Noncommutative Dynamics and E-semigroups

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Preface

These days, the term Noncommutative Dynamics has several interpretations. It is used in this book to refer to a set of phenomena associated with the dynamical evolution of quantum systems of the simplest kind that involve rigorous mathematical structures associated with infinitely many degrees of freedom. The dynamics of such a system is represented by a one-parameter group of automorphisms of a noncommutative algebra of observables, and we focus primarily on the most concrete case in which that algebra consists of all bounded operators on a Hilbert space.

If one introduces a natural causal structure into such a dynamical system, then a pair of one-parameter semigroups of endomorphisms emerges, and it is useful to think of this pair as representing the past and future with respect to the given causality. These are both E_0-semigroups, and to a great extent the problem of understanding such causal dynamical systems reduces to the problem of understanding E_0-semigroups. The nature of these connections is discussed at length in Chapter 1. The rest of the book elaborates on what the author sees as the important aspects of what has been learned about E_0-semigroups during the past fifteen years. Parts of the subject have evolved into a satisfactory theory with effective tools; other parts remain quite mysterious.

Like von Neumann algebras, E_0-semigroups divide naturally into three types: I, II, III. The type I examples are now known to be classified to cocycle conjugacy by their numerical index. It is also known that examples of type II and III exist in abundance (there are uncountably many cocycle conjugacy classes of each type), but we are a long way from a satisfactory understanding: we have surely not seen all the examples of type II or III, and we still lack effective cocycle conjugacy invariants for distinguishing between the ones we have seen.

This subject makes significant contact with several areas of current interest, including quantum field theory, the dynamics of open quantum systems, and probability theory, both commtative and noncommutative. Indeed, Powers' first examples of type III E_0-semigroups were based on a construction involving quasi-free states of the C^*-algebra associated with the infinite-dimensional canonical anticommutation relations. More recently, the product systems constructed by Tsirelson are based on subtle properties of "noises" of various types, both Gaussian and non-Gaussian, that bear some relation to Brownian motion and white noise. When combined with appropriate results from the theory of E_0-semigroups, the examples of product systems based on Bessel processes give rise to a continuum of examples of E_0-semigroups of type II, and an E_0-semigroup that cannot be paired with itself. The Tsirelson–Vershik product systems discussed in Chapter 14 lead to a continuum of type III examples that are mutually non-cocycle-conjugate.

It appears to me that the current state of knowledge about these matters can be likened to the state of knowledge of von Neumann algebras in the late sixties, in the period of time after Powers' proof that there are uncountably many nonisomorphic type III factors but before the revolutionary developments of the seventies, which began with the discovery, based on the Tomita–Takesaki theory, that a type III factor is an object that carries with it an intrinsic dynamical group, and culminated with Connes's classification of amenable factors. I believe that there are exciting developments in the future of E_0-semigroups as well.

The book contains new material as well as reformulations of results scattered throughout the literature. For example, we have based our discussion of dilation theory on certain aspects of noncommutative dynamics that are common to all dynamical systems, allowing us to deduce the existence of dilations of quantum dynamical semigroups from very general considerations involving continuous free products of C^*-algebras. We have freed the discussion of the interaction inequality of Chapter 12 from the context of semigroups of endomorphisms in order to place it in an appropriate general context, in which the central result becomes an assertion about the convergence of eigenvalue lists along a tower of type I factors in $\mathcal{B}(H)$. Chapter 13 contains a technically complete discussion of Powers' examples of type III E_0-semigroups that brings out the role of Toeplitz and Hankel operators and quasi–continuous functions, and provides a new concrete criterion for the absence of units. Finally, the theory of spectral C^*-algebras presented in Chapter 4 has been simplified and rewritten from scratch.

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Contents

CHAPTER 1

Dynamical Origins

In this chapter we give an overview of some applications of the theory of semigroups of endomorphisms of type I factors to certain concrete issues of noncommutative dynamics that are associated with the time evolution of quantum systems. The chapter is expository in nature—discussions of all technical issues being postponed—in order to provide a context for the general theory that is developed throughout the sequel. A systematic development of the material described here can be found in Part 4.

1.1. The Flow of Time in Quantum Theory

We begin with a discussion some basic aspects of probability theory, focusing on the limitations of the classical view of stationary stochastic processes as families of random variables. The observables of probability theory are random variables, that is to say, real-valued measurable functions defined on a probability space $(\Omega, \mathcal{F}, P)$. Every random variable $X : \Omega \to \mathbb{R}$ gives rise to a probability measure μ_X defined on the Borel sets of the real line, by using X to push forward the ambient probability measure P,

$$\mu_X(S) = P\{\omega \in \Omega : X(\omega) \in S\},$$

S being a Borel subset of $\mathbb{R}$. It is the probability measure μ_X that governs the statistics of repeated observations of X. If one makes many repeated samplings of X, one finds that the probability of finding that a particular measurement lies in an interval $I = (a, b)$ is approximately $\mu_X(I)$. Similarly, the expected value of any bounded measurable function of X can be computed in terms of μ_X via the formula

$$E(f(X)) = \int_{-\infty}^{\infty} f(t)\, d\mu_X(t),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function and $f(X)$ denotes the random variable $\omega \mapsto f(X(\omega))$.

Similarly, any n-tuple of random variables $\bar{X} = (X_1, \dots, X_n)$ gives rise to a probability measure $\mu_{\bar{X}}$ defined on Borel subsets S of $\mathbb{R}^n$ by

$$\mu_{\bar{X}}(S) = P\{\omega \in \Omega : (X_1(\omega), \dots, X_n(\omega)) \in S\}.$$

The measure $\mu_{\bar{X}}$ is called the *joint distribution* of $\bar{X}$. Two n-tuples of random variables $\bar{X} = (X_1, \dots, X_n)$ and $\bar{Y} = (Y_1, \dots, Y_n)$ (perhaps acting on two different probability spaces) are considered equivalent if they have the same joint distribution: $\mu_{\bar{X}} = \mu_{\bar{Y}}$. This simply means that $\bar{X}$ and $\bar{Y}$ carry the same statistical information.

The simplest example of dynamical behavior in probability theory is described as follows. A *stationary stochastic process* is a family of random variables $\{X_t : t \in \mathbb{R}\}$ defined on a common probability space $(\Omega, \mathcal{F}, P)$ whose joint distributions are translation-invariant in the following sense: For every $(t_1, \dots, t_n) \in \mathbb{R}^n$ and every $\lambda \in \mathbb{R}$, the joint distributions of $(X_{t_1+\lambda}, \dots, X_{t_n+\lambda})$ and $(X_{t_1}, \dots, X_{t_n})$ are the same. One also requires that the process should be continuous in the time parameter in the natural sense, one formulation of which is that for every bounded measurable function $f : \mathbb{R} \to \mathbb{R}$, the random variable $f(X_t)$ should move continuously in the L^2 norm in that for every $t_0 \in \mathbb{R}$,

$$\lim_{t \to t_0} \int_\Omega |f(X_t) - f(X_{t_0})|^2 \, dP = 0.$$

Two stationary processes $\{X_t : t \in \mathbb{R}\}$ and $\{Y_t : t \in \mathbb{R}\}$ are considered isomorphic if for any n and any n-tuple $t_1, \dots, t_n \in \mathbb{R}$, $(X_{t_1}, \dots, X_{t_n})$ and $(Y_{t_1}, \dots, Y_{t_n})$ have the same joint distribution.

