11

Commensurable Arithmetic Groups and Volumes

In this chapter, we return to considering arithmetic Kleinian and Fuchsian groups and the related quaternion algebras. Recall that the wide commensurability classes of arithmetic Kleinian groups are in one-to-one correspondence with the isomorphism classes of quaternion algebras over a number field with one complex place which are ramified at all the real places. There is a similar one-to-one correspondence for arithmetic Fuchsian groups. Thus, for a suitable quaternion algebra A, let $\mathcal{C}(A)$ denote the (narrow) commensurability class of associated arithmetic Kleinian or Fuchsian groups. In this chapter, we investigate how the elements of $\mathcal{C}(A)$ are distributed and, in particular, determine the maximal elements of $\mathcal{C}(A)$ of which there are infinitely many. Since these groups are all of finite covolume, their volumes are, of course, commensurable. As a starting point to determining these volumes, a formula for the groups $P\rho(\mathcal{O}^1)$, where $\mathcal{O}$ is a maximal order in A, is obtained in terms of the number-theoretic data defining the number field and the quaternion algebra. This relies critically on the fact that the Tamagawa number of the quotient $A^1_{\mathcal{A}}/A^1_k$ of the idèle group $A^1_{\mathcal{A}}$, is 1, as discussed in Chapter 7. From this formula, one can determine the covolumes of the maximal elements of $\mathcal{C}(A)$ and show that all of these volumes are integral multiples of a single number. Much of this chapter is based on work of Borel.

11.1 Covolumes for Maximal Orders

In this section, we determine a formula for the covolumes of the groups $P\rho(\mathcal{O}^1)$, where $\mathcal{O}$ is a maximal order in A.

First consider the Kleinian case. Thus, k is a number field with exactly one complex place and A a quaternion algebra over k which is ramified at all real places. Let $\mathcal{O}$ be a maximal order in A. Then in the group of idèles $A^1_{\mathcal{A}}$ as described in Theorem 8.1.2, let U be the open subgroup

$$\mathrm{SL}(2,\mathbb{C}) \times \prod_{v \in \mathrm{Ram}_\infty(A)} A^1_v \times \prod_{v \in \Omega_f} \mathcal{O}^1_v \cong \mathrm{SL}(2,\mathbb{C}) \times \prod_{v \text{ real}} \mathcal{H}^1 \times \prod_{\mathcal{P} \in \Omega_f} \mathcal{O}^1_{\mathcal{P}}. \tag{11.1}$$

Note that by choosing $\mathcal{O}$ to be maximal, all $\mathcal{O}_{\mathcal{P}}$ are maximal by Corollary 6.2.8. Now as we have seen in Theorem 8.1.2, following Theorem 7.6.3, the Tamagawa volume of $U/\mathcal{O}^1$ is 1. All components in U apart from the first are compact. Thus, if ρ is the projection of $\mathcal{O}^1$ into the first component, then the Tamagawa volume of $U/\mathcal{O}^1$ is the product of the local Tamagawa volumes of $\mathrm{SL}(2,\mathbb{C})/\rho(\mathcal{O}^1)$ and of the factors $\mathcal{H}^1$ and $\mathcal{O}^1_{\mathcal{P}}$ (see Exercise 8.1, No. 3). In Chapter 7, we determined the local Tamagawa volumes of these factors. Thus, $\mathrm{Vol}(\mathcal{H}^1) = 4\pi^2$ and

$$\mathrm{Vol}(\mathcal{O}^1_{\mathcal{P}}) = \begin{cases} D_{k_{\mathcal{P}}}^{-3/2}(1 - N(\mathcal{P})^{-2}) & \text{if } \mathcal{P} \notin \mathrm{Ram}_f(A) \\ D_{k_{\mathcal{P}}}^{-3/2}(1 - N(\mathcal{P})^{-2})(N(\mathcal{P}) - 1)^{-1} & \text{if } \mathcal{P} \in \mathrm{Ram}_f(A) \end{cases}$$

(see Lemmas 7.5.7 and 7.5.8). Here $D_{k_{\mathcal{P}}}$ is the discriminant of the local field extension $k_{\mathcal{P}} \mid \mathbb{Q}_p$, where $p \mid \mathcal{P}$ and the product $\prod_{\mathcal{P}} D_{k_{\mathcal{P}}} = \Delta_k$, the absolute discriminant of k. Also, as earlier, $N(\mathcal{P})$ is the cardinality of $|R_{\mathcal{P}}/\pi_{\mathcal{P}} R_{\mathcal{P}}| = |R/\mathcal{P}|$.

Thus for the Tamagawa measure,

$$\begin{aligned} \mathrm{Vol}(\mathrm{SL}(2,\mathbb{C})/\rho(\mathcal{O}^1)) &= \prod_{v \text{ real}} (\mathrm{Vol}(\mathcal{H}^1))^{-1} \prod_{\mathcal{P}} (\mathrm{Vol}(\mathcal{O}^1_{\mathcal{P}}))^{-1} \\ &= \frac{|\Delta_k|^{3/2} \zeta_k(2) \prod_{\mathcal{P} | \Delta(A)} (N(\mathcal{P}) - 1)}{(4\pi^2)^{[k:\mathbb{Q}]-2}}. \end{aligned} \tag{11.2}$$

The Dedekind zeta function ζ_k of the field k is defined, for $\Re(s) > 1$, by $\zeta_k(s) = \sum_I \frac{1}{N(I)^s}$, where the sum is over all ideals I in R_k. It has an Euler product expansion $\zeta_k(s) = \prod_{\mathcal{P}} (1 - N(\mathcal{P})^{-s})^{-1}$, where the product is over all prime ideals, and this is used in deriving formula (11.2). Recall also that $\Delta(A)$ is the (reduced) discriminant of the quaternion algebra, which is the ideal defined as the product of those primes ramified in A.

In the same way, for arithmetic Fuchsian groups, k is a totally real field, A is ramified at all real places except one and $\mathcal{O}$ is a maximal order in A.

Then for Tamagawa measures,

$$\mathrm{Vol}(\mathrm{SL}(2,\mathbb{R})/\rho(\mathcal{O}^1)) = \frac{\Delta_k^{3/2}\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1)}{(4\pi^2)^{[k:\mathbb{Q}]-1}}. \tag{11.3}$$

In both the Kleinian and Fuchsian cases, we need to determine the scaling factor which relates the Tamagawa volume to the hyperbolic volume. This is done by measuring both volumes for a single suitable group, which, in the Fuchsian case, is taken to be $\mathrm{SL}(2,\mathbb{Z})$, so we start with that.

The hyperbolic plane $\mathbf{H}^2$ can be identified with the symmetric space $\mathrm{SO}(2,\mathbb{R})\backslash\mathrm{SL}(2,\mathbb{R})$. Specifically, taking the upper half-space model of $\mathbf{H}^2$, the continuous map $\phi : \mathrm{SL}(2,\mathbb{R}) \to \mathbf{H}^2$, given by $\phi(\gamma) = \bar{\gamma}(i)$, where $\bar{\gamma}$ is the image of γ in $\mathrm{PSL}(2,\mathbb{R})$, maps the compact subgroup $\mathrm{SO}(2,\mathbb{R})$ onto the stabiliser of i. Thus taking the Tamagawa measure on $\mathrm{SL}(2,\mathbb{R})$ and the hyperbolic measure on $\mathbf{H}^2$, we obtain a compatible measure, as described in §7.5, on $\mathrm{SO}(2,\mathbb{R})$, given by the volume of this compact group. Then, if Γ is a torsion-free discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ of finite covolume, we have

$$\mathrm{Vol}(\mathbf{H}^2/\bar{\Gamma}) \times \mathrm{Vol}(\mathrm{SO}(2,\mathbb{R})) = \mathrm{Vol}(\mathrm{SL}(2,\mathbb{R})/\Gamma) \tag{11.4}$$

with respect to these compatible measures.

Now let Γ be a torsion-free subgroup of finite index in $\mathrm{SL}(2,\mathbb{Z})$ and note that

$$[\mathrm{SL}(2,\mathbb{Z}) : \Gamma] = 2[\mathrm{PSL}(2,\mathbb{Z}) : \bar{\Gamma}].$$

Thus from (11.4),

$$\mathrm{Vol}(\mathbf{H}^2/\mathrm{PSL}(2,\mathbb{Z})) \times \mathrm{Vol}(\mathrm{SO}(2,\mathbb{R})) = 2\mathrm{Vol}(\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})). \tag{11.5}$$

The volume on the right-hand side here is the Tamagawa volume, which, from (11.3), is seen to be $\zeta_{\mathbb{Q}}(2) = \pi^2/6$, whereas the hyperbolic volume, $\mathrm{Vol}(\mathbf{H}^2/\mathrm{PSL}(2,\mathbb{Z})) = \pi/3$. Thus,

$$\mathrm{Vol}(\mathrm{SO}(2,\mathbb{R})) = \pi.$$

More generally, this argument shows that for any arithmetic group Γ contained in $\mathrm{SL}(2,\mathbb{R})$,

$$\text{Hyperbolic Vol}\left(\mathbf{H}^2/\bar{\Gamma}\right) = \frac{\text{Tamagawa Vol}\,(\mathrm{SL}(2,\mathbb{R})/\Gamma)}{\pi} \times \begin{cases} 1 & \text{if } -1 \notin \Gamma \\ 2 & \text{if } -1 \in \Gamma. \end{cases}$$

Orders always contain -1, so we deduce the following:

Theorem 11.1.1 *Let k be a totally real number field, A be a quaternion algebra over k which is ramified at all real places except one and $\mathcal{O}$ be a maximal order in A. Then the hyperbolic covolume of the Fuchsian group $P\rho(\mathcal{O}^1)$ is*

$$\frac{8\pi\Delta_k^{3/2}\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1)}{(4\pi^2)^{[k:\mathbb{Q}]}}. \tag{11.6}$$

The Kleinian case is similar using the Picard group $\mathrm{PSL}(2, O_1)$ ($O_1 = \mathbb{Z}[i]$) to obtain the scaling factor. However, first we discuss some general connections between the values of $\zeta_k(2)$, where k is quadratic imaginary, and the values of the Lobachevski function. Remember that the values of the Lobachevski function can be used to measure hyperbolic volumes in $\mathbf{H}^3$, particularly of polyhedra.

Recall, from §1.7, the Lobachevski function $\mathcal{L}$ is defined for $\theta \neq n\pi$ by

$$\mathcal{L}(\theta) = -\int_0^\theta \ln|2\sin u|\, du$$

and admits a continuous extension to $\mathbb{R}$. Now $\mathcal{L}$ has period π and is an odd function. It then has a uniformly convergent Fourier series expansion

$$\mathcal{L}(\theta) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n^2}.$$

There are rational linear relationships between the values of $\mathcal{L}(\theta)$ which arise by using the following identity:

$$2\sin nu = \prod_{j=0}^{n-1} 2\sin\left(u + \frac{\pi j}{n}\right). \tag{11.7}$$

(See Exercise 11.1, No. 3). Thus in the integral definition,

$$-n\int_0^\theta \ln|2\sin nu|\, du = -\sum_{j=0}^{n-1} n\int_0^\theta \ln\left|2\sin\left(u + \frac{\pi j}{n}\right)\right| du$$

which yields, by a change of variable,

$$-\int_0^{n\theta} \ln|2\sin u|\, du = -\sum_{j=0}^{n-1} n\int_{\frac{\pi j}{n}}^{\frac{\pi j}{n}+\theta} \ln|2\sin u|\, du.$$

Thus

$$\mathcal{L}(n\theta) = n\sum_{j=0}^{n-1} \mathcal{L}\left(\theta + \frac{\pi j}{n}\right) - n\sum_{j=0}^{n-1} \mathcal{L}\left(\frac{\pi j}{n}\right).$$

Since $\mathcal{L}$ is odd of period π, the last term in this expression is zero.

Lemma 11.1.2

$$\mathcal{L}(n\theta) = n\sum_{j \pmod n} \mathcal{L}\left(\theta + \frac{\pi j}{n}\right)$$

where the sum is over a complete set of residues mod n.

Consider, on the other hand, the unique non-principal character χ of the quadratic extension $\mathbb{Q}(\sqrt{-d}) \mid \mathbb{Q}$ (see Exercise 11.1, No. 4). Thus χ induces a mod $|D|$ character, also denoted $\chi : \mathbb{Z} \to \mathbb{R}$, where D is the discriminant of $\mathbb{Q}(\sqrt{-d})$. This χ is of period $|D|$, is totally multiplicative and is defined on primes by $\chi(p) = 0, \pm 1$ according as to whether p ramifies, decomposes or is inert respectively, in the extension $\mathbb{Q}(\sqrt{-d}) \mid \mathbb{Q}$ (see Lemma 0.3.10). Also note that $\chi(-1) = -1$. Thus χ is a real character taking only the values 0 and ± 1. The associated L-series is given by

$$L(s,\chi) = \sum \frac{\chi(n)}{n^s},$$

which has an Euler product expansion $\prod_p (1 - \frac{\chi(p)}{p^s})^{-1}$ for $\Re(s) > 1$. From the Euler product expansion for $\zeta_{\mathbb{Q}(\sqrt{-d})}(s)$, it follows easily that

$$\zeta_{\mathbb{Q}(\sqrt{-d})}(s) = \zeta_{\mathbb{Q}}(s)\ L(s,\chi). \tag{11.8}$$

Now let us return to determining the scaling factor in the volume formula for arithmetic Kleinian groups. Thus consider $k = \mathbb{Q}(i)$ so that $D = -4$ and $\chi(n) = 0$ if $2 \mid n$, $\chi(n) = \pm 1$ according as to whether $n \equiv \pm 1 (\text{mod } 4)$. Hence for all n, $\chi(n) = \sin(2n\pi/4)$. Thus,

$$L(2,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^2} = \sum_{n=1}^{\infty} \frac{\sin(2n\pi/4)}{n^2} = 2\mathcal{L}(\pi/4). \tag{11.9}$$

This can be generalised (see Exercise 11.1, No. 5). Thus from (11.2), (11.8) and (11.9), the Tamagawa volume of $\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,O_1)$ can be expressed in terms of the Lobachevski function and we now turn to determining the hyperbolic volume.

A fundamental region in $\mathbf{H}^3$ for the action of $\mathrm{PSL}(2,O_1)$ is

$$\{(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}^+ \mid x^2 + y^2 + t^2 \geq 1, x \leq 1/2, y \leq 1/2, x + y \geq 0\}$$

whose projection on the (x,y)-plane is shown in Figure 11.1 (see §1.4.1 and Figure 1.1). This is made up of four congruent tetrahedra, each with one ideal vertex at ∞. The tetrahedron with vertices O, A, B, and ∞ has three right dihedral angles at the edges OA, OB, and $A\infty$ and other dihedral angles $\alpha = \pi/4$ at the edge $O\infty$, $\pi/2 - \alpha = \pi/4$ at the edge $B\infty$ and $\gamma = \pi/3$ at the edge AB. The hyperbolic volume of such a tetrahedron (see (1.18)) is given by

$$\frac{1}{4}\Big[\mathcal{L}(\gamma+\alpha)+\mathcal{L}(\alpha-\gamma)+2\mathcal{L}\Big(\frac{\pi}{2}-\alpha\Big)\Big] = \frac{1}{4}\Big[\mathcal{L}\Big(\frac{\pi}{3}+\frac{\pi}{4}\Big)+\mathcal{L}\Big(\frac{\pi}{4}-\frac{\pi}{3}\Big)+2\mathcal{L}\Big(\frac{\pi}{4}\Big)\Big].$$

By Lemma 11.1.2, this equals $\frac{1}{4}[\frac{1}{3}\mathcal{L}(\frac{3\pi}{4}) + \mathcal{L}(\frac{\pi}{4})] = \frac{1}{6}\mathcal{L}(\frac{\pi}{4})$.

