3. State Variable Equation of a Continuous System

3.1 Introduction

This chapter introduces the concept of state space and the solution of state variable equations or state equations. This chapter is fundamentally important to the remainder of this book. This is because throughout this book, the three-dimensional governing equations of elasticity will be represented in the form of state equations rather than in their usual form and, therefore, their solutions will be sought on the basis of the theories of state equations.

The term 'state space' is often used in connection with linear control system where the principal concern is the relationship between inputs (or source) and outputs (or responses). In practice, these systems may be electrical, hydraulic, mechanical, pneumatic, thermal, or mixtures of these. For example, the state of a continuous system can best be represented by a single-input, single-output, linear electrical network whose structure is known. The input to and the output of the network are both functions of time. Since the network is known, complete knowledge of the input over a time interval is sometimes sufficient to determine the output over the same time interval.

Another example of the state of a continuous system is a set of linear differential equations with constant coefficients. Once the form of the complete solution is obtained in terms of a set of arbitrary constants, these constants can then be determined by the fact that the system must satisfy initial boundary conditions. The boundary conditions can be termed as the initial state of the system. Hence, the state of the system separates the future from the past, so that the state contains all the relevant information concerning the past history of the system required to determine the responses to any input.

For three-dimensional analyses of laminated plates and shells, the use of state equations has many advantages. For example, if we take the displacements and transverse stresses at the bottom surface of a laminated plate as the initial state of the system, after introducing boundary conditions, the displacements and stresses at the top surface of the plate may be found and the displacements and stresses at an arbitrary interface of the laminate can be traced as the past history of the system.

In this chapter, we will focus only on the fundamental aspects of state variable equations, including some commonly used solution methods. The theories presented in this chapter have been well documented in the literature, *e.g.*, in Elgerd (1967) and Derusso *et al.* (1998), where rigorous mathematical proofs of these theories are provided. Application of the theories to three-dimensional elasticity and the deduction of state equations for laminated plates and shells will be discussed in detail in subsequent chapters.

3.2 Concept of State and State Variables

Consider a spring−damper−mass system shown in Figure 3.1. The differential equation of motion of the system can be derived by means of Newton's second law.

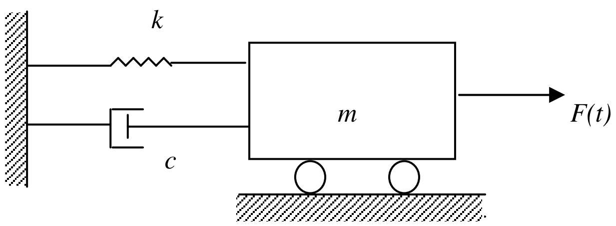

Figure 3.1. Spring−damper−mass system

The equation is a second-order linear ordinary differential equation with constant coefficients, as shown below:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \tag{3.1}$$

where the dots designate time derivatives; t denotes time. The three constant coefficients m, c and k represent the system parameters. In general, the future position $x(t)$ of the mass is not uniquely determinable unless the position and the velocity of the mass at an arbitrary time instant t_0 are known. Hence, the state of the system at t_0 is the minimum amount of information (minimum set of initial conditions) that, together with the input $F(t)$, determines uniquely the response of the system for all $t \ge t_0$. The state of a system is often represented by a column vector called the state vector. Each component of the vector is called a state variable. The system shown in Figure 3.1 has two state variables, *i.e.*, the position and velocity of the mass.

Equation (3.1) can be further transformed into a first-order linear differential equation system by letting

$$\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= \dot{x}(t) \end{aligned} \tag{3.2}$$

The linear differential equation (3.1) of second order can be converted to the following matrix form:

$$\frac{d}{dt}\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix}\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} \tag{3.3}$$

It can be seen that the matrix equation (3.3) is now a first-order differential equation system in terms of the state vector of the system shown in Figure 3.1. The equation governs the behaviour of the state of the system and is called the state variable equation or state equation of the system.

In view of these results, we may now assume n independent variables, x_i (i=1,2,…,n) as carriers of the full information about a transient state of a system. Initially, at t_0=0, the state of the system can be represented by $x_1(0)$, $x_2(0),..., x_n(0)$, from which the updated state expressed by $x_1(t), x_2(t),..., x_n(t)$ can be continuously obtained. We define the state of the system by the n-dimensional vector $\{\mathbf{x}(t)\}$, which has as its components the n state variables $x_1(t), x_2(t),..., x_n(t)$. The corresponding state equation can be written as (see equation (3.3))

$$\{\dot{\mathbf{x}}(t)\} = [\mathbf{A}]\{\mathbf{x}(t)\} + [\mathbf{B}]\{\mathbf{u}(t)\} \tag{3.4}$$

where [**A**] is an $n \times n$ constant matrix called the coefficient matrix or system matrix in the literature. [**B**] is an $n \times 1$ column vector. $\{\mathbf{u}(\mathrm{t})\}$ is the input of the system.

A linear system that can be defined by equation (3.4) is called a linear time-invariant system since the system is characterised by the two matrices, [**A**] and [**B**], that are independent of time. In contrast, we may have a complex linear system in which the two matrices of its state equation are time-dependent. In this case, equation (3.4) becomes

$$\{\dot{\mathbf{x}}(t)\} = [\mathbf{A}(t)]\{\mathbf{x}(t)\} + [\mathbf{B}(t)]\{\mathbf{u}(t)\} \tag{3.5}$$

A system defined by equation (3.5) is called a linear time-varying system and is a general form of equation (3.4).

A linear time-varying system is usually considerably more difficult to solve than a linear time-invariant one of the same order. However, the solution of equation (3.4) can be used to form an approximate solution of equation (3.5). This will be discussed later in this chapter after the solution to equation (3.4) has been formed.

3.3 Solutions for a Linear Time-invariant System

If equation (3.4) were a scalar differential equation, *i.e.*, n=1, we would have

$$\dot{x}(t) = ax(t) + bu(t) \tag{3.6}$$

where a and b are constant. Based on the classical solution method of a linear differential equation, the solution of equation (3.6) can be written as

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(t)d\tau \tag{3.7}$$

where $t_0 = 0$ is assumed. In analogy with the scalar case, we first try to obtain a solution for a homogeneous system defined by

$$\{\dot{\mathbf{x}}(t)\} = [\mathbf{A}]\{\mathbf{x}(t)\} \tag{3.8}$$

by assuming a solution in the form of a vector power series in t as follows:

$$\{\mathbf{x}(t)\} = \{\mathbf{a}_0\} + \{\mathbf{a}_1\}t + \{\mathbf{a}_2\}t^2 + ... + \{\mathbf{a}_k\}t^k + ... \tag{3.9}$$

The vector coefficients $\{\mathbf{a}_k\}$ can be determined after we substitute the assumed solution into equation (3.8). Thus,

$$\begin{aligned}&\{\mathbf{a}_1\} + 2\{\mathbf{a}_2\}t + 3\{\mathbf{a}_3\}t^2 + ...\\ &\qquad = [\mathbf{A}](\{\mathbf{a}_0\} + \{\mathbf{a}_1\}t + \{\mathbf{a}_2\}t^2 + ... + \{\mathbf{a}_k\}t^k + ...)\end{aligned} \tag{3.10}$$

Comparing the vector coefficients for equal powers of t yields

$$\begin{aligned}&\{\mathbf{a}_1\} = [\mathbf{A}]\{\mathbf{a}_0\}\\ &2\{\mathbf{a}_2\} = [\mathbf{A}]\{\mathbf{a}_1\}\\ &.........\\ &k\{\mathbf{a}_k\} = [\mathbf{A}]\{\mathbf{a}_{k-1}\}\end{aligned} \tag{3.11}$$

From these comparisons, we have

$$\begin{aligned}&\{\mathbf{a}_1\} = [\mathbf{A}]\{\mathbf{a}_0\}\\ &\{\mathbf{a}_2\} = \frac{1}{2}[\mathbf{A}]\{\mathbf{a}_1\} = \frac{1}{2}[\mathbf{A}][\mathbf{A}]\{\mathbf{a}_0\} = \frac{1}{2}[\mathbf{A}]^2\{\mathbf{a}_0\}\\ &.........\end{aligned} \tag{3.12}$$

$$\{\mathbf{a}_k\} = \frac{1}{k!}[\mathbf{A}]^k \{\mathbf{a}\}_0$$

The vector coefficient $\{\mathbf{a}_0\}$ must be equal to $\{\mathbf{x}(0)\}$ in order for the solution (3.9) to approach the proper initial state for vanishing t. After substituting $\{\mathbf{a}_0\}$ into equation (3.12), all the coefficients are thus known, and the solution of the homogeneous equation is

$$\begin{aligned}\{\mathbf{x}(t)\} &= \{\mathbf{x}(0)\} + [\mathbf{A}]\{\mathbf{x}(0)\}t + \cdots + \frac{1}{k!}[\mathbf{A}]^k\{\mathbf{x}(0)\}t^k + \cdots \\ &= ([\mathbf{I}] + [\mathbf{A}]t + \frac{1}{2}[\mathbf{A}]^2 t^2 + \cdots + \frac{1}{k!}[\mathbf{A}]^k t^k + \cdots)\{\mathbf{x}(0)\}\end{aligned} \tag{3.13}$$