It is a nontrivial fact that stationary stochastic processes can always be obtained from a one-parameter group of measure preserving transformations. In more detail, let $(\Omega, \mathcal{F}, P)$ be a probability space with the special feature that Ω is a standard Borel space, $\mathcal{F}$ is the σ-algebra of all Borel subsets of Ω, and P is a probability measure. By a flow on Ω we mean a one parameter group $\{T_t : t \in \mathbb{R}\}$ of Borel isomorphisms $T_t : \Omega \to \Omega$ that is jointly measurable in that

$$(t, \omega) \in \mathbb{R} \times \Omega \mapsto T_t \omega \in \Omega$$

should be a measurable map. A flow is measure-preserving if for every $t \in \mathbb{R}$ and every $S \in \mathcal{F}$ we have $P(T_t(S)) = P(S)$. If one is given a measure-preserving flow $\{T_t : t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, P)$ and a fixed random variable $X : \Omega \to \mathbb{R}$, one can define an associated stationary random process $\{X_t : t \in \mathbb{R}\}$ as follows:

$$X_t(\omega) = X(T_t \omega), \qquad t \in \mathbb{R}, \quad \omega \in \Omega.$$

The fact we alluded to above is that every stationary stochastic process is isomorphic to one obtained in this way.

What is important here is that processes obtained by the above construction have measurable sample paths. More precisely, for every $\omega \in \Omega$ there is an associated sample path, namely, the real-valued Borel function $\hat{\omega} : \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{\omega}(t) = X_t(\omega) = X(T_t \omega), \qquad t \in \mathbb{R}.$$

It is also true that one may arrange things in such a way that the sample paths distinguish points of Ω in the sense that $\hat{\omega}_1 = \hat{\omega}_2 \implies \omega_1 = \omega_2$ for every $\omega_1, \omega_2 \in \Omega$. In effect, this realizes the stochastic process in function-theoretic terms as a probability measure defined on a space of measurable functions of a real variable. Since we do not require the details of this realization, we will not pursue it here.

However, we do want to point out that the above description of stationary stochastic processes is inadequate for important examples such as white noise, whose "sample paths" cannot be realized as functions but merely as distributions. In order to give a precise definition of stationary random distributions such as white noise, it is necessary to reformulate the idea of a stochastic process as follows.

Consider the Schwartz space $\mathcal{S}$ of all smooth rapidly decreasing real-valued functions of a real variable (which one thinks of as time). The dual of $\mathcal{S}$ is the space of all tempered distributions on $\mathbb{R}$, and we write it as Ω. Being the dual of a separable σ-normed Fréchet space, Ω has a natural weak*-topology, and this topology determines a natural σ-algebra of subsets $\mathcal{F}$ of Ω, with respect to which $(\Omega, \mathcal{F})$ becomes a standard Borel space. A fundamental theorem of Minlos [**GV64**] asserts that for every continuous positive definite function $\phi : \mathcal{S} \to \mathbb{C}$ satisfying $\phi(0) = 1$, there is a unique probability measure P defined on $(\Omega, \mathcal{F})$ such that

$$\phi(f) = \int_\Omega e^{i\omega(f)}\, dP(\omega), \qquad f \in \mathcal{S}.$$

An example of such a function ϕ is the Gaussian characteristic function

$$\phi(f) = e^{-\int_{\mathbb{R}} |f(t)|^2\, dt}, \qquad f \in \mathcal{S}.$$

White noise is defined as the probability space $(\Omega, \mathcal{F}, P)$ obtained from the measure P associated with this characteristic function by way of Minlos' theorem.

The additive group $\mathbb{R}$ acts naturally as translation operators on the function space $\mathcal{S}$, and hence on its dual Ω. Moreover, since the characteristic function ϕ above is obviously invariant under time translations, so is the probability measure P. Thus we have a natural action of the additive group of $\mathbb{R}$ as measure-preserving transformations of this probability space $(\Omega, \mathcal{F}, P)$. In this sense, white noise is a stationary random distribution. A "sample path" of white noise is simply a distribution $\omega \in \Omega$, chosen according to the dictates of the probability measure P.

There is no sharp time value of this random process that corresponds to $X_t(\omega)$ in the classical setting. However, with every open interval $(a, b) \subseteq \mathbb{R}$ one can associate an *algebra* of bounded random variables. Indeed, in order to define this algebra one considers the space of all test functions $f \in \mathcal{S}$ that have compact support in (a, b), and every such function f is associated with a bounded random variable

$$X_f(\omega) = e^{i\omega(f)}, \qquad \omega \in \Omega.$$

The weak*-closed subalgebra of $L^\infty(\Omega, \mathcal{F}, P)$ generated by these functions X_f is an abelian von Neumann algebra $\mathcal{A}_{(a,b)}$. It is easy to verify that for white noise the algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ are probabilistically independent when the intervals I and J are disjoint. Moreover, the action of translation by t carries the algebra $\mathcal{A}_I$ to A_{I+t} for every interval $I = (a, b)$ and every $t \in \mathbb{R}$. These algebras $\{\mathcal{A}_I : I = (a, b) \subseteq \mathbb{R}\}$ are the proper replacement for the family of random variables $\{X_t : t \in \mathbb{R}\}$ in the classical theory of stationary stochastic processes. Typically—and in particular for the case of white noise—the intersection of the algebras corresponding to all neighborhoods of a fixed point $t_0 \in \mathbb{R}$ will be the trivial one-dimensional subalgebra of constant functions in $L^\infty(\Omega, \mathcal{F}, P)$. In this precise sense, sharp time values of such random distributions do not exist.

On the other hand, for every $t \in \mathbb{R}$ there is a subalgebra $\mathcal{A}_{(-\infty,t)}$ of $L^\infty(\Omega, \mathcal{F}, P)$ that represents the "past" of white noise up to time t. The union of these algebras as t varies is weak*-dense in $L^\infty(\Omega, \mathcal{F}, P)$, and their intersection is the trivial subalgebra of constant functions.

We now recall the basic mathematical setting of quantum physics. The observables of quantum theory are self-adjoint operators acting on a separable Hilbert space H. Observables such as linear or angular momentum arise as generators of one-parameter unitary groups and are therefore unbounded and only densely defined. However, there is no essential loss in restricting attention to bounded functions of these unbounded operators, much as we did above in the probabilistic setting. Here, of course, one has to make use of the spectral theorem in order to define bounded functions of unbounded self-adjoint operators. We do so freely.

The quantum analogue of the distribution of a random variable requires specifying not only an observable X but also a unit vector $\xi \in H$. Once X and ξ are fixed, there is a unique probability measure $\mu_{X,\xi}$ defined on the real line by specifying its integral with respect to bounded Borel functions $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$\int_{-\infty}^{\infty} f(t)\, d\mu_{X,\xi}(t) = \langle f(X)\xi, \xi \rangle .$$

For a Borel set $S \subseteq \mathbb{R}$, one thinks of $\mu_{X,\xi}(S)$ as representing the probability of finding an observed value of X in the set S, given that the system is in the pure state associated with ξ.

Given two observables X, Y and a unit vector ξ, there is no "joint distribution" $\mu_{X,Y,\xi}$ defined on $\mathbb{R}^2$. On the level of physics, this phenomenon is associated with the theory of measurement and is a consequence of the uncertainty principle. From the point of view of operator theory, since the operators X and Y normally fail to commute, there is no way of using them to define a spectral measure on $\mathbb{R}^2$. This nonexistence of joint distributions is one of the fundamental differences between quantum theory and probability theory.

