

5

Applications

The invariant trace field and quaternion algebra of a finite-covolume Kleinian group was introduced in Chapter 3 accompanied by methods to enable the computation of these invariants to be made. Such computations were carried out in Chapter 4 for a variety of examples. We now consider some general applications of these invariants to problems in the geometry and topology of hyperbolic 3-manifolds. Generally, these have the form that special properties of the invariants have geometric consequences for the related manifolds or groups. In some cases, to fully exploit these applications, the existence of manifolds or groups whose related invariants have these special properties requires the construction of arithmetic Kleinian groups, and these cases will be revisited in later chapters.

5.1 Discreteness Criteria

In general, proving a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is discrete is very difficult. In this section, we prove a result that guarantees discreteness under certain conditions on the invariant trace field. This result can be thought of as a generalization of a classical result in number theory.

Recall from Exercise 0.1, No.6 that if p is a monic irreducible polynomial over \mathbb{Z} of degree n with roots $\alpha_1, \dots, \alpha_n$, then

$$p(z) = \prod_{i=1}^n (z - \alpha_i) = z^n - s_1 z^{n-1} + \dots + (-1)^k s_k z^{n-k} + \dots + (-1)^n s_n$$

where s_i is the i th symmetric polynomial in $\alpha_1, \dots, \alpha_n$. As a consequence, we deduce the following easy lemma whose proof is left as an exercise below (Exercise 5.1, No.1).

Lemma 5.1.1 *There are only finitely many algebraic integers z of bounded degree such that z and all Galois conjugates of z are bounded.*

In what follows, \mathbf{c} denotes complex conjugation.

Theorem 5.1.2 *Let Γ be a finitely generated subgroup of $\mathrm{PSL}(2, \mathbb{C})$ such that the following three conditions all hold.*

1. $\Gamma^{(2)}$ is irreducible.
2. $\mathrm{tr}(\Gamma)$ consists of algebraic integers.
3. For each embedding $\sigma : k\Gamma \rightarrow \mathbb{C}$ such that $\sigma \neq \mathrm{Id}$ or \mathbf{c} , the set $\{\sigma(\mathrm{tr}(f)) : f \in \Gamma^{(2)}\}$ is bounded.

Then Γ is discrete.

Proof: Note that since Γ is finitely generated, so is $\Gamma^{(2)}$ and so from §3.5, all traces in $\Gamma^{(2)}$ are obtained from integral polynomials in a finite number of traces. Thus $k\Gamma$ is a finite extension of \mathbb{Q} .

It suffices to prove that the finite index subgroup $\Gamma^{(2)}$ is discrete. Suppose that this is not the case and let f_n be a sequence of distinct elements converging to the identity in $\Gamma^{(2)}$. Since $\Gamma^{(2)}$ is irreducible, choose g_1 and g_2 in $\Gamma^{(2)}$ such that g_1 and g_2 have no common fixed point in their action on \mathbb{C} . If $z_n = \mathrm{tr}(f_n)$ and $z_{n,i} = \mathrm{tr}([f_n, g_i])$, then

$$\beta(f_n) = z_n^2 - 4 \rightarrow 0 \quad \text{and} \quad \gamma(f_n, g_i) = z_{n,i} - 2 \rightarrow 0$$

for $i = 1, 2$ as $n \rightarrow \infty$. Hence we may assume that $|z_n| < K$ for some fixed constant K . Next by condition 3, $|\sigma(z_n)| < K_\sigma$ for each embedding $\sigma \neq \mathrm{Id}$ or \mathbf{c} of $k\Gamma$, where K_σ is a constant which depends only on σ .

Let $R = \max\{K, K_\sigma\}$, where σ ranges over all embeddings $\sigma \neq \mathrm{Id}$ or \mathbf{c} of $k\Gamma$. Then the algebraic integers z_n are of bounded degree and they and all of their Galois conjugates are bounded in absolute value by R . By Lemma 5.1.1, the z_n assume only finitely many values. Thus for large n , $\beta(f_n) = 0$ and f_n is parabolic with a single fixed point w_n .

Next we can apply the above argument to the algebraic integers $z_{n,i}$ to conclude that $\gamma(f_n, g_i) = 0$ for $i = 1, 2$ and large n . This then implies that g_1 and g_2 each have w_n as a common fixed point for large n , contradicting condition 1. \square

To apply Theorem 5.1.2 to specific examples, we give an equivalent condition to condition 3, which, in view of the Hilbert symbol representation of $A\Gamma$ in §3.6, can be readily checked. This is the content of the following lemma.

Lemma 5.1.3 *With Γ as described in Theorem 5.1.2 satisfying conditions 1 and 2, condition 3 is equivalent to the following requirement:*

3'. All embeddings σ , apart from the identity and \mathbf{c} , complex conjugation, are real and $A\Gamma$ is ramified at all real places.