Obviously, the expression within the parenthesis is an $n \times n$ matrix that is termed the matrix exponential and is denoted by the symbol $e^{[\mathbf{A}]t}$ because of its similarity with the infinite power series for a scalar exponential. Thus,

$$e^{[\mathbf{A}]t} = [\mathbf{I}] + [\mathbf{A}]t + \frac{1}{2}[\mathbf{A}]^2 t^2 + \ldots + \frac{1}{k!}[\mathbf{A}]^k t^k + \ldots \tag{3.14}$$

The solution of equation (3.8) can now be written in the following compact form that is similar to its scalar counterpart:

$$\{\mathbf{x}(t)\} = e^{[\mathbf{A}]t}\{\mathbf{x}(0)\} \tag{3.15a}$$

If $t_0 \neq 0$, equation (3.15a) can be represented by the general form below, with $t_0 = 0$ as a special case:

$$\{\mathbf{x}(t)\} = e^{[\mathbf{A}](t-t_0)}\{\mathbf{x}(t_0)\} \tag{3.15b}$$

This can readily be verified by introducing $t - t_0$ rather than t in the deduction process. The matrix exponential is known in the literature by the names transition matrix, transfer matrix or fundamental matrix.

From equation (3.14), it can also be shown that

$$\begin{aligned}\frac{d}{dt}e^{[\mathbf{A}]t} &= [\mathbf{A}] + [\mathbf{A}]^2 t + \frac{1}{2!}[\mathbf{A}]^3 t^2 + \cdots + \frac{1}{(k-1)!}[\mathbf{A}]^k t^{k-1} + \cdots \\ &= [\mathbf{A}]([\mathbf{I}] + [\mathbf{A}]t + \frac{1}{2!}[\mathbf{A}]^2 t^2 + \cdots + \frac{1}{k!}[\mathbf{A}]^k t^k + \cdots) \\ &= ([\mathbf{I}] + [\mathbf{A}]t + \frac{1}{2!}[\mathbf{A}]^2 t^2 + \cdots + \frac{1}{k!}[\mathbf{A}]^k t^k + \cdots)[\mathbf{A}] \\ &= [\mathbf{A}]e^{[\mathbf{A}]t} = e^{[\mathbf{A}]t}[\mathbf{A}]\end{aligned} \tag{3.16}$$

Now consider the solution of equation (3.4). Again, if this were a scalar system in the form of, *e.g.*, equation (3.6), an integrating factor e^{-at} would be introduced. For the matrix equation, on the basis of the solution to the homogeneous case, the $n \times n$ integrating factor $e^{-[\mathbf{A}]t}$ will be used. Note that this matrix is actually the inverse of $e^{[\mathbf{A}]t}$. Pre-multiplying both sides of equation (3.4) by the integrating factor, we have

$$e^{-[\mathbf{A}]t}\{\dot{\mathbf{x}}(t)\} = e^{-[\mathbf{A}]t}[\mathbf{A}]\{\mathbf{x}(t)\} + e^{-[\mathbf{A}]t}[\mathbf{B}]\{\mathbf{u}(t)\} \tag{3.17}$$

or

$$e^{-[\mathbf{A}]t}\{\dot{\mathbf{x}}(t)\} - e^{-[\mathbf{A}]t}[\mathbf{A}]\{\mathbf{x}(t)\} = e^{-[\mathbf{A}]t}[\mathbf{B}]\{\mathbf{u}(t)\} \tag{3.18}$$

From equation (3.16), we know that the left-hand side of equation (3.18) is exactly the derivative of $e^{-[\mathbf{A}]t}\{\mathbf{x}(t)\}$ with respect to *t*. Hence, equation (3.18) can be written as

$$\frac{d}{dt}\{e^{-[\mathbf{A}]t}\{\mathbf{x}(t)\}\} = e^{-[\mathbf{A}]t}[\mathbf{B}]\{\mathbf{u}(t)\} \tag{3.19}$$

Integrate both sides of equation (3.19)

$$\int_0^t d\{e^{-[\mathbf{A}]\tau}\{\mathbf{x}(\tau)\}\} = \int_0^t e^{-[\mathbf{A}]\tau}[\mathbf{B}]\{\mathbf{u}(\tau)\}d\tau \tag{3.20}$$

This yields

$$e^{-[\mathbf{A}]t}\{\mathbf{x}(t)\} - \{\mathbf{x}(0)\} = \int_0^t e^{-[\mathbf{A}]\tau}[\mathbf{B}]\{\mathbf{u}(\tau)\}d\tau \tag{3.21a}$$

or

$$e^{-[\mathbf{A}]t}\{\mathbf{x}(t)\} = \{\mathbf{x}(0)\} + \int_0^t e^{-[\mathbf{A}]\tau}[\mathbf{B}]\{\mathbf{u}(\tau)\}d\tau \tag{3.21b}$$

Finally, pre-multiplying both sides of equation (3.21b) by $e^{[\mathbf{A}]t}$, we obtain the following solution:

$$\{\mathbf{x}(t)\} = e^{[\mathbf{A}]t}\{\mathbf{x}(0)\} + e^{[\mathbf{A}]t}\int_0^t e^{-[\mathbf{A}]\tau}[\mathbf{B}]\{\mathbf{u}(\tau)\}d\tau \tag{3.22a}$$

or

$$\{\mathbf{x}(t)\} = e^{[\mathbf{A}]t}\{\mathbf{x}(0)\} + \int_0^t e^{[\mathbf{A}](t-\tau)}[\mathbf{B}]\{\mathbf{u}(\tau)\}d\tau \tag{3.22b}$$

Equation (3.22) is the complete solution of equation (3.4), which is once again similar to the solution form of its scalar counterpart. From the viewpoint of computing, equation (3.22) does not introduce any additional work since the calculation of $e^{[\mathbf{A}]t}$ has already been required in the solution of the homogeneous equation.

3.3.1 The Eigenvalue Problem

It has been shown that the solution of a state equation requires the calculation of the exponential matrix $e^{[\mathbf{A}]t}$. From matrix algebra, we know that the eigenvalues and eigenvectors of $[\mathbf{A}]$ can be used to form an explicit expression for the exponential matrix. To show this, the basic theories of an eigenvalue problem are introduced very briefly in this section without mathematical proofs. Numerical examples are used instead to verify the theories and show their applications in connection with exponential matrices. Further details of these theories can be found, for example, in Hohn (1973)

An eigenvalue problem of a square matrix is defined as follows: given a real or complex $n \times n$ matrix $[\mathbf{A}]$, for what non-zero vectors $\{\mathbf{X}\}$ and for what λ, is it true that

$$[\mathbf{A}]\{\mathbf{X}\} = \lambda\{\mathbf{X}\} \tag{3.23}$$

or, for what vectors $\{\mathbf{X}\}$, are $[\mathbf{A}]\{\mathbf{X}\}$ and $\{\mathbf{X}\}$ proportional?

A non-zero vector $\{\mathbf{X}\}$ that satisfies equation (3.23) is called an eigenvector of $[\mathbf{A}]$ and the associated value of λ is called an eigenvalue of $[\mathbf{A}]$.

Equation (3.23) holds if and only if

$$([\mathbf{A}] - \lambda[\mathbf{I}])\{\mathbf{X}\} = \{\mathbf{0}\} \tag{3.24}$$

This equation represents a homogeneous system of n simultaneous equations that has a non-trivial solution for $\{\mathbf{X}\}$ if and only if

$$\det([\mathbf{A}] - \lambda[\mathbf{I}]) = 0 \tag{3.25}$$

or, equivalently

$$\det\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \tag{3.26}$$

where the a_{ij} are elements of matrix [**A**]. This equation is called the characteristic equation of [**A**]. The determinant of it will expand into a polynomial of degree n in terms of the variable λ. This polynomial is called the characteristic polynomial and has n roots, $\lambda_1, \lambda_2, \cdots, \lambda_n$ (not necessarily all different or real). These roots are called the eigenvalues of [**A**]. If λ_i is one of these eigenvalues, we solve the equation

$$([\mathbf{A}]-\lambda_i[\mathbf{I}])\{\mathbf{X}\}=\{\mathbf{0}\} \tag{3.27}$$

for $\{\mathbf{X}\}$. A non-trivial solution of $\{\mathbf{X}\}$ is an eigenvector of [**A**] that is associated with the eigenvalue λ_i. For an eigenvalue, there are an infinite number of $\{\mathbf{X}\}$ that can satisfy equation (3.27). However, these eigenvectors are linearly dependent and can be normalised to a unit eigenvector that is unique. Details of these are shown in the following examples.