Turning now to dynamics, consider the way the flow of time acts on the algebra of observables. Every symmetry of quantum theory corresponds to either a $*$-automorphism or a $*$-antiautomorphism of the algebra $\mathcal{B}(H)$ of all bounded operators on H. If we are given a one-parameter group of such symmetries, then since each one of them is the square of another, it follows that all of the symmetries must be $*$-automorphisms. Thus, the flow of time on a quantum sytem is given by a one-parameter family $\alpha = \{\alpha_t : t \in \mathbb{R}\}$ of automorphisms of $\mathcal{B}(H)$ such that $\alpha_s \circ \alpha_t = \alpha_{s+t}$, and which satisfies the natural continuity condition: For every $A \in \mathcal{B}(H)$ and every pair of vectors $\xi, \eta \in H$ the function $t \in \mathbb{R} \mapsto \langle \alpha_t(A)\xi, \eta \rangle$ is continuous.

Let us consider the possibilities: How does one classify one-parameter groups of automorphisms of $\mathcal{B}(H)$? In the late 1930s, Eugene Wigner proved that every such one-parameter group is implemented by a strongly continuous one-parameter unitary group $U = \{U_t : t \in \mathbb{R}\}$ in the sense that

$$\alpha_t(A) = U_t A U_t^*, \qquad A \in \mathcal{B}(H), \quad t \in R.$$

Earlier, Marshall Stone had shown that a strongly continuous one-parameter unitary group U is the Fourier transform of a unique spectral measure E defined on the Borel subsets of the real line as follows:

$$U_t = \int_{-\infty}^{\infty} e^{i\lambda t}\, dE(\lambda).$$

Equivalently, Stone's theorem implies that for the unbounded self-adjoint operator $X = \int_{\mathbb{R}} \lambda\, dE(\lambda)$, we have $U_t = e^{itX}$. Thus Wigner's result implies that every one-parameter group α of automorphisms of $\mathcal{B}(H)$ corresponds to an observable X as follows:

$$\alpha_t(A) = e^{itX} A e^{-itX}, \qquad A \in \mathcal{B}(H), \quad t \in \mathbb{R}.$$

The operator X is not uniquely determined by the group α, since replacing X with a scalar translate of the form $X + \lambda\mathbf{1}$ with $\lambda \in \mathbb{R}$ does not change α. However, X is uniquely determined by α up to such scalar perturbations.

Two one-parameter groups α and β of $*$-automorphisms (acting on $\mathcal{B}(H)$ and $\mathcal{B}(K)$ respectively) are said to be *conjugate* if there is a $*$-isomorphism θ of $\mathcal{B}(H)$ on $\mathcal{B}(K)$ such that

$$\theta(\alpha_t(A)) = \beta_t(\theta(A)), \qquad A \in \mathcal{B}(H), \quad t \in \mathbb{R}.$$

Recalling that such a $*$-isomorphism θ must be implemented by a unitary operator $W : H \to K$ by way of $\theta(A) = WAW^*$, we see that Wigner's theorem completely settles the classification issue for one-parameter groups of automorphisms of $\mathcal{B}(H)$. Indeed, using that result we may find unbounded self-adjoint operators X, Y on the respective Hilbert spaces such that α and β are given by $\alpha_t(A) = e^{itX}Ae^{-itX}$ and $\beta_t(B) = e^{itY}Be^{-itY}$, $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$, $t \in \mathbb{R}$. It is an elementary—though nontrivial—exercise to show that $\theta(A) = WAW^*$ implements a conjugacy of α and β if and only if there is a real scalar λ such that $WXW^* = Y + \lambda\mathbf{1}$. Thus, the classification of one-parameter groups of automorphisms is reduced to the classification of unbounded self-adjoint operators up to unitary equivalence. By the spectral theorem, this is equivalent to the classification up to unitary equivalence of spectral measures on the real line; and the latter problem is completely understood in terms of the multiplicity theory of Hahn and Hellinger [**Arv98**].

These remarks imply that the most basic aspect of quantum dynamics, namely the structure and classification of one-paramter groups of $*$-automorphisms of $\mathcal{B}(H)$, is completely understood. We have seen all of the possibilities, and they are described by self-adjoint operators (or spectral measures on the line) and their multiplicity theory in an explicit way.

1.2. Causality and Interactions

We now show that by introducing a natural notion of causality into such dynamical systems, one encounters entirely new phenomena. These "causal" dynamical systems acting on $\mathcal{B}(H)$ are only partially understood. We have surely not seen all of them, and we have only partial information about how to classify the ones we have seen.

We are concerned with one-parameter groups of $*$-automorphisms of the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H that carry a particular kind of causal structure. More precisely, A *history* is a pair (U, M) consisting of a one-parameter group $U = \{U_t : t \in \mathbb{R}\}$ of unitary operators acting on a separable infinite-dimensional Hilbert space H, together with a type I subfactor $M \subseteq \mathcal{B}(H)$ that is invariant under the automorphisms $\gamma_t(X) = U_t X U_t^*$ for negative t and that has the following two properties

(i) (irreducibility)

$$\left(\bigcup_{t\in\mathbb{R}} \gamma_t(M)\right)'' = \mathcal{B}(H),$$

(ii) (trivial infinitely remote past)

$$\bigcap_{t\in\mathbb{R}} \gamma_t(M) = \mathbb{C}\cdot\mathbf{1}.$$

We find it useful to think of the group $\{\gamma_t : t \in \mathbb{R}\}$ as representing the flow of time in the Heisenberg picture, and the von Neumann algebra M as representing bounded observables that are associated with the "past". However, we focus attention on purely mathematical issues concerning the dynamical properties of histories, with problems concerning their existence and construction, and especially with the issue of nontriviality (to be defined momentarily). Two histories (U, M) (acting on H) and $(\tilde{U}, \tilde{M})$ (acting on $\tilde{H}$) are said to be *isomorphic* if there is a $*$-isomorphism $\theta : \mathcal{B}(H) \to \mathcal{B}(K)$ such that $\theta(M) = \tilde{M}$ and $\theta \circ \gamma_t = \tilde{\gamma}_t \circ \theta$ for every $t \in \mathbb{R}$, γ,

$\tilde{\gamma}$ denoting the automorphism groups associated with U, $\tilde{U}$. The basic problems addressed in this book all bear some relation to the problem of classifying histories. We have already alluded to the fact that the results are far from complete.

An E_0-*semigroup* is a one-parameter semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of unit-preserving $*$-endomorphisms of a type I_∞ factor M, which is continuous in the natural sense. The subfactors $\alpha_t(M)$ decrease as t increases, and α is called *pure* if $\cap_t \alpha_t(M) = \mathbb{C}\mathbf{1}$. There are two E_0-semigroups α^-, α^+ associated with any history, α^- being the one associated with the "past" by restricting γ_{-t} to M for $t \geq 0$ and α^+ being the one associated with the "future" by restricting γ_t to the commutant M' for $t \geq 0$.

By an *interaction* we mean a history with the additional property that there are normal states ω_-, ω_+ of M, M', respectively, such that ω_- is invariant under the action of α^- and ω_+ is invariant under the action of α^+. Both α^- and α^+ are pure E_0-semigroups, and when a pure E_0-semigroup has a normal invariant state, then that state is uniquely determined; see Section 2.9 below. Thus ω_- (resp. ω_+) is the unique normal invariant state of α^- (resp. α^+).

