Preface to the English Edition

The author is very pleased that his book, first published in Russian in 2000 by MCCME Publishers, is now appearing in English under the auspices of such a truly classical publishing house as Springer-Verlag.

In this edition several pertinent remarks by the referees (to whom the author expresses his gratitude) were taken into account, and new exercises were added (mostly) to the first half of the book, thus achieving a better balance with the second half. Besides, some typos and minor errors, noticed in the Russian edition, were corrected. We are extremely grateful to all our readers who assisted us in this tiresome bug hunt. We are especially grateful to A. De Paris, I. S. Krasil'schik, and A. M. Verbovetski, who demonstrated their acute eyesight, truly of *degli Lincei* standards.

The English translation was carried out by A. B. Sossinsky (Chapters 1–8), I. S. Krasil'schik (Chapter 9), and S. V. Duzhin (Chapters 10–11) and reduced to a common denominator by the first of them; A. M. Astashov prepared new versions of the figures; all the TEX-nical work was done by M. M. Vinogradov.

In the process of preparing this edition, the author was supported by the Istituto Nazionale di Fisica Nucleare and the Istituto Italiano per gli Studi Filosofici. It is only thanks to these institutions, and to the efficient help of Springer-Verlag, that the process successfully came to its end in such a short period of time.

Jet Nestruev
Moscow–Salerno
April 2002

Preface

The limits of my language
are the limits of my world.
— L. Wittgenstein

This book is a self-contained introduction to smooth manifolds, fiber spaces, and differential operators on them, accessible to graduate students specializing in mathematics and physics, but also intended for readers who are already familiar with the subject. Since there are many excellent textbooks in manifold theory, the first question that should be answered is, Why another book on manifolds?

The main reason is that the good old differential calculus is actually a particular case of a much more general construction, which may be described as the *differential calculus over commutative algebras*. And this calculus, in its entirety, is just the consequence of properties of arithmetical operations. This fact, remarkable in itself, has numerous applications, ranging from delicate questions of algebraic geometry to the theory of elementary particles. Our book explains in detail why the differential calculus on manifolds is simply an aspect of commutative algebra.

In the standard approach to smooth manifold theory, the subject is developed along the following lines. First one defines the notion of smooth manifold, say M. Then one defines the algebra $\mathcal{F}_M$ of smooth functions on M, and so on. In this book this sequence is reversed: We begin with a certain commutative $\mathbb{R}$-algebra[1] $\mathcal{F}$, and then define the manifold $M = M_{\mathcal{F}}$

[1] Here and below $\mathbb{R}$ stands for the real number field. Nevertheless, and this is very important, nothing prevents us from replacing it by an arbitrary field (or even a ring) if this is appropriate for the problem under consideration.

as the $\mathbb{R}$-spectrum of this algebra. (Of course, in order that $M_{\mathcal{F}}$ deserve the title of a smooth manifold, the algebra $\mathcal{F}$ must satisfy certain conditions; these conditions appear in Chapter 3, where the main definitions mentioned here are presented in detail.)

This approach is by no means new: It is used, say, in algebraic geometry. One of its advantages is that from the outset it is not related to the choice of a specific coordinate system, so that (in contrast to the standard analytical approach) there is no need to constantly check that various notions or properties are independent of this choice. This explains the popularity of this viewpoint among mathematicians attracted by sophisticated algebra, but its level of abstraction discourages the more pragmatically inclined applied mathematicians and physicists.

But what is really new in this book is the motivation of the algebraic approach to smooth manifolds. It is based on the fundamental notion of *observable*, which comes from physics. It is this notion that creates an intuitively clear environment for the introduction of the main definitions and constructions. The concepts of *state of a physical system* and *measuring device* endow the very abstract notions of *point of the spectrum* and *element of the algebra* $\mathcal{F}_M$ with very tangible physical meanings.

One of the fundamental principles of contemporary physics asserts that *what exists is only that which can be observed.* In mathematics, which is not an experimental science, the notion of observability was never considered seriously. And so the discussion of any *existence problem* in the formalized framework of mathematics has nothing to do with reality. A present-day mathematician studies sets supplied with various structures without ever specifying (distinguishing) individual elements of those sets. Thus their observability, which requires some *means of observation*, lies beyond the limits of formal mathematics.

This state of affairs cannot satisfy the working mathematician, especially one who, like Archimedes or Newton, regards his science as natural philosophy. Now, physicists, for example, in their study of quantum phenomena, come to the conclusion that it is impossible in principle to completely distinguish the observer from the observed. Hence any adequate mathematical description of quantum physics must include, as an inherent part, an appropriate formalization of observability.

Scientific observation relies on measuring devices, and in order to introduce them into mathematics, it is natural to begin with its classical parts, i.e., those coming from classical physics. Thus we begin with a detailed explanation of why the classical measuring procedure can be translated into mathematics as follows:

Physics lab	$\longrightarrow$	Commutative unital $\mathbb{R}$-algebra A
Measuring device	$\longrightarrow$	Element of the algebra A
State of the observed physical system	$\longrightarrow$	Homomorphism of unital $\mathbb{R}$-algebras $h\colon A \to \mathbb{R}$
Output of the measuring device	$\longrightarrow$	Value of this function $h(a)$, $a \in A$

In the framework of this approach, smooth (i.e., differentiable) manifolds appear as $\mathbb{R}$-spectra of a certain class of $\mathbb{R}$-algebras (the latter are therefore called *smooth*), and their elements turn out to be the smooth functions defined on the corresponding spectra. Here the $\mathbb{R}$-spectrum of some $\mathbb{R}$-algebra A is the set of all its unital homomorphisms into the $\mathbb{R}$-algebra $\mathbb{R}$, i.e., the set that is "visible" by means of this algebra. Thus smooth manifolds are "worlds" whose observation can be carried out by means of smooth algebras. Because of the algebraic universality of the approach described above, "nonsmooth" algebras will allow us to observe "nonsmooth worlds" and study their singularities by using the differential calculus. But this differential calculus is not the naive calculus studied in introductory (or even "advanced") university courses; it is a much more sophisticated construction.