Example 3.1. Find the eigenvalues and eigenvectors of [**A**], where

$$[\mathbf{A}]=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

Using equation (3.26)

$$\det([\mathbf{A}]-\lambda[\mathbf{I}])=\det\begin{bmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{bmatrix}$$
$$=(2-\lambda)^2-1=(1-\lambda)(3-\lambda)=0$$

So the eigenvalues are λ_1 =1 and λ_2 =3.

Substituting λ_1=1 into $([\mathbf{A}]-\lambda[\mathbf{I}])\{\mathbf{X}\}=\{\mathbf{0}\}$ yields

$$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\{\mathbf{0}\}$$

or in scalar form, a system that reduces to the following single equation

$$x_1+x_2=0$$

which has the complete solution

$$\mathbf{v}_1=\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=t\begin{Bmatrix}-1\\ 1\end{Bmatrix} \qquad \text{for any } t\neq 0$$

Similarly, substituting λ_2=3 into $([\mathbf{A}]-\lambda[\mathbf{I}])\{\mathbf{X}\}=\{\mathbf{0}\}$ yields

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \mathbf{0}$$

with the complete solution

$$\mathbf{v}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = t \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \text{for any } t \neq 0$$

The two complete solutions are the two eigenvectors $\{\mathbf{v}_1\}$ and $\{\mathbf{v}_2\}$ of $[\mathbf{A}]$ associated with $\lambda_1=1$ and $\lambda_2=3$, respectively. Each of the eigenvectors can be normalised to obtain the unit eigenvectors:

$$\overline{\mathbf{v}}_1 = \begin{Bmatrix} -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{Bmatrix} \quad \textit{and} \quad \overline{\mathbf{v}}_2 = \begin{Bmatrix} \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{Bmatrix}$$

These eigenvectors determine an orthogonal transformation of co-ordinates that can be used to diagonalise matrix $[\mathbf{A}]$. The matrix consists of the two eigenvectors and is called the modal matrix of $[\mathbf{A}]$. That is

$$[\mathbf{M}] = [\{\mathbf{v}_1\} \quad \{\mathbf{v}_2\}] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \mathrm{t}$$

or

$$[\mathbf{M}] = [\{\overline{\mathbf{v}}_1\} \quad \{\overline{\mathbf{v}}_2\}] = \begin{bmatrix} \dfrac{-1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}$$

For an $n \times n$ matrix with distinct eigenvalues, there are always n independent eigenvectors that form an $n \times n$ non-singular modal matrix. For a matrix having repeated eigenvalues, it is still possible to find n independent eigenvectors if certain conditions are satisfied.

Example 3.2. Find a set of linearly independent eigenvectors for $[\mathbf{A}]$, where

$$[\mathbf{A}] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Using equation (3.26)

$$\det([\mathbf{A}]-\lambda[\mathbf{I}])=\det\begin{bmatrix}2-\lambda & 1 & 1\\ 1 & 2-\lambda & 1\\ 0 & 0 & 1-\lambda\end{bmatrix}=(1-\lambda)^2(3-\lambda)=0$$

So the eigenvalues are 1, 1 and 3, a double root at λ=1 and a single root at λ=3.

Substituting λ=1 into $([\mathbf{A}]-\lambda[\mathbf{I}])\{\mathbf{X}\}=\{\mathbf{0}\}$ yields

$$x_1+x_2+x_3=0$$

Two independent solutions can be generated by letting $x_2=1, x_3=0$ and $x_2=0, x_3=1$, respectively. Thus, we obtain two independent eigenvectors that are associated with λ=1. They are

$$\{\mathbf{v}_1\}=\begin{Bmatrix}-1\\ 1\\ 0\end{Bmatrix}\quad\text{and}\quad\{\mathbf{v}_2\}=\begin{Bmatrix}-1\\ 0\\ 1\end{Bmatrix}$$

Substituting λ=3 into $([\mathbf{A}]-\lambda[\mathbf{I}])\{\mathbf{X}\}=\{\mathbf{0}\}$ leads to the following equation system:

$$\begin{aligned}-x_1+x_2+x_3&=0\\ 2x_3&=0\end{aligned}$$

with a solution that is the eigenvector associated with λ=3:

$$\{\mathbf{v}_3\}=\begin{Bmatrix}1\\ 1\\ 0\end{Bmatrix}$$

The modal matrix of [**A**] is then

$$[\mathbf{M}]=\left[\{\mathbf{v}_1\}\quad\{\mathbf{v}_2\}\quad\{\mathbf{v}_3\}\right]=\begin{bmatrix}-1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

Example 3.3. Find the eigenvectors for [**A**], where

$$[\mathbf{A}]=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

Using equation (3.26)

$$\det([\mathbf{A}]-\lambda[\mathbf{I}])=\det\begin{bmatrix}1-\lambda & 1\\ 0 & 1-\lambda\end{bmatrix}=(1-\lambda)^2=0$$

So the eigenvalues are $\lambda_1=1$ and $\lambda_2=1$.

Substituting $\lambda=1$ into $([\mathbf{A}]-\lambda[\mathbf{I}])\{\mathbf{X}\}=\{\mathbf{0}\}$ yields

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\mathbf{0}$$

or in scalar form, a system that reduces to the following single equation for an arbitrary value of x_1:

$$x_2=0$$

which has the complete solution

$$\mathbf{v}_1=\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=t\begin{Bmatrix}1\\ 0\end{Bmatrix}\qquad \text{for any } t\neq 0$$

Evidently, there is only one independent eigenvector generated from the solution of $([\mathbf{A}]-\lambda[\mathbf{I}])\{\mathbf{X}\}=\{\mathbf{0}\}$.

Example 3.3 shows that the number of independent eigenvectors of a matrix having repeated eigenvalues may be smaller than the order of the matrix. In this situation, the eigenvalue problem of the matrix is considerably more difficult to solve than a matrix with distinct eigenvalues.

3.3.2 Diagonalisation of Square Matrices

If the modal matrix [**M**] of an $n\times n$ matrix [**A**] consists of n independent eigenvectors, it can be shown that the inverse $[\mathbf{M}]^{-1}$ exists. This is always the case if the eigenvalues of [**A**] are distinct. Considering all the eigenvectors and the associated eigenvalues of the matrix, the assemblage of equation (3.23) applying to each of the individual eigenvalues leads to

$$[\mathbf{A}][\mathbf{M}]=[\mathbf{M}][\mathbf{\Lambda}] \tag{3.28}$$

where

$$[\mathbf{\Lambda}]=\begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_2 & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \lambda_n\end{bmatrix}$$

is a diagonal matrix composed of the n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since $[\mathbf{M}]^{-1}$ exists, [**A**] can be diagonalised by

$$[\mathbf{\Lambda}] = [\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}] \tag{3.29}$$

Higher powers of [**A**] can also been diagonalised in a similar manner, *i.e.*,

$$\begin{aligned}
[\mathbf{\Lambda}]^2 &= [\mathbf{\Lambda}][\mathbf{\Lambda}] \\
&= ([\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}])([\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}]) \\
&= [\mathbf{M}]^{-1}[\mathbf{A}]^2[\mathbf{M}] \\
[\mathbf{\Lambda}]^3 &= [\mathbf{\Lambda}][\mathbf{\Lambda}][\mathbf{\Lambda}] \\
&= ([\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}])([\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}])([\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}]) \\
&= [\mathbf{M}]^{-1}[\mathbf{A}]^3[\mathbf{M}] \\
&\vdots \\
[\mathbf{\Lambda}]^k &= [\mathbf{\Lambda}][\mathbf{\Lambda}]\cdots[\mathbf{\Lambda}] \\
&= ([\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}])([\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}])\cdots([\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}]) \\
&= [\mathbf{M}]^{-1}[\mathbf{A}]^k[\mathbf{M}]
\end{aligned} \tag{3.30}$$

A transformation of the type $[\mathbf{B}] = [\mathbf{Q}]^{-1}[\mathbf{A}][\mathbf{Q}]$**,** where [**A**] and [**B**] are square matrices and [**Q**] is a non-singular square matrix, is called a similarity transformation.

Example 3.4. For the matrix and the modal matrix of example 3.2, verify that $[\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}]$ is a diagonal matrix with its elements equal to the eigenvalues of $[\mathbf{A}]$.

$$[\mathbf{A}] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\mathbf{M}] = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Since

$$[\mathbf{M}]^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

then

$$[\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

In the case where matrix [**A**] has repeated eigenvalues and the number of independent eigenvectors is less than *n*, it is impossible to diagonalize the matrix. However, it can be shown that in this situation, the matrix can be transformed by means of a similarity transformation to a Jordan canonical matrix that has following properties:

(1) The principal diagonal elements of the matrix are the *n* eigenvalues of [**A**].