Proof: If condition 3' holds and $\sigma : k\Gamma \rightarrow \mathbb{R}$, then there exists $\tau : A\Gamma \rightarrow \mathcal{H}$, Hamilton's quaternions, such that $\sigma(\text{tr } f) = \text{tr } (\tau(f))$ for each $f \in \Gamma^{(2)}$. Since $\det(f) = 1$, $\tau(f) \in \mathcal{H}^1$, so that $\text{tr } (\tau(f)) \in [-2, 2]$.

Conversely, suppose condition 3 holds and $\sigma : k\Gamma \rightarrow \mathbb{C}$. Let $f \in \Gamma^{(2)}$ have eigenvalues λ and λ^{-1} and μ be an extension of σ to $k\Gamma(\lambda)$. Then $\sigma(\text{tr } f^n) = \mu(\lambda)^n + \mu(\lambda)^{-n}$. Thus

$$|\sigma(\text{tr } f^n)| \geq ||\mu(\lambda)|^n - |\mu(\lambda)|^{-n}|.$$

So, if $\sigma(\text{tr } f^n)$ is bounded, then $|\mu(\lambda)| = 1$ so that $\sigma(\text{tr } f) = \mu(\lambda) + \mu(\lambda)^{-1}$ is a real number in the interval $[-2, 2]$. Now choose an irreducible subgroup $\langle g_1, g_2 \rangle$ of $\Gamma^{(2)}$ such that g_1 is not parabolic. Then

$$A\Gamma \cong \left(\frac{\text{tr }^2 g_1 (\text{tr }^2 g_1 - 4), \text{tr } [g_1, g_2] - 2}{k\Gamma} \right)$$

by (3.38). Since $\sigma(\text{tr } f) \in [-2, 2]$ for all f , it follows that $A\Gamma$ is ramified at all real places (see Theorem 2.5.1). \square

Exercise 5.1

1. Prove Lemma 5.1.1.
2. State and prove the corresponding result to Theorem 5.1.2 for finitely generated subgroups of $\text{PSL}(2, \mathbb{R})$.
3. Let $\Gamma = \langle f, g \rangle$ be a subgroup of $\text{PSL}(2, \mathbb{C})$ where g has order 2 and f has order 3. Let $\gamma = \text{tr } [f, g] - 2$ be a non-real algebraic integer, with minimum polynomial $p(x)$ all of whose roots, except γ and $\bar{\gamma}$ lie in the interval $(-3, 0)$. Prove that Γ is a discrete group.
4. Let $\Gamma = \langle f, g \rangle$, where f has order 6 and g has order 2, with $\gamma = \text{tr } [f, g] - 2$ satisfying the polynomial $x^3 + x^2 + 2x + 1$. Prove that Γ is discrete.
5. Let $\Gamma = \langle x_1, x_2, x_3 \rangle$ be a non-elementary subgroup of $\text{PSL}(2, \mathbb{R})$ such that $o(x_i) = 2$ for $i = 1, 2, 3$ and $o(x_1 x_2 x_3)$ is odd ($\neq 1$). Let $x = \text{tr } x_1 x_2$, $y = \text{tr } x_2 x_3$, $z = \text{tr } x_3 x_1$. If x, y and z are totally real algebraic integers with $x, y \neq 0, \pm 2$, and for every embedding σ of $\mathbb{Q}(\text{tr } \Gamma)$ such that $\sigma|_{k\Gamma} \neq \text{Id}$, then $|\sigma(x)| < 2$, prove that Γ is discrete and cocompact in $\text{PSL}(2, \mathbb{R})$.

5.2 Bass's Theorem

One of the first applications of number-theoretic methods in 3-manifold topology arises directly from Bass-Serre theory of group actions on trees. To state Bass's theorem, we introduce the following definition.

Definition 5.2.1 Let $\overline{\mathbb{Q}}$ denote the algebraic closure of \mathbb{Q} in \mathbb{C} and let $\Gamma < \mathrm{SL}(2, \overline{\mathbb{Q}})$. Then Γ is said to have integral traces if for all $\gamma \in \Gamma$, $\mathrm{tr}(\gamma)$ is an algebraic integer. Otherwise, we say Γ has non-integral trace. We also use this terminology for Γ a subgroup of $\mathrm{PSL}(2, \overline{\mathbb{Q}})$.

It is not difficult to show that the property of having integral traces is preserved by commensurability (see Exercise 5.2, No. 1). The following theorem of Bass is the main result of this section.

Theorem 5.2.2 Let $M = \mathbf{H}^3/\Gamma$ be a finite-volume hyperbolic 3-manifold for which Γ has non-integral trace. Then M contains a closed embedded essential surface.

