general problems such as nonlinear constrained optimization and semidefinite programs via the use of Propositions 2.8.1 and 2.8.2 proven in [17]. The presentation here follows the work developed in [17].

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