2
Complex Numbers

2.1 Are Complex Numbers Necessary?

Much of mathematics is concerned with various kinds of equations, of which equations with numerical solutions are the most elementary. The most fundamental set of numbers is the set $\mathbb{N} = \{1, 2, 3, \dots\}$ of **natural numbers**. If a and b are natural numbers, then the equation $x + a = b$ has a solution *within the set of natural numbers* if and only if $a < b$. If $a \geq b$ we must *extend* the number system to the larger set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3 \dots\}$ of **integers**. Here we get a bonus, for the equation $x + a = b$ has a solution $x = b - a$ in $\mathbb{Z}$ *for all a and b in $\mathbb{Z}$*.

If $a, b \in \mathbb{Z}$ and $a \neq 0$, then the equation $ax + b = 0$ has a solution in $\mathbb{Z}$ if and only if a *divides* b. Otherwise we must once again extend the number system to the larger set $\mathbb{Q}$ of **rational numbers**. Once again we get a bonus, for the equation $ax + b = 0$ has a solution $x = -b/a$ in $\mathbb{Q}$ for all $a \neq 0$ in $\mathbb{Q}$ and all b in $\mathbb{Q}$.

When we come to consider a **quadratic** equation $ax^2 + bx + c = 0$ (where $a, b, c \in \mathbb{Q}$ and $a \neq 0$) we encounter our first real difficulty. We may safely assume that a, b and c are integers: if not, we simply multiply the equation by a suitable positive integer. The standard solution to the equation is given by the familiar formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

Let us denote $b^2 - 4ac$, the **discriminant** of the equation, by Δ. If Δ is the square of an integer (what is often called a **perfect square**) then the equation has rational solutions, and if Δ is positive then the two solutions are in the extended set $\mathbb{R}$ of real numbers. But if $\Delta < 0$ then there is no solution even within $\mathbb{R}$.

We have already carried out three extensions (to $\mathbb{Z}$, to $\mathbb{Q}$, to $\mathbb{R}$) from our starting point in natural numbers, and there is no reason to stop here. We can modify the standard formula to obtain

$$x = \frac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a},$$

where $4ac - b^2 > 0$. If we postulate the existence of $\sqrt{-1}$, then we get a "solution"

$$x = \frac{-b \pm \sqrt{-1}\sqrt{4ac - b^2}}{2a}.$$

Of course we know that there is no real number $\sqrt{-1}$, but the idea seems in a way to work. If we look at a specific example,

$$x^2 + 4x + 13 = 0,$$

and decide to write i for $\sqrt{-1}$, the formula gives us two solutions $x = -2 + 3i$ and $x = -2 - 3i$. If we use normal algebraic rules, replacing i^2 by -1 whenever it appears, we find that

$$\begin{aligned}(-2+3i)^2 + 4(-2+3i) + 13 &= (-2)^2 + 2(-2)(3i) + (3i)^2 - 8 + 12i + 13 \\ &= 4 - 12i - 9 - 8 + 12i + 13 \text{ (since } i^2 = -1) \\ &= 0,\end{aligned}$$

and the validity of the other root can be verified in the same way. We can certainly agree that if there is a number system containing "numbers" $a + bi$, where $a, b \in \mathbb{R}$, then they will add and multiply according to the rules

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i \tag{2.1}$$

$$(a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i. \tag{2.2}$$

We shall see shortly that there is a way, closely analogous to our picture of real numbers as points on a line, of visualising these new **complex** numbers.

Can we find equations that require us to extend our new complex number system (which we denote by $\mathbb{C}$) still further? No, in fact we cannot: the important **Fundamental Theorem of Algebra**, which we shall prove in Chapter 7, states that, for all $n \geq 1$, every polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

with coefficients $a_0, a_1, \ldots, a_n$ in $\mathbb{C}$ and $a_n \neq 0$, has all its roots within $\mathbb{C}$. This is one of many reasons why the number system $\mathbb{C}$ is of the highest importance in the development and application of mathematical ideas.