In particular, it follows from this uniqueness that if one is given two interactions (U, M) and $(\tilde{U}, \tilde{M})$ with respective pairs of normal states ω_+, ω_- and $\tilde{\omega}_+$, $\tilde{\omega}_-$, then an isomorphism of histories $\theta : (U, M) \to (\tilde{U}, \tilde{M})$ must associate $\tilde{\omega}_+$, $\tilde{\omega}_-$ with ω_+, ω_- in the sense that if θ_+ (resp. θ_-) denotes the restriction of θ to M (resp. M'), then one has $\tilde{\omega}_\pm \circ \theta_\pm = \omega_\pm$.

Remark 1.2.1. Since the state space of a unital C^*-algebra is weak*-compact, the Markov–Kakutani fixed point theorem implies that every E_0-semigroup has invariant states. But there is no reason to expect that there is a *normal* invariant state. Indeed, there are examples of pure E_0-semigroups that have no normal invariant states (see Theorem 7.3.4 and Proposition 8.11.1). Notice too that ω_-, for example, is defined *only* on the algebra M of the past. Of course, ω_- has many extensions to normal states of $\mathcal{B}(H)$, but none of these normal extensions need be invariant under the action of the group γ. In fact, we will see that if there is a *normal* γ-invariant state defined on all of $\mathcal{B}(H)$, then the interaction must be trivial.

In order to discuss the dynamics of interactions we must introduce a C^*-algebra of "local observables." For every compact interval $[s, t] \subseteq \mathbb{R}$ there is an associated von Neumann algebra

$$\mathcal{A}_{[s,t]} = \gamma_t(M) \cap \gamma_s(M)'. \tag{1.1}$$

Notice that since $\gamma_s(M) \subseteq \gamma_t(M)$ are both type I factors, so is the relative commutant $\mathcal{A}_{[s,t]}$. It is clear that $\mathcal{A}_I \subseteq \mathcal{A}_J$ if $I \subseteq J$, and for adjacent intervals $[r, s]$, $[s, t]$, $r \leq s \leq t$, we have

$$\mathcal{A}_{[r,t]} = \mathcal{A}_{[r,s]} \otimes \mathcal{A}_{[s,t]}, \tag{1.2}$$

in the sense that the two factors $\mathcal{A}_{[r,s]}$ and $\mathcal{A}_{[s,t]}$ mutually commute and generate $\mathcal{A}_{[r,t]}$ as a von Neumann algebra. The automorphism group γ permutes the algebras $\mathcal{A}_I$ covariantly,

$$\gamma_t(\mathcal{A}_I) = \mathcal{A}_{I+t}, \qquad t \in \mathbb{R}. \tag{1.3}$$

Finally, we define the local C^*-algebra $\mathcal{A}$ to be the *norm* closure of the union of all the $\mathcal{A}_I$, $I \subseteq \mathbb{R}$. The algebra $\mathcal{A}$ is strongly dense in $\mathcal{B}(H)$, and it is invariant under the action of the automorphism group γ.

REMARK 1.2.2. It may be of interest to compare the local structure of the C^*-algebra $\mathcal{A}$ to its commutative counterpart, namely, the local algebras associated with a stationary random distribution with independent values at every point [**GV64**]. More precisely, suppose that we are given a random distribution ϕ, i.e., a linear map from the space of real-valued test functions on $\mathbb{R}$ to the space of real-valued random variables on some probability space (Ω, P). With every compact interval $I = [s,t]$ with $s < t$ one may consider the weak*-closed subalgebra $\mathcal{A}_I$ of $L^\infty(\Omega, P)$ generated by random variables of the form $e^{i\phi(f)}$, f ranging over all test functions supported in I. When the random distribution ϕ is stationary and has independent values at every point, this family of subalgebras of $L^\infty(\Omega, P)$ has properties analogous to (1.2) and (1.3), in that there is a one-parameter group of measure-preserving automorphisms $\gamma = \{\gamma_t : t \in \mathbb{R}\}$ of $L^\infty(\Omega, P)$ that satisfies (1.3), and instead of (1.2) we have the assertion that the algebras $\mathcal{A}_{[r,s]}$ and $\mathcal{A}_{[s,t]}$ are *probabilistically independent* and generate $\mathcal{A}_{[r,t]}$ as a weak*-closed algebra.

One should keep in mind, however, that this commutative analogy has serious limitations. For example, we have already pointed out that in the case of interactions there is typically no normal γ-invariant state on $\mathcal{B}(H)$, and there is no reason to expect any normal state of $\mathcal{B}(H)$ to decompose as a product state relative to the decompositions of (1.2).

There is also some common ground with the Boolean algebras of type I factors of Araki and Woods [**AW69**], but here too there are significant differences. For example, the local algebras of (1.1) and (1.2) are associated with intervals (and more generally with finite unions of intervals), but not with more general Borel sets as in [**AW69**]. Moreover, here the translation group acts as automorphisms of the given structure, whereas in [**AW69**] there is no assumption of "stationarity" with respect to translations.

The C^*-algebra $\mathcal{A}$ of local observables is important because it provides a way of comparing ω_- and ω_+. Indeed, both states ω_- and ω_+ extend *uniquely* to γ-invariant states $\bar{\omega}_-$ and $\bar{\omega}_+$ of $\mathcal{A}$. We sketch the proof for ω_-.

PROPOSITION 1.2.3. *There is a unique γ-invariant state $\bar{\omega}_-$ of $\mathcal{A}$ such that*

$$\bar{\omega}_- \restriction_{\mathcal{A}_I} = \omega_- \restriction_{\mathcal{A}_I}$$

for every compact interval $I \subseteq (-\infty, 0]$.

PROOF. For existence of the extension, choose any compact interval $I = [a,b]$ and any operator $X \in \mathcal{A}_I$. Then for sufficiently large $s > 0$ we have $I - s \subseteq (-\infty, 0]$ and for these values of s, $\omega_-(\gamma_{-s}(X))$ does not depend on s because ω_- is invariant under the action of $\{\gamma_t : t \leq 0\}$. Thus we can define $\bar{\omega}_-(X)$ unambiguously by

$$\bar{\omega}_-(X) = \lim_{t \to -\infty} \omega_-(\gamma_t(X)).$$

This defines a positive linear functional $\bar{\omega}_-$ on the unital $*$-algebra $\cup_I \mathcal{A}_I$, and now we extend $\bar{\omega}_-$ to all of $\mathcal{A}$ by norm-continuity. The extended state is clearly invariant under the action of γ_t, $t \in \mathbb{R}$.

The uniqueness of the state $\bar{\omega}_-$ is apparent. $\square$

It is clear from the proof of Proposition 1.2.3 that these extensions of ω_- and ω_+ are *locally normal* in the sense that their restrictions to any localized subalgebra $\mathcal{A}_I$ define normal states on that type I factor.

The local C^*-algebra $\mathcal{A}$ has a definite "state of the past" and a definite "state of the future" in the following sense.