(2) All the elements below the principal matrix diagonal are zero.

(3) A certain number of unit elements are contained in the superdiagonal (the elements immediately to the right of the principal diagonal) when the adjacent elements in the principal diagonal are equal.

The details of this transformation are not discussed here because they are less relevant to the solution of the state space equation of three-dimensional elasticity discussed in the remainder of this book. Readers may refer to Derusso *et al.* (1998) and Hildebrand (1952).

3.3.3 Calculation of $e^{[\mathbf{A}]t}$ by Series Expansions

It has been shown by equation (3.22) that the solution of a state equation depends on the calculation of the exponential matrix $e^{[\mathbf{A}]t}$. In general, equation (3.14) can always be used to calculate $e^{[\mathbf{A}]t}$. This is the most straightforward and simplest way and requires only powers of [**A**]. An example is given below to show this solution.

Example 3.5. Calculate $e^{[\mathbf{A}]t}$, where

$$[\mathbf{A}] = \begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix}$$

The powers of [**A**] are

$$[\mathbf{A}]^2 = \begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 16 & -5 \\ 0 & 1 \end{bmatrix}$$

$$[\mathbf{A}]^3 = [\mathbf{A}]^2[\mathbf{A}] = \begin{bmatrix} 16 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -64 & 21 \\ 0 & -1 \end{bmatrix}$$

$$[\mathbf{A}]^4 = [\mathbf{A}]^3[\mathbf{A}] = \begin{bmatrix} -64 & 21 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 256 & -85 \\ 0 & 1 \end{bmatrix}$$

$$\cdots\cdots\cdots$$

Substituting these into equation (3.14) yields

$$e^{[\mathbf{A}]t} = \begin{bmatrix} 1 - 4t + \dfrac{16t^2}{2!} - \dfrac{64t^3}{3!} + \dfrac{256t^4}{4!} - \cdots & t - \dfrac{5t^2}{2!} + \dfrac{21t^3}{3!} - \dfrac{85t^4}{4!} + \cdots \\ 0 & 1 - t + \dfrac{t^2}{2!} - \dfrac{t^3}{3!} + \dfrac{t^4}{4!} - \cdots \end{bmatrix}$$

In this case, it is possible to recognise the series for each element in $e^{[\mathbf{A}]t}$. In fact

$$e^{[\mathbf{A}]t} = \begin{bmatrix} e^{-4t} & \dfrac{e^{-t} - e^{-4t}}{3} \\ 0 & e^{-t} \end{bmatrix}$$

In most cases, however, it will not be possible to obtain a closed form of solution on the basis of observation. Truncated series expansions have to be used instead to form an approximate solution.

As a special case of exponential matrices, if [**A**] is a diagonal matrix, the exponential matrix $e^{[\mathbf{A}]t}$ equals the matrix that results from taking the exponential of each element in [**A**]. That is, if

$$[\mathbf{A}] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{3.31}$$

$$e^{[\mathbf{A}]t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \tag{3.32}$$

This can be verified directly by introducing equation (3.31) into equation (3.14). For a non-diagonal matrix, other methods have to be used to obtain a closed form solution. Some of these methods are introduced below.

3.3.4 Calculation of $e^{[\mathbf{A}]t}$ by Matrix Transformation

The matrix transformation method uses a similarity transformation to transfer the exponential of a general square matrix [**A**] of order n to the exponential of a diagonal matrix $[\mathbf{\Lambda}]$ that consists of the n eigenvalues of [**A**]. Here we assume that [**A**] has n eigenvalues with n independent eigenvectors. Some of the eigenvalues may be repeated. Thus, the modal matrix, [**M**], of [**A**] is non-singular and its inverse exists.

Multiplying both sides of equation (3.14) by [**M**] and then its inverse $[\mathbf{M}]^{-1}$ yields

$$\begin{aligned}[\mathbf{M}]^{-1}e^{[\mathbf{A}]t}[\mathbf{M}] &= [\mathbf{M}]^{-1}[\mathbf{I}][\mathbf{M}]+[\mathbf{M}]^{-1}[\mathbf{A}][\mathbf{M}]t \\ &\quad +\frac{1}{2}[\mathbf{M}]^{-1}[\mathbf{A}]^2[\mathbf{M}]t^2+\ldots \\ &\quad +\frac{1}{k!}[\mathbf{M}]^{-1}[\mathbf{A}]^k[\mathbf{M}]t^k+\ldots\end{aligned} \tag{3.33}$$

Using equations (3.29) and (3.30) and considering equation (3.14) again, we obtain

$$\begin{aligned}[\mathbf{M}]^{-1}e^{[\mathbf{A}]t}[\mathbf{M}] &= [\mathbf{I}]+[\mathbf{\Lambda}]t+\frac{1}{2}[\mathbf{\Lambda}]^2t^2+\cdots+\frac{1}{k!}[\mathbf{\Lambda}]^kt^k+\cdots \\ &= e^{[\mathbf{\Lambda}]t}\end{aligned} \tag{3.34}$$

Hence,

$$e^{[\mathbf{A}]t} = [\mathbf{M}]e^{[\mathbf{\Lambda}]t}[\mathbf{M}]^{-1} \tag{3.35}$$

where, according to equation (3.32),

$$e^{[\mathbf{\Lambda}]t} = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n}\end{bmatrix}$$

Example 3.6. Calculate the $e^{[\mathbf{A}]t}$ in example 3.5 using the matrix transformation method.

Since

$$[\mathbf{A}] = \begin{bmatrix} -4 & 1 \\ 0 & -1\end{bmatrix}$$

the eigenvalues of the matrix are found from

$$\det([\mathbf{A}]-\lambda[\mathbf{I}])=\det\begin{bmatrix}-4-\lambda & 1\\ 0 & -1-\lambda\end{bmatrix}$$
$$=(4+\lambda)(1+\lambda)=0$$

The eigenvalues are λ_1=–1 and λ_2=–4.

Substituting λ=–1 into $([\mathbf{A}]-\lambda[\mathbf{I}])\{\mathbf{X}\}=\{\mathbf{0}\}$ yields

$$\begin{bmatrix}-3 & 1\\ 0 & 0\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\{\mathbf{0}\}$$

Hence

$$\{\mathbf{v}_1\}=\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{Bmatrix}1\\ 3\end{Bmatrix}$$

is an eigenvector associated with the first eigenvalue.

Substituting λ=–4 into $([\mathbf{A}]-\lambda[\mathbf{I}])\{\mathbf{X}\}=\{\mathbf{0}\}$ leads to

$$\begin{bmatrix}0 & 1\\ 0 & 3\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\{\mathbf{0}\}$$

and

$$\{\mathbf{v}_2\}=\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{Bmatrix}1\\ 0\end{Bmatrix}$$

is an eigenvector associated with the second eigenvalue. Since the two eigenvalues are distinct, the two eigenvectors are linear independent and the modal matrix consisting of the two eigenvectors, *i.e.*,

$$[\mathbf{M}]=\begin{bmatrix}1 & 1\\ 3 & 0\end{bmatrix}$$

is non-singular. The inverse of the modal matrix is

$$[\mathbf{M}]^{-1}=\begin{bmatrix}0 & \dfrac{1}{3}\\ 1 & -\dfrac{1}{3}\end{bmatrix}$$

From equation (3.35), the exponential of [**A**] can be calculated by the following matrix transformation.

$$e^{[\mathbf{A}]t} = [\mathbf{M}]e^{[\Lambda]t}[\mathbf{M}]^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-4t} & \dfrac{e^{-t} - e^{-4t}}{3} \\ 0 & e^{-t} \end{bmatrix}$$

which is identical to the solution obtained in example 3.5.

Example 3.7. Find $e^{[\mathbf{A}]t}$, using the same [**A**] as the one used in example 3.4, where

$$[\mathbf{A}] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = 1 \qquad \lambda_3 = 3$$

$$[\mathbf{M}] = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[\mathbf{M}]^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$[\Lambda] = \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

$$e^{[\mathbf{A}]t} = [\mathbf{M}]e^{[\Lambda]t}[\mathbf{M}]^{-1}$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{e^t+e^{3t}}{2} & \frac{-e^t+e^{3t}}{2} & \frac{-e^t+e^{3t}}{2} \\ \frac{-e^t+e^{3t}}{2} & \frac{e^t+e^{3t}}{2} & \frac{-e^t+e^{3t}}{2} \\ 0 & 0 & e^t \end{bmatrix}$$

It can be seen from the two examples that matrix transformation is a very convenient method for finding the exponential of a square matrix, although the method requires a complete eigenvector analysis.