Now $\mathbf{H}^3$ is the symmetric space $\mathrm{SU}(2,\mathbb{C})\backslash\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SU}(2,\mathbb{C})$ has a measure compatible with the Tamagawa measure on $\mathrm{SL}(2,\mathbb{C})$ and the

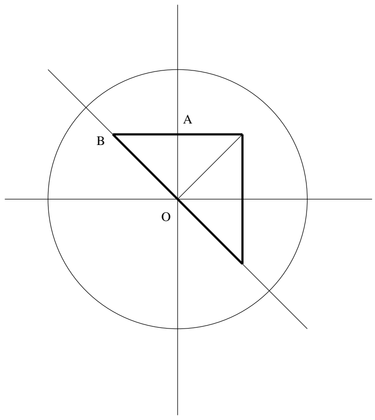

FIGURE 11.1.

hyperbolic measure on $\mathbf{H}^3$. Using $\mathrm{SL}(2, O_1)$ in this case, just as we used $\mathrm{SL}(2, \mathbb{Z})$ in the Fuchsian case, we obtain

$$\mathrm{Vol}(\mathrm{SU}(2, \mathbb{C})) = 8\pi^2.$$

More generally, for arithmetic $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$,

$$\text{Hyperbolic Vol}\left(\mathbf{H}^3/\bar{\Gamma}\right) = \frac{\text{Tamagawa Vol}\,(\mathrm{SL}(2,\mathbb{C})/\Gamma)}{8\pi^2} \times \begin{cases} 1 & \text{if } -1 \notin \Gamma \\ 2 & \text{if } -1 \in \Gamma. \end{cases}$$

Theorem 11.1.3 *Let k be a number field with exactly one complex place, A be a quaternion algebra over k ramified at all real places and $\mathcal{O}$ be a maximal order in A. Then if ρ is a k-representation of A to $M_2(\mathbb{C})$ then*

$$\text{Hyperbolic Vol}(\mathbf{H}^3/P\rho(\mathcal{O}^1)) = \frac{4\pi^2|\Delta_k|^{3/2}\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1)}{(4\pi^2)^{[k:\mathbb{Q}]}}. \tag{11.10}$$

It should be immediately noted that the volume formulas (11.6) and (11.10) for maximal orders depend only on the data defining k and A and not on the particular choice of maximal order. This will have important consequences for the distribution of arithmetic groups, which will be discussed in the next section along with several other consequences and examples.

Exercise 11.1

1. Let A be any quaternion algebra over $\mathbb{Q}$ which is ramified at ∞. Let p be a finite prime at which A is not ramified and let $\mathcal{O}$ be a maximal R_S-order in A where $S = \{p\}$. Show that the Tamagawa volume of $\mathrm{SL}(2, \mathbb{Q}_p)/\rho(\mathcal{O}^1)$, *where $\rho : A \to M_2(\mathbb{Q}_p)$, is $\frac{(1-p^{-2})}{24}\prod_{q|\Delta(A)}(q-1)$. (See Exercise 8.1, No. 4.)*

2. Hilbert Modular Groups. Let k be a totally real field with ring of integers R and let $A = M_2(k)$. The group $\mathrm{SL}(2, R)$ *is a Hilbert modular group.*
(a) Show that $\mathrm{SL}(2, R)$ *embeds, via the diagonal map ρ, as a discrete subgroup of finite covolume in* $\mathrm{SL}(2, \mathbb{R})^n$*, where $n = [k : \mathbb{Q}]$.*
(b) Determine the Tamagawa volume of $\mathrm{SL}(2, \mathbb{R})^n/\rho(SL(2, R))$.

3. By factoring the polynomial $z^n - 1$ over $\mathbb{C}$, establish the identity at (11.7).

4. Let C_f denote the cyclotomic field $\mathbb{Q}(e^{2\pi i/f})$. It is known that the smallest f such that $\mathbb{Q}(\sqrt{-d}) \subset C_f$ is given by $f = |D|$, where D is the discriminant of $\mathbb{Q}(\sqrt{-d})$. Since C_f is an abelian extension of $\mathbb{Q}$, there is a natural epimorphism $\mu : \mathrm{Gal}(C_f \mid \mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\sqrt{-d}) \mid \mathbb{Q})$. Thus the non-trivial homomorphism $\chi : \mathrm{Gal}(\mathbb{Q}(\sqrt{-d}) \mid \mathbb{Q}) \to \{\pm 1\} \subset S^1$ gives rise to a homomorphism $\chi \circ \mu : \mathrm{Gal}(C_f \mid \mathbb{Q}) \to \{\pm 1\}$. Since the Galois group $\mathrm{Gal}(C_f \mid \mathbb{Q})$ can be identified with the group of units $\mathbb{Z}_f^*$, we can extend $\chi \circ \mu$ to a character, still denoted $\chi : \mathbb{Z} \to \mathbb{R}$, by requiring that $\chi(d) = 0$ for $(d, f) > 1$. This is the $\mathrm{mod}(|D|)$ character described in this section. Prove that it has the properties stated; that is, it has period $|D|$, is totally multiplcative, $\chi(p) = 0, \pm 1$ according as to whether p ramifies, decomposes or is inert in $\mathbb{Q}(\sqrt{-d}) \mid \mathbb{Q}$ and that $\chi(-1) = -1$.

5. Following on from Exercise 4, define the Gauss sum

$$\mathcal{G}(\chi, n) = \sum_{r \bmod |D|} \chi(r)\xi^{nr},$$

where $\xi = e^{2\pi i/|D|}$. Assuming the result on Gauss sums that states that $\mathcal{G}(\chi, n) = \chi(n)\mathcal{G}(\chi, 1)$, for all n, prove that

$$\sum_{r \bmod |D|} \chi(r)\mathcal{L}\left(\frac{r\pi}{|D|}\right) = \sqrt{D}L(2, \chi).$$

Deduce that the covolume of the Bianchi group $\mathrm{PSL}(2, O_d)$ *can be expressed in the terms of the Lobachevski function with the two expressions for the covolume being related via*

$$\frac{\zeta_{\mathbb{Q}(\sqrt{-d})}(2)|D|^{3/2}}{4\pi^2} = \frac{|D|}{12} \sum_{r \bmod |D|} \chi(r)\mathcal{L}\left(\frac{r\pi}{|D|}\right).$$

6. In this section, the scaling factor relating the hyperbolic volume measure to the Tamagawa measure for arithmetic Kleinian groups was established using the Picard group. Confirm that the $\mathrm{Vol}(\mathrm{SU}(2, \mathbb{C})) = 8\pi^2$ *using, instead, the Bianchi group* $\mathrm{SL}(2, O_3)$.

11.2 Consequences of the Volume Formula

In this section, we give a variety of consequences of the volume formulas for arithmetic Kleinian groups $P\rho(\mathcal{O}^1)$, where $\mathcal{O}$ is a maximal order, and discuss the computations involved in determining estimations for these volumes.

11.2.1 Arithmetic Kleinian Groups with Bounded Covolume

Theorem 11.2.1 (Borel) *Let $K > 0$. There are only finitely many conjugacy classes of arithmetic Kleinian groups Γ such that $\mathrm{Vol}(\mathbf{H}^3/\Gamma) < K$.*

Proof: We first obtain a bound on the number of generators of such groups Γ. By the results of Thurston, each member of the set of hyperbolic 3-orbifolds whose volume is bounded by K is obtained by Dehn surgery on a finite number of hyperbolic 3-orbifolds (see Theorem 1.5.9). Thus if $n(K)$ is the maximum rank of those fundamental groups of this finite number of orbifolds, then Γ can be generated by at most $n(K)$ generators. Thus $[\Gamma : \Gamma^{(2)}] \leq 2^{n(K)}$. Now $\Gamma^{(2)} \subset P\rho(\mathcal{O}^1)$ by Corollary 8.3.3, where $\mathcal{O}$ is a maximal order in a quaternion algebra A over a number field k and some k-representation $\rho : A \to M_2(\mathbb{C})$. Thus Covol $(P\rho(\mathcal{O}^1)) < K.2^{n(K)}$. For each $\Gamma^{(2)}$, Γ is contained in the normaliser of $\Gamma^{(2)}$ in $\mathrm{PSL}(2,\mathbb{C})$ and so, for each $\Gamma^{(2)}$, there are finitely many possibilities for Γ. Since all k-representations of A in $M_2(\mathbb{C})$ are conjugate and there are finitely many conjugacy classes of maximal orders in A, it suffices to show that there are finitely many fields k and quaternion algebras A such that

$$\frac{4\pi^2|\Delta_k|^{3/2}\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1)}{(4\pi^2)^{[k:\mathbb{Q}]}} \leq K.2^{n(K)}.$$

Clearly, for each k, there can only be finitely many quaternion algebras A over k such that this bound holds, since in these cases where A is ramified at all real places, A is determined by $\Delta(A)$ by Theorem 7.3.6 and there are only finitely many $\Delta(A)$ such that $\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1)$ is bounded. Thus, since $\zeta_k(2) \geq 1$, it remains to show that there are finitely many fields k with one complex place such that $|\Delta_k|^{3/2}/(4\pi^2)^{[k:\mathbb{Q}]}$ is bounded.

We now assume results of Odlyzko relating the magnitude of discriminants of number fields to their degree over $\mathbb{Q}$. He has shown that if $[k:\mathbb{Q}]$ is large enough, then $|\Delta_k| \geq 19^2(50)^{[k:\mathbb{Q}]-2}$. With this we have,

$$\frac{|\Delta_k|^{3/2}}{(4\pi^2)^{[k:\mathbb{Q}]-2}} \geq K_0\left(\frac{50^{3/2}}{4\pi^2}\right)^{[k:\mathbb{Q}]-2}.$$

Thus the degree of the field must be bounded, so that the discriminants are bounded and there are only finitely many fields of bounded discriminant (see Theorem 0.2.8) $\square$

Note that a consequence of Theorem 11.2.1 is the following:

Corollary 11.2.2 *There are only finitely many arithmetic hyperbolic 3-orbifolds obtained by Dehn surgery on a cusped hyperbolic 3-orbifold.*

In fact, the proof of Theorem 11.2.1 can be used to give a practical way to determine which surgeries on a cusped hyperbolic 3-manifold yield arithmetic orbifolds. In the remainder of this section we indicate how this goes in the case of a knot in S^3. Thus, let $M = S^3 \setminus K$ be a 1-cusped finite-volume hyperbolic 3-manifold, where we fix a framing for a torus cusp cross-section. Assume that M has volume V and that (p,q)-Dehn surgery, yielding $M(p,q)$, is arithmetic. Let $k = kM(p,q)$ be the invariant trace field, and n its degree over $\mathbb{Q}$. Since M is a knot complement in S^3, $H_1(M(p,q);\mathbb{Z})$ is cyclic. In particular, if $\Gamma = \pi_1^{\mathrm{orb}}(M(p,q))$, then $[\Gamma : \Gamma^{(2)}] \leq 2$. By Corollary 8.3.5, Γ is arithmetic if and only if $\Gamma^{(2)}$ is derived from a quaternion algebra, with k coinciding with the centre of the invariant quaternion algebra. Thus putting these statements together with the fact that $\mathrm{Vol}(M(p,q)) < V$ (see §1.5.3), we deduce the existence of a maximal order $\mathcal{O}$ in $A\Gamma$ with,

$$2V > 2\mathrm{Covol}(\Gamma)) \geq \mathrm{Covol}(\Gamma^{(2)}) \geq \mathrm{Covol}(P\rho(\mathcal{O}^1)$$

Thus Theorem 11.1.3 yields

$$2V > \frac{4\pi^2|\Delta_k^{3/2}|\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1)}{(4\pi^2)^n}.$$

Now $\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1) \geq 1$, and so if Γ is arithmetic, we have

$$V > \frac{2\pi^2|\Delta_k^{3/2}|}{(4\pi^2)^n}.$$

We appeal once more to bounds of Odlyzko relating discriminant bounds to degree, where he has shown that

$$|\Delta_k| > A^{n-2}B^2e^{-E}$$

where $A = 24.987$, $B = 13.157$ and $E = 6.9334$ for all n. Combining these estimates, we obtain

$$\frac{V}{2\pi^2} > \Big(\frac{A^{3/2}}{4\pi^2}\Big)^n(0.0000044),$$

which, on simplifying and taking natural logs, implies that, if $M(p,q)$ is arithmetic, n satisfies

$$8.11 + (0.87)\log V > n.$$

Thus for example, if M is the figure 8 knot complement, so that $V = 2.029883\ldots$, and if $M(p,q)$ is arithmetic, then the above estimate shows that the degree of the invariant trace field of $M(p,q)$ is at most 9. From §4.8 and Exercises 9.8, Nos. 1 and 2, the orbifolds $M(4,0)$, $M(5,0)$, $M(6,0)$, $M(8,0)$ and $M(12,0)$ are arithmetic with fields of degrees 2,4,2,4 and 4 respectively, and the manifold $M(5,1)$ is arithmetic with degree 4. Also from Exercise 8.3, No. 4, the orbifolds $M(0,2)$ and $M(0,3)$, which are covered by Jørgensen's fibre bundles, are arithmetic with fields of degree 4.

In fact other techniques can be brought to bear in getting even more control over which surgeries are arithmetic (see later in this chapter and §12.3).

11.2.2 Volumes for Eichler Orders

Let A be a quaternion algebra over a field k defining a commensurability class $\mathcal{C}(A)$ of arithmetic Kleinian or Fuchsian groups. If $\Gamma_1, \Gamma_2 \in \mathcal{C}(A)$, then their generalised index $[\Gamma_1 : \Gamma_2] \in \mathbb{Q}$ is well-defined by

$$[\Gamma_1 : \Gamma_2] = \frac{[\Gamma_1 : \Gamma_1 \cap \Gamma_2]}{[\Gamma_2 : \Gamma_1 \cap \Gamma_2]}. \tag{11.11}$$

Notice that if $\mathcal{O}$ is not a maximal order in A, it remains true that the Tamagawa volume of $U/\mathcal{O}^1$ is still 1 (see Theorem 8.1.2). For any order $\mathcal{O}$, $\mathcal{O}_\mathcal{P}$ is maximal for all but a finite number of $\mathcal{P}$ (see §6.3). Thus the same analysis as applied in the case of a maximal order, in particular (11.1), can be applied to any order. Thus if $\mathcal{O}_1$ and $\mathcal{O}_2$ are two orders in A then

$$[P\rho(\mathcal{O}_1^1) : P\rho(\mathcal{O}_2^1)] = \frac{\mathrm{Covol}(\mathcal{O}_2^1)}{\mathrm{Covol}(\mathcal{O}_1^1)} = \prod_\mathcal{P} \frac{\mathrm{Vol}((\mathcal{O}_2)^1_\mathcal{P})}{\mathrm{Vol}((\mathcal{O}_1)^1_\mathcal{P})} = \prod_\mathcal{P} [(\mathcal{O}_1)^1_\mathcal{P} : (\mathcal{O}_2)^1_\mathcal{P}]$$

where these products, as noted above, are finite.

Let $\mathcal{O}$ be an Eichler order of level N in A, so that $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$ where $\mathcal{O}_1$ and $\mathcal{O}_2$ are maximal orders (see §6.6). Suppose that $\mathcal{P}^n | N$ so that, in $A_\mathcal{P} \cong M_2(k_\mathcal{P})$, we can take $(\mathcal{O}_1)_\mathcal{P} = M_2(R_\mathcal{P})$ and

$$\mathcal{O}_\mathcal{P} = \left\{ \begin{pmatrix} a & b \\ \pi^n c & d \end{pmatrix} \mid a,b,c,d \in R_\mathcal{P} \right\}.$$

Then,

$$[(\mathcal{O}_1)^1_\mathcal{P} : \mathcal{O}^1_\mathcal{P}] = N(\mathcal{P})^{n-1}(N(\mathcal{P}) + 1) \tag{11.12}$$

(see Exercise 11.2, No. 2). Thus if $N = \mathcal{P}_1^{n_1}\mathcal{P}_2^{n_2}\cdots\mathcal{P}_r^{n_r}$, then

$$[P\rho(\mathcal{O}_1^1) : P\rho(\mathcal{O}^1)] = \prod_{i=1}^{r} N(\mathcal{P}_i)^{n_i-1}(N(\mathcal{P}_i)+1).$$

11.2.3 Arithmetic Manifolds of Equal Volume

In the Kleinian cases, arithmetic groups defined by quaternion algebras over the same field will have commensurable covolumes. However, by allowing the finite ramification of the quaternion algebras to vary, we can obtain families of mutually non-commensurable groups (see Theorem 8.4.1). Thus in particular, we can obtain cocompact and non-cocompact groups with identical covolumes and, indeed, compact and non-compact manifolds with identical volumes.

For example, let d be a square-free integer such that $d \equiv -1 (\text{mod } 8)$ and let $k = \mathbb{Q}(\sqrt{-d})$. Let $A_1 = M_2(k)$ and take the maximal order $\mathcal{O}_1 = M_2(O_d)$ in A_1. By our choice of d, 2 decomposes in $k \mid \mathbb{Q}$ so that there are two primes $\mathcal{P}_2'$ and $\mathcal{P}_2''$ with $N(\mathcal{P}_2') = N(\mathcal{P}_2'') = 2$. Let A_2 be the quaternion algebra over k which is ramified at $\mathcal{P}_2'$ and $\mathcal{P}_2''$ and let $\mathcal{O}_2$ be a maximal order in A_2. Then $P\rho(\mathcal{O}_2^1)$ has the same covolume as the Bianchi group $\mathrm{PSL}(2, O_d)$, but, of course, $P\rho(\mathcal{O}_2^1)$ is cocompact.