It is to the foundations of this calculus that the second part of this book is devoted. In Chapter 9 we "discover" the notion of differential operator *over a commutative algebra* and carefully analyze the main notion of the classical differential calculus, that of the derivative (or more precisely, that of the tangent vector). Moreover, in this chapter we deal with the other simplest constructions of the differential calculus from the new point of view, e.g., with tangent and cotangent bundles, as well as jet bundles. The latter are used to prove the equivalence of the algebraic and the standard analytic definitions of differential operators for the case in which the basic algebra is the algebra of smooth functions on a smooth manifold. As an illustration of the possibilities of this "algebraic differential calculus," at the end of Chapter 9 we present the construction of the Hamiltonian formalism over an arbitrary commutative algebra.

In Chapters 10 and 11 we study fiber bundles and vector bundles from the algebraic point of view. In particular, we establish the equivalence of the category of vector bundles over a manifold M and the category of finitely generated projective modules over the algebra $C^\infty(M)$. Chapter 11 is concluded by a study of jet modules of an arbitrary vector bundle and an explanation of the universal role played by these modules in the theory of differential operators.

Thus the last three chapters acquaint the reader with some of the simplest and most thoroughly elaborated parts of the new approach to the differential calculus, whose complete logical structure is yet to be determined. In fact, one of the main goals of this book is to show that the discovery of the differential calculus by Newton and Leibniz is quite similar to the discovery of the New World by Columbus. The reader is invited to continue the expedition into the internal areas of this beautiful new world, differential calculus.

Looking ahead beyond the (classical) framework of this book, let us note that the mechanism of *quantum observability* is in principle of cohomological nature and is an appropriate specification of those natural observation methods of solutions to (nonlinear) partial differential equations that have appeared in the *secondary differential calculus* and in the fairly new branch of mathematical physics known as *cohomological physics.*

The prerequisites for reading this book are not very extensive: a standard advanced calculus course and courses in linear algebra and algebraic structures. So as not to deviate from the main lines of our exposition, we use certain standard elementary facts without providing their proofs, namely, partition of unity, Whitney's immersion theorem, and the theorems on implicit and inverse functions.

* * *

In 1969 Alexandre Vinogradov, one of the authors of this book, started a seminar aimed at understanding the mathematics underlying quantum field theory. Its participants were his mathematics students, and several young physicists, the most assiduous of whom were Dmitry Popov, Vladimir Kholopov, and Vladimir Andreev. In a couple of years it became apparent that the difficulties of quantum field theory come from the fact that physicists express their ideas in an inadequate language, and that an adequate language simply does not exist (see the quotation preceding the Preface). If we analyze, for example, what physicists call the covariance principle, it becomes clear that its elaboration requires a correct definition of differential operators, differential equations, and, say, second-order differential forms.

For this reason in 1971 a mathematical seminar split out from the physical one, and began studying the structure of the differential calculus and searching for an analogue of algebraic geometry for systems of (nonlinear) partial differential equations. At the same time, the above-mentioned author began systematically lecturing on the subject.

At first, the participants of the seminar and the listeners of the lectures had to manage with some very schematic summaries of the lectures and their own lecture notes. But after ten years or so, it became obvious that all these materials should be systematically written down and edited. Thus Jet Nestruev was born, and he began writing an infinite series of books entitled *Elements of the Differential Calculus*. Detailed contents of the first installments of the series appeared, and the first one was written. It contained, basically, the first eight chapters of the present book.

Then, after an interruption of nearly fifteen years, due to a series of objective and subjective circumstances, work on the project was resumed, and the second installment was written. Amalgamated with the first one, it constitutes the present book. This book is a self-contained work, and we have consciously made it independent of the rest of the Nestruev project. In it the reader will find, in particular, the definition of differential operators on a manifold. However, Jet Nestruev has not lost the hope to explain, in the not too distant future, what a system of partial differential equations is, what a second-order form is, and some other things as well. The reader who wishes to have a look ahead without delay can consult the references appearing on page 217. A more complete bibliography can be found in [8].

Unlike a well-known French general, Jet Nestruev is a civilian and his personality is not veiled in military secrecy. So it is no secret that this book was written by A. M. Astashov, A. B. Bocharov, S. V. Duzhin, A. B. Sossinsky, A. M. Vinogradov, and M. M. Vinogradov. Its conception and its main original observations are due to A. M. Vinogradov. The figures were drawn by A. M. Astashov. It is a pleasure for Jet Nestruev to acknowledge the role of I. S. Krasil'schik, who carefully read the whole text of the book and made several very useful remarks, which were taken into account in the final version.

During the final stages, Jet Nestruev was considerably supported by *Istituto Italiano per gli Studi Filosofici* (Naples), *Istituto Nazionale di Fisica Nucleare* (Italy), and INTAS (grant 96-0793).

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Moscow–Pereslavl-Zalesski–Salerno
April 2000

Springer

http://www.springer.com/978-0-387-95543-8

Smooth Manifolds and Observables
Nestruev, J.
2003, XIV, 222 p., Hardcover
ISBN: 978-0-387-95543-8