3.3.5 Calculation of $e^{[\mathbf{A}]t}$ by the Cayley−Hamilton Method

Consider a polynomial in [**A**] of the form

$$[\mathbf{N}(\mathbf{A})]=[\mathbf{A}]^n+c_1[\mathbf{A}]^{n-1}+c_2[\mathbf{A}]^{n-2}+\cdots+c_{n-1}[\mathbf{A}]+c_n[\mathbf{I}] \tag{3.36}$$

where [**A**] is an $n\times n$ matrix having n distinct eigenvalues. Introducing equations (3.29) and (3.30) into equation (3.36) yields

$$\begin{aligned}[\mathbf{N}(\mathbf{A})]=[\mathbf{M}][\mathbf{\Lambda}]^n[\mathbf{M}]^{-1}+c_1[\mathbf{M}][\mathbf{\Lambda}]^{n-1}[\mathbf{M}]^{-1} \\ +c_2[\mathbf{M}][\mathbf{\Lambda}]^{n-2}[\mathbf{M}]^{-1}+\cdots+c_{n-1}[\mathbf{M}][\mathbf{\Lambda}][\mathbf{M}]^{-1}+c_n[\mathbf{I}]\end{aligned} \tag{3.37}$$

where [**Λ**] is a diagonal matrix composed of the n distinct eigenvalues, $\lambda_1,\lambda_2,\ldots,\lambda_n$, of [**A**]. Equation (3.37) can be further rewritten as

$$[\mathbf{N}(\mathbf{A})]=[\mathbf{M}]\begin{bmatrix} N(\lambda_1) & 0 & 0 & 0 \\ 0 & N(\lambda_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & N(\lambda_n) \end{bmatrix}[\mathbf{M}]^{-1} \tag{3.38}$$

where

$$N(\lambda_i)=\lambda_i^n+c_1\lambda_i^{n-1}+c_2\lambda_i^{n-2}+\cdots+c_{n-1}\lambda+c_n \tag{3.39}$$

is a polynomial obtained by replacing [**A**] with λ_i as the variable in equation (3.36).

If the polynomial is identical to the characteristic polynomial, *i.e.*, the expansion of $\det([\mathbf{A}]-\lambda_i[\mathbf{I}])$, $N(\lambda_i)=0$. Hence,

$$[\mathbf{N}(\mathbf{A})]=[\mathbf{0}] \tag{3.40}$$

Equation (3.40) is known as the Cayley−Hamilton theorem. The theorem states that matrix [**A**] satisfies its characteristic polynomial. Equations (3.36)−(3.40) are based on the assumption that [**A**] has distinct eigenvalues. In fact, it can be shown that this theorem holds true for any square matrix (Hohn, 1973)

An important application of the Cayley−Hamilton theorem is in the representation of high powers of a matrix that reduces any polynomial of the $n \times n$ matrix [**A**] to a linear combination of [**I**], [**A**], $[\mathbf{A}]^2$, $\cdots$, $[\mathbf{A}]^{n-1}$.

Let us write $[\mathbf{N}(\mathbf{A})] = [\mathbf{0}]$ in the form

$$[\mathbf{A}]^n + c_1[\mathbf{A}]^{n-1} + c_2[\mathbf{A}]^{n-2} + \cdots + c_{n-1}[\mathbf{A}] + c_n[\mathbf{I}] = [\mathbf{0}] \tag{3.41}$$

so that, if we have already computed $[\mathbf{A}]^2$, $[\mathbf{A}]^3 \cdots$, $[\mathbf{A}]^{n-1}$, we can express $[\mathbf{A}]^n$ as a linear combination of these:

$$[\mathbf{A}]^n = -c_1[\mathbf{A}]^{n-1} - c_2[\mathbf{A}]^{n-2} - \cdots - c_{n-1}[\mathbf{A}] - c_n[\mathbf{I}] \tag{3.42}$$

Multiplying both sides of equation (3.42) by [**A**] and substituting from it for $[\mathbf{A}]^n$ on the right, we obtain

$$\begin{aligned}[\mathbf{A}]^{n+1} &= (c_1^2 - c_2)[\mathbf{A}]^{n-1} + (c_1c_2 - c_3)[\mathbf{A}]^{n-2} + \cdots \\ &\quad + (c_1c_{n-1} - c_n)[\mathbf{A}] + c_1c_n[\mathbf{I}]\end{aligned} \tag{3.43}$$

By continuing this process, any positive integral power of [**A**] can be expressed as a linear combination of [**I**], [**A**], $[\mathbf{A}]^2$, $\cdots$, $[\mathbf{A}]^{n-1}$.

Application of the above process to the exponential function of [**A**] gives

$$\begin{aligned}e^{[\mathbf{A}]t} &= [\mathbf{I}] + [\mathbf{A}]t + \frac{1}{2!}[\mathbf{A}]^2 t^2 + \ldots + \frac{1}{k!}[\mathbf{A}]^k t^k + \ldots \\ &= \alpha_0(t)[\mathbf{I}] + \alpha_1(t)[\mathbf{A}] + \alpha_2(t)[\mathbf{A}]^2 + \cdots + \alpha_{n-1}(t)[\mathbf{A}]^{n-1}\end{aligned} \tag{3.44}$$

where the $\alpha_i(t)$, i=0,1,2,…n−1, are unknown scalar functions of t. Since the polynomial used to derive equation (3.44) is the characteristic polynomial of [**A**], it is quite obvious that equation (3.44) holds true when [**A**] is replaced by one of its eigenvalues, *i.e.,*

$$e^{\lambda t} = \alpha_0(t) + \alpha_1(t)\lambda + \alpha_2(t)\lambda^2 + \cdots + \alpha_{n-1}(t)\lambda^{n-1} \tag{3.45}$$

at $\lambda = \lambda_i$, where λ_i is the ith eigenvalue of [**A**]. This can be verified easily by using equation (3.39) in the deduction process.

If [**A**] has n distinct eigenvalues, equation (3.45) provides n simultaneous linear algebra equations, from which the n unknown functions, $\alpha_i(t)$, can be obtained. If

λ_i is a repeated eigenvalue of [**A**] with multiplicity m, only one linearly independent equation can be obtained by substituting the eigenvalue into equation (3.45). The remaining $m-1$ linear equations, which must be found in order to solve for the n unknown $\alpha_i(t)$, can be obtained by differentiating both sides of equation (3.45) ($m-1$) times with respect to λ. Therefore, the $m-1$ linear equations are found as follows:

$$
\begin{aligned}
& t\, e^{\lambda_i t} = \sum_{k=1}^{n-1} k\alpha_k(t)\lambda_i^{k-1} \\
& t^2 e^{\lambda_i t} = \sum_{k=2}^{n-1} k(k-1)\alpha_k(t)\lambda_i^{k-2} \\
& \ldots\ldots \\
& t^{m-1} e^{\lambda_i t} = \sum_{k=m-1}^{n-1} k(k-1)(k-2)\cdots(k-m+2)\alpha_k(t)\lambda_i^{k-m+1}
\end{aligned}
\tag{3.46}
$$

For instance, if λ_1 is an r-times repeated eigenvalue of [**A**] and the remaining $n-r$ eigenvalues, $\lambda_{r+1}, \lambda_{r+2} \ldots \lambda_n$, are distinct, the α_i(t) in equation (3.44) can be solved from the following linear algebra equation system:

$$
\begin{aligned}
& \alpha_0(z) + \alpha_1(z)\lambda_1 + \alpha_2(z)\lambda_1^2 + \ldots\ldots + \alpha_{n-1}(z)\lambda_1^{n-1} = e^{\lambda_1 t} \\
& \frac{d}{d\lambda_1}[\alpha_0(z) + \alpha_1(z)\lambda_1 + \alpha_2(z)\lambda_1^2 + \ldots\ldots + \alpha_{n-1}(z)\lambda_1^{n-1}] = \frac{d}{d\lambda_1} e^{\lambda_1 t} \\
& \frac{d^2}{d\lambda_1^2}[\alpha_0(z) + \alpha_1(z)\lambda_1 + \alpha_2(z)\lambda_1^2 + \ldots\ldots + \alpha_{n-1}(z)\lambda_1^{n-1}] = \frac{d^2}{d\lambda_1^2} e^{\lambda_1 t} \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \frac{d^{r-1}}{d\lambda_1^{r-1}}[\alpha_0(z) + \alpha_1(z)\lambda_1 + \alpha_2(z)\lambda_1^2 + \ldots\ldots + \alpha_{n-1}(z)\lambda_1^{n-1}] = \frac{d^{r-1}}{d\lambda_1^{r-1}} e^{\lambda_1 t} \\
& \alpha_0(z) + \alpha_1(z)\lambda_{r+1} + \alpha_2(z)\lambda_{r+1}^2 + \ldots\ldots + \alpha_{n-1}(z)\lambda_{r+1}^{n-1} = e^{\lambda_{r+1} t} \\
& \alpha_0(z) + \alpha_1(z)\lambda_{r+2} + \alpha_2(z)\lambda_{r+2}^2 + \ldots\ldots + \alpha_{n-1}(z)\lambda_{r+2}^{n-1} = e^{\lambda_{r+2} t} \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \alpha_0(z) + \alpha_1(z)\lambda_n + \alpha_2(z)\lambda_n^2 + \ldots\ldots + \alpha_{n-1}(z)\lambda_n^{n-1} = e^{\lambda_n t}
\end{aligned}
\tag{3.47}
$$