EXERCISES

2.1. One way of proving that the set $\mathbb{C}$ "exists" is to define it as the set of all 2×2 matrices

$$M(a,b) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

where $a, b \in \mathbb{R}$.

a) Determine the sum and product of $M(a,b)$ and $M(c,d)$.

b) Show that

$$M(a,0) + M(b,0) = M(a+b,0), \quad M(a,0)M(b,0) = M(ab,0).$$

Thus $\mathbb{C}$ contains the real numbers "in disguise" as 2×2 diagonal matrices. Identify $M(a,0)$ with the real number a.

c) With this identification, show that $M(0,1)^2 = -1$. Denote $M(0,1)$ by i.

d) Show that $M(a,b) = a + bi$.

2.2. Determine the roots of the equation $x^2 - 2x + 5 = 0$.

2.2 Basic Properties of Complex Numbers

We can visualise a complex number $z = x + yi$ as a point (x, y) on the plane. Real numbers x appear as points $(x, 0)$ on the x-axis, and numbers yi as points $(0, y)$ on the y-axis. Numbers yi are often called **pure imaginary**, and for this reason the y-axis is called the **imaginary axis**. The x-axis, for the same reason, is referred to as the **real axis**. It is important to realise that these terms are used for historical reasons only: within the set $\mathbb{C}$ the number $3i$ is no more "imaginary" than the number 3.

If $z = x + iy$, where x and y are real, we refer to x as the **real part** of z and write $x = \operatorname{Re} z$. Similarly, we refer to y as the **imaginary part** of z, and write $y = \operatorname{Im} z$. Notice that *the imaginary part of z is a real number.*

The number $\bar{z} = x - iy$ is called the **conjugate** of z. It is easy to verify that, for all complex numbers z and w,

$$\bar{\bar{z}} = z, \quad \overline{z+w} = \bar{z} + \bar{w}, \quad \overline{zw} = \bar{z}\bar{w}, \tag{2.3}$$

and

$$z + \bar{z} = 2\operatorname{Re} z, \quad z - \bar{z} = 2i \operatorname{Im} z \tag{2.4}$$

and so, from (2.5),

$$|z+w|^2 \leq |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2 .$$

The result now follows by taking square roots.

For Part (iv), we observe first that

$$|z| = \big|(z-w)+w\big| \leq |z-w| + |w|$$

and deduce that

$$|z-w| \geq |z| - |w| . \tag{2.6}$$

Similarly, from

$$|w| = \big|z-(z-w)\big| \leq |z| + |z-w|$$

we deduce that

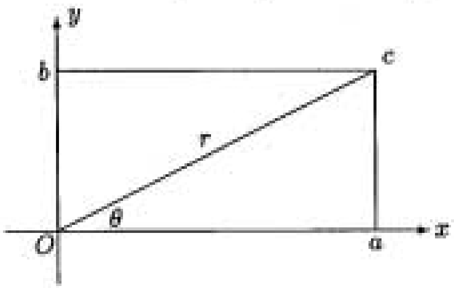
$$|z-w| \geq |w| - |z| . \tag{2.7}$$

Hence, since for a real number x we have that $|x| = \max\{x, -x\}$, we deduce from (2.6) and (2.7) that

$$|z-w| \geq \big||z|-|w|\big| .$$

□

The correspondence between complex numbers $c = a + bi$ and points (a, b) in the plane is so close that we shall routinely refer to "the point c", and we shall refer to the plane as the **complex plane**, or as the **Argand**[1] **diagram**. The point c lies on the circle $x^2 + y^2 = r^2$, where $r = |c| = \sqrt{a^2+b^2}$.