These groups just considered will, in general, have torsion so that their quotients are orbifolds rather than manifolds. By dropping to suitable torsion-free subgroups of finite index we can obtain compact and non-compact manifolds with identical volumes. More specifically, let $\mathcal{P}$ be a prime ideal in O_d, where d is chosen as above and $\mathcal{P}$ is such that $(\mathcal{P}, 2d) = 1$. Let the orders $\mathcal{O}_1$ and $\mathcal{O}_2$ in the quaternion algebras A_1 and A_2 be as described above. The groups $\mathcal{O}_1^1$ and $\mathcal{O}_2^1$ embed densely in the groups $(\mathcal{O}_1)_{\mathcal{P}}{}^1, (\mathcal{O}_2)_{\mathcal{P}}{}^1$, these latter groups both being isomorphic to $\mathrm{SL}(2, R_{\mathcal{P}})$. Let $\Gamma_1(\mathcal{P})$ and $\Gamma_2(\mathcal{P})$ be the images in $P(\mathcal{O}_1^1)$ and $P\rho(\mathcal{O}_2^1)$, respectively, of the principal congruence subgroups of level $\mathcal{P}$. Then by Lemma 6.5.6, $[P(\mathcal{O}_1^1) : \Gamma_1(\mathcal{P})] = [P\rho(\mathcal{O}_2^1 : \Gamma_2(\mathcal{P})]$ and, since $\mathcal{P}$ is unramified in $k \mid \mathbb{Q}$, $\Gamma_1(\mathcal{P})$ and $\Gamma_2(\mathcal{P})$ are torsion free. Thus using the infinitely many choices of d, and for each d, the infinitely many choices of $\mathcal{P}$, we have proved the following:

Theorem 11.2.3 *There are infinitely many pairs of compact and non-compact hyperbolic 3-manifolds with the same volume.*

Alternatively, consider the following specific examples of manifolds considered in §4.8.2. These manifolds N_n $(n \geq 4)$ are the n-fold cyclic covers of the hyperbolic orbifolds O_n obtained by $(n, 0)$ filling on the figure 8 knot complement: the Fibonacci manifolds. The hyperbolic structure can be obtained by completing an incomplete hyperbolic structure on the union of two ideal tetrahedra which are parametrised by the complex numbers z, w

with positive imaginary parts (see §1.7). The parameters must satisfy the gluing consistency condition: $zw(z-1)(w-1) = 1$; and the filling condition on the meridian gives $(w(1-z))^n = 1$. If T_z and T_w denote the tetrahedra parametrised by z and w respectively, then the volume of O_n is the sum of the volumes of T_z and T_w. Recall that the ideal tetrahedron T_z, with all vertices on the sphere at infinity, has volume given in terms of the Lobachevski function by

$$\mathcal{L}(\arg z) + \mathcal{L}\Big(\arg \frac{z-1}{z}\Big) + \mathcal{L}\Big(\arg \frac{1}{1-z}\Big).$$

For $n \geq 5$, let $\phi = 2\pi/n$ and choose ψ such that $0 < \psi < \pi$ and $\cos\psi = \cos\phi - 1/2$. Then the equations in z and w above admit the solution $z = -e^{-i\psi}(1 - e^{i(\psi-\phi)})$ and $w = (1 - e^{i(\psi-\phi)})^{-1}$.

Consider the case $n = 6$, so that $\phi = \pi/3$ and $\psi = \pi/2$. Then a simple calculation yields

$$\mathrm{Vol}(O_6) = 2[\mathcal{L}(\pi/12) + \mathcal{L}(5\pi/12)] = \frac{8}{3}\mathcal{L}(\pi/4)$$

where the last equality is obtained using Lemma 11.1.2, with $\theta = \pi/12$ and $n = 3$. Hence, $\mathrm{Vol}(N_6) = 16\mathcal{L}(\pi/4)$. Note that the covolume of $\mathrm{PSL}(2, O_1)$ has already been calculated to be $2\mathcal{L}(\pi/4)/3$ (see §11.1). It is well-known that the Borromean rings complement is an arithmetic hyperbolic manifold whose fundamental group is of index 24 in $\mathrm{PSL}(2, O_1)$ (see §9.2). Thus the volume of the compact manifold N_6 is the same as that of the non-compact hyperbolic manifold, which is the Borromean rings complement. This is just one case in an infinite family of examples. The Borromean rings arise from the closed 3-braid given by $(\sigma_1\sigma_2^{-1})^3$. By the methods given above of calculating tetrahedral parameters, it can be shown that the volume of the complement of the closed 3-braid $(\sigma_1\sigma_2^{-1})^n$ is equal to the volume of the Fibonacci manifold N_{2n} for each $n \geq 3$.

11.2.4 Estimating Volumes

The formulas (11.6) and (11.10) can be used to obtain numerical estimates for the covolumes of these groups. From a knowledge of the invariants k and A, most of the terms are readily determined, but some estimation is necessary to evaluate $\zeta_k(2)$. Via the Euler product, this depends on a knowledge of the prime ideals in R_k, and at a fairly crude level, the primes of "small" norm allow estimates to be obtained as follows:

$$\zeta_k(2) = \prod_{\mathcal{P}} (1 - N(\mathcal{P})^{-2})^{-1} = \prod_p \Big(\prod_{\mathcal{P}|p} (1 - N(\mathcal{P})^{-2})^{-1}\Big).$$

Note that

$$\prod_{\mathcal{P}|p} (1 - N(\mathcal{P})^{-2})^{-1} \leq (1 - p^{-2})^{-[k:\mathbb{Q}]}.$$

If p_0 is a fixed prime, take logs and do some estimating to obtain

$$\prod_{p \geq p_0} \Bigl(\prod_{\mathcal{P}|p} (1 - N(\mathcal{P})^{-2})^{-1} \Bigr) \leq \exp\Bigl([\mathrm{k} : \mathbb{Q}]\Bigl(\frac{1}{2(\mathrm{p}_0 - 1)}\Bigr)\Bigr) \quad (:= \mathrm{k}(\mathrm{p}_0)) \tag{11.13}$$

(see Exercise 11.2, No. 4). This then yields

$$\prod_{p < p_0} \Bigl(\prod_{\mathcal{P}|p} (1 - N(\mathcal{P})^{-2})^{-1} \Bigr) \leq \zeta_k(2) \leq \prod_{p < p_0} \Bigl(\prod_{\mathcal{P}|p} (1 - N(\mathcal{P})^{-2})^{-1} \Bigr) k(p_0). \tag{11.14}$$

In order to implement such estimates, a knowledge of the prime ideals in R_k is necessary. These may be obtained via Kummer's Theorem (see Theorem 0.3.9). Thus if $k = \mathbb{Q}(\theta)$ and $R_k = \mathbb{Z}[\theta]$, then the factorisation of the minimum polynomial of θ mod p, p a prime in $\mathbb{Z}$, reflects the decomposition of the prime ideal pR_k into prime ideals in R_k. If such an integral basis does not exist, more sophisticated versions of Kummer's Theorem or other methods may need to be applied. The literature contains tables from which this information may be obtained and there are expert systems, such as Pari, via which many number-theoretic computations, such as the evaluation of $\zeta_k(2)$, can be performed. For applications, see §11.2.5, §11.3.4 and §11.6.

11.2.5 A Tetrahedral Group

In this subsection, we consider, again, certain examples of arithmetic Kleinian groups to exhibit methods of obtaining yet more information on these groups, making use of the comparison between the numerical estimates as outlined above and volume estimates obtained by other means, notably the Lobachevski function. Thus consider the tetrahedral group Γ whose related Coxeter symbol is given in Figure 11.2. This group is arith-

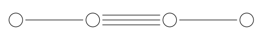

FIGURE 11.2.

metic (see Examples 8.3.8 and 8.4.3). The number field $k = \mathbb{Q}(t)$, where $p(t) = t^4 - 2t^3 + t - 1 = 0$. Thus $[k : \mathbb{Q}] = 4$ and $\Delta_k = -275$ (see Appendix 13.1 for all tetrahedral groups). Furthermore, the quaternion algebra A is only ramified at the real places. Note that the Fibonacci group F_{10}, considered in §4.8.2, is also arithmetic with the same invariants and so is commensurable with Γ (see Example 8.4.3).

Now suppose that $\mathcal{O}$ is a maximal order in this quaternion algebra A. To estimate the covolume of $P\rho(\mathcal{O}^1)$, we need to find the approximate value of $\zeta_k(2)$. In this case, $R_k = \mathbb{Z}[t]$ so we can apply Kummer's Theorem to determine the prime ideals in k. Thus factorising the polynomial $p(t)$ modulo primes such as 2, 3 and 5 we obtain $2R_k = \mathcal{P}_2$, $3R_k = \mathcal{P}_3'\mathcal{P}_3''$, $5R_k = \mathcal{P}_5^2$. Thus R_k has one prime of norm 16, two of norm 9, one of norm 25, and so on, so that $\zeta_k(2)$ can be evaluated.

The upshot of this is that, by these methods or by using one of the packages which make such calculations, we obtain the evaluation $\zeta_k(2) \approx 1.05374$. Thus from (11.10), the covolume of $P\rho(\mathcal{O}^1) \approx 0.07810$. Now the tetrahedral group Γ has a simple presentation (given in §4.7.2) from which one deduces that $\Gamma = \Gamma^{(2)}$. By Corollary 8.3.3, $\Gamma^{(2)} \subset P\rho(\mathcal{O}^1)$ for some order $\mathcal{O}$ which can be taken to be maximal. On the other hand, the covolume of Γ is twice the volume of the tetrahedron. The volume of a compact tetrahedron can be expressed as sums and differences of volumes of ideal tetrahedra, whose volumes, as we have already seen, can be expressed by values of the Lobachevski function (see §1.7). For the nine compact tetrahedra, these volumes have been computed and the volume of the one under consideration is 0.03905 (see §10.4.2 and Appendix 13.1). Thus we deduce that the tetrahedral group $\Gamma = P\rho(\mathcal{O}^1)$, where $\mathcal{O}$ is a maximal order in A. Furthermore, up to conjugacy, it does not matter which maximal order we choose, as the type number of A is 1, for recall from Theorem 6.7.6 that the type number divides h_∞ as defined at (6.13). For the field under consideration, the class number h is 1 and the group of units $R_k^* = \langle t, t-1, -1\rangle \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$. The two real embeddings correspond to the two real roots of the polynomial p and these lie in the intervals $(0,1)$ and $(2,3)$. It follows that $h_\infty = 1$.

What about the Fibonacci group F_{10}? Recall that F_{10} is a normal subgroup of index 5 in the orbifold fundamental group H_5 with presentation

$$H_5 = \langle X, Y, T \mid T^5 = 1, TXT^{-1} = XYX, TYT^{-1} = YX\rangle, \text{(see §4.8.2)}.$$

Note that $H_5^{(2)} = H_5$ so that, by the above remarks $H_5 \subset P\rho(\mathcal{O}^1) = \Gamma$. Thus $F_{10} \subset \Gamma$ and one can determine the index as follows: Note that Γ contains a finite subgroup isomorphic to A_5 as the stabiliser of a vertex. Hence H_5 has index at least 12 in Γ. Suppose $[\Gamma : H_5] = n$. Then H_5 determines, via the action on cosets, a permutation representation of Γ onto a transitive subgroup of S_n so that the pull-back of the stabiliser of 1 is H_5. Conversely, by examining transitive permutation representations of Γ into subgroups of S_n, one can determine subgroups of index n. Presentations of such subgroups can then be deduced from that of Γ and the permutation representation. In this way, one can attempt to determine the index n. There are expert group theory systems such as Magma and Gap with routines that determine presentations of subgroups of low index by this method. Using this, we obtain $[\Gamma : F_{10}] = 60$. We present this as an alternative to the method used in §11.2.4 for determining the relationship between Γ and $P\rho(\mathcal{O}^1)$. Indeed using the methods of §11.2.4, it is possible to calculate their volumes in terms of values of Lobachevski functions. One can then confirm the computed index by comparing these volumes.

Exercise 11.2

1. *Obtain an estimate of the covolume of* $\mathrm{PSL}(2, O_7)$.

2. *Establish the formula at (11.12).*

3. *Recall that the Fibonacci manifold* N_{10} *is not arithmetic (see Theorem 4.8.2). Show that its volume is* $10[3\mathcal{L}(\pi/5) + \mathcal{L}(2\pi/5)]$.

4. *Show how to obtain the estimate at (11.13).*

5. *The tetrahedral group* Γ *whose associated Coxeter symbol is given Figure 11.3 is arithmetic and the tetrahedron has volume approximately 0.22223*

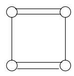

FIGURE 11.3.

(see Appendix 13.1). Let $\mathcal{O}$ *be a maximal order in the associated quaternion algebra. Determine the covolume of* $P\rho(\mathcal{O}^1)$ *and, hence, the relationship between* Γ *and* $P\rho(\mathcal{O}^1)$.

11.3 Fuchsian Groups

As in the preceding section, we exhibit some consequences, this time for arithmetic Fuchsian groups, of the volume formula (11.6) for a maximal order defining an arithmetic Fuchsian group.

11.3.1 Arithmetic Fuchsian Groups with Bounded Covolume

Theorem 11.3.1 *Let* $K > 0$. *There are only finitely many conjugacy classes of arithmetic Fuchsian groups* Γ *such that* $\mathrm{Vol}(\mathbf{H}^2/\Gamma) < K$.

Proof: For arithmetic Fuchsian groups, the same argument as that employed in Theorem 11.2.1 holds, except for the first part. For any finite covolume Fuchsian group Γ, $\mathrm{Vol}(\mathbf{H}^2/\Gamma) = 2\pi|\chi(\Gamma)|$, where $\chi(\Gamma)$ is the rational Euler characteristic of the group Γ. By the structure theorem for finitely generated Fuchsian groups, putting a bound on $|\chi(\Gamma)|$ bounds the number of generators of Γ. The remainder of the argument is the same. For these cases where k is totally real, Odlyzko has again given estimates relating the growth of Δ_k to the degree $[k : \mathbb{Q}]$ for all degrees $[k : \mathbb{Q}]$ (see discussion following Corollary 11.2.2). Precisely, Odlyzko has shown that for all such k,

$$\Delta_k \geq 2.439 \times 10^{-4} \times (29.099)^{[k:\mathbb{Q}]}.$$

□

Corollary 11.3.2 *For each signature of a Fuchsian group of finite covolume, there are only finitely many points in the moduli space of that signature which represent arithmetic Fuchsian groups.*

Theorem 11.3.1 also shows that there can only be finitely many arithmetic Fuchsian triangle groups (see also §11.3.3).

11.3.2 Totally Real Fields

In the Fuchsian case, all hyperbolic volumes are rational multiples of π. Thus the volume formula (11.6) for maximal orders implies the following result, originally due to Klingen and Siegel:

Theorem 11.3.3 *If k is a totally real field, then*

$$\frac{\Delta_k^{1/2}\zeta_k(2)}{(4\pi^2)^{[k:\mathbb{Q}]}}$$

is rational.

We defined the Dedekind zeta function earlier for $\Re(s) > 1$, but it admits a meromorphic extension to the whole complex plane. It then satisfies a functional equation, which, in particular shows that

$$\zeta_k(-1) = \frac{\zeta_k(2)\Delta_k^{3/2}}{(-2\pi^2)^{[k:\mathbb{Q}]}}.$$

The above theorem can thus be restated as follows:

Theorem 11.3.4 *If k is a totally real field, then $\zeta_k(-1)$ is rational.*

11.3.3 Fuchsian Triangle Groups

Let Δ be an (e_1, e_2, e_3)-Fuchsian triangle group. Recall that

$$k\Delta = \mathbb{Q}(\cos 2\pi/e_1, \cos 2\pi/e_2, \cos 2\pi/e_3, \cos \pi/e_1 \cos \pi/e_2 \cos \pi/e_3)$$

and §8.3, where we determined necessary and sufficient conditions for Δ to be arithmetic. In §11.3.1, we used estimates due to Odlyzko to show that there are only finitely many arithmetic triangle groups, a result originally due to Takeuchi, who also enumerated them. In this subsection, we show, without reference to Odlyzko's results, that there are only finitely many arithmetic triangle groups, and obtain bounds on the e_i from which a search can be made to determine all arithmetic triangle groups. The steps of this search will be outlined and the results tabulated in Appendix 13.3.

For any positive integer N, let

$$R_N = \mathbb{Q}(\cos 2\pi/N) \subset \mathbb{Q}(e^{2\pi i/N}) = C_N.$$

Lemma 11.3.5 *If* $(N, M) > 2$, *then* $\mathbb{Q}(\cos 2\pi/N, \cos 2\pi/M) = R_{[N,M]}$.