Example 3.8. Find $e^{[\mathbf{A}]t}$, where [**A**] is the same as the one used in example 3.6.

$$
[\mathbf{A}] = \begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix} \qquad \lambda_1 = -1 \qquad \lambda_2 = -4
$$

The two linear equations obtained by substituting $\lambda_1 = -1$ and $\lambda_2 = -4$ into equation (3.45) are

$$\alpha_0(t) - \alpha_1(t) = e^{-t}$$
$$\alpha_0(t) - 4\alpha_1(t) = e^{-4t}$$

Solving for $\alpha_0(t)$ and $\alpha_1(t)$ yields

$$\alpha_0(t) = \frac{1}{3}(4e^{-t} - e^{-4t}) \qquad \alpha_1(t) = \frac{1}{3}(e^{-t} - e^{-4t})$$

Using equation (3.44)

$$e^{[\mathbf{A}]t} = \alpha_0(t)[\mathbf{I}] + \alpha_1(t)[\mathbf{A}]$$
$$= \frac{1}{3}(4e^{-t} - e^{-4t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{3}(e^{-t} - e^{-4t})\begin{bmatrix} -4 & -1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-4t} & \dfrac{e^{-t} - e^{-4t}}{3} \\ 0 & e^{-t} \end{bmatrix}$$

which is identical to the one obtained using the matrix transformation method in example 3.6.

Example 3.9. Find $e^{[\mathbf{A}]t}$, where $[\mathbf{A}]$ and its eigenvalues are

$$[\mathbf{A}] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = 1 \qquad \lambda_3 = 3$$

Since $\lambda=1$ is a repeated eigenvalue with multiplicity 2, equation (3.47) is used to calculate the exponential. Thus, after taking the differential with respect to λ, we obtain the following two equations for $\lambda_1 = \lambda_2 = 1$:

$$\alpha_0(t) + \alpha_1(t) + \alpha_2(t) = e^{t}$$
$$\alpha_1(t) + 2\alpha_2(t) = te^{t}$$

For $\lambda_3 = 3$:

$$\alpha_0(t) + 3\alpha_1(t) + 9\alpha_2(t) = e^{3t}$$

Solving the three linear algebra equations for $\alpha_0(t)$, $\alpha_1(t)$ and $\alpha_2(t)$, we obtain

$$\begin{Bmatrix} \alpha_0(t) \\ \alpha_1(t) \\ \alpha_2(t) \end{Bmatrix} = \frac{1}{4} \begin{Bmatrix} e^{3t} + 3e^t - 6te^t \\ -2e^{3t} + 2e^t + 8te^t \\ e^{3t} - e^t - 2te^t \end{Bmatrix}$$

Hence

$$\begin{aligned} e^{[\mathbf{A}]t} &= \alpha_0(t)[\mathbf{I}] + \alpha_1(t)[\mathbf{A}] + \alpha_2(t)[\mathbf{A}]^2 \\ &= \alpha_0(t)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha_1(t)\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &\qquad + \alpha_2(t)\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{e^t + e^{3t}}{2} & \dfrac{-e^t + e^{3t}}{2} & \dfrac{-e^t + e^{3t}}{2} \\ \dfrac{-e^t + e^{3t}}{2} & \dfrac{e^t + e^{3t}}{2} & \dfrac{-e^t + e^{3t}}{2} \\ 0 & 0 & e^t \end{bmatrix} \end{aligned}$$

which is the same as found in example 3.7 by means of the matrix transformation method. The advantage of using the Cayley−Hamilton method is that a full eigenvector analysis is not required.

3.3.6 Calculation of $e^{[\mathbf{A}]t}$ by the Time Step Integration Method

It has been seen that if the dimension of an exponential matrix is small, it is quite straightforward to use the closed form solutions discussed in the preceding sections. In the cases of matrices of large order, numerical solutions may be more effective and efficient.

The numerical method introduced here is called the time step integration method. The method was proposed by Zhong and Williams (1994) to solve a linear time-invariant structural dynamic systems. The method is based on the division of a time domain [0, T] into a number of small time intervals. As a result, the exponential matrix can be represented by the product of a number of exponential matrices relative to these small time intervals, *i.e.*,

$$
\begin{aligned}
e^{[\mathbf{A}]T} &= \exp(\overbrace{[\mathbf{A}]\frac{T}{m} + [\mathbf{A}]\frac{T}{m} + \cdots + [\mathbf{A}]\frac{T}{m}}^{m}) \\
&= \left\{ \exp([\mathbf{A}]\frac{T}{m}) \right\}^m
\end{aligned}
\tag{3.48}
$$

where m is a sufficiently large integer and T/m is the small time interval. As a consequence the following truncated Taylor series expansion can be used for each of the time intervals:

$$
\begin{aligned}
\exp([\mathbf{A}]\frac{T}{m}) &\approx [\mathbf{I}] + [\mathbf{A}]\frac{T}{m} + \frac{([\mathbf{A}]T/m)^2}{2!} \\
&\quad + \frac{([\mathbf{A}]T/m)^3}{3!} + \frac{([\mathbf{A}]T/m)^4}{4!} \\
&= [\mathbf{I}] + [\mathbf{T}_a]
\end{aligned}
\tag{3.49}
$$

$$
[\mathbf{T}_a] = [\mathbf{A}]\frac{T}{m} + \frac{([\mathbf{A}]T/m)^2[[\mathbf{I}] + ([\mathbf{A}]T/m)/3 + ([\mathbf{A}]T/m)^2/12]}{2} \tag{3.50}
$$

Using equations (3.48) and (3.50), the exponential matrix can be calculated approximately as follows:

$$
\begin{aligned}
e^{[\mathbf{A}]T} &= \left\{ \exp([\mathbf{A}]\frac{T}{m}) \right\}^m \approx ([\mathbf{I}] + [\mathbf{T}_a])^{2^N} \\
&= \left[([\mathbf{I}] + [\mathbf{T}_a])([\mathbf{I}] + [\mathbf{T}_a]) \right]^{2^{(N-1)}}
\end{aligned}
\tag{3.51a}
$$

A recursive factorisation process has been developed by Zhong and Williams (1994) to calculate the exponential matrix. The process starts from equation (3.50). Then, the following iteration is executed:

$$
\mathit{for}\ \ (\mathit{iter} = 0;\ \mathit{iter} < N)\ \ [\mathbf{T}_a] = 2 \times [\mathbf{T}_a] + [\mathbf{T}_a] \times [\mathbf{T}_a] \tag{3.51b}
$$

Then

$$
e^{[\mathbf{A}]T} = ([\mathbf{I}] + [\mathbf{T}_a]) \tag{3.51c}
$$

In order to obtain an accurate time integration, the value of N should be chosen so that $\max\{\mathrm{abs}(\lambda_i) \times T/m)\}$ <<1. In Zhong and Williams (1994), 20 was recommended as the minimum value of N, where numerical tests were also carried out to show the precision and robustness of the method. The time step integration method will be used in Chapter 9 to calculate exponential matrices of large order resulting from a state space finite element analysis.