If $c \neq 0$ there is a unique θ in the interval $(-\pi, \pi]$ such that

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} , \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}} ,$$

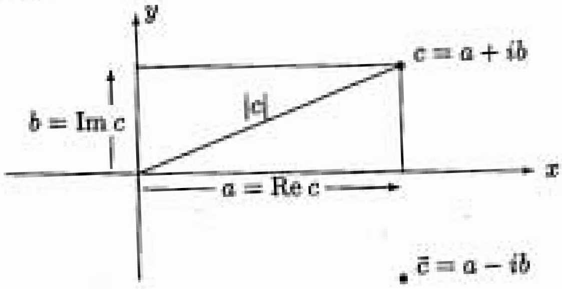
and we can write

$$c = r(\cos\theta + i\sin\theta) . \tag{2.8}$$

[1] Jean-Robert Argand, 1768–1822.

Note also that $\bar{z} = z$ if and only if z is real, and $\bar{z} = -z$ if and only if z is pure imaginary.

The following picture of a complex number $c = a + ib$ is very useful.



The product $c\bar{c}$ is the non-negative real number $a^2 + b^2$. Its square root $\sqrt{c\bar{c}} = \sqrt{a^2 + b^2}$, the distance of the point (a, b) from the origin, is denoted by $|c|$ and is called the **modulus** of c. If c is real, then the modulus is simply the **absolute value** of c. Some of the following results are familiar in the context of real numbers:

Theorem 2.1

Let z and w be complex numbers. Then:

(i) $|\text{Re}\, z| \le |z|$, $|\text{Im}\, z| \le |z|$, $|\bar{z}| = |z|$;

(ii) $|zw| = |z||w|$;

(iii) $|z + w| \le |z| + |w|$;

(iv) $|z - w| \ge \big||z| - |w|\big|$.

Proof

(i) is immediate.

(ii) By (2.3),

$$|zw|^2 = (zw)(\overline{zw}) = (z\bar{z})(w\bar{w}) = \big(|z||w|\big)^2$$

and Part (ii) follows immediately.

For Part (iii), observe that

$$|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}\,. \tag{2.5}$$

Now,

$$z\bar{w} + w\bar{z} = z\bar{w} + \overline{z\bar{w}} = 2\text{Re}\,(z\bar{w}) \le 2|z\bar{w}| = 2|z||\bar{w}| = 2|z||w|\,,$$

This amounts to describing the point (a, b) by means of **polar coordinates**, and $r(\cos\theta + i\sin\theta)$ is called the **polar form** of the complex number. By some standard trigonometry we see that

$$\begin{aligned}
&(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\
&= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\\
&= \cos(\theta+\phi) + i\sin(\theta+\phi)\,. \qquad (2.9)
\end{aligned}$$

Looking ahead to a notation that we shall justify properly in Chapter 3, we note that, if we extend the series definition of the exponential function to complex numbers, we have, for any real θ,

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \cdots\\
&= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right)\\
&= \cos\theta + i\sin\theta\,.
\end{aligned}$$

We may therefore write (2.9) in the easily remembered form

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}\,.$$

From well known properties of sin and cos we deduce that

$$e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta\,,$$

and **Euler's**[2] **formulae** follow easily:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})\,, \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})\,. \qquad (2.10)$$

With the exponential notation, the polar form for the non-zero complex number $c = a + bi$ is written as $re^{i\theta}$, where $a = r\cos\theta$, $b = r\sin\theta$. The positive number r is the **modulus** $|c|$ of c, and θ is the **argument**, written $\arg c$, of c. The polar form of $\bar{c}$ is $re^{-i\theta}$.

The periodicity of sin and cos imply that $e^{i\theta} = e^{i(\theta+2n\pi)}$ for every integer n, and so, more precisely, we specify $\arg c$ by the property that $\arg c = \theta$, where $c = re^{i\theta}$ and $-\pi < \theta \le \pi$. We call $\arg c$ the **principal argument** if there is any doubt.