PROPOSITION 1.2.4. *For every $X \in \mathcal{A}$ and every normal state ρ of $\mathcal{B}(H)$ we have*

$$\lim_{t\to-\infty} \rho(\gamma_t(X)) = \bar{\omega}_-(X), \qquad \lim_{t\to+\infty} \rho(\gamma_t(X)) = \bar{\omega}_+(X).$$

PROOF. Consider the first limit formula. The set of all $X \in \mathcal{A}$ for which this formula holds is clearly closed in the operator norm; hence it suffices to show that it contains $\mathcal{A}_I$ for every compact interval $I \subseteq \mathbb{R}$.

We will make use of the fact (discussed more fully in Section 2.9) that if ρ is any normal state of M and A is an operator in M, then

$$\lim_{t\to-\infty} \rho(\gamma_t(A)) = \omega_-(A);$$

see Corollary 2.9.6. If we choose a real number T sufficiently negative that $I + T \subseteq (-\infty, 0]$, the preceding remark implies that for the operator $A = \gamma_T(X) \in M$ we have $\lim_{t\to-\infty} \rho(\gamma_t(A)) = \omega_-(A)$, and hence

$$\lim_{t\to-\infty} \rho(\gamma_t(X)) = \lim_{t\to-\infty} \rho(\gamma_{t-T}(\gamma_T(X))) = \omega_-(\gamma_T(X)) = \bar{\omega}_-(X).$$

The proof of the second limit formula is similar. $\square$

DEFINITION 1.2.5. The interaction (U, M), with past and future states ω_- and ω_+, is said to be trivial if $\bar{\omega}_- = \bar{\omega}_+$.

More generally, the norm $\|\bar{\omega}_- - \bar{\omega}_+\|$ gives some measure of the "strength" of the interaction, and of course we have $0 \leq \|\bar{\omega}_- - \bar{\omega}_+\| \leq 2$.

If there is a normal state ρ of $\mathcal{B}(H)$ that is invariant under the action of γ, then since ω_- (resp. ω_+) is the unique normal invariant state of α_- (resp. α_+), we must have $\rho\restriction_M = \omega_-$, $\rho\restriction_{M'} = \omega_+$, and hence $\bar{\omega}_- = \bar{\omega}_+ = \rho\restriction_{\mathcal{A}}$ by the uniqueness part of Proposition 1.2.3. In particular, *if the interaction is nontrivial then neither $\bar{\omega}_-$ nor $\bar{\omega}_+$ can be extended from $\mathcal{A}$ to a normal state of its strong closure $\mathcal{B}(H)$.*

Thus, whatever (normal) state ρ one chooses to watch evolve over time on operators in $\mathcal{A}$, it settles down to become $\bar{\omega}_+$ in the distant future, it must have come from $\bar{\omega}_-$ in the remote past, and the limit states do not depend on the choice of ρ. For a trivial interaction, nothing happens over the long term: For fixed X and ρ the function $t \in \mathbb{R} \mapsto \rho(\gamma_t(X))$ starts out very near some value (namely, $\bar{\omega}_-(X)$), exhibits transient fluctuations over some period of time, and then settles down near the same value again. For a nontrivial interaction, there will be a definite change from the limit at $-\infty$ to the limit at $+\infty$ (for some choices of $X \in \mathcal{A}$).

A number of questions arise naturally; for example: (1) How does one determine whether a given interaction is nontrivial? (2) How does one construct examples of interactions? The first question asks for a method of computing, or at least estimating, the quantity $\|\bar{\omega}_- - \bar{\omega}_+\|$ in terms of concrete data associated with the normal states ω_- and ω_+. We provide a general solution of (1) that involves an inequality that we feel is of some interest in its own right. The results are summarized in the paragraphs to follow. We also describe an effective partial solution of (2) in Section 1.4, after outlining some of the basic theory of E_0-semigroups.

By an *eigenvalue list* we mean a decreasing sequence of nonnegative real numbers $\lambda_1 \geq \lambda_2 \geq \cdots$ with finite sum. Every normal state ω of a type I factor is associated with a positive operator of trace 1, whose eigenvalues counting multiplicity can be arranged into an eigenvalue list that will be denoted by $\Lambda(\omega)$. If the factor is finite-dimensional, we still consider $\Lambda(\omega)$ to be an infinite list by adjoining zeros in the obvious way. Given two eigenvalue lists $\Lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots\}$ and $\Lambda' = \{\lambda_1' \geq \lambda_2' \geq \cdots\}$, we will write

$$\|\Lambda - \Lambda'\| = \sum_{k=1}^{\infty} |\lambda_k - \lambda_k'|$$

for the ℓ^1-distance from one list to the other. A classical result implies that if ρ and σ are normal states of a type I factor M, then we have

$$\|\Lambda(\rho) - \Lambda(\sigma)\| \leq \|\rho - \sigma\|$$

(see Chapter 12 for more detail).

THEOREM 1.2.6 (Interaction inequality). *Let (U, M) be an interaction with past and future states ω_-, ω_+ on M, M', respectively, and let $\bar{\omega}_-$ and $\bar{\omega}_+$ denote their extensions to γ-invariant states of $\mathcal{A}$. Then*

$$\|\bar{\omega}_- - \bar{\omega}_+\| \geq \|\Lambda(\omega_- \otimes \omega_-) - \Lambda(\omega_+ \otimes \omega_+)\|.$$

REMARK 1.2.7. Theorem 1.2.6 is proved in Chapter 12. Notice the tensor product of states on the right. For example, $\Lambda(\omega_- \otimes \omega_-)$ is obtained from the eigenvalue list $\Lambda(\omega_-) = \{\lambda_1 \geq \lambda_2 \geq \cdots\}$ of ω_- by rearranging the doubly infinite sequence of all products $\lambda_i \lambda_j$, $i, j = 1, 2, \cdots$, into decreasing order. It can be an unpleasant combinatorial chore to calculate $\Lambda(\omega_- \otimes \omega_-)$ even when $\Lambda(\omega_-)$ is relatively simple and finitely nonzero; but we also show in Chapter 12 that if A and B are two positive trace class operators such that $\Lambda(A \otimes A) = \Lambda(B \otimes B)$, then $\Lambda(A) = \Lambda(B)$. Thus we have the following conclusion:

COROLLARY 1.2.8. *Let (U, M), ω_-, ω_+ be as in Theorem 1.2.6, and let Λ_- and Λ_+ be the eigenvalue lists of ω_- and ω_+, respectively. If $\Lambda_- \neq \Lambda_+$, then the interaction is nontrivial.*

Thus, if we are given two E_0-semigroups α^- and α^+ that are pure and have normal invariant states ω_- and ω_+, respectively, and *if* there exists a history having past and future semigroups conjugate respectively to α^- and α^+, then such a history will be a nontrivial interaction whenever the eigenvalue lists of ω_- and ω_+ are different. There may or may not be such a history, and we turn now to a discussion of this existence issue for such pairs of E_0-semigroups, namely, problem (2). It will be convenient to first summarize some of the basic results in the theory of E_0-semigroups.

1.3. Semigroups of Endomorphisms

The purpose of this section is to introduce some of the basic notions in the theory of E_0-semigroups without technicalities, including cocycle conjugacy, the numerical index, and the CAR/CCR flows, in order to describe the role of E_0-semigroups in noncommutative dynamics. These topics will be developed in more detail in Chapter 2.