Proof: Note that, $\mathrm{Gal}(C_{[N,M]} \mid \mathbb{Q}) \cong \mathbb{Z}^*_{[N,M]}$ and $R_{[N,M]}$ is the fixed field of the subgroup $\{\pm 1\}$. The field $\mathbb{Q}(\cos 2\pi/N, \cos 2\pi/M)$ is the fixed field of the subgroup consisting of elements $a \in \mathbb{Z}^*_{[N,M]}$, where $a \equiv \pm 1(\mathrm{mod}\ N)$ and $a \equiv \pm 1(\mathrm{mod}\ M)$. The simultaneous congruences $a \equiv 1(\mathrm{mod}\ N)$ and $a \equiv -1(\mathrm{mod}\ M)$ have a solution if and only if $(N, M) \mid 2$ and the result follows. $\square$

The discriminant of a cyclotomic field is well-known and can be deduced from the discussion on discriminants given in Chapter 0 (see Exercise 0.1, No. 9, and Exercise 0.3, No. 5). Recall also, from Theorem 0.2.10, that if $k \mid \ell$ is an extension of number fields, then

$$|\Delta_k| = |\Delta_\ell|^{[k:\ell]} N_{\ell|\mathbb{Q}}(\delta_{k|\ell}). \tag{11.15}$$

Applying this to the extension $C_N \mid R_N$, we obtain

$$\Delta_{R_N} = \begin{cases} \dfrac{N^{\phi(N)/2}}{\prod_{p|N} p^{\phi(N)/2(p-1)}} & \text{if } N \neq p^\alpha, 2p^\alpha \\ (p^{\alpha p^{\alpha-1}(p-1) - p^{\alpha-1} - 1})^{1/2} & \text{if } N = p^\alpha, 2p^\alpha, p \neq 2 \\ 2^{(\alpha-1)2^{\alpha-2}-1} & \text{if } N = 2^\alpha, \alpha \geq 2 \\ 1 & \text{if } N = 2. \end{cases} \tag{11.16}$$

Using (11.16), simple lower bounds for Δ_{R_N} of the form

$$\Delta_{R_N} \geq (aN)^{\phi(N)/4} \tag{11.17}$$

for varying values of a and large enough values of N are readily obtained. With a view to the enumeration methods below, we state these results. Thus we list the values of a and the corresponding values of N for which (11.17) holds:

$a = 1 \quad N \neq 2, 3, 4, 6, 10, 14, 18$
$a = 3 \quad N \geq 27, N \neq 30, 60$
$a = 4 \quad N \geq 27, N \neq 28, 30, 36, 42, 54, 60, 84$
$a = 5 \quad N \geq 27, N \neq 28, 30, 34, 36, 38, 40, 42, 48, 50, 54, 60, 84, 120.$

Note that we include in these results the cases where $N = 2N'$ with N' odd, in which case $R_N = R_{N'}$.

Now consider the invariant trace field $k\Delta$ which contains the subfield $\mathbb{Q}(\cos 2\pi/e_1, \cos 2\pi/e_2, \cos 2\pi/e_3)$. Let e denote the least common multiple of e_1, e_2 and e_3, and split into three cases:

Case A. At least two of $(e_1, e_2), (e_2, e_3), (e_3, e_1)$ is > 2.
By Lemma 11.3.5, $k\Delta \supset R_e$.

Case B. Exactly one of $(e_1, e_2), (e_2, e_3), (e_3, e_1)$ is > 2.
Suppose $(e_i, e_j) > 2$ and let $f = [e_i, e_j]$. Let g denote the greater of f, e_k. Then $k\Delta \supset R_g$.

Case C. All three of $(e_1, e_2), (e_2, e_3), (e_3, e_1)$ divide 2.
Let h denote the biggest of e_1, e_2, e_3. Then $k\Delta \supset R_h$.

Now suppose that Δ is arithmetic. The index $[\Delta : \Delta^{(2)}] := i(e_1, e_2, e_3) = 1, 2, 4$ according as to whether at most one (respectively exactly two, respectively three) of e_1, e_2 and e_3 are even. By Corollary 8.3.3, $\Delta^{(2)} \subset P\rho(\mathcal{O}^1)$, where $\mathcal{O}$ is a maximal order in the quaternion algebra A. Thus from the formula for the covolume of a maximal order at (11.6), we have

$$\begin{aligned} i(e_1, e_2, e_3)\Big(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\Big) \quad &\geq \quad \frac{8\zeta_{k\Delta}(2)\Delta_{k\Delta}^{3/2}}{(4\pi^2)^{[k\Delta:\mathbb{Q}]}} \prod_{\mathcal{P}|\Delta(A)} (N\mathcal{P} - 1) \\ &\geq \quad \frac{8\Delta_{k\Delta}^{3/2}}{(4\pi^2)^{[k\Delta:\mathbb{Q}]}} \end{aligned} \tag{11.18}$$

since $\zeta_{k\Delta}(2) > 1$ and $(N\mathcal{P} - 1) \geq 1$. Furthermore, let N represent any one of e, g and h as described in the three cases A, B and C above. Note that $[R_N : \mathbb{Q}] = \phi(N)/2$ unless $N = 2$. Then,

$$\Delta_{k\Delta} \geq \Delta_{R_N}^{[k\Delta:R_N]} \geq (aN)^{\frac{\phi(N)}{4}[k\Delta:R_N]} = (aN)^{\frac{[k\Delta:\mathbb{Q}]}{2}} \tag{11.19}$$

using (11.15) and (11.17) and subject to the restrictions on the values of N as listed following (11.17). Using all of the estimates in (11.18) we obtain

$$4 > 8\left(\frac{(aN)^{3/4}}{4\pi^2}\right)^{[k\Delta:\mathbb{Q}]} \tag{11.20}$$

for $N \geq 19$ when $a = 1$. However, if $N^{3/4} \geq 4\pi^2$, this inequality fails to hold and no such group can be arithmetic. This then implies that there are finitely many arithmetic Fuchsian triangle groups, and for these, $N \leq 134$.

Theorem 11.3.6 *There are finitely many arithmetic Fuchsian triangle groups* (e_1, e_2, e_3) *and all* $e_i \leq 134$.

We now make use of the other values of a to reduce the bound on N, so that a search for all arithmetic triangle groups can be made. The values $a = 3, 4, 5$ employed in (11.20) imply, respectively, that $N \leq 44, 33, 26$. Combined with the restrictions for which (11.17) holds, this yields that N must belong to the following set:

$$\mathcal{I} = \{2, 3, \ldots, 26, 28, 30, 36, 42, 60\}.$$

From this, a complete list of all triangle groups which are eligible to be arithmetic can be determined and whether or not they are arithmetic, tested using the condition given in Theorem 8.3.11. This condition is that $\sigma(\lambda) < 0$, where

$$\lambda = 4\cos^2 \pi/e_1 + 4\cos^2 \pi/e_2 + 4\cos^2 \pi/e_3 + 8\cos \pi/e_1 \cos \pi/e_2 \cos \pi/e_3 - 4$$

and σ is any non-identity element of $\mathrm{Gal}(k\Delta \mid \mathbb{Q})$. Not surprisingly, this has a simplifying geometric interpretation as follows.

If α, β and γ are three angles in the interval $(0, \pi)$ such that one of the triples $(\alpha, \beta, \gamma), (\alpha, \pi - \beta, \pi - \gamma)$, $(\pi - \alpha, \beta, \pi - \gamma)$ and $(\pi - \alpha, \pi - \beta, \gamma)$ has angle sum less than π, then we say that the triple (α, β, γ) satisfies *the angle sum condition*. In that case, we can construct a hyperbolic triangle whose angles are those in one of these triples. This hyperbolic triangle will have inscribed radius R which satisfies

$$\tanh^2 R = \frac{\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2\cos\alpha \cos\beta \cos\gamma - 1}{2(1 \pm \cos\alpha)(1 \pm \cos\beta)(1 \pm \cos\gamma)},$$

necessarily positive. Thus if there exists an element x in the appropriate Galois group $\mathbb{Z}_N^*$, $x \neq \pm 1$ such that the angle sum condition holds for $\pm x\pi/e_1$, $\pm x\pi/e_2$ and $\pm x\pi/e_3$, with $\pm$ chosen so that, mod 2π, the angles lie in the interval $(0, \pi)$, this will violate the condition of Theorem 8.3.11 and the group will fail to be arithmetic. We also note that this is a necessary and sufficient condition, because if it fails to hold, then all four triples have angle sum greater than π so that this defines a triangle on the 2-sphere. The group generated by reflections in the sides is then a subgroup of $\mathrm{SU}(2, \mathbb{C})$, which is the required ramification condition for the quaternion algebra to define an arithmetic group.

Testing some list of eligible triples eventually has to be done, but we can use the arithmetic data already employed, but more accurately, to reduce the list. This is illustrated here with the one case where $N = 36$.

Case A. In these cases, $k\Delta = R_e$ or R_{2e}, so that using (11.16), the right-hand side of (11.18) can be determined precisely for $k\Delta = R_{36}$ or R_{72} as approximately 2.98 or $(2.4)^{12}$. Thus for $k\Delta = R_{72}$, (11.18) can never hold, and for $k\Delta = R_{36}$, we must have $i(e_1, e_2, e_3) = 4$. Thus the possible triangle groups which satisfy (11.18) have $i(e_1, e_2, e_3) = 4$, and $k\Delta = R_{36}$

are $(6,36,36), (12,12,18), (12,18,36)$ and $(18,36,36)$. For these four triples, the angle sum condition is easily seen to be satisfied for respectively $x = 11, 11, 11, 5$.

Case B. In these cases, $k\Delta \supset R_f.R_{e_k}$, the compositum of R_f and R_{e_k}. Furthermore, since $(\Delta_{R_f}, \Delta_{R_{e_k}}) = 1$ by the description from (11.16),

$$\Delta_{R_f.R_{e_k}} = (\Delta_{R_f})^{[R_{e_k}:\mathbb{Q}]}(\Delta_{R_{e_k}})^{[R_f:\mathbb{Q}]}$$

(see Exercise 0.3, No 5). Now apart from the case where $N = 2$, $\Delta_{R_N} = A_N^{\phi(N)/2}$ for some $A_N \geq 1$. If $\{N, M\} = \{f, e_k\}$, we have for $N = 36$, $A_{36} = 6\sqrt{3}$, and the right-hand side of (11.18) becomes

$$\left(\frac{(6\sqrt{3}A_M)^{3/2}}{4\pi^2}\right)^{[k\Delta:\mathbb{Q}]}.$$

It follows that either $M = 2$ or $A_M = 1$, in which case $M = 3, 4, 6$. Enumerating the relevant triples as earlier yields the following six triples: $(2,4,36)$, $(2,6,36)$, $(2,12,36)$, $(2,12,18)$, $(2,18,36)$ and $(2,36,36)$ all of which can be ruled out by the angle sum condition.

Case C. One of e_1, e_2 or e_3 is 36 and, arguing as in the preceding case, shows that the other two must belong to the set $\{2,3,4,6\}$. However, then there are no such triples satisfying the conditions of Case C.

We thus conclude that there are no arithmetic triangle groups (e_1, e_2, e_3) where e, g and h, as defined in Cases A,B and C, are equal to 36.

By these methods, all arithmetic Fuchsian triangle groups can be enumerated and the results are given in Appendix 13.3. Once the triple is determined, the corresponding Hilbert symbol for the related quaternion algebra can be calculated as shown in §8.3 and, thus its isomorphism class determined. This then enables the arithmetic Fuchsian triangle groups to be gathered together into commensurability classes. These classes and the arithmetic data are also given in Appendix 13.3.

11.3.4 Signatures of Arithmetic Fuchsian Groups

For Fuchsian groups, all volumes are rational multiples of 2π, which simplifies volume calculations. Furthermore, since the rational multiple is just the negative of the Euler characteristic, any prime power appearing in the denominator will be the order of an element in the group. Thus if the covolume of $P\rho(\mathcal{O}^1)$, where $\mathcal{O}$ is a maximal order in a quaternion algebra over k, is $2\pi q$, then the denominator of q can only be divisible by prime powers p^n such that $\mathbb{Q}(\cos 2\pi/p^n) \subset k$.

Consider the following example, which arises in the extremal examples to be considered in §12.3. Let $k = \mathbb{Q}(t)$, where $p(t) = t^5+t^4-5t^3-3t^2+2t+1 =$

0. This field is totally real. Let A be a quaternion algebra over k which has four real ramified places and no finite ramification. For a maximal order $\mathcal{O}$ in A, we can estimate the covolume of $P\rho(\mathcal{O}^1)$ and, in this case, determine the group's signature. Since k has degree 5 over $\mathbb{Q}$, k contains no proper subfields other than $\mathbb{Q}$. Furthermore, k has discriminant 36497, which is prime, and hence consideration of the traces of elements of finite order shows that the group $P\rho(\mathcal{O}^1)$ can only contain elements of orders 2 and 3. Hence its covolume is of the form $2\pi a/b$, where $b \mid 6$. Now $R_k = \mathbb{Z}[t]$, and applying Kummer's Theorem to $p(t)$, we find that there are only two prime ideals in R_k of norm ≤ 17 and these have norms 3 and 13. Thus using the crude estimate at (11.14) with $p_0 = 17$ and substituting the upper and lower estimates in the volume formula (11.6) gives

$$0.65\pi \leq \mathrm{Covol}(P\rho(\mathcal{O}^1)) \leq 0.882\pi.$$

Thus $P\rho(\mathcal{O}^1)$ has covolume $2\pi/3$ and so must have signature $(0; 2, 2, 3, 3)$.

In this discussion, it is worth noting that we have actually defined five distinct quaternion algebras A. For, since k is not a Galois extension of $\mathbb{Q}$ the five real embeddings yield five different subfields of $\mathbb{R}$. Thus the five different choices of ramification set yield five distinct commensurability classes of arithmetic Fuchsian groups (see the discussion in §8.4). For all of these, and for each maximal order $\mathcal{O}$, $P\rho(\mathcal{O}^1)$ has signature $(0; 2, 2, 3, 3)$. However, the type number in each case can, and indeed does, vary. To establish this, we again use (6.12) and Theorem 6.7.6. The class number is 1 and the free basis of R_k^* can be computed to be $r_1 = t, r_2 = 1 + t, r_3 = 5 + 14t + 7t^2 - 4t^3 - 2t^4$ and $r_4 = 6 + 18t + 12t^2 - 5t^3 - 3t^4$. Each real embedding σ_i correspomds to a root t_i of the minimum polynomial and we order these such that $t_1 < t_2 < t_3 < t_4 < t_5$. We tabulate the signs of the basis elements at these embeddings.

	r_1	r_2	r_3	r_4
σ_1	$-$	$-$	$-$	$-$
σ_2	$-$	$+$	$-$	$+$
σ_3	$-$	$+$	$+$	$+$
σ_4	$+$	$+$	$+$	$+$
σ_5	$+$	$+$	$-$	$+$

From this it follows that if A is unramified at σ_1, then the type number is 2, whereas if A is unramified at $\sigma_i, i = 2, 3, 4, 5$, then the type number is 1.

Exercise 11.3

1. *Show, using the triangle group* $(0; 2, 3, 8)$*, that* $\zeta_k(-1) = 1/12$ *for* $k = \mathbb{Q}(\sqrt{2})$.

2. *Use Odlyzko's estimate to show that for any arithmetic Fuchsian group of signature* $(1; 2; 0)$*, the defining totally real field has degree no greater than 8 over* $\mathbb{Q}$.

3. Let $k = \mathbb{Q}(t)$, *where* $t^4 + t^3 - 3t^2 - t + 1 = 0$. *Let* A *be a quaternion algebra over* k *which is ramified at three real places and at one non-dyadic prime ideal* $\mathcal{P}$. *Let* $\mathcal{O}$ *be a maximal order in* A. *If it is known that* $P\rho(\mathcal{O}^1)$ *has signature* $(0; 2, 2, 5, 5)$, *determine* $N(\mathcal{P})$.

11.4 Maximal Discrete Groups

Let A be a quaternion algebra defining a commensurability class $\mathcal{C}(A)$ of either arithmetic Kleinian groups or arithmetic Fuchsian groups. In this section, a subset of $\mathcal{C}(A)$ is described which almost consists of the maximal elements of $\mathcal{C}(A)$. Here, "almost" means that this subset certainly contains all of the maximal elements but can also contain some other groups which may not be maximal. However, this subset is sufficiently close to the set of all maximal groups that, from it, effective results on maximal groups can be obtained, and detailed information on the distribution of the covolumes of members of $\mathcal{C}(A)$ will be deduced from it in the next section.

This subset of mainly maximal groups is obtained via local-global arguments, by prescribing that local groups should be maximal. Thus let us recall the local cases. Let A be as above and let $\mathcal{O}$ be a maximal order in A. When $\mathcal{P} \in \text{Ram}_f(A)$, $\mathcal{O}_\mathcal{P}$ is the unique maximal order in $A_\mathcal{P}$ and so the normaliser $N(\mathcal{O}_\mathcal{P}) = A^*_\mathcal{P}$. Thus, at this prime, $P(N(\mathcal{O}_\mathcal{P}))$ is certainly a maximal subgroup of $A^*_\mathcal{P}$. (Here, and throughout this section, $P(G)$ denotes the factor group $G/Z(G)$.)