Multiplication for complex numbers is easy if they are in polar form:

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = (r_1r_2)e^{i(\theta_1+\theta_2)}\,.$$

We already know that $|c_1c_2| = |c_1|\,|c_2|$, and we now deduce that

$$\arg(c_1c_2) \equiv \arg c_1 + \arg c_2 \pmod{2\pi}\,.$$

[2] Leonhard Euler, 1707–1783.

By this we mean that the difference between $\arg(c_1c_2)$ and $\arg c_1 + \arg c_2$ is an integral multiple of 2π.

The results extend:

$$|c_1c_2\dots c_n| = |c_1|\,|c_2|\dots|c_n|\,,$$

$$\arg c_1c_2\dots c_n \equiv \arg c_1 + \arg c_2 + \cdots + \arg c_n \pmod{2\pi}\,,$$

and, putting $c_1 = c_2 = \cdots = c_n$, we also deduce that, for all positive integers n,

$$|c^n| = |c|^n, \quad \arg c^n \equiv n \arg c \pmod{2\pi}\,.$$

Example 2.2

Determine the modulus and argument of c^5, where $c = 1 + i\sqrt{3}$.

Solution

An easy calculation gives $|c| = 2$, $\arg c = \theta$, where $\cos\theta = 1/2$, $\sin\theta = \sqrt{3}/2$; hence $\arg\theta = \pi/3$. It follows that $|c^5| = 2^5 = 32$, while $\arg(c^5) \equiv 5\pi/3 \equiv -\pi/3$. Here we need to make an adjustment in order to arrive at the principal argument. The **standard form** of c^5, by which we mean the form $a+ib$, where a and b are real, is $32\big(\cos(-\pi/3) + i\sin(-\pi/3)\big) = 16(1 - i\sqrt{3})$. □

Remark 2.3

For a complex number $c = a + bi = re^{i\theta}$ it is true that $\tan\theta = b/a$, but it is *not* always true that $\theta = \tan^{-1}(b/a)$. For example, if $c = -1 - i$, then $\theta = -3\pi/4 \neq \tan^{-1} 1$. It is much safer – indeed essential – to find θ by using $\cos\theta = a/r$, $\sin\theta = b/r$.

Finding the reciprocal of a non-zero complex number c is again easy if the number is in polar form: the reciprocal of $re^{i\theta}$ is $(1/r)e^{-i\theta}$. In the standard form $c = a + bi$ the reciprocal is less obvious:

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}\,.$$

The technique of multiplying the denominator of a fraction by its conjugate is worth noting:

Example 2.4

Express

$$\frac{3+7i}{2+5i}$$

in standard form.

Solution

$$\frac{3+7i}{2+5i} = \frac{(3+7i)(2-5i)}{(2+5i)(2-5i)} = \frac{1}{29}(41-i)\,.$$

□

Again, the fact that every complex number has a square root is easily seen from the polar form: $\sqrt{r}e^{i(\theta/2)}$ is a square root of $re^{i\theta}$. From this we may deduce that every quadratic equation

$$az^2 + bz + c = 0\,,$$

where $a, b, c \in \mathbb{C}$ and $a \neq 0$ has a solution in $\mathbb{C}$. The procedure, by "completing the square", and the resulting formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are just the same as for real quadratic equations.

Example 2.5

Find the roots of the equation

$$z^2 + 2iz + (2-4i) = 0\,.$$

Solution

By the standard formula, the solution of the equation is

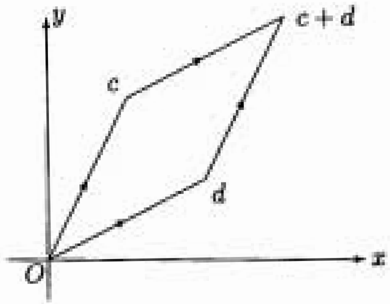
$$\frac{1}{2}\Big(-2i \pm \sqrt{(-2i)^2 - 4(2-4i)}\Big) = \frac{1}{2}\Big(-2i \pm \sqrt{-12+16i}\Big) = -i \pm \sqrt{-3+4i}\,.$$