Before embarking on the proof of this theorem, we deduce a succinct version of Theorem 5.2.2 in the closed setting (see §1.5).

Corollary 5.2.3 If $M = \mathbf{H}^3/\Gamma$ is non-Haken, then Γ has integral traces.

We also remark that having integral traces is equivalent to having an “integral representation” in the following sense. Let \mathbb{A} denote the ring of all algebraic integers in $\overline{\mathbb{Q}}$.

Lemma 5.2.4 Let Γ be a finitely generated non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$. Then Γ has integral traces if and only if Γ is conjugate in $\mathrm{SL}(2, \mathbb{C})$ to a subgroup of $\mathrm{SL}(2, \mathbb{A})$.

Proof: One way is obvious, so we assume that Γ has integral traces. Since Γ is finitely generated, the trace field of Γ is a finite extension k of \mathbb{Q} . Let $A_0\Gamma$ be the quaternion algebra generated over k by elements of Γ and $\mathcal{O}\Gamma$ the R_k -module generated by the elements of Γ . Then $\mathcal{O}\Gamma$ is an order of $A_0\Gamma$ (see Exercise 3.2, No. 1). By choosing a suitable quadratic extension L , $A_0(\Gamma) \otimes_k L \cong M_2(L)$ (Corollary 2.1.9, Corollary 3.2.4), and so by the Skolem Noether Theorem, we may conjugate so that $A_0\Gamma \subset M_2(L)$. The order $\mathcal{O}\Gamma \otimes_{R_k} R_L$ is then conjugate to a suborder of $M_2(R_L; J)$ where J is a fractional ideal as defined at (2.5) (see Lemma 2.2.8 and Theorem 2.2.9). Now pass to a finite extension H say, of L to make the ideal J principal. There is always such a finite extension and the Hilbert Class field is such an extension. A further conjugation of $M_2(R_H; J)$ shows that Γ is contained in $\mathrm{SL}(2, R_H)$. This completes the proof. \square

In light of this lemma, a reformulation of Theorem 5.2.2 is as follows:

Theorem 5.2.5 *Let $M = \mathbf{H}^3/\Gamma$ be a finite-volume hyperbolic 3-manifold not containing any closed embedded essential surface. Then Γ is conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{A})$.*

The proof of Theorem 5.2.2 requires some information about the tree of $\mathrm{SL}(2)$ over a \mathcal{P} -adic field K , as developed by Serre. This tree can alternatively be described in terms of maximal orders and, in this vein, is discussed in Chapter 6. The actions of the groups $\mathrm{SL}(2, K)$ and $\mathrm{GL}(2, K)$ on this tree play a critical role in obtaining the description of maximal arithmetic Kleinian and Fuchsian groups via local-global arguments and so a comprehensive treatment of these actions is given in §11.4. Thus the basic results recalled in the next subsection will be developed more fully later as indicated.

5.2.1 Tree of $\mathrm{SL}(2, K_{\mathcal{P}})$

Let K be a finite extension of \mathbb{Q}_p with valuation v and uniformizing parameter π , valuation ring R and unique prime ideal \mathcal{P} . Let V denote the vector space K^2 . Recall from §2.2 that a lattice L in V is a finitely generated R -submodule which spans V . Define an equivalence relation on the set of lattices of V : $L \sim L'$ if and only if $L' = xL$ for some $x \in K^*$. Let Λ denote the equivalence class of L . These equivalence classes form the vertices of a combinatorial graph \mathcal{T} where two vertices Λ and Λ' are connected by an edge if there are representative lattices L and L' , where $L' \subset L$ and $L/L' \cong R/\pi R$. Serre proved that \mathcal{T} is a tree; that is, it is connected and simply connected (see Theorem 6.5.3 for a proof), and each vertex has valency $N\mathcal{P} + 1$ (see Exercise 5.2, No. 3).

The obvious action of $\mathrm{GL}(2, K)$ on the set of lattices in V determines an action on \mathcal{T} , which is transitive on vertices (see Corollary 2.2.10). The action of $\mathrm{SL}(2, K)$ on \mathcal{T} is without inversion and the vertices fall into two orbits. Thus the stabiliser of a vertex under the action of $\mathrm{SL}(2, K)$ is conjugate either to $\mathrm{SL}(2, R)$ or to

$$\left\{ \begin{pmatrix} a & \pi b \\ \pi^{-1}c & d \end{pmatrix} \in \mathrm{SL}(2, K) \mid a, b, c, d \in R \right\}.$$

Lemma 5.2.6 *If G is a subgroup of $\mathrm{SL}(2, K)$ which fixes a vertex then the traces of the elements of G lie in R .*

With this, we state the following version of the arboreal splitting theorem of Serre:

Theorem 5.2.7 *Let G be a subgroup of $\mathrm{SL}(2, K)$ which is not virtually solvable and contains an element g for which $v(\mathrm{tr} g) < 0$. Then G has a non-trivial splitting as the fundamental group of a graph of groups.*

Note that if G satisfies Theorem 5.2.7 and the centre $Z(G)$ is non-trivial, then since the centre of an amalgamated product is contained in the amalgamating group, it follows that $G/Z(G)$ also splits as a free product with amalgamation.