When $\mathcal{P} \in \Omega_f \setminus \text{Ram}_f(A)$, then $A_\mathcal{P} \cong M_2(k_\mathcal{P})$ and $\mathcal{O}_\mathcal{P}$ is conjugate to $M_2(R_\mathcal{P})$. This situation was discussed in §6.5 and we extend that discussion here, in particular with respect to the tree $\mathcal{T}_\mathcal{P}$ of maximal orders.

Thus, for the moment, let $K = k_\mathcal{P}$ and $R = R_\mathcal{P}$. Let V be a two-dimensional space over K so that $\text{End}(V) = M_2(K)$. The vertices of the tree $\mathcal{T}$ are represented by equivalence classes Λ of complete R-lattices L or by the corresponding maximal orders $\text{End}(L)$. The geometric edges are represented by pairs $\{\Lambda_1, \Lambda_2\}$ of vertices at distance 1. An edge can also be represented by an Eichler order of level $\mathcal{P}$ (see §6.1.1 and §6.6.6), which is $\text{End}(L_1) \cap \text{End}(L_2)$ where L_1 and L_2 are representatives of the equivalence classes Λ_1 and Λ_2 respectively of lattices.

The group $\text{SL}(2, K)$ acts on the tree $\mathcal{T}$ by translating the lattices or conjugating the maximal orders, thus preserving distances. Under this action, there are two orbits of vertices, as will now be shown. Take any pair of adjacent vertices Λ and Λ' represented by lattices L and L', respectively, where L has basis $\{e_1, e_2\}$ and L' has basis $\{\pi e_1, e_2\}$. If L_1 is any other lattice, then the proof of Theorem 2.2.9 shows that L_1 has a basis $\{\pi^a(e_1 - \gamma e_2), \pi^b e_2\}$ for some $a, b \in \mathbb{Z}$ and $\gamma \in K$. Let $a - b = 2n + \epsilon$, where $\epsilon = 0, 1$. Then the element $\left(\begin{smallmatrix} \pi^{-n} & 0 \\ \gamma\pi^n & \pi^n \end{smallmatrix}\right) \in \text{SL}(2, K)$ maps L_1 to a multiple of L or L' according as to whether $\epsilon = 0$ or 1. Thus all vertices in the same orbit are an even distance from each other in the tree $\mathcal{T}$ (cf. Exercise 6.5, No. 4 and see also Exercise 11.4, No. 1).

Now let us consider the action of $\mathrm{GL}(2, K)$ on the tree $\mathcal{T}$. It acts transitively on the vertices by Theorem 6.5.3 and an element of $\mathrm{GL}(2, K)$ is called *even* or *odd* according to whether it leaves the two orbits of vertices of $\mathcal{T}$ under $\mathrm{SL}(2, K)$ invariant or interchanges them (see also Exercise 11.4, No. 2). Since the centre of $\mathrm{GL}(2, K)$ acts trivially on $\mathcal{T}$, this terminology also applies to elements of $\mathrm{PGL}(2, K)$. Under this action, the stabiliser of a vertex, given by a maximal order $\mathcal{O}$, can be identified with $P(N(\mathcal{O}))$, which is $P(\mathcal{O}^*)$ since $N(\mathcal{O}) = K^*\mathcal{O}^*$ in this case (see Exercise 6.5, No. 1). Thus this group $P(N(\mathcal{O}))$ is a maximal compact open subgroup of $\mathrm{PGL}(2, K)$.

The odd element $\left(\begin{smallmatrix} 0 & \pi \\ 1 & 0 \end{smallmatrix}\right)$ maps the lattice L with basis $\{e_1, e_2\}$ to the lattice L' with basis $\{\pi e_1, e_2\}$ and maps L' to πL and so lies in the stabiliser of the geometric edge $\{\Lambda, \Lambda'\}$. Identifying this edge with the Eichler order $\mathcal{E} = \mathrm{End}(L) \cap \mathrm{End}(L')$, its stabiliser under the action of $\mathrm{PGL}(2, K)$ will be $P(N(\mathcal{E}))$, which is also a maximal compact open subgroup. Now since $\mathrm{PSL}(2, K) \subset \mathrm{PGL}(2, K)$ and $\mathrm{PSL}(2, K)$ acts transitively on the edges of $\mathcal{T}$, $\mathrm{PGL}(2, K)$ has two conjugacy classes of maximal compact open subgroups. Also, any maximal compact open subgroup of $\mathrm{PGL}(2, K)$ is conjugate to the stabiliser of an edge if and only if it contains odd elements. It follows that the stabiliser of a vertex, $P(N(\mathcal{O}))$, only fixes the vertex $\mathcal{O}$. For, since it fixes $\mathcal{O}$ it must consist of even elements only. If it fixed another point of the tree, it would fix the unique path between these points, and hence would fix an edge. However, that would force this maximal compact open subgroup to contain odd elements. Thus it fixes a unique vertex.

Now let us return to the global situation and suppose that $\Gamma \in \mathcal{C}(A)$, where A is defined over the number field k and $\mathcal{O}$ is a maximal order in A. Thus Γ is commensurable with $P\rho(\mathcal{O}^1)$, where ρ is a k-representation of A into $M_2(\mathbb{C})$ or $M_2(\mathbb{R})$. Now $P\rho(N(\mathcal{O}))$ lies in the normaliser of the finite-covolume group $P\rho(\mathcal{O}^1)$ so that $[P\rho(N(\mathcal{O})) : P\rho(\mathcal{O}^1)] < \infty$. Thus Γ is commensurable with $P\rho(N(\mathcal{O}))$, and therefore Γ lies in the commensurator, which is $P\rho(A^*)$ (see Theorem 8.4.4). Therefore, we can drop ρ and consider $\mathcal{C}(A)$ as consisting of the groups in $P(A^*)$ which are commensurable with $P(N(\mathcal{O}))$.

Lemma 11.4.1 *Let $\mathcal{D}$ be any order in the quaternion algebra A. Then,*

$$N(\mathcal{D}) = \{x \in A^* \mid x \in N(\mathcal{D}_\mathcal{P}) \ \forall \ \mathcal{P} \in \Omega_f\}.$$

Proof: It is clear that for each $x \in N(\mathcal{D})$, $x \in N(\mathcal{D}_\mathcal{P})$ for all $\mathcal{P}$. Now suppose that $x \in A^*$ is such that $x \in N(\mathcal{D}_\mathcal{P})$ for all $\mathcal{P}$. Let $x\mathcal{D}x^{-1} = \mathcal{D}'$. Thus $\mathcal{D}'_\mathcal{P} = \mathcal{D}_\mathcal{P}$ for all $\mathcal{P}$ and therefore, $\mathcal{D} = \mathcal{D}'$ by Lemma 6.2.7. □

Now for $\Gamma \in \mathcal{C}(A)$, the closure of Γ in $P(A^*_\mathcal{P})$ is a compact open subgroup of $P(A^*_\mathcal{P})$ which coincides with $P(N(\mathcal{O}_\mathcal{P}))$ for almost all $\mathcal{P} \in \Omega_f$ and will be contained in a maximal compact open subgroup of $P(A^*_{\mathcal{P}})$ for each $\mathcal{P}$. Thus consider the following family of groups in $\mathcal{C}(A)$:

Let $\mathcal{O}$ be a maximal order in A and let S be a finite set of primes disjoint from $\mathrm{Ram}_f(A)$. For each $\mathcal{P} \in S$, choose a maximal order $(\mathcal{O}_{\mathcal{P}})'$ such that $d(\mathcal{O}_{\mathcal{P}}, (\mathcal{O}_{\mathcal{P}})') = 1$, where d is the distance in the tree $\mathcal{T}_{\mathcal{P}}$. Let $\mathcal{O}'$ be the maximal order in A such that $\mathcal{O}'_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}}$ for $\mathcal{P} \notin S$ and $\mathcal{O}'_{\mathcal{P}} = (\mathcal{O}_{\mathcal{P}})'$ for $\mathcal{P} \in S$ (see Lemma 6.2.7). Let $\mathcal{E} = \mathcal{O} \cap \mathcal{O}'$. Thus $\mathcal{E}$ is an Eichler order except in the cases where $S = \emptyset$ when $\mathcal{E} = \mathcal{O}$, a maximal order.

Definition 11.4.2
$$\Gamma_{S,\mathcal{O}} = P(N(\mathcal{E})).$$

Notice that, in this definition, $\mathcal{E}$ is not uniquely defined by the set S because of the choices involved. However, from Exercise 11.4, No. 1, $\mathcal{O}^1_{\mathcal{P}}$ acts transitively on the edges of the tree adjacent to $\mathcal{O}_{\mathcal{P}}$. Suppose then that the Eichler orders $\mathcal{E}' = \mathcal{O} \cap \mathcal{O}'$ and $\mathcal{E}'' = \mathcal{O} \cap \mathcal{O}''$ are obtained from the two choices of orders $(\mathcal{O}_{\mathcal{P}})'$ and $(\mathcal{O}_{\mathcal{P}})''$, respectively, for each $\mathcal{P} \in S$. There exist elements $x_{\mathcal{P}} \in \mathcal{O}^1_{\mathcal{P}}$ such that $x_{\mathcal{P}}(\mathcal{O}_{\mathcal{P}})' x_{\mathcal{P}}^{-1} = (\mathcal{O}_{\mathcal{P}})''$ for each $\mathcal{P} \in S$. Now use the Strong Approximation Theorem 7.7.5 taking, in the statement of that theorem, $S = \Omega_\infty$. Then A^1_k is dense in the restricted product of the groups $A^1_{\mathcal{P}}$, $\mathcal{P} \in \Omega_f$, restricted with respect to the compact open subgroups $\mathcal{O}^1_{\mathcal{P}}$. Thus there exists an element $x \in A^1_k$ which is arbitrarily close to $x_{\mathcal{P}}$ for each $\mathcal{P} \in S$ and lies in $\mathcal{O}^1_{\mathcal{P}}$ otherwise. Thus $x \in \mathcal{O}^1$ and, by construction, $x\mathcal{E}'x^{-1} = \mathcal{E}''$. Thus the group $\Gamma_{S,\mathcal{O}}$ is defined, up to conjugacy, by $\mathcal{O}$ and the set S.

Theorem 11.4.3 *Let $\Gamma \in \mathcal{C}(A)$. Let $S(\Gamma)$ be the set of primes $\mathcal{P}$ such that Γ has an element which is odd at $\mathcal{P}$. Then there exists a maximal order $\mathcal{O}$ such that Γ is conjugate to a subgroup of $\Gamma_{S(\Gamma),\mathcal{O}}$ with equality if Γ is maximal.*

Proof: Let $\mathcal{D}$ be a maximal order so that the closure of Γ in $P(A^*_{\mathcal{P}})$ is a compact open subgroup which is equal to $P(N(\mathcal{D}_{\mathcal{P}}))$ for all $\mathcal{P} \in \Omega_f \setminus T$, where T is a finite set. Since all elements of $P(N(\mathcal{D}_{\mathcal{P}}))$ are even, $S(\Gamma)$ will be a finite set. By default, it is disjoint from $\mathrm{Ram}_f(A)$ since odd and even are not defined in these cases. For each $\mathcal{P} \in T$, we choose a maximal compact open subgroup of $P(A^*_{\mathcal{P}})$ containing the closure of Γ. If $\mathcal{P} \in T \setminus S(\Gamma)$, choose $P(N((\mathcal{D}_{\mathcal{P}})'))$, where $(\mathcal{D}_{\mathcal{P}})'$ is a maximal order. If $\mathcal{P} \in S(\Gamma)$, then the maximal compact open subgroup will be of the form $P(N(\mathcal{F}_{\mathcal{P}}))$, where $\mathcal{F}_{\mathcal{P}} = (\mathcal{D}_{\mathcal{P}})'' \cap (\mathcal{D}_{\mathcal{P}})'''$ is an Eichler order of level $\mathcal{P}$, with $(\mathcal{D}_{\mathcal{P}})''$ and $(\mathcal{D}_{\mathcal{P}})'''$ maximal orders in $A_{\mathcal{P}}$.

Let $\mathcal{O}$ (resp. $\mathcal{O}'$) be the maximal order such that $\mathcal{O}_{\mathcal{P}} = \mathcal{D}_{\mathcal{P}}$ (resp. $\mathcal{O}'_{\mathcal{P}} = \mathcal{D}_{\mathcal{P}}$) for $\mathcal{P} \notin T$, $\mathcal{O}_{\mathcal{P}} = (\mathcal{D}_{\mathcal{P}})'$ (resp. $\mathcal{O}'_{\mathcal{P}} = (\mathcal{D}_{\mathcal{P}})'$) for $\mathcal{P} \in T \setminus S(\Gamma)$ and $\mathcal{O}_{\mathcal{P}} = (\mathcal{D}_{\mathcal{P}})''$ (resp. $\mathcal{O}'_{\mathcal{P}} = (\mathcal{D}_{\mathcal{P}})'''$) for $\mathcal{P} \in S(\Gamma)$. Let $\mathcal{E} = \mathcal{O} \cap \mathcal{O}'$. Then by

Lemma 11.4.1, $\Gamma \subset P(N(\mathcal{E}))$. Since $P(N(\mathcal{E}))$ is of finite covolume, there will be equality if Γ is maximal. $\square$

Note that if $\mathcal{O}$ and $\mathcal{O}'$ are conjugate maximal orders, then $\Gamma_{S,\mathcal{O}}$ and $\Gamma_{S,\mathcal{O}'}$ are conjugate. Thus it suffices to consider only one maximal order from each type. As has already been shown in Theorem 6.7.6, the type number is finite. In fact, the groups $\Gamma_{S,\mathcal{O}}$ described above for the conjugacy classes of maximal orders can all be described in terms of a single maximal order $\mathcal{O}$ as follows. Fix a maximal order $\mathcal{O}$. From a different conjugacy class of maximal orders, we can choose a representative $\mathcal{O}'$ such that there is a finite set of primes S' disjoint from $\mathrm{Ram}_f(A)$ such that $\mathcal{O}'_\mathcal{P} = \mathcal{O}_\mathcal{P}$ for $\mathcal{P} \notin S'$ and $d(\mathcal{O}_\mathcal{P}, \mathcal{O}'_\mathcal{P}) = 1$ for $\mathcal{P} \in S'$ by Corollary 6.7.8. Thus a group of the form $\Gamma_{S,\mathcal{O}'}$ can alternatively be described by the sets S and S' relative to the fixed maximal order $\mathcal{O}$. Again, using the Strong Approximation Theorem, this would depend, up to conjugation, only on the sets S and S'.

If, in Definition 11.4.2, $S = \emptyset$, then the group $\Gamma_{\emptyset,\mathcal{O}}$ is maximal. To see this, applying the proof of Theorem 11.4.3 to the group $\Gamma_{\emptyset,\mathcal{O}}$, for each $\mathcal{P} \in T$, yields a unique maximal compact open subgroup containing the closure of $P(N(\mathcal{O}))$ since $P(N(\mathcal{O}_\mathcal{P}))$ has a unique fixed point in $\mathcal{T}_\mathcal{P}$.

However, when $S \neq \emptyset$, the group $\Gamma_{S,\mathcal{O}}$ may not be maximal. This could happen when, for some $\mathcal{P} \in S$, no element of $\Gamma_{S,\mathcal{O}}$ is odd at $\mathcal{P}$. Thus $\Gamma_{S,\mathcal{O}}$ would be a proper subgroup of $\Gamma_{S',\mathcal{O}}$ where $S' = S \setminus \{\mathcal{P}\}$ (see below and Exercise 11.4, No. 4).

Nonetheless, we can establish that there are infinitely many conjugacy classes of maximal elements in $\mathcal{C}(A)$. We first show that for each prime ideal $\mathcal{P}$, there are groups in $\mathcal{C}(A)$ which contain an element odd at $\mathcal{P}$. To do this, choose $c \in k$ such that $c \in \mathcal{P} \setminus \mathcal{P}^2$ and c is positive at all real ramified places of A. Such a c exists by the Approximation Theorem 7.2.6. By the Norm Theorem 7.4.1, let $x \in A^*$ be such that $n(x) = c$. Let T be the finite set of primes such that $P(x) \notin P(N(\mathcal{O}_\mathcal{P}))$. For each such $\mathcal{P}$, choose $(\mathcal{O}_\mathcal{P})'$ adjacent to $\mathcal{O}_\mathcal{P}$ and hence, as earlier, obtain the maximal order $\mathcal{O}'$ and Eichler order $\mathcal{E} = \mathcal{O} \cap \mathcal{O}'$. For each $\mathcal{P} \in T$, let $h_\mathcal{P} \in A^1_\mathcal{P}$ be such that $h_\mathcal{P} x \in N(\mathcal{E}_\mathcal{P})$. By the Strong Approximation Theorem, there exists $h \in A^1$ such that h is arbitrarily close to $h_\mathcal{P}$ for all $\mathcal{P} \in T$ and $h \in \mathcal{O}^1_\mathcal{P}$ otherwise. Let $g = hx \in A^*$, and, by construction, $P(g)$ lies in the arithmetic group $P(N(\mathcal{E})) = \Gamma_{T,\mathcal{O}}$. Also by construction, $P(g)$ is odd at $\mathcal{P}$ (see Exercise 11.4 No. 2). Thus for any prime $\mathcal{P}$, we have constructed a group in $\mathcal{C}(A)$ which is odd at $\mathcal{P}$.