Observe now that $(1+2i)^2 = -3+4i$, and so the solution is

$$z = -i \pm (1+2i) = 1+i \quad \text{or} \quad -1-3i\,.$$

□

The addition of complex numbers has a strong geometrical connection, being in effect vector addition:

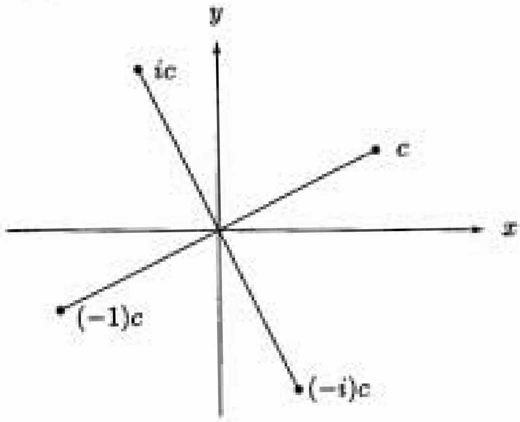


The geometrical aspect of complex multiplication becomes apparent if we use the polar form: if we multiply c by $re^{i\theta}$ we multiply $|c|$ by a factor of r, and add θ to $\arg c$. Of special interest is the case where $r = 1$, when multiplication by $e^{i\theta}$ corresponds simply to a rotation by θ. In particular:

– multiplication by $-1 = e^{i\pi}$ rotates by π;

– multiplication by $i = e^{i\pi/2}$ rotates by $\pi/2$;

– multiplication by $-i = e^{-i\pi/2}$ rotates by $-\pi/2$;



Example 2.6

Find the real and imaginary parts of $c = 1/(1 + e^{i\theta})$.

Solution

One way is to use the standard method of multiplying the denominator by its conjugate, obtaining

$$c = \frac{1+e^{-i\theta}}{(1+e^{i\theta})(1+e^{-i\theta})} = \frac{(1+\cos\theta) - i\sin\theta}{2+2\cos\theta},$$

and hence

$$\operatorname{Re} c = \frac{1}{2}, \quad \operatorname{Im} c = \frac{-\sin\theta}{2+2\cos\theta}.$$

More ingeniously, we can multiply the numerator and denominator by $e^{-i(\theta/2)}$, obtaining

$$c = \frac{e^{-i(\theta/2)}}{e^{-i(\theta/2)} + e^{i(\theta/2)}} = \frac{\cos(\theta/2) - i\sin(\theta/2)}{2\cos(\theta/2)},$$

and hence

$$\operatorname{Re} c = \frac{1}{2}, \quad \operatorname{Im} c = -\frac{1}{2}\tan(\theta/2).$$

The verification that the two answers for the imaginary part are actually the same is a simple trigonometrical exercise. □

Example 2.7

Sum the (finite) series

$$C = 1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta,$$

where θ is not an integral multiple of 2π.

Solution

Consider the series

$$Z = 1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta}.$$

This is a geometric series with common ratio $e^{i\theta}$. The formula for a sum of a geometric series works just as well in $\mathbb{C}$ as in $\mathbb{R}$, and so

$$\begin{aligned}
Z &= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} = \frac{e^{i(n+\frac{1}{2})\theta} - e^{-\frac{1}{2}i\theta}}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}} \\
&= \frac{e^{i(n+\frac{1}{2})\theta} - e^{-\frac{1}{2}i\theta}}{2i\sin\frac{1}{2}\theta} \quad \text{(by the Euler formula (2.10))} \\
&= \frac{-i\big(\cos(n+\frac{1}{2})\theta + i\sin(n+\frac{1}{2})\theta\big) + i\big(\cos\frac{1}{2}\theta - i\sin\frac{1}{2}\theta\big)}{2\sin\frac{1}{2}\theta} \quad \text{(since } 1/i = -i\text{)} \\
&= \frac{\big(\sin(n+\frac{1}{2})\theta + \sin\frac{1}{2}\theta\big) + i\big(\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta\big)}{2\sin\frac{1}{2}\theta}.
\end{aligned}$$