3.4 Solutions for a Time-varying System

In the cases where matrices [**A**] and [**B**] are not constant but vary with *t*, a closed form solution can still be obtained sometimes. We can find the solution by starting with the simplest case, *i.e.*, the homogeneous, scalar differential equation:

$$\dot{x}(t) = a(t)x(t) \tag{3.52}$$

The solution to the above equation is

$$x(t) = e^{d(t)} x(t_0) \tag{3.53}$$

where $x(t_0)$ is the value of $x(t)$ at t_0 and

$$d(t) = \int_{t_0}^{t} a(\tau)d\tau \tag{3.54}$$

In analogy with the scalar case, we try to seek the solution for the matrix system

$$\{\dot{\mathbf{x}}(t)\} = [\mathbf{A}(t)]\{\mathbf{x}(t)\} \tag{3.55}$$

by assuming a solution in the form of the serious expansion of equation (3.13). Thus

$$\begin{aligned}\{\mathbf{x}(t)\} = \{[\mathbf{I}] + \int_{t_0}^{t}[\mathbf{A}(\tau)]d\tau + \frac{1}{2!}\left[\int_{t_0}^{t}[\mathbf{A}(\tau)]d\tau\right]^2 \\ + \frac{1}{3!}\left[\int_{t_0}^{t}[\mathbf{A}(\tau)]d\tau\right]^3 + \ldots + \frac{1}{n!}\left[\int_{t_0}^{t}[\mathbf{A}(\tau)]d\tau\right]^n + \ldots\}\{\mathbf{x}(t_0)\}\end{aligned} \tag{3.56}$$

After substituting the assumed solution into equation (3.55), we find that equation (3.56) is the solution of equation (3.55) if and only if

$$[\mathbf{A}(t)]\int_{t_0}^{t}[\mathbf{A}(\tau)]d\tau = \int_{t_0}^{t}[\mathbf{A}(\tau)]d\tau[\mathbf{A}(t)] \tag{3.57}$$

We thus conclude that the solution of equation (3.55) cannot be expressed in an explicit form unless equation (3.57) is satisfied. In the general case, however, equation (3.57) may not necessarily be satisfied and the solution of equation (3.55) must be sought numerically. In this book we use a successive approximation method that solves equation (3.55) by solving a series of state equations relative to a set of linear time-invariant systems. Hence, the solutions presented in Section 3.3 for time-invariant systems can readily be applied here in a straightforward manner.

3.4.1 The Successive Approximation Method for Homogeneous Equations

The successive approximation method is based on the division of the time domain $[t_0, t]$ into *N* small time intervals, *i.e.*, $[t_0, t_1]$, $[t_1, t_2]$, $\ldots$, and $[t_{N-1}, t_N]$. Each of these time intervals is sufficiently small and approaches zero uniformly as *N* approaches infinity. At an arbitrarily selected time interval, equation (3.55) can be represented approximately by

$$\{\dot{\mathbf{x}}(t)\} = [\mathbf{A}(\frac{t_j + t_{j-1}}{2})]\{\mathbf{x}(t)\} = [\tilde{\mathbf{A}}_j]\{\mathbf{x}(t)\} \qquad t_{j-1} \le t \le t_j \tag{3.58}$$

where the system matrix [**A**] is calculated by choosing *t* as the mean value of the small time interval. Hence, $[\tilde{\mathbf{A}}_j]$ becomes a constant matrix and can be a satisfactory approximation to [**A**(*t*)] if $[t_{j-1}, t_j]$ is sufficiently small. Obviously, equation (3.58) is now representing the state space equation of a time-invariant system. The solution of the equation is

$$\{\mathbf{x}(t)\} = e^{[\tilde{\mathbf{A}}_j](t-t_{j-1})}\{\mathbf{x}(t_{j-1})\} \qquad t_{j-1} \le t \le t_j \tag{3.59}$$

where the exponential matrix can be calculated by means of the methods introduced in Section 3.3. For each of these time intervals, a solution in the form of equation (3.59) can be obtained and the state vectors at each of the time divisions are expressed as

$$\begin{aligned}
&\{\mathbf{x}(t_N)\} = e^{[\tilde{\mathbf{A}}_N](t_N - t_{N-1})}\{\mathbf{x}(t_{N-1})\} \\
&\{\mathbf{x}(t_{N-1})\} = e^{[\tilde{\mathbf{A}}_{N-1}](t_{N-1} - t_{N-2})}\{\mathbf{x}(t_{N-2})\} \\
&\cdots\cdots \\
&\{\mathbf{x}(t_j)\} = e^{[\tilde{\mathbf{A}}_j](t_j - t_{j-1})}\{\mathbf{x}(t_{j-1})\} \\
&\cdots\cdots \\
&\{\mathbf{x}(t_1)\} = e^{[\tilde{\mathbf{A}}_1](t_1 - t_0)}\{\mathbf{x}(t_0)\}
\end{aligned} \tag{3.60}$$

Considering the continuity conditions at these time divisions and using equation (3.60) recursively, the following recursive relationship can be formed:

$$
\begin{aligned}
\{\mathbf{x}(t_N)\} &= e^{[\tilde{\mathbf{A}}_N](t_N - t_{N-1})}\{\mathbf{x}(t_{N-1})\} \\
&= e^{[\tilde{\mathbf{A}}_N](t_N - t_{N-1})} e^{[\tilde{\mathbf{A}}_{N-1}](t_{N-1} - t_{N-2})}\{\mathbf{x}(t_{N-2})\} \\
&= \cdots\cdots \\
&= e^{[\tilde{\mathbf{A}}_N](t_N - t_{N-1})} e^{[\tilde{\mathbf{A}}_{N-1}](t_{N-1} - t_{N-2})} \cdots e^{[\tilde{\mathbf{A}}_j](t_j - t_{j-1})}\{\mathbf{x}(t_{j-1})\} \\
&= \cdots\cdots \\
&= e^{[\tilde{\mathbf{A}}_N](t_N - t_{N-1})} e^{[\tilde{\mathbf{A}}_{N-1}](t_{N-1} - t_{N-2})} \cdots e^{[\tilde{\mathbf{A}}_1](t_1 - t_0)}\{\mathbf{x}(t_0)\} \\
&= \left\{\prod_{\mathrm{j=N}}^{1} e^{[\tilde{\mathbf{A}}_j](t_j - t_{j-1})}\right\}\{\mathbf{x}(t_0)\}
\end{aligned}
\tag{3.61a}
$$

Equation (3.61a) represents the state vector at t_N in terms of the vector at t_0. At an arbitrary time instant t^* within the *i*th time interval $[t_{i-1}, t_i]$, the state vector can be expressed as

$$
\begin{aligned}
\{\mathbf{x}(t^*)\} &= e^{[\tilde{\mathbf{A}}_i](t^* - t_{i-1})}\{\mathbf{x}(t_{i-1})\} \\
&= e^{[\tilde{\mathbf{A}}_i](t^* - t_{i-1})} e^{[\tilde{\mathbf{A}}_{i-1}](t_{i-1} - t_{i-2})}\{\mathbf{x}(t_{i-2})\} \\
&= \cdots\cdots \\
&= e^{[\tilde{\mathbf{A}}_i](t^* - t_{i-1})} e^{[\tilde{\mathbf{A}}_{i-1}](t_{i-1} - t_{i-2})} \cdots e^{[\tilde{\mathbf{A}}_1](t_1 - t_0)}\{\mathbf{x}(t_0)\} \\
&= e^{[\tilde{\mathbf{A}}_i](t^* - t_{i-1})}\left\{\prod_{\mathrm{j=i-1}}^{1} e^{[\tilde{\mathbf{A}}_j](t_j - t_{j-1})}\right\}\{\mathbf{x}(t_0)\}
\end{aligned}
\tag{3.61b}
$$

Equation (3.61b) can be used to predict the state of a linear time-varying system at any time instant, as long as the state of the system at t_0 is known.

3.4.2 The Successive Approximation Method for Non-homogeneous Equations

The application of the successive approximation method for the solution of a time-varying system defined by a non-homogeneous state equation is a direct extension of the solution presented in Section 3.4.1 for a homogeneous case. By applying the approximation to both [**A**] and [**B**] for each of the small time intervals, $[t_0, t_1]$, $[t_1, t_2]$, $\ldots$, and $[t_{N-1}, t_N]$, the time-varying non-homogeneous state equation (3.5) is expressed approximately by

$$
\begin{aligned}
\{\dot{\mathbf{x}}(t)\} &= [\mathbf{A}(\frac{t_j + t_{j-1}}{2})]\{\mathbf{x}(t)\} + [\mathbf{B}(\frac{t_j + t_{j-1}}{2})]\{\mathbf{u}(t)\} \\
&= [\tilde{\mathbf{A}}_j]\{\mathbf{x}(t)\} + [\tilde{\mathbf{B}}_j]\{\mathbf{u}(t)\} \\
& \qquad t_{j-1} \le t \le t_j \qquad\qquad j = 1,2,\cdots N
\end{aligned} \tag{3.62}
$$

where both $[\tilde{\mathbf{A}}_j]$ and $[\tilde{\mathbf{B}}_j]$ are constant matrices and can be very close to $[\mathbf{A}(t)]$ and $[\mathbf{B}(t)]$, respectively, if $[t_{j-1}, t_j]$ is sufficiently small. Equation (3.62) is now reduced to a non-homogeneous state equation relative to a time-invariant system whose solution is

$$
\begin{aligned}
\{\mathbf{x}(t)\} &= e^{[\tilde{\mathbf{A}}_j](t-t_{j-1})}\{\mathbf{x}(t_{j-1})\} \\
&+ e^{[\tilde{\mathbf{A}}_j](t-t_{j-1})}\int_{t_{j-1}}^{t} e^{-[\tilde{\mathbf{A}}_j](\tau-t_{j-1})}[\tilde{\mathbf{B}}]\{\mathbf{u}(\tau)\}d\tau \qquad t_{j-1} \le t \le t_j
\end{aligned} \tag{3.63}
$$