If Γ contains an element odd at $\mathcal{P}$, then Γ cannot be conjugate to a subgroup of $\Gamma_{S,\mathcal{O}}$ if $\mathcal{P} \notin S$, because if $\mathcal{P} \notin S$, $\Gamma_{S,\mathcal{O}} \subset P(N(\mathcal{O}_\mathcal{P}))$, where $\mathcal{O}_\mathcal{P}$ is maximal and consists entirely of even elements. Hence, the following is readily shown (see Exercise 11.4, No. 3):

Theorem 11.4.4 $\mathcal{C}(A)$ *contains infinitely many non-conjugate maximal elements.*

In the above construction of a group containing an element odd at $\mathcal{P}$, that element may, of course, also be odd at other primes. This will depend (see Exercise 11.4, No. 2) on its norm, and, hence on the choice of $c \in k$, as described in the discussion prior to Theorem 11.4.4. Thus to show that the group $\Gamma_{S,\mathcal{O}}$, where $S = \{\mathcal{P}\}$, contains an element odd at $\mathcal{P}$, it is necessary and sufficient that there exist an element $c \in k$ such that $\nu_{\mathcal{P}}(c)$ is odd, c is positive at all real ramified places of A, and $\nu_{\mathcal{Q}}(c)$ is even for all $\mathcal{Q} \in \Omega_f \setminus \{\mathrm{Ram}_f(A), \mathcal{P}\}$. Such elements may not exist and so $\Gamma_{S,\mathcal{O}}$ will not be maximal, as $\Gamma_{S,\mathcal{O}} \subset \Gamma_{\emptyset,\mathcal{O}}$ (see Exercise 11.4, No. 4). On the other hand, they will always exist when $A = M_2(\mathbb{Q}(\sqrt{-d}))$ and O_d is a principal ideal domain (cf. Exercise 11.4, No. 5).

Exercise 11.4

1. *If K is a $\mathcal{P}$-adic field with ring of integers R, show that in the tree $\mathcal{T}$ of maximal orders in $A = M_2(K)$, the group $\mathcal{O}^1$, for $\mathcal{O}$ a maximal order, acts transitively on the edges of $\mathcal{T}$ adjacent to $\mathcal{O}$ and deduce that A^1 acts transitively on the edges of $\mathcal{T}$.*

2. *With notation as in No. 1, show that $x \in \mathrm{GL}(2, K)$ is even or odd according as to whether $\nu(\det(x))$ is even or odd, where ν is the logarithmic valuation on K.*

3. *Complete the proof of Theorem 11.4.4.*

4. *Let $k = \mathbb{Q}(\sqrt{-6})$ with ring of integers O_6. Show that there is no element $c \in k$ such that $\nu_{\mathcal{P}}(c)$ is odd for $\mathcal{P} = \mathcal{P}_2$ and $\nu_{\mathcal{Q}}(c)$ is even for all other primes $\mathcal{Q}$ (here ν denotes the logarithmic valuation). Deduce that for $A = M_2(\mathbb{Q}(\sqrt{-6}))$ and $\mathcal{O} = M_2(O_6)$, $S = \{\mathcal{P}_2\}$, the group $\Gamma_{S,\mathcal{O}}$ is not maximal.*

5. *In the case where $A = M_2(\mathbb{Q})$, describe precisely the maximal elements in $\mathcal{C}(A)$ up to conjugacy.*

11.5 Distribution of Volumes

The maximal elements of $\mathcal{C}(A)$ as described in the preceding section are among the groups $P(N(\mathcal{O}))$ where $\mathcal{O}$ is either a maximal order or an Eichler order of level $\mathcal{P}_1\mathcal{P}_2\cdots\mathcal{P}_r$ where these $\mathcal{P}_i$ are distinct primes, not belonging to $\mathrm{Ram}_f(A)$. These groups are, of course, finite extensions of the groups $P(\mathcal{O}^1)$ whose covolumes, in the case of maximal orders, have been determined in §11.1. For the case of Eichler orders, see §11.2.2. In this section, the distribution of the covolumes of elements of $\mathcal{C}(A)$ will be determined.

For groups $\Gamma_1, \Gamma_2 \in \mathcal{C}(A)$, we continue the practice of using generalised indices $[\Gamma_1 : \Gamma_2]$ introduced at (11.11).

Theorem 11.5.1 *For $\mathcal{O}$ a maximal order in A,*

$$[\Gamma_{\emptyset,\mathcal{O}} : \Gamma_{S,\mathcal{O}}] = 2^{-m} \prod_{\mathcal{P} \in S} (N(\mathcal{P}) + 1) \tag{11.21}$$

for some $0 \leq m \leq |S|$. *Also, if $\mathcal{O}'$ is another maximal order,*

$$[\Gamma_{\emptyset,\mathcal{O}} : \Gamma_{\emptyset,\mathcal{O}'}] = 1. \tag{11.22}$$

Proof: Let $\mathcal{O}$ and $\mathcal{O}'$ be maximal orders of A such that $\mathcal{O} \cap \mathcal{O}'$ is an Eichler order of level $\prod \mathcal{P}$ for $\mathcal{P} \in S$.

Now $\Gamma_1 = \Gamma_{\emptyset,\mathcal{O}} \cap \Gamma_{S,\mathcal{O}}$ consists of those elements of $\Gamma_{\emptyset,\mathcal{O}} = P(N(\mathcal{O}))$ whose action on $\mathcal{T}_{\mathcal{P}}$, $\mathcal{P} \in S$, is to fix pointwise the edge $(\mathcal{O}_{\mathcal{P}}, \mathcal{O}'_{\mathcal{P}})$. Thus the index $[\Gamma_{\emptyset,\mathcal{O}} : \Gamma_1]$ is no greater than $\prod_{\mathcal{P} \in S}(N(\mathcal{P}) + 1)$. Thus, either by using the Strong Approximation Theorem or the result from §11.2.2 which gives $[P(\mathcal{O}^1) : P((\mathcal{O} \cap \mathcal{O}')^1)] = \prod_{\mathcal{P} \in S}(N(\mathcal{P}) + 1)$, we obtain

$$[\Gamma_{\emptyset,\mathcal{O}} : \Gamma_1] = \prod_{\mathcal{P} \in S} (N(\mathcal{P}) + 1). \tag{11.23}$$

Note that $\Gamma_1 \subset P(N(\mathcal{E}))$, where $\mathcal{E} = \mathcal{O} \cap \mathcal{O}'$ and $P(N(\mathcal{E})) = \Gamma_{S,\mathcal{O}}$. If $\gamma \in P(N(\mathcal{E}))$ fixes the edge $(\mathcal{O}_{\mathcal{P}}, \mathcal{O}'_{\mathcal{P}})$ pointwise for each $\mathcal{P} \in S$, then $\gamma \in \Gamma_1$. Thus $[P(N(\mathcal{E})) : \Gamma_1] \leq 2^{|S|}$. As discussed at the end of the preceding section, there may or may not be elements in $\Gamma_{S,\mathcal{O}}$ which are odd only at $\mathcal{P}$ for each $\mathcal{P} \in S$. Thus (11.21) follows.

Recall from Corollary 6.7.8 that for any two maximal orders $\mathcal{O}$ and $\mathcal{O}'$ we can assume that, up to conjugacy, for the groups $\Gamma_{\emptyset,\mathcal{O}}$ and $\Gamma_{\emptyset,\mathcal{O}'}$, the orders can be chosen so that $\mathcal{O} \cap \mathcal{O}'$ is an Eichler order of level $\mathcal{P}$ for $\mathcal{P} \in$ some finite set S'. Thus (11.22) follows from (11.23). $\square$

Theorem 11.5.2 *Let e be the number of primes in k dividing* 2 *and not contained in* $\text{Ram}_f(A)$. *Let $\mathcal{O}$ be a maximal order in A and let $\Gamma \in \mathcal{C}(A)$. Then the covolume of Γ is an integral multiple of* $2^{-e}\text{Covol}(\Gamma_{\emptyset,\mathcal{O}})$. *Furthermore,* $\text{Covol}(\Gamma) = \text{Covol}(\Gamma_{\emptyset,\mathcal{O}})$ *if and only if Γ is conjugate to $\Gamma_{\emptyset,\mathcal{O}'}$ for some maximal order $\mathcal{O}'$ and* $\text{Covol}(\Gamma) > \text{Covol}(\Gamma_{\emptyset,\mathcal{O}})$ *in all other cases.*

Proof: By Theorem 11.4.3, $\text{Covol}(\Gamma)$ is an integral multiple of $\text{Covol}(\Gamma_{S,\mathcal{O}})$ for some maximal order $\mathcal{O}$. The right-hand side of (11.21) is a multiple of $\prod_{\mathcal{P} \in S} \frac{N(\mathcal{P})+1}{2}$. If $\mathcal{P}$ is non-dyadic, then $(N(\mathcal{P}) + 1)/2 \in \mathbb{Z}$ and so $\text{Covol}(\Gamma_{S,\mathcal{O}})$ will be an integral multiple of $2^{-e}\text{Covol}(\Gamma_{\emptyset,\mathcal{O}})$. From (11.22), $\text{Covol}(\Gamma_{\emptyset,\mathcal{O}})$ does not depend on the choice of maximal order. For all choices of $\mathcal{P}$, dyadic or otherwise, $(N(\mathcal{P}) + 1)/2 > 1$, so that when $S \neq \emptyset$, $\text{Covol}(\Gamma_{S,\mathcal{O}}) > \text{Covol}(\Gamma_{\emptyset,\mathcal{O}})$. $\square$

Exercise 11.5

1. Determine the covolume of the maximal Kleinian groups commensurable with $\text{PSL}(2, O_3)$.

2. Prove that there are two conjugacy classes in $\mathrm{PGL}(2, \mathbb{R})$ *of Fuchsian groups with signature* $(0; 2, 2, 2, 3)$ *which are commensurable with the Fuchsian triangle group* Γ_0 *of signature* $(0; 2, 3, 8)$ *but are not conjugate to subgroups of* Γ_0.

3. Let A *be a quaternion algebra over a field* k *defining arithmetic Kleinian groups such that* A *has type number* > 1. *Let* $\mathcal{O}$ *and* $\mathcal{O}'$ *be non-conjugate maximal orders. Show that* $\Gamma_{\emptyset,\mathcal{O}}$ *and* $\Gamma_{\emptyset,\mathcal{O}'}$ *cannot be isomorphic.*

11.6 Minimal Covolume

From the preceding section, the minimal covolume of an arithmetic Kleinian or Fuchsian group in $\mathcal{C}(A)$ is achieved by $\Gamma_{\emptyset,\mathcal{O}} = P(N(\mathcal{O}))$, where $\mathcal{O}$ is a maximal order. It should be noted here that in the Fuchsian case, the above discussion refers to subgroups of $\mathrm{PGL}(2, \mathbb{R})$ rather than $\mathrm{PSL}(2, \mathbb{R})$, so that Fuchsian here should be interpreted as discrete in $\mathrm{PGL}(2, \mathbb{R})$. Since the covolume of $P(\mathcal{O}^1)$ was computed in §11.1, it remains to determine the index

$$[\Gamma_{\emptyset,\mathcal{O}} : P(\mathcal{O}^1)].$$

Here, as earlier, the embedding ρ has been dropped. Following Borel, we introduce some intermediate groups and simplify our notation in order to analyse this index. First we gather together the necessary terminology:

- R_k^* = Group of units of R_k
- $R_{k,\infty}^*$ = Subgroup of units which are positive at all real ramified places of A
- r_f = Number of places in $\mathrm{Ram}_f(A)$
- R_f = Ring of elements in k which are integral at all finite places of k not in $\mathrm{Ram}_f(A)$
- R_f^* = Group of units of R_f
- $R_{f,\infty}^*$ = Subgroup of units which are positive at all real ramified places of A
- I_k = Group of fractional ideals of k
- P_k = Subgroup of principal fractional ideals
- $P_{k,\infty}$ = Subgroup of principal fractional ideals which have a generator which is positive at all real ramified places of A
- M_1 = Subgroup of I_k generated by $P_{k,\infty}$ and those ideals $\mathcal{P} \in \mathrm{Ram}_f(A)$

- $J_1 = I_k/M_1$
- $J_2 =$ Image of P_k in J_1
- ${}_2J_1 =$ Kernel of the mapping $y \mapsto y^2$ in J_1

Throughout the remainder of this section, $\mathcal{O}$ will be a fixed maximal order in the quaternion algebra A over k. Recall that A satisfies the Eichler condition.

Theorem 11.6.1 (Eichler)

$$n(\mathcal{O}^*) = R^*_{k,\infty}.$$

Proof: Clearly $n(\mathcal{O}^*) \subset R^*_{k,\infty}$, so suppose that $t \in R^*_{k,\infty}$. By the Norm Theorem 7.4.1, there exists $\alpha \in A^*$ such that $n(\alpha) = t$. For all but a finite set S of primes, $\alpha, \alpha^{-1} \in \mathcal{O}_{\mathcal{P}}$. For $\mathcal{P} \in S$, it is not difficult to see that there exists $\gamma_{\mathcal{P}} \in \mathcal{O}^*_{\mathcal{P}}$ such that $n(\gamma_{\mathcal{P}}) = t$ (see Exercise 6.7, No. 1). By the Strong Approximation Theorem, A^1_k is dense in the restricted product of the $A^1_{\mathcal{P}}$, $\mathcal{P} \in \Omega_f$. Thus there exists $\beta \in A^1_k$ such that β is arbitrarily close to $\alpha^{-1}\gamma_{\mathcal{P}}$ for $\mathcal{P} \in S$ and lies in $\mathcal{O}^1_{\mathcal{P}}$ otherwise. Thus $n(\alpha\beta) = t$ and both $\alpha\beta, \beta^{-1}\alpha^{-1} \in \mathcal{O}_{\mathcal{P}}$ for all $\mathcal{P}$. Thus $\alpha\beta \in \mathcal{O}^*$. $\square$

By local-global arguments, one readily establishes that

$$\mathcal{O}^1 = \{\alpha \in N(\mathcal{O}) \mid n(\alpha) = 1\}, \tag{11.24}$$
$$\mathcal{O}^* = \{\alpha \in N(\mathcal{O}) \mid n(\alpha) \in R^*_k\}. \tag{11.25}$$

On the basis of this, we adopt the following:

Notation 11.6.2

- $\Gamma_{\mathcal{O}} = \Gamma_{\emptyset,\mathcal{O}} = P(N(\mathcal{O}))$
- $\Gamma_{R_f} = P(A_{R_f})$ where $A_{R_f} = \{\alpha \in N(\mathcal{O}) \mid n(\alpha) \in R^*_f\}$
- $\Gamma_{\mathcal{O}^*} = P(\mathcal{O}^*)$
- $\Gamma_{\mathcal{O}^1} = P(\mathcal{O}^1)$

Thus, from (11.24) and (11.25)

$$\Gamma_{\mathcal{O}} \supset \Gamma_{R_f} \supset \Gamma_{\mathcal{O}^*} \supset \Gamma_{\mathcal{O}^1}. \tag{11.26}$$

Now $\Gamma_{\mathcal{O}^1}$ is a normal subgroup of $\Gamma_{\mathcal{O}}$, and as an arithmetic Kleinian or Fuchsian group, we already know that $\Gamma^{(2)}_{\mathcal{O}} \subset \Gamma_{\mathcal{O}^1}$.

Theorem 11.6.3

1. $\Gamma_{\mathcal{O}}/\Gamma_{\mathcal{O}^1}$ *is an elementary abelian 2-group*
2. $\Gamma_{R_f}/\Gamma_{\mathcal{O}^1} \cong R^*_{f,\infty}/(R^*_f)^2$

Proof: For Part *2*, let $\bar{n} : \Gamma_{R_f} \to R^*_{f,\infty}/(R^*_f)^2$ be defined by $\bar{n}(P(\alpha)) = n(\alpha)(R^*_f)^2$. This is then a well-defined homomorphism. By a similar argument to that used in Theorem 11.6.1, $\bar{n}$ is onto (see Exercise 11.6, No. 1). The result then easily follows. □

Corollary 11.6.4

$$[\Gamma_{R_f} : \Gamma_{\mathcal{O}^1}] \leq 2^{r_1+r_2+r_f}.$$

The Dirichlet Unit Theorem (see Theorem 0.4.2) can be extended to cover groups of S-units from which the above corollary follows. Thus this divisor of the index $[\Gamma_{\mathcal{O}} : \Gamma_{\mathcal{O}^1}]$ can be calculated starting from a knowledge of the group of units R^*_k. Finally, we need to determine $[\Gamma_{\mathcal{O}} : \Gamma_{R_f}]$ and this depends on the class number of k.