Hence, equating real parts, we deduce that

$$C = \frac{\sin(n+\frac{1}{2})\theta + \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} \quad (\theta \neq 2k\pi)\,.$$

As a bonus, our method gives us (if we equate imaginary parts) the result that

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}\,.$$

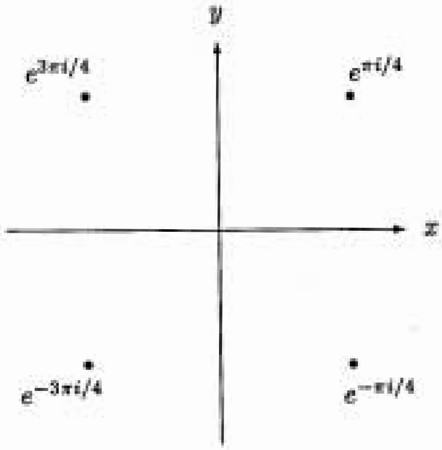
□

Example 2.8

Find all the roots of the equation $z^4 + 1 = 0$. Factorise the polynomial in $\mathbb{C}$, and also in $\mathbb{R}$.

Solution

$z^4 = -1 = e^{i\pi}$ if and only if $z = e^{\pm\pi i/4}$ or $e^{\pm 3\pi i/4}$. The roots all lie on the unit circle, and are equally spaced.



In $\mathbb{C}$ the factorisation is

$$z^4 + 1 = (z - e^{\pi i/4})(z - e^{-\pi i/4})(z - e^{3\pi i/4})(z - e^{-3\pi i/4})\,.$$

Combining conjugate factors, we obtain the factorisation in $\mathbb{R}$:

$$\begin{aligned} z^4 + 1 &= \big(z^2 - 2z\cos(\pi/4) + 1\big)\big(z^2 - 2z\cos(3\pi/4) + 1\big) \\ &= (z^2 - z\sqrt{2} + 1)(z^2 + z\sqrt{2} + 1)\,. \end{aligned}$$

The strong connections between the operations of complex numbers and the geometry of the plane enable us to specify certain important geometrical objects by means of complex equations. The most obvious case is that of the circle $\{z : |z-c| = r\}$ with centre c and radius $r \geq 0$. This easily translates to the familiar form of the equation of a circle: if $z = x + iy$ and $c = a + ib$, then $|z-c| = r$ if and only if $|z-c|^2 = r^2$, that is, if and only if $(x-a)^2+(y-b)^2 = r^2$. The other form, $x^2 + y^2 + 2gx + 2fy + c = 0$, of the equation of the circle can be rewritten as $z\bar{z} + hz + \overline{hz} + c = 0$, where $h = g - if$. More generally, we have the equation

$$Az\bar{z} + Bz + \overline{Bz} + C = 0\,, \tag{2.11}$$

where $A\,(\neq 0)$ and C are real, and B is complex. The set

$$\{z \in \mathbb{C} : Az\bar{z} + Bz + \overline{Bz} + C = 0\}$$

is:

(C1) a circle with centre $-\bar{B}/A$ and radius R, where $R^2 = (B\bar{B} - AC)/A^2$ if $B\bar{B} - AC \geq 0$;

(C2) empty if $B\bar{B} - AC < 0$.

If $A = 0$, the equation reduces to

$$Bz + \overline{Bz} + C = 0\,, \tag{2.12}$$

and this (provided $B \neq 0$) is the equation of a straight line: if $B = B_1 + iB_2$ and $z = x + iy$ the equation becomes

$$B_1x - B_2y + C = 0\,.$$

Theorem 2.9

Let c, d be distinct complex numbers, and let $k > 0$. Then the set

$$\{z : |z-c| = k|z-d|\}$$

is a circle unless $k = 1$, in which case the set is a straight line, the perpendicular bisector of the line joining c and d.