The most obvious notion of equivalence for E_0-semigroups is conjugacy. Two E_0-semigroups α and β, acting on M and N, respectively, are *congugate* if there exists a $*$-isomorphism $\theta : M \to N$ that intertwines their actions in that $\beta_t \circ \theta = \theta \circ$

α_t for every $t \geq 0$. In case $M = \mathcal{B}(H)$ and $N = \mathcal{B}(K)$, then such isomorphisms θ are implemented by unitary operators $U : H \to K$ by way of $\theta(A) = UAU^*$, $A \in \mathcal{B}(H)$. Thus, conjugate E_0-semigroups are correctly regarded as *indistinguishable*. It is not possible to list representatives of all unitary equivalence classes of representations of C^*-algebras that are not of type I in a measurable way, and for similar reasons one cannot expect to classify E_0-semigroups up to conjugacy.

Notice that for every E_0-semigroup α acting on M, we have a decreasing one-parameter family $M_t = \alpha_t(M)$, $t \geq 0$, of type I subfactors of M. The intersection $M_\infty = \cap_{t\geq 0}\alpha_t(M)$ is called the *tail* von Neumann algebra of α. The tail von Neumann algebra may or may not be a factor, and even when it is a factor it may *not* be of type I. But in all cases $\alpha_t(M_\infty) = M_\infty$ for every $t \geq 0$; hence α acts as a semigroup of *automorphisms* of its tail von Neumann algebra. By adjoining inverses in an obvious way we obtain a one-parameter group or automorphisms of the tail von Neumann algebra M_∞. This W*-dynamical system is an important conjugacy invariant of α; unfortunately, we know very little about what the possibilities are. An E_0-semigroup α is called *pure* if its tail von Neumann algebra is trivial: $\cap_{t\geq 0}\alpha_t(M) = \mathbb{C}\mathbf{1}$. Pure E_0-semigroups will be the focus of much of the following discussion.

The useful notion of equivalence for E_0-semigroups is cocycle conjugacy, described as follows. Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on M. A cocycle for α is a family of unitary operators $U = \{U_t : t \geq 0\}$ in M that is strongly continuous in the parameter t and satisfies the cocycle equation

$$U_{s+t} = U_s\alpha_t(U_t), \qquad s, t \geq 0. \tag{1.4}$$

Equation (1.4) implies that the family of endomorphisms

$$\beta_t(A) = U_t\alpha_t(A)U_t^*, \qquad t \geq 0, \quad A \in \mathcal{B}(H),$$

also satisfies the semigroup property $\beta_{s+t} = \beta_s \circ \beta_t$; hence $\beta = \{\beta_t : t \geq 0\}$ is another E_0-semigroup. Such a β is called a *cocycle perturbation* of α. Two E_0-semigroups α, β acting on M, N, respectively, are said to be *cocycle conjugate* if β is conjugate to a cocycle perturbation of α.

The fundamental problem in the theory of E_0-semigroups is their classification up to cocycle conjugacy. We will develop methods to deal with this problem, and give solutions in special cases. But it remains very much an open problem. For example, it is undoubtedly true that every E_0-semigroup can be perturbed by a cocycle into a pure E_0-semigroup, but this has not been proved in general. See Remark 4.10.4 below.

We now describe the index invariant of E_0-semigroups. The index is an integer (or a generalized integer), it is stable under cocycle conjugacy, it can be computed for concrete examples, and it is defined as follows. Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting concretely on $\mathcal{B}(H)$. By a *unit* for α we mean a semigroup of bounded operators $T = \{T_t : t \geq 0\}$ on H that is continuous in the strong operator topology, satisfies $T_0 = \mathbf{1}$, and obeys

$$\alpha_t(A)T_t = T_tA, \qquad t \geq 0, \quad A \in \mathcal{B}(H). \tag{1.5}$$

Notice that the trivial semigroup $T_t = 0$, $t \geq 0$, does not qualify as a unit for α. The set of all units of α is written $\mathcal{U}_\alpha$.

It is a notable fact that there are E_0-semigroups α that have no units. However, assuming that $\mathcal{U}_\alpha \neq \emptyset$, we construct a Hilbert space associated with $\mathcal{U}_\alpha$ as follows.

We claim that there is a unique function $c : \mathcal{U}_\alpha \times \mathcal{U}_\alpha \to \mathbb{C}$ such that

$$e^{c(S,T)t}\mathbf{1} = T_t^* S_t, \qquad S, T \in \mathcal{U}_\alpha, \quad t \geq 0. \tag{1.6}$$

Indeed, if $S = \{S_t : t \geq 0\}$ and $T = \{T_t : t \geq 0\}$ are two units of α, then for every $t \geq 0$ and for every operator $A \in \mathcal{B}(H)$, (1.5) implies

$$T_t^* S_t A = T_t^* \alpha_t(A) S_t = (\alpha_t(A^*) T_t)^* S_t = (T_t A^*)^* S_t = A T_t^* S_t,$$

and therefore $T_t^* S_t$ commutes with all operators on H. It follows that there is a unique complex number $f(t)$ such that

$$T_t^* S_t = f(t)\mathbf{1}.$$

Obviously, $f : [0, \infty) \to \mathbb{C}$ is continuous, it satisfies $f(0) = 1$, and a straightforward computation shows that $f(s+t) = f(s)f(t)$ for all $s, t \geq 0$. It follows that there is a unique complex number $c(S,T)$ such that

$$f(t) = e^{c(S,T)t}, \qquad t \geq 0.$$

The function $c : \mathcal{U}_\alpha \times \mathcal{U}_\alpha \to \mathbb{C}$ is called the *covariance function* of the E_0-semigroup α. The defining equation (1.6) implies that $e^{tC(\cdot,\cdot)}$ is a positive definite function for every $t > 0$, and that $\overline{c(S,T)} = c(T,S)$. It follows that the covariance function is conditionally positive definite.

Thus, there is a Hilbert space associated with the covariance function, and the index of α is defined as the dimension of this Hilbert space. In more detail, let $\mathbb{C}_0(\mathcal{U}_\alpha)$ denote the vector space of all complex functions $\lambda : \mathcal{U}_\alpha \to \mathbb{C}$ that vanish off some finite subset of $\mathcal{U}_\alpha$ and sum to zero in the sense that

$$\sum_{x \in \mathcal{U}_\alpha} \lambda(x) = 0.$$

Since the covariance function is conditionally positive definite, it defines a positive semidefinite inner product $\langle \,, \rangle$ on $\mathbb{C}_0(\mathcal{U}_\alpha)$ by way of

$$\langle \lambda, \mu \rangle = \sum_{x,y \in \mathcal{U}_\alpha} c(x,y)\lambda(x)\overline{.\mu(y)}.$$

After passing to an appropriate quotient vector space one obtains an inner product space, and the completion of the latter is a Hilbert space $H(\mathcal{U}_\alpha)$.

Definition 1.3.1. Let α be an E_0-semigroup. If $\mathcal{U}_\alpha \neq \emptyset$, then the index of α is defined as $\operatorname{ind}(\alpha) = \dim(H(\mathcal{U}_\alpha))$. In case $\mathcal{U}_\alpha = \emptyset$, the index of α is defined as the cardinality of the continuum.