For this, first of all recall some notation and results from §6.7. Thus $\mathcal{LR}(\mathcal{O})$ is the group of two-sided ideals of $\mathcal{O}$ in A and the norm maps this group isomorphically onto $\mathcal{D}I^2_k$, the subgroup of fractional ideals of k generated by the the prime ideals $\mathcal{P} \in \mathrm{Ram}_f(A)$ and the squares of all prime ideals in k, by Lemma 6.7.5.

Now, $\alpha \in N(\mathcal{O})$ if and only if the principal ideal $\mathcal{O}\alpha$ is two-sided. Thus if $P\mathcal{LR}(\mathcal{O})$ denotes the subgroup of principal two-sided ideals of $\mathcal{O}$, then n maps $P\mathcal{LR}(\mathcal{O})$ isomorphically onto $P_{k,\infty} \cap \mathcal{D}I^2_k$ (see Exercise 11.6, No. 3).

Theorem 11.6.5 *With the notation as given in this section,*

$$[\Gamma_{\mathcal{O}} : \Gamma_{R_f}] = [{}_2J_1 : J_2]. \tag{11.27}$$

If k has class number 1*, then* $\Gamma_{\mathcal{O}} = \Gamma_{R_f}$.

Proof: When $\alpha \in N(\mathcal{O})$, then $\alpha \in \mathcal{O}^*_{\mathcal{P}}$ for all but a finite set S of primes. For $\mathcal{P} \in S \setminus \mathrm{Ram}_f(A)$, $\alpha \in t_{\mathcal{P}}\mathcal{O}^*_{\mathcal{P}}$ for some $t_{\mathcal{P}} \in k^*_{\mathcal{P}}$ (see Exercise 6.5, No. 1). Let $\mathcal{M}(\alpha)$ be the ideal of k defined locally by requiring that $\mathcal{M}(\alpha)_{\mathcal{P}} = R_{\mathcal{P}}$ if $\mathcal{P} \notin S$ or $\mathcal{P} \in \mathrm{Ram}_f(A)$, and $\mathcal{M}(\alpha)_{\mathcal{P}} = t_{\mathcal{P}}R_{\mathcal{P}}$ if $\mathcal{P} \in S \setminus \mathrm{Ram}_f(A)$. Then $\mathcal{M}(\alpha)$ is uniquely defined and $n(\alpha)R_k = \mathcal{M}(\alpha)^2 L$, where $L \in \mathcal{D}$.

Define $\tau : N(\mathcal{O}) \to J_1$ by $\tau(\alpha) = \mathcal{M}(\alpha)M_1$. By the above, $\mathcal{M}(\alpha)^2 \in M_1$ so that $\tau(\alpha) \in {}_2J_1$. Furthermore, if $JM_1 \in {}_2J_1$, then there exists $t \in k_\infty$ such that $tR_k = J^2L$, where $L \in \mathcal{D}$. Thus since $tR_k \in P_{k,\infty} \cap \mathcal{D}I^2_k$ by the remarks preceding this theorem, there exists $\alpha \in N(\mathcal{O})$ such that $\tau(\alpha) = JM_1$.

Now if $\alpha = t\beta$, where $t \in k^*$ and $\beta \in N(\mathcal{O})$ is such that $n(\beta) \in R^*_f$, then $\tau(\alpha) = tR_kM_1 \in J_2$. Conversely, if $\alpha \in N(\mathcal{O})$ is such that $\tau(\alpha) \in J_2$, then $n(\alpha)R_k = a^2R_kL$ for $a \in k^*$ and $L \in \mathcal{D}$. Thus $\alpha = a\,(a^{-1}\alpha)$, where $a \in k^*$

and $a^{-1}\alpha \in N(\mathcal{O})$ with $n(a^{-1}\alpha) \in R_f^*$. Thus $[N(\mathcal{O}) : k^* A_{R_f}] = [{}_2J_1 : J_2]$ and (11.27) follows. If k has class number 1, then $J_1 = J_2$. $\square$

Corollary 11.6.6 *The smallest covolume of a group in the commensurability class $\mathcal{C}(A)$ of an arithmetic Kleinian group is*

$$\frac{4\pi^2|\Delta_k|^{3/2}\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1)}{(4\pi^2)^{[k:\mathbb{Q}]}[R_{f,\infty}^* : (R_f^*)^2][{}_2J_1 : J_2]}. \tag{11.28}$$

The above formula uses (11.10), and a similar formula holds for Fuchsian groups using (11.6).

Examples 11.6.7

1. Consider again the Coxeter group with symbol shown in Figure 11.4 We

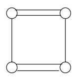

FIGURE 11.4.

   have already seen that this tetrahedral group is arithmetic with quaternion algebra A defined over $\mathbb{Q}(\sqrt{-7})$ and ramified at the primes $\mathcal{P}_2$ and $\mathcal{P}_2'$ of norm 2 (see Exercise 8.3, No. 5). The volume of the tetrahedron can be calculated as discussed in §1.7 and is approximately 0.2222287, so that the covolume of the tetrahedral group is twice that. Also, an approximation to $\zeta_k(2)$ for $k = \mathbb{Q}(\sqrt{-7})$ can be obtained yielding approximately $\zeta_k(2) = 1.8948415$ (see Exercise 11.2 No. 5). Thus from (11.10), the covolume of $P\rho(\mathcal{O}^1)$, for $\mathcal{O}$ a maximal order, is approximately 0.8889149.

   Now k has class number 1 and $[R_f^* : (R_f^*)^2] = 8$ so that the minimum covolume in the commensurability class is approximately 0.1111144. Note that the tetrahedron is symmetric and so the tetrahedral group admits an obvious extension of order 4. As the type number of A is 1, this extended group must coincide with the group $P\rho(N(\mathcal{O}))$. In this way, all entries in the table in Appendix 13.1 can be completed (see Exercise 11.6, No. 5).

2. Illustrating the type of calculations involved in determining minimal volume orbifolds or manifolds globally within certain classes, to be discussed in the next section, we consider here the problem of identifying the smallest volume orbifold arising from quaternion algebras A defined over the cubic field $k = \mathbb{Q}(t)$, where $t^3 + t + 1 = 0$. This field has discriminant -31. By Theorem 0.5.3, the class number $h = 1$, so that

$[{}_2J_1 : J_2] = 1$ by Theorem 11.6.5. As a cubic field with one real place, A will be ramified at an odd number of finite places. Furthermore, since $-1 \in R_k^*$, the index $[R_{f,\infty}^* : R_f^{*2}]$ is precisely 2^{1+r_f}. Thus the minimum volume for an orbifold in the commensurability class defined by A is

$$\mathrm{Vol}(\mathbf{H}^3/\Gamma_{\mathcal{O}}) = \frac{31^{3/2}\zeta_k(2)}{2(4\pi^2)^2} \prod_{\mathcal{P}|\Delta(A)} \frac{N(\mathcal{P})-1}{2}.$$

With t as described above, $\{1, t, t^2\}$ can easily be shown to be an integral basis. Thus using Kummer's Theorem, k has primes of norm 3, 8, 9 and 11 but not of norms 5 or 7. It now follows that the minimum volume orbifold will be obtained by choosing $\mathrm{Ram}_f(A)$ to consist of the prime of norm 3, yielding an orbifold of volume

$$\frac{31^{3/2}\zeta_k(2)}{2(4\pi^2)^2} \approx 0.065965277$$

where Pari has been used to obtain a value for $\zeta_k(2)$.

3. We now consider one example of the application of these results in the Fuchsian case. Let $k = \mathbb{Q}(\sqrt{3}, \sqrt{5})$ and let A be ramified at the three non-identity real places and at the unique prime $\mathcal{P}_2$ over 2. We will determine the signature of $\Gamma_{\mathcal{O}}^+$, where $\Gamma_{\mathcal{O}}^+ = \Gamma_{\mathcal{O}} \cap \mathrm{PSL}(2,\mathbb{R})$. Recall that $\Gamma_{\mathcal{O}}$ is the maximal group in the commensurability class in $\mathrm{PGL}(2,\mathbb{R})$. We also define $\Gamma_{R_f}^+ = \Gamma_{R_f} \cap \mathrm{PSL}(2,\mathbb{R})$. Now k is the compositum of $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$ which have coprime discriminants. Thus $\Delta_k = \Delta^2_{\mathbb{Q}(\sqrt{3})} \times \Delta^2_{\mathbb{Q}(\sqrt{5})}$ (see Exercise 0.3, No. 5). Using this decomposition and the fact that k is a Galois extension, the small primes in k are readily determined so that we obtain an estimation of $\zeta_k(2)$. Note also that $N(\mathcal{P}_2) = 4$. Thus from (11.6), we obtain that $\mathrm{Covol}(\Gamma_{\mathcal{O}^1}) = 2\pi q$ where q is a rational close to 1.2. Now, by arguing as in §11.3.4, we see that $\Gamma_{\mathcal{O}^1}$ can only have elements of finite orders 2, 3, or 5. Thus $\mathrm{Covol}(\Gamma_{\mathcal{O}^1}) = 2\pi(6/5)$. As k has class number one, $\Gamma_{\mathcal{O}}^+ = \Gamma_{R_f}^+$. Also $R_f^* = \langle -1, 2+\sqrt{3}, (1+\sqrt{5})/2, 4+\sqrt{15}, 1+\sqrt{3}\rangle$. Thus $[R_{f,+}^* : (R_f^*)^2] = 4$. (For the definition of $R_{f,+}^*$ and its significance, see Exercise 11.6, No. 7.) Thus $[\Gamma_{R_f}^+ : \Gamma_{\mathcal{O}^1}] = 4$ and $\mathrm{Covol}(\Gamma_{\mathcal{O}}^+) = 2\pi(3/10)$. Now if $\Gamma_{\mathcal{O}}^+$ were to be an arithmetic triangle group defined over $\mathbb{Q}(\sqrt{3}, \sqrt{5})$, then from the form of the invariant trace field of a triangle group given at Exercise 4.9, No. 1 it would have to have elements of orders 5 and 12 (alternatively see Appendix 13.3). Thus a simple calculation shows that $\Gamma_{\mathcal{O}}^+$ has signature $(0; 2, 2, 2, 5)$.

Exercise 11.6

1. *Prove that $\bar{n}$ in the proof of Theorem 11.6.3 is onto.*
2. *Establish (11.24) and (11.25).*

3. Prove that n *maps* $P\mathcal{LR}(\mathcal{O})$ *isomorphically onto* $P_{k,\infty} \cap \mathcal{D}I_k^2$ *(see §6.7 and Theorem 11.6.5).*

4. Determine how the triangle group $(0; 2, 4, 8)$ *lies relative to the groups of minimal covolume in its commensurability class.*

5. Show that among the groups commensurable with any of the cocompact tetrahedral groups, the maximal group corresponding to the Coxeter symbol at Figure 11.4, is the group of minimal covolume; that is, establish the information in column "Min Vol" of Appendix 13.1 from the other information in the table.

6. Let $k = \mathbb{Q}(\sqrt{-d})$, $A = M_2(k)$ *and* $\mathcal{O} = M_2(O_d)$*. Let* C *denote the class group of* k *and* C_2 *the subgroup of exponent* 2*. Show that* $\Gamma_{\mathcal{O}}/\Gamma_{\mathcal{O}^*} \cong C_2$*. Let* t *be the number of distinct rational primes dividing* Δ_k*. Show that* $C_2 \cong \mathbb{Z}_2^{t-1}$*. If* $p_i \mid d$ *and* $-d = p_i q_i$*, let* $\sigma_{p_i} = \left(\begin{smallmatrix} \sqrt{-d} & p_i \\ b_i p_i & a_i\sqrt{-d} \end{smallmatrix}\right)$ *where* $a_i q_i = b_i p_i = 1$*. If* $d \equiv 1 (\text{mod } 4)$ *and* $-1 - d = 2q$*, let* $\sigma_2 = \left(\begin{smallmatrix} 1+\sqrt{-d} & 2 \\ 2b & a(-1+\sqrt{-d}) \end{smallmatrix}\right)$ *where* $aq - 2b = 1$*. Prove that* $\Gamma_{\mathcal{O}}$ *is generated by* $\Gamma_{\mathcal{O}^*}$ *and these elements* $P\sigma_{p_i}$*.*

7. In the case of Fuchsian groups, let $\Gamma_{\mathcal{O}}^+ = \Gamma \cap \mathrm{PSL}(2, \mathbb{R})$ *and* $\Gamma_{R_f}^+ = \Gamma_{R_f} \cap \mathrm{PSL}(2, \mathbb{R})$*. Prove that*

$$\frac{\Gamma_{R_f}^+}{\Gamma_{\mathcal{O}^1}} \cong \frac{R_{f,+}^*}{(R_f^*)^2}$$

where $R_{f,+}^*$ *is the group of totally positive units in* R_f*.*

11.7 Minimum Covolume Groups

In the preceding section, the minimum covolume of a group within the commensurability class of an arithmetic Kleinian or Fuchsian group was determined. This has been utilised to determine the minimum covolume arithmetic Kleinian group and the minimum volume arithmetic hyperbolic 3-manifold. These results are too detailed to include here, but in this section, we give a flavour of the ideas behind the arguments by considering minimum covolume arithmetic Kleinian groups within some restricted classes.

The cocompact Kleinian group of smallest known covolume is the order 2 extension of the the tetrahedral group with symbol given in Figure 11.5 which is also known to be arithmetic (see Example 8.3.8). This group has covolume approximately 0.0390502 (see §11.2.5). In this case, if we restrict to the class of arithmetic Kleinian groups, it has been proved that this is the arithmetic Kleinian group of minimal covolume. Below we discuss two results which establish the minimal volume arithmetic orbifolds within

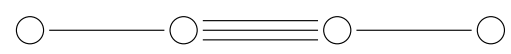

FIGURE 11.5.

certain classes. The proofs of these results give an indication of the arguments used in establishing that the group described above is, indeed, the arithmetic Kleinian group of minimal covolume.

Let $Q = \mathbf{H}^3/\Gamma_Q$ denote the orbifold obtained from the quotient of $\mathbf{H}^3$ by the orientation-preserving subgroup, Γ_Q, in the Coxeter group with diagram at Figure 11.5. As is easily seen from the presentation of Γ_Q given in (4.7), $\Gamma_Q = \Gamma_Q^{(2)}$, so that Γ_Q is derived from a quaternion algebra (see Definition 8.3.4).

Theorem 11.7.1 Γ_Q *is the unique minimal covolume arithmetic Kleinian group derived from a quaternion algebra.*

Proof: Note that the volume of Q is approximately $0.0781\ldots$ (see §11.2.5). We now assume that there exists a Kleinian group Γ, derived from a quaternion algebra whose volume is less than 0.079. Thus there is a maximal order $\mathcal{O}$ in a suitable quaternion algebra A for which Theorem 11.1.3 gives

$$0.079 > \mathrm{Covol}(\mathbf{H}^3/P\rho(\mathcal{O}^1)) = \frac{|\Delta_k^{3/2}|\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1)}{(4\pi^2)^{n-1}}. \quad (11.29)$$

Following the discussion after Corollary 11.2.2, using the values for A, B and E given there, we get the following estimate:

$$0.079 > \left(\frac{A^{3/2}}{4\pi^2}\right)^n (0.0000044)(4\pi^2).$$

Rewriting gives

$$455 > \left(\frac{A^{3/2}}{4\pi^2}\right)^n,$$

so that this gives $(3.16)^n < 455$; that is, $n \leq 5$.

Given this bound on degree, we now turn to bounding the discriminants for these degrees making use of tables of such discriminants in low degrees. Returning to the volume formula and the estimate (11.29), we have

$$0.079 > \frac{|\Delta_k^{3/2}|}{(4\pi^2)^{n-1}}.$$

In degree 5, the minimal discriminant of a field with exactly one complex place is -4511. However, $4511^{3/2}/(4\pi^2)^4$ is approximately 0.12 and thus larger than 0.079. Hence this and therefore all degree 5 fields are eliminated.

In degree 4, the three smallest discriminants are -275, -283 and -331, each of which corresponds to a unique field. The first case is the discriminant of the invariant trace field of Q, and so we expect small volumes there. In the third case, $331^{3/2}/(4\pi^2)^3$ is approximately 0.098 and so can be eliminated. The estimate in the second case gives approximately 0.077. However, using the volume formula, with a value of $\zeta_k(2) = 1.05694057\ldots$ computed, say from Pari, we get that the volume of $P\rho(\mathcal{O}^1)$ for a maximal order $\mathcal{O}$, is at least $0.08178735\ldots$, which again exceeds the bound.