Proof

We begin with some routine algebra:

$$\begin{aligned}\{z : |z-c| = k|z-d|\} &= \{z : |z-c|^2 = k^2|z-d|^2\}\\ &= \{z : (z-c)(\bar{z}-\bar{c}) = k^2(z-d)(\bar{z}-\bar{d})\}\\ &= \{z : z\bar{z} - c\bar{z} - \bar{c}z + c\bar{c} = k^2(z\bar{z} - d\bar{z} - \bar{d}z + d\bar{d})\}\\ &= \{z : Az\bar{z} + Bz + \overline{Bz} + C = 0\}\,,\end{aligned}$$

where $A = k^2 - 1$, $B = \bar{c} - k^2\bar{d}$, $C = k^2 d\bar{d} - c\bar{c}$.

If $k = 1$ we have the set

$$\{z : (\bar{c} - \bar{d})z + (c - d)\bar{z} + (d\bar{d} - c\bar{c}) = 0\}$$

and this is a straight line. Geometrically, it is clear that it is the perpendicular bisector of the line joining c and d.

If $A \neq 1$ then the set is a circle with centre

$$-\frac{\bar{B}}{A} = \frac{k^2 d - c}{k^2 - 1}, \tag{2.13}$$

for we can show that $B\bar{B} - AC > 0$:

$$\begin{aligned} B\bar{B} - AC &= (c - k^2 d)(\bar{c} - k^2\bar{d}) - (k^2 - 1)(k^2 d\bar{d} - c\bar{c}) \\ &= c\bar{c} - k^2 c\bar{d} - k^2\bar{c}d + k^4 d\bar{d} - k^4 d\bar{d} + k^2 d\bar{d} + k^2 c\bar{c} - c\bar{c} \\ &= k^2(c\bar{c} - c\bar{d} - \bar{c}d + d\bar{d}) \\ &= k^2(c - d)(\bar{c} - \bar{d}) = k^2|c - d|^2 > 0\,. \end{aligned}$$

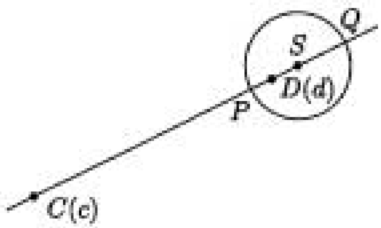
The radius of the circle is R, where

$$R^2 = \frac{B\bar{B} - AC}{A^2} = \frac{k^2|c - d|^2}{(k^2 - 1)^2}\,. \tag{2.14}$$

□

Remark 2.10

The circle $\{z : |z - c| = k|z - d|\}$ has PQ as diameter. If S is the centre of the circle, then



From (2.13) and (2.14) we see that

$$|SC|.|SD| = \left|c - \frac{k^2 d - c}{k^2 - 1}\right| \left|d - \frac{k^2 d - c}{k^2 - 1}\right| = \frac{k^2|c - d|^2}{(k^2 - 1)^2} = R^2\,. \tag{2.15}$$

We say that the points C and D are **inverse points** with respect to the circle. We shall return to this idea in Chapter 11.

Remark 2.11

The observation that C and D are inverse points is the key to showing that *every* circle can be represented as $\{z : |z-c| = k|z-d|\}$. Suppose that Σ is a circle with centre a and radius R. Let $c = a+t$, where $0 < t < |a|$, and let $d = a + (R^2/t)$. Then c and d are inverse points with respect to Σ. For every point $z = a + Re^{i\theta}$ on Σ,

$$\frac{z-c}{z-d} = \left|\frac{z-c}{\bar{z}-d}\right| = \left|\frac{Re^{i\theta}-t}{Re^{-i\theta}-(R^2/t)}\right| = \left|\frac{te^{i\theta}}{R}\right| \left|\frac{R-te^{-i\theta}}{te^{-i\theta}-R}\right| = \frac{t}{R},$$

and so $|z-c| = (t/R)|z-d|$. The answer is not unique.