At each of the time instants, $t_1, t_2.\cdots t_N$, we have

$$
\{\mathbf{x}(t_j)\} = [\mathbf{D}_j]\{\mathbf{x}(t_{j-1})\} + \{\mathbf{H}_j\} \tag{3.64}
$$

where

$$
\begin{aligned}
[\mathbf{D}_j] &= e^{[\tilde{\mathbf{A}}_j](t_j - t_{j-1})} \\
\{\mathbf{H}_j\} &= e^{[\tilde{\mathbf{A}}_j](t_j - t_{j-1})}\int_{t_{j-1}}^{t_j} e^{-[\tilde{\mathbf{A}}_j](\tau-t_{j-1})}[\tilde{\mathbf{B}}]\{\mathbf{u}(\tau)\}d\tau
\end{aligned} \tag{3.65}
$$

are two constant matrices. By following the same procedure as used to derive the recursive formulation for a homogeneous case, we have

$$
\begin{aligned}
\{\mathbf{x}(t_N)\} &= [\mathbf{D}_N]\{\mathbf{x}(t_{N-1})\} + \{\mathbf{H}_N\} \\
&= [\mathbf{D}_N]\{[\mathbf{D}_{N-1}]\{\mathbf{x}(t_{N-2})\} + \{\mathbf{H}_{N-1}\}\} + \{\mathbf{H}_N\} \\
&= [\mathbf{D}_N][\mathbf{D}_{N-1}]\{\mathbf{x}(t_{N-2})\} + [\mathbf{D}_N]\{\mathbf{H}_{N-1}\} + \{\mathbf{H}_N\} \\
&= \cdots\cdots \\
&= [\mathbf{D}_N][\mathbf{D}_{N-1}][\mathbf{D}_{N-2}]\cdots[\mathbf{D}_{N-j}]\{\mathbf{x}(t_{N-j-1})\} \\
&\quad + [\mathbf{D}_N][\mathbf{D}_{N-1}]\cdots[\mathbf{D}_{N-j+1}]\{\mathbf{H}_{N-j}\} \\
&\quad + [\mathbf{D}_N][\mathbf{D}_{N-1}]\cdots[\mathbf{D}_{N-j+2}]\{\mathbf{H}_{N-j+1}\} + \cdots \\
&\quad + [\mathbf{D}_N]\{\mathbf{H}_{N-1}\} + \{\mathbf{H}_N\} \\
&= \cdots\cdots \\
&= [\mathbf{\Psi}]\{\mathbf{x}(t_0)\} + \{\mathbf{\Omega}\}
\end{aligned} \tag{3.66a}
$$

where

$$
\begin{aligned}
&[\mathbf{\Psi}] = \prod_{j=N}^{1} [\mathbf{D}]_j \\
&[\mathbf{\Omega}] = \sum_{i=2}^{N} \left(\prod_{j=N}^{i} [\mathbf{D}_j] \right) \{\mathbf{H}_{i-1}\} + \{\mathbf{H}_N\}
\end{aligned}
\tag{3.66b}
$$

At an arbitrary time instant t^* within the ith time interval $[t_{i-1}, t_i]$, the state vector can be found in a similar way to that described for the homogeneous case. The state vector is

$$
\begin{aligned}
\{\mathbf{x}(t^*)\} &= [\mathbf{D}_i^*]\{\mathbf{x}(t_{i-1})\} + \{\mathbf{H}_i^*\} \\
&= [\mathbf{D}_i^*]\{[\mathbf{D}_{i-1}]\{\mathbf{x}(t_{i-2})\}\} + \{\mathbf{H}_{i-1}\} + \{\mathbf{H}_i^*\} \\
&= [\mathbf{D}_i^*][\mathbf{D}_{i-1}]\{\mathbf{x}(t_{i-2})\} + [\mathbf{D}_i^*]\{\mathbf{H}_{i-1}\} + \{\mathbf{H}_i^*\} \\
&= \cdots\cdots \\
&= [\mathbf{D}_i^*][\mathbf{D}_{i-1}][\mathbf{D}_{i-2}]\cdots[\mathbf{D}_{i-j}]\{\mathbf{x}(t_{i-j-1})\} \\
&\quad + [\mathbf{D}_i^*][\mathbf{D}_{i-1}]\cdots[\mathbf{D}_{i-j+1}]\{\mathbf{H}_{i-j}\} \\
&\quad + [\mathbf{D}_i^*][\mathbf{D}_{i-1}]\cdots[\mathbf{D}_{i-j+2}]\{\mathbf{H}_{i-j+1}\} + \cdots \\
&\quad + [\mathbf{D}_i^*]\{\mathbf{H}_{i-1}\} + \{\mathbf{H}_i^*\} \\
&= \cdots\cdots \\
&= [\mathbf{\Psi}^*]\{\mathbf{x}(t_0)\} + \{\mathbf{\Omega}^*\}
\end{aligned}
\tag{3.67a}
$$

where

$$
\begin{aligned}
&[\mathbf{\Psi}^*] = [\mathbf{D}_i^*] \prod_{j=i-1}^{1} [\mathbf{D}_j] \\
&[\mathbf{\Omega}^*] = [\mathbf{D}_i^*] \sum_{k=2}^{i-1} \left(\prod_{j=i-1}^{k} [\mathbf{D}_j] \right) \{\mathbf{H}_{k-1}\} + \{\mathbf{H}_i^*\} \\
&[\mathbf{D}_i^*] = e^{[\tilde{\mathbf{A}}_i](t^* - t_{i-1})}, \\
&\{\mathbf{H}_i^*\} = e^{[\tilde{\mathbf{A}}_i](t^* - t_{i-1})} \int_{t_{i-1}}^{t^*} e^{-[\tilde{\mathbf{A}}_i](\tau - t_{i-1})} [\tilde{\mathbf{B}}]\{\mathbf{u}(\tau)\} d\tau
\end{aligned}
\tag{3.67b}
$$

Equations (3.66) and (3.67) provide approximate solutions of a non-homogeneous time-varying state equation. The accuracy of the solutions depends on the length of the small time intervals. The successive approximation method is particularly useful when a three-dimensional shell problem is solved by means of the state space method.

3.5 State Variable Equation of Elasticity

From Section 3.2, it has been seen that the mechanical behaviour of the spring–damper–mass system shown in Figure 3.1 can be described by a state equation of time co-ordinate t. The state variables of the equation are the displacement and its derivative with respect to t. In analogy with this, the state of an elastic body in a three-dimensional space can also be described by a state equation with respect to one of the three orthogonal co-ordinates, *e.g.*, the z co-ordinate in the rectangular co-ordinate system. The state vector, therefore, may include all the displacements and their derivatives with respect to the z co-ordinate. The state equation of this form has been used to solve shell problems (see Soldatos and Hadjigeorgiou 1990). Since the state vector contains the derivatives of displacements with respect to z, it is natural to convert these derivatives to relative strains and then stresses. In consequence, displacements and the stresses relative to the z-direction may also be used as the state variables in the state vector. In the case of a plate bending problem, for instance, if the z-direction is taken as the transverse direction, the state vector will contain three displacements of the plate and the three transverse stresses (Fan and Ye, 1990a, 1990b). It is also possible to construct the state equation in other different forms, for example, in the one used by Tarn and Wang (2001), where the transverse stresses in the state vector are all multiplied by the transverse co-ordinate. The first application of the state variable equation to the solution of three-dimensional elasticity appears to be the work of Vlasov (1957) where the method was called the method of initial functions (MIF). The equation was solved by means of a Maclaurin series expansion for stresses and displacements (also see Iyengar *et al.,* 1975; Faraji and Archer, 1985, 1989; Chandrashekhara and Rao, 1998). At the beginning of the last decade, Fan and Ye (1990a,b) started applying the state space method systematically to the solutions of various plates and shells composed of laminated composite materials. These solutions include stress, free vibration, forced vibration, stability analyses of laminates subjected to complex loading and boundary conditions. The volume of research publications in this area has increased significantly over the last few years.

Apart from the application in three-dimensional elasticity, the state variable equation method has been used also for the solution of two-dimensional plate bending problems by Forsberg (1964), Khdeir *et al.* (1989), Librescu *et al.* (1989), Khdeir and Reddy (1990), Nosier and Reddy (1992) and Timarci and Soldatos (1994), where the transverse deflection and up to its fourth order differential are used as the state variables.

Other applications of the state variable equation in elasticity include the state space approach to generalised thermo-elasticity by Anwar and Sherief (1988) where a thermal shock problem in a half-space domain was considered and the state variables were temperature, displacements and their gradients (also see, for example, Bahar and Hetnarski, 1978, 1980).

As a primary objective of this book, from the next chapter we will be focusing on the state variable equations of laminated composite plates and shells.

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