We will see in Chapter 3 that when $\mathcal{U}_\alpha \neq \emptyset$, the Hilbert space $H(\mathcal{U}_\alpha)$ is either finite-dimensional, or infinite-dimensional and separable. Thus the possible values of $\operatorname{ind}(\alpha)$ are $0, 1, 2, \ldots, \infty = \aleph_0$, together with the uncountable exceptional value $2^{\aleph_0}$ associated with the case $\mathcal{U}_\alpha = \emptyset$. It is a straightforward exercise to verify that the index is stable under cocycle perturbations, and thus *the index defines a numerical invariant for cocycle conjugacy.*

The key property of the index is its additivity with respect to tensor products. Given two E_0-semigroups α and β acting, respectively, on $\mathcal{B}(H)$ and $\mathcal{B}(K)$, and given $t \geq 0$, there is a unique endomorphism of $\mathcal{B}(H \otimes K)$ that carries $A \otimes B$ to $\alpha_t(A) \otimes \beta_t(B)$, $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$, and these endomorphisms give rise to an E_0-semigroup $\alpha \otimes \beta$ acting on $\mathcal{B}(H \otimes K)$. We will show in Chapter 3 that

$$\operatorname{ind}(\alpha \otimes \beta) = \operatorname{ind}(\alpha) + \operatorname{ind}(\beta). \tag{1.7}$$

Formula (1.7) makes the following assertions. If $\mathcal{U}_{\alpha\otimes\beta}$ is empty, then either $\mathcal{U}_\alpha$ or $\mathcal{U}_\beta$ is empty, and both sides of (1.7) are the cardinality of the continuum. If $\mathcal{U}_{\alpha\otimes\beta}$ is nonempty, then so are both $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$; the indices of all three E_0-semigroups take values in the set $\{0, 1, 2, \dots, \infty = \aleph_0\}$; and (1.7) has an obvious meaning.

Finally, we briefly describe the simplest examples of E_0-semigroups and their indices, the CAR/CCR flows of index n. These examples can be defined in terms of the canonical commutation relations, or in terms of the canonical anticommutation relations, or in terms of Gaussian random processes, according to taste. For this discussion we use the commutation relations as follows. Let n be a positive integer or $\infty = \aleph_0$, let C be a Hilbert space of dimension n, and consider $H = L^2([0,\infty)) \otimes C$. Viewing elements of H as vector-valued functions $\xi : x \in [0,\infty) \mapsto \xi(x) \in C$, let $S = \{S_t : t \geq 0\}$ be the natural shift semigroup

$$S_t\xi(x) = \begin{cases} \xi(x-t), & x \geq t, \\ 0, & 0 < x < t. \end{cases}$$

Then S is a strongly continuous semigroup of isometries in $\mathcal{B}(H)$ having multiplicity n, which is *pure* in the sense that $\cap_{t\geq 0} S_t H = \{0\}$.

Using bosonic second quantization, we construct an E_0-semigroup out of the operator semigroup S as follows. A *Weyl system* over H is a strongly continuous function $\xi \in H \mapsto W_\xi \in \mathcal{B}(K)$ from H with its metric topology to unitary operators on some Hilbert space K that satisfies the canonical commutation relations in Weyl's form

$$W_\xi W_\eta = e^{i\omega(\xi,\eta)} W_{\xi+\eta}, \qquad \xi, \eta \in H, \tag{1.8}$$

where ω is the symplectic form on $H \times H$ associated with the imaginary part of the inner product

$$\omega(\xi, \eta) = \Im\langle \xi, \eta\rangle.$$

The Fock representation of the commutation relations is uniquely characterized as a Weyl system $W = \{W_\xi : \xi \in H\}$ acting on K with the property that there is a vacuum vector: a unit vector $v \in K$ with the property

$$\langle W_\xi v, v\rangle = e^{-\|\xi\|^2}, \qquad \xi \in H,$$

and that is cyclic in the sense that K is the closed linear span of $\{W_\xi v : \xi \in H\}$.

One can write down the Fock representation of the CCRs in a very explicit way (see Section 2.1), but we will not have to do so for this discussion. This Weyl system is irreducible in that $\{W_\xi : \xi \in H\}' = \mathbb{C}\mathbf{1}$, and hence the set of all finite linear combinations of the Weyl operators is a dense $*$-subalgebra of $\mathcal{B}(K)$. There is a necessarily unique E_0-semigroup α acting on $\mathcal{B}(K)$ that is defined by its action on the Weyl operators as follows

$$\alpha_t(W_\xi) = W_{S_t\xi}, \qquad \xi \in H, \quad t \geq 0.$$

This E_0-semigroup is called *the CAR/CCR flow of index* $n = 1, 2, \dots, \infty$.

This terminology is justified by the fact that the index of a CAR/CCR flow can be calulated, and is the multiplicity $\dim C$ of the shift semigroup $\{S_t : t \geq 0\}$ acting on the one-particle space $H = L^2([0,\infty)) \otimes C$. These calculations are carried out in Section 2.6 below. Moreover, the general results of Chapter 6 imply that *all* of the most tractible E_0-semigroups must be conjugate to cocycle perturbations of CAR/CCR flows, and are therefore completely classified up to cocycle conjugacy by their numerical index in the sense that two such E_0-semigroups α, β are cocycle conjugate iff $\mathrm{ind}(\alpha) = \mathrm{ind}(\beta)$. It is fair to say that these are the only E_0-semigroups that are well understood.

1.4. Existence of Dynamics

We now summarize results on the existence of histories and interactions with specified properties, and how they are constructed from pairs of E_0-semigroups.

Flows on spaces are described infinitesimally by vector fields. Flows on Hilbert spaces—that is to say, one-paramter unitary groups - are described infinitesimally by unbounded self-adjoint operators. In practice, one is usually presented with a symmetric operator A that is not known to be self-adjoint, much like being presented with a differential equation that is not known to posses solutions for all time, and one wants to know if there is a one-paramter unitary group that can be associated with it. More precisely, one wants to know if A can be *extended* to a self-adjoint operator.

This problem of the existence of dynamics was solved by von Neumann as follows. Every densely defined symmetric operator A has an adjoint A^* with dense domain $\mathcal{D}^*$, and using A^* one defines two *deficiency spaces* $\mathcal{E}_-$, $\mathcal{E}_+$ by

$$\mathcal{E}_\pm = \{\xi \in \mathcal{D}^* : A^*\xi = \pm i\xi\}.$$

von Neumann's result is that A has self-adjoint extensions iff $\dim \mathcal{E}_- = \dim \mathcal{E}_+$ (see Section XII.4 of [**DS58**]). Moreover, when $\mathcal{E}_-$ and $\mathcal{E}_+$ have the same dimension, von Neumann also showed that for every unitary operator from $\mathcal{E}_-$ to $\mathcal{E}_+$ there is an associated self-adjoint extension of A, and that this association is a bijection that parameterizes the set of all self-adjoint extensions of A.

We now describe an analogous result, which locates the obstruction to the existence of dynamics for pairs of E_0-semigroups of the simplest kind in terms of their numerical index. This is a consequence of more general results, closer in spirit to the results of von Neumann cited in the preceding paragraph, that will be proved in Chapter 3.