EXERCISES

2.3. Show that $\mathrm{Re}\,(iz) = -\mathrm{Im}\,z$, $\mathrm{Im}\,(iz) = \mathrm{Re}\,z$.

2.4. Write each of the following complex numbers in the standard form $a+bi$, where $a, b \in \mathbb{R}$:

a) $(3+2i)/(1+i)$;

b) $(1+i)/(3-i)$;

c) $(z+2)/(z+1)$, where $z = x+yi$ with x, y in $\mathbb{R}$.

2.5. Calculate the modulus and principal argument of

a)	$1-i$	b)	$-3i$
c)	$3+4i$	d)	$-1+2i$

2.6 Show that, for every pair c, d of non-zero complex numbers,

$$\arg(c/d) \equiv \arg c - \arg d \pmod{2\pi}.$$

2.7. Express $1+i$ in polar form, and hence calculate $(1+i)^{16}$.

2.8. Show that $(2+2i\sqrt{3})^9 = -2^{18}$. (Don't use the binomial theorem!)

2.9. Let $n \in \mathbb{Z}$. Show that, if $n = 4q + r$, with $0 \le r \le 3$, then

$$i^n = \begin{cases} 1 & \text{if } r=0 \\ i & \text{if } r=1 \\ -1 & \text{if } r=2 \\ -i & \text{if } r=3. \end{cases}$$

2.10. Calculate $\sum_{r=0}^{100} i^r$.

2.11. Show by induction that, for all $z \neq 1$,

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{1 - (n+1)z^n + nz^{n+1}}{(1-z)^2}.$$

Deduce that, if $|z| < 1$,

$$\sum_{n=1}^{\infty} nz^{n-1} = \frac{1}{(1-z)^2}.$$

2.12. Let z_1, z_2 be complex numbers such that $|z_1| > |z_2|$. Show that, for all $n \geq 2$,

$$n\left|\frac{z_2}{z_1}\right|^{n-1} < \frac{|z_1|}{|z_1| - |z_2|}.$$

2.13. Prove that, if $z_1, z_2 \in \mathbb{C}$, then

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Deduce that, for all c, d in $\mathbb{C}$,

$$|c + \sqrt{c^2 - d^2}| + |c - \sqrt{c^2 - d^2}| = |c + d| + |c - d|.$$

2.14. Sum the series

$$\cos\theta + \cos 3\theta + \cdots + \cos(2n+1)\theta.$$

2.15. Let $\gamma = \rho e^{i\theta}$ ($\notin \mathbb{R}$) be a root of $P(z) = 0$, where

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

and where $a_0, a_1, \ldots, a_n$ are *real*. Show that $\bar{\gamma}$ is also a root, and deduce that $z^2 - 2\rho\cos\theta + \rho^2$ is a factor of $P(z)$.

2.16. Determine the roots of the equations

a) $z^2 - (3-i)z + (4-3i) = 0$;

b) $z^2 - (3+i)z + (2+i) = 0$.

2.17. Give geometrical descriptions of the sets

$$\text{a)} \quad \{z : |2z+3| \leq 1\} \qquad \text{b)} \quad \{z : |z| \geq |2z+1|\}.$$

2.18. Determine the roots of $z^5 = 1$, and deduce that

$$z^5 - 1 = (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right).$$

Deduce that

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}, \quad \cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = -\frac{1}{4},$$

and hence show that

$$\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}, \quad \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}.$$

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Complex Analysis
Howie, J.M.
2003, XI, 260 p., Softcover
ISBN: 978-1-85233-733-9