Thus we need to consider groups arising from algebras over the quartic field k of discriminant -275. We do this below, and uniqueness will also follow easily from this.

In degree 3, the discriminant bound shows that we need only consider the fields of discriminants -23 and -31. The latter case was considered in Examples 11.6.7 and the minimal volume obtained was approximately $0.0659\ldots$. However, from the analysis there, $[\Gamma_{\mathcal{O}} : P\rho(\mathcal{O}^1)] = 2^{1+r_f} \geq 4$, so that the covolume of $P\rho(\mathcal{O}^1)$ for any maximal order must exceed 0.079. The remaining field k in this case is the invariant trace field of the Weeks manifold and a generator of the field is a complex root of $x^3 - x^2 + 1$. As noted in §9.8.2, such a root is a generator for the ring of integers. Hence we can apply Theorem 0.3.9 to determine primes of small norm in k. As is easily checked, the smallest such norm is 5. Arguing as in Example 11.6.7, we see that the volume of $P\rho(\mathcal{O}^1)$ for a maximal order $\mathcal{O}$ is greater than $23^{3/2} \times 4/(4\pi^2)^2 > 0.35$, which eliminates this case.

In degree 2, the only case for which the trivial estimate $|\Delta_k^{3/2}|/(4\pi^2)$ does not exceed 0.079 is $k = \mathbb{Q}(\sqrt{-3})$. In this case, using a value $\zeta_k(2) = 1.285190955$, we obtain a volume of $P\rho(\mathcal{O}^1)$ for a maximal order $\mathcal{O}$ of approximately 0.169156.

It remains to consider groups arising from algebras over the quartic field k of discriminant -275. In §11.2.5, we noted that k has no primes of norm ≤ 5, so that the only possible quaternion algebra over k yielding a group $P\rho(\mathcal{O}^1)$ within the volume bound must have no finite ramification. This is the invariant quaternion algebra of Γ_Q and $\text{Covol}(P\rho(\mathcal{O}^1)) = \text{Vol Q}$. Additionally, it was shown in §11.2.5 that the type number of this quaternion algebra is 1, so that there is only one such group up to conjugacy. $\square$

As mentioned above, the smallest covolume arithmetic Kleinian group is the degree 2 extension of the group Γ_Q. The bulk of the proof of this result, due to Chinburg and Friedman, is taken up in handling orbifolds not derived from quaternion algebras. Recall from §11.5 and §11.6, that for the minimum covolume group $\Gamma_{\mathcal{O}}$ in a commensurability class,

$$[\Gamma_{\mathcal{O}} : P\rho(\mathcal{O}^1)] = [R^*_{f,\infty} : R^{*2}_f][{}_2J_1 : J_2].$$

Thus it is necessary to gain some general information on the magnitude of these indices. By Theorem 11.6.3, these indices are powers of 2 and patently involve the structure of the group of units and the class group of k. The initial strategy in the general proof is similar to that in Theorem 11.7.1; roughly speaking, as the degree and discriminant increase, the volume is expected to increase. Fields with primes of norm 2, candidates for belonging to the ramified set in the algebra, give particular problems. There are many other technical difficulties in the proof of Chinburg and Friedman. As a sample of some of the ideas employed, we include the following result which deals with a restricted and, hence, simpler case.

Theorem 11.7.2 *The smallest covolume arithmetic Kleinian group which can be defined over a quadratic field is* $\mathrm{PGL}(2, O_3)$. *The smallest covolume cocompact arithmetic Kleinian group defined over a quadratic field is the group which is the extension of the tetrahedral group described in Example 11.6.7, No. 1.*

Proof: The group $\mathrm{PGL}(2, O_3)$ is an extension of $\mathrm{PSL}(2, O_3)$ and so, for example, from calculations using Corollary 11.2.4, has covolume $\mu_0 = 0.084578$ approximately. The group described in Example 11.6.7 has the form $P\rho(N(\mathcal{O}))$, where $\mathcal{O}$ is a maximal order in the quaternion algebra A defined over $\mathbb{Q}(\sqrt{-7})$ with ramification at the two primes $\mathcal{P}_2'$ and $\mathcal{P}_2''$ of norm 2. It is the minimum covolume group in $\mathcal{C}(A)$ and has covolume $\mu_1 = 0.1111144$ approximately.

Let Γ be an arithmetic Kleinian group defined over a quadratic field $k = \mathbb{Q}(\sqrt{-d})$. Note that r_f is even. By Corollary 11.6.6, the minimum volume μ in the wide commensurability class of Γ is given by

$$\mu = \frac{|\Delta_k|^{3/2}\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N\mathcal{P}-1)}{4\pi^2 2^{r_f+1}[{}_2J_1 : J_2]} \geq \frac{|\Delta_k|^{3/2}}{8\pi^2}\prod_{\mathcal{P}|\Delta(A)}\left(\frac{N\mathcal{P}-1}{2}\right)\frac{\zeta_k(2)}{h_k}$$

since, in these cases, $[{}_2J_1 : J_2] \mid h_k$, the class number. Note that

$$\prod_{\mathcal{P}|\Delta(A)}\left(\frac{N\mathcal{P}-1}{2}\right) \geq \frac{1}{4}\prod_{\mathcal{P}|\Delta(A),\mathcal{P}\nmid 2}\left(\frac{N\mathcal{P}-1}{2}\right).$$

Thus

$$\mu \geq \frac{|\Delta_k|^{3/2}}{32\pi^2}\frac{\zeta_k(2)}{h_k}.$$

With a number of results of this type, it is necessary to appeal to some deep result from number theory to reduce the proof to manageable proportions. This result is no exception and we now quote an estimate relating the discriminant, class number and $\zeta_k(2)$. The Brauer-Siegel Theorem gives asymptotic estimates over suitable sequences of number fields relating the class number h_k, discriminant and the regulator R. The regulator was defined in Exercise 0.4, No. 7 and is 1 for quadratic imaginary fields. The proof of the Brauer-Siegel Theorem proceeds by estimating the residues at poles of generalised zeta functions. In the process, the following inequality is obtained as a special case:

$$|\Delta_k|\ \zeta_k(2) \geq \frac{h_k R}{2w}(2\pi)^{[k:\mathbb{Q}]} \tag{11.30}$$

where w is the order of the group of roots of unity in R_k^*.

Now assume that $|\Delta_k| \geq 13$, so that $w = 2$. Using the estimate at (11.30), we thus obtain that $\mu \geq \frac{|\Delta_k|^{1/2}}{32} > 0.112 > \mu_1$.

It remains to consider the cases where $k = \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-7})$, $\mathbb{Q}(\sqrt{-11})$, all of which have class number 1. If Γ is not cocompact, $\mu = \frac{|\Delta_k|^{3/2}\,\zeta_k(2)}{8\pi^2}$ in these cases.

Field	$\zeta_k(2)$	Smallest norms of prime ideals
$\mathbb{Q}(\sqrt{-1})$	1.50670301	$2, 5$
$\mathbb{Q}(\sqrt{-2})$	1.75141751	$2, 3$
$\mathbb{Q}(\sqrt{-3})$	1.28519096	$3, 4$
$\mathbb{Q}(\sqrt{-7})$	1.89484145	$2, 2$
$\mathbb{Q}(\sqrt{-11})$	1.49613186	$3, 3$

Using this table of values, a simple calculation in each of the five cases gives that the minimum occurs when $k = \mathbb{Q}(\sqrt{-3})$. This minimum is achieved for the group $\mathrm{PGL}(2, O_3)$.

It thus suffices now to assume that Γ is cocompact, so that $r_f \geq 2$. The minimum values in the cocompact cases will then be attained by considering primes of small norm in each of the five cases. Again, a calculation using the above table gives that the minimum is attained when $k = \mathbb{Q}(\sqrt{-7})$, with A ramified at the two primes of norm 2. The bound is achieved by the group described in Example 11.6.7, No. 1. $\square$

For hyperbolic 3-manifolds, the manifold of smallest known volume is the Week's manifold, which is arithmetic (see §9.8.2). It is given by a torsion-free subgroup of index 3 in $\Gamma_{\mathcal{O}^1}$ where $\mathcal{O}$ is a maximal order in the quaternion algebra A defined over the degree 3 field of discriminant -23 with A ramified at the one real place and at the unique prime of norm 5. Its volume is approximately 0.94270736. It has been proved that this gives the arithmetic hyperbolic 3-manifold of minimal volume. To deal with manifolds, we must get control over torsion in arithmetic Kleinian groups; this will be discussed in Chapter 12.

We close this section by commenting more generally on the state of knowledge on small-volume hyperbolic 3-orbifolds amd manifolds. From Theorem 1.5.9, there is a smallest-volume hyperbolic 3-orbifold and hyperbolic 3-manifold. It is conjectured that the minimal-volume arithmetic hyperbolic 3-orbifold and 3-manifold are actually the minimal-volume hyperbolic 3-orbifold and 3-manifold. Much work has been done on this, and the current evidence strongly suggests that, indeed, this is the case. For example, recently, inspired by work of Gabai, Meyerhoff and Thurston, Przeworski has given the best current lower bound in the manifold case as $> 0.27\ldots$. In addition, the programme initiated by Culler and Shalen together with Hersonsky uses topological information to help in estimating the volume. At present, this work has culminated in showing that the closed hyperbolic 3-manifold of smallest volume has $b_1 \leq 2$, where b_1 is the rank of the first homology with coefficients in $\mathbb{Q}$. In the orbifold case, the work of Gehring and Martin has shown that if there is a smaller orbifold than the minimal-volume arithmetic hyperbolic 3-orbifold, then it can only have at most 2− and 3-torsion (with some additional control on 3-torsion).

In the cusped case, we have the following more complete results which are summarised in the following theorem. Note that all the examples are arithmetic.

Theorem 11.7.3

1. *(Meyerhoff) The orientable cusped hyperbolic* 3*-orbifold of smallest volume is* $\mathbf{H}^3/\mathrm{PGL}(2, O_3)$.
2. *(Cao and Meyerhoff) The smallest-volume cusped orientable hyperbolic 3-manifolds are the figure* 8 *knot complement and its sister manifold.*
3. *(Adams) The smallest-volume cusped hyperbolic 3-manifold is the Giesking manifold which is a twofold quotient of the figure* 8 *knot complement by an orientation-reversing involution.*

Exercise 11.7

1. Let $k = \mathbb{Q}(t)$*, where* t *is a complex root of* $x^3 - 7 = 0$*. Compute the minimal covolume in the commensurability class of arithmetic Kleinian groups determined by the quaternion algebra* A *over* k *ramified at the unique real place and the place of norm 2.*

2. Let $k = \mathbb{Q}(t)$*, where* t *satisfies* $t^4 - 5t^3 + 10t^2 - 6t + 1 = 0$*.*
(a) Show that $d_k = -331$*, that* $h = 1$ *and that* k *has units of all possible signatures.*
(b) Let $A = \left(\frac{-1,-1}{k}\right)$*. What is the minimal volume in the commensurability class?*
(c) Show that there is a unique prime of norm 5 *in* k*. Let* ν *be the place associated to this prime and let* $S = \{\nu\}$ *. What is the covolume of the maximal group* $\Gamma_{S,\mathcal{O}}$*?*

11.8 Further Reading

Most of the fundamental results in this chapter can be found in the seminal paper of Borel (1981) This applies to the volume formulas (11.6), (11.10), (11.28), the finiteness result in §11.2.1, the results on maximal groups in §11.4, the distribution of volumes in §11.5 and the minimal covolume in a commensurability class in §11.6. The translation from Borel's description to one involving Eichler orders is straightforward and appears in Chinburg and Friedman (1999). There is an extended discussion of Eichler orders and their normalisers and related groups in Vignéras (1980a). The derivation of the volume formula given in §11.1 follows that given in Vignéras (1980a). In the Fuchsian and other cases, it is derived in Shimizu (1965). For the particular cases of the Bianchi groups, various methods of derivation are possible (e.g., Elstrodt et al. (1998)). The main features of the Lobachevski function and its use in computing volumes of ideal tetrahedra are given by Milnor in Thurston (1979).

From calculations on the conductor of the extension $\mathbb{Q}(\sqrt{-d})$, it follows that the smallest cyclotomic field containing $\mathbb{Q}(\sqrt{-d})$ is $\mathbb{Q}(\xi_{|D|})$ where D is the discriminant (e.g., Janusz (1996)), where the results on Gauss sums used in §11.1 can also be found. The proof of Theorem 11.2.1 following Borel (1981) uses the geometric result of Thurston and Jorgensen, (Thurston (1979), Gromov (1981)), given in Chapter 1, on obtaining manifolds and orbifolds by Dehn surgery, and the number-theoretic results of Odlyzko giving lower bounds on discriminants in terms of the degree of the field (Odlyzko (1975), Martinet (1982)). The Fuchsian case was proved by a similar method in Takeuchi (1983). Finiteness within certain subclasses (e.g., two-generator groups), can be obtained without using bounds on the covolume, as was discussed in §11.2. See also Takeuchi (1977a), Takeuchi (1983), Maclachlan and Rosenberger (1983) and Maclachlan and Martin (1999).

The existence of arithmetic compact and non-compact hyperbolic 3-manifolds of the same volume and even non-arithmetic manifolds of the same volume as constructed at the end of §11.2.3 is given in Mednykh and Vesnin (1995) (see also Reid (1995)).

The values of the zeta function for quadratic extensions of $\mathbb{Q}$ and also for quadratic extensions of quadratic extensions of $\mathbb{Q}$ can be obtained by using the Epstein-zeta function as in Zagier (1986). The detailed analysis of the tetrahedral groups in Maclachlan and Reid (1989) uses this. There are subsequent applications to other groups having extremal geometric properties in Gehring et al. (1997).

The rationality of $\zeta_k(-1)$ for k totally real was proved in Siegel (1969) following earlier work in Klingen (1961). Using triangle groups, specific values of $\zeta_k(2)$ for certain totally real fields can be computed (see Takeuchi (1977b)). Extending these ideas to fields with one complex place is carried out in Zagier (1986). The first finiteness result for classes of Fuchsian groups referred to triangle groups and was obtained by Takeuchi (1977a), who enumerated them also and went on to classify them into commensurability classes (Takeuchi (1977b)). Other small covolume groups were considered in Takeuchi (1983), Maclachlan and Rosenberger (1983), Maclachlan and Rosenberger (1992b), Sunaga (1997a), Sunaga (1997b) and Nakinishi et al. (1999).

As indicated above, determining the maximal groups in $\mathcal{C}(A)$ is carried out in Borel (1981), where an extended version of Theorem 11.4.4, requiring that the groups be torsion free, and so correspond to manifolds, is also proved. These results and the further discussion in §11.5 and §11.6 are widely used in the determination of arithmetic hyperbolic 3-orbifolds and manifolds of minimal volume noted below and also in analysing the smaller covolume groups in the commensurability classes of arithmetic Fuchsian triangle groups (see Takeuchi (1977b), Maclachlan and Rosenberger (1992a)). The maximal extensions of the Bianchi groups described in Exercise 11.6, No. 6 have been used in Vulakh (1994), Vinberg (1990), Shaiheev (1990), Elstrodt et al. (1983) and James and Maclachlan (1996).

The determination of the minimal covolume arithmetic Kleinian group is due to Chinburg and Friedman (1986). Their proof makes use, in particular, of the Brauer-Siegel Theorem, which is described in Lang (1970) where the inequality (11.30) can be found. The cocompact part of Theorem 11.7.2 appears in Maclachlan and Reid (1989). The determination of the arithmetic hyperbolic 3-manifold of minimum volume is due to Chinburg et al. (2001).

The investigation of specific arithmetic Kleinian groups frequently leads to problems in computational algebraic number theory. Recent books by Cohen (1993) and Pohst and Zassenhaus (1989) discuss many aspects of this theory and indicate the availability and utility of packages such as Pari. As mentioned earlier, this package is incorporated into the program Snap to investigate the arithmetic invariants and the arithmeticity of specific hyperbolic 3-manifolds (Goodman (2001)).

The proof of the first part of Theorem 11.7.3. appears in Meyerhoff (1986), the second in Cao and Meyerhoff (2001) and the last in Adams (1987). The middle part is in a recent preprint. The general case is, as indicated, under active investigation and Przeworski's work is in a preprint based on Gabai et al. (2002). The foundational work of Culler and Shalen is in Culler and Shalen (1992) and is extended in Culler et al. (1998). The result ascribed to Gehring and Martin appears in Gehring and Martin (1998) and builds on earlier work of these authors. Indeed, these papers mentioned here are only a sample of the many recent publications involved in attempts to settle these minimality problems.

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