Let M be a type I subfactor of $\mathcal{B}(H)$, and let α, β be two E_0-semigroups acting, respectively, on M and its commutant M'. We seek conditions under which there is a one-parameter unitary group $U = \{U_t : t \in \mathbb{R}\}$ acting on H whose associated automorphism group $\gamma_t(A) = U_t A U_t^*$ has α as its past and β as its future in the sense that

$$\gamma_{-t} \restriction_M = \alpha_t, \qquad \gamma_t \restriction_{M'} = \beta_t, \qquad t \geq 0. \tag{1.9}$$

It is a simple matter to write down pairs of E_0-semigroups that act, respectively, on a type I subfactor and its commutant. For example, given an arbitrary pair of E_0-semigroups α (acting on $M = \mathcal{B}(K)$) and β (acting on $N = \mathcal{B}(K)$), consider the Hilbert space $H = K \otimes L$, and the type I subfactor $\tilde{M} = \mathcal{B}(K) \otimes \mathbf{1}$. The commutant of $\tilde{M}$ is $\mathbf{1} \otimes \mathcal{B}(L)$, and the two E_0-semigroups

$$\tilde{\alpha}_t(A \otimes \mathbf{1}) = \alpha_t(A) \otimes \mathbf{1}, \qquad \tilde{\beta}_t(\mathbf{1} \otimes B) = \mathbf{1} \otimes \beta_t(B), \tag{1.10}$$

acting on $\tilde{M}$ and its commutant are conjugate, respectively, to the original pair α and β. Conversely, since every type I subfactor $M \subseteq \mathcal{B}(H)$ corresponds in this way to a decomposition $H = K \otimes L$ of H into a tensor product in which M is identified with $\mathcal{B}(K) \otimes \mathbf{1}$, formula (1.10) describes the most general way that a pair of E_0-semigroups can act on a type subfactor and its commutant.

Notice too that if we are given a pair α, β acting on M and M', respectively, with the property that there exists an automorphism group γ satisfying (1.9), then $\cap_{t<0}\gamma_t(M) = \mathbb{C}\mathbf{1}$ iff α is a *pure* E_0-semigroup, and $\cap_{t>0}\gamma_t(M') = \mathbb{C}\mathbf{1}$ iff β is a pure E_0-semigroup. Since $\cap_{t>0}\gamma_t(M')$ is the commutant of $\cup_{t>0}\gamma_t(M)$, we conclude that (γ, M) *defines a history if and only if both α and β are pure E_0-semigroups.* Similarly, one finds that (γ, M) *defines an interaction iff both α and β are pure E_0-semigroups having normal invariant states.* These remarks show that the existence problem for both histories and interactions reduces to the problem of determining when a given pair α, β of E_0-semigroups acting on M and its commutant M' can be extended to an automorphism group γ in the sense specified by (1.9).

The following result is a counterpart for noncommutative dynamics of von Neumann's theorem on the existence of self-adjoint extensions of symmetric operators in terms of deficiency indices.

THEOREM 1.4.1. *Let α and β be two E_0-semigroups, acting on $\mathcal{B}(H)$ and $\mathcal{B}(K)$, respectively, each of which is a cocycle perturbation of a CCR/CAR flow. There is a one-parameter group $\gamma = \{\gamma_t : t \in \mathbb{R}\}$ of automorphisms of $\mathcal{B}(H \otimes K)$ that satisfies*

$$\gamma_{-t}(A \otimes \mathbf{1}) = \alpha_t(A) \otimes \mathbf{1}, \qquad \gamma_t(\mathbf{1} \otimes B) = \mathbf{1} \otimes \beta_t(B), \qquad t \geq 0$$

for all $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$ if and only if α and β have the same numerical index. In particular, for any two pure E_0-semigroups α, β that are cocycle perturbations of the CAR/CCR flow of index $n = 1, 2, \ldots, \infty$, there is a history (U, M) whose past and future semigroups are conjugate, respectively, to α and β.

Significantly, there are many extensions of a fixed compatible pair α, β to automorphism groups γ that satisfy (1.10). Such extensions can be parameterized in a way that is analogous to von Neumann's parameterization of the self-adjoint extensions of symmetric operators; this more precise result is Theorem 3.5.5.

Theorem 1.4.1 implies that any two pure E_0-semigroups that are cocycle perturbations of the same CAR/CCR flow can be assembled into a history. Significantly, it is possible to find cocycle perturbations of any CAR/CCR flow that have absorbing states with specified eigenvalue lists. Our results here are incomplete, but are effective for eigenvalue lists that contain only a finite number of nonzero terms (see Theorem 11.3.1). When Theorem 11.3.1 is combined with the results of the preceding discussion, one obtains the following:

THEOREM 1.4.2. *Let $n = 1, 2, \ldots, \infty$ and let Λ_- and Λ_+ be two eigenvalue lists, each of which has only finitely many nonzero terms. There is an interaction (U, M) whose past and future states ω_-, ω_+ have eigenvalue lists Λ_- and Λ_+, and whose past and future E_0-semigroups are both cocycle perturbations of the CAR/CCR flow of index n.*

We conjecture that the finiteness hypothesis above can be dropped.

Naturally, one might expect that an interaction of the kind described in Theorem 1.4.2 should be nontrivial when $\Lambda_- \neq \Lambda_+$. That is true, but the fact is subtle, involving the interaction inequality in an essential way. To illustrate the point, let $n = 1, 2, \ldots, \infty$ be a positive integer and choose a pair of distinct eigenvalue lists Λ_- and Λ_+ each of which has only finitely many nonzero terms. We see from Theorem 1.4.2 that there are interactions whose past and future E_0-semigroups are cocycle perturbations of the CAR/CCR flow of index n, and whose past and future states have eigenvalue lists Λ_- and Λ_+ respectively. Theorem 1.2.6 implies that

$$\|\bar{\omega}_- - \bar{\omega}_+\| \geq \|\Lambda_- \otimes \Lambda_- - \Lambda_+ \otimes \Lambda_+\|,$$

and Remark 1.2.7 implies that the right side of this inequality is nonzero. Thus all such interactions are nontrivial.

As a somewhat more concrete application, we prove the following result which implies that "strong" interactions exist.

THEOREM 1.4.3. *Let $n = 1, 2, \dots, \infty$ and choose $\epsilon > 0$. There is an interaction (U, M) such that α^- and α^+ are cocycle perturbations of the CAR/CCR flow of index n, for which*

$$\|\bar{\omega}_- - \bar{\omega}_+\| \geq 2 - \epsilon.$$

PROOF. Choose positive integers $p < q$ and consider the eigenvalue lists

$$\Lambda_- = \{1/p, 1/p, \dots, 1/p, 0, 0, \dots\}$$
$$\Lambda_+ = \{1/q, 1/q, \dots, 1/q, 0, 0, \dots\},$$

where $1/p$ is repeated p times and $1/q$ is repeated q times. Theorem 1.4.2 implies that there is an interaction (U, M) whose past and future E_0-semigroups are cocycle-conjugate to the CAR/CCR flow of index n, for which $\Lambda(\omega_-) = \Lambda_-$ and $\Lambda(\omega_+) = \Lambda_+$. By Theorem 1.2.6,

$$\|\bar{\omega}_+ - \bar{\omega}_-\| \geq \|\Lambda(\omega_+ \otimes \omega_+) - \Lambda(\omega_- \otimes \omega_-)\|.$$

If we neglect zeros, the eigenvalue list of $\omega_- \otimes \omega_-$ consists of the single eigenvalue $1/p^2$, repeated p^2 times, and that of $\omega_+ \otimes \omega_+$ consists of $1/q^2$ repeated q^2 times. Thus

$$\|\Lambda(\omega_+ \otimes \omega_+) - \Lambda(\omega_- \otimes \omega_-)\| = p^2(1/p^2 - 1/q^2) + (q^2 - p^2)/q^2 = 2 - 2p^2/q^2,$$

and the inequality of Theorem 1.4.3 follows whenever $q > p\sqrt{2/\epsilon}$. $\square$

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