5.2.2 Non-integral Traces

The proof of Theorem 5.2.2 can now be completed. The trace field, k , of Γ is a finite extension of \mathbb{Q} . By Corollary 3.2.4, we can assume that $\hat{\Gamma}$ is a subgroup of $\mathrm{SL}(2, L)$, where $[L : k] \leq 2$. Having non-integral traces means that there is an L -prime \mathcal{P} and an element $\hat{\gamma} \in \hat{\Gamma}$ such that $v_{\mathcal{P}}(\mathrm{tr} \hat{\gamma}) < 0$. By using the injection $i_{\mathcal{P}} : L \rightarrow L_{\mathcal{P}} = K$, we inject $\hat{\Gamma}$ into $\mathrm{SL}(2, K)$, and we are in the situation of Theorem 5.2.7. Thus $\hat{\Gamma}$ and, hence, Γ split as described there. By Theorem 1.5.3, we deduce the existence of an embedded incompressible surface. Furthermore, in the case when M has toroidal boundary components, since the traces of parabolic elements are ± 2 , we see that any $\mathbb{Z} \oplus \mathbb{Z}$ subgroup will lie in a vertex stabilizer. From this, we deduce from Theorem 1.5.3 that the incompressible surface may be chosen to be closed and not boundary parallel. \square

Examples 5.2.8 1. In §4.7.3, we calculated the trace field of the polyhedral groups of prisms obtained by truncating a super-ideal vertex of a tetrahedron. Further we also calculated the traces of certain elements in these groups and showed that the groups Γ_{6p} , p a prime, as described in §4.7.3, had non-integral traces. Since this is preserved by commensurability, any hyperbolic 3-manifold arising from a torsion-free subgroup of such a group is therefore Haken. (For other examples of this type, see Exercise 5.2, No. 6 and §10.4.)

2. Here we consider a Dehn surgery example, the details of which require machine calculation. The surgery will be carried out on a two-bridge knot complement and we first collect some general information from the discussion in Chapter 4. Recall that for odd coprime integers p and q , the knot complement (p/q) has fundamental group Γ with presentation on two meridional generators

$$\langle u, v \mid uw = vw, \quad w = v^{e_1} u^{e_2} \dots u^{e_{p-1}} \rangle$$

where e_i are defined in §4.5. From the two-bridge representation, we can take u and $\ell = w\tilde{w}^{-1}u^{-2\sigma}$ as meridian and longitude of a peripheral subgroup where $\tilde{w} = v^{-e_1}u^{-e_2} \dots u^{-e_{p-1}}$ and $\sigma = \sum e_i$.

Any representation ρ of Γ into $\mathrm{SL}(2, \mathbb{C})$ can be conjugated so that

$$\rho(u) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(v) = \begin{pmatrix} x & 0 \\ r & x^{-1} \end{pmatrix}.$$

The group relation in Γ determines a single polynomial equation in x, r from which the character variety of Γ can be determined since $\text{tr}(\rho(u)) = x + x^{-1}$ and $\text{tr}(\rho(uv)) = r + \text{tr}^2(\rho(u)) - 2$ (see Exercise 3.5, No 6 and 4.4, No. 4 for the figure 8 knot group).

Now consider the knot 5_2 , which has the two-bridge representation $(7/3)$ as discussed in §4.5. With the framing defined by u and ℓ , performing $(10, 1)$ surgery on 5_2 produces a compact hyperbolic 3-manifold M whose volume is approximately 2.362700793 (see §1.7).

This manifold will correspond to a point on the character variety which will, in addition, satisfy a further polynomial in x, r given by the trace of the Dehn surgery equation (cf. §4.8.3). From the resultant of the two two-variable polynomials, we obtain that the trace field of M is $\mathbb{Q}(s)$, where $s = \text{tr} \rho(uv)$ satisfies

$$s^4 - 4s^3 + 5s^2 + s - 5.$$

This field $\mathbb{Q}(s)$ has one complex place, and so is the invariant trace field, and has discriminant -2151 . Again the resultant shows that the square of the trace of the image of the meridian, which is the core curve of the Dehn surgery, is non-integral, as it satisfies

$$2x^4 - 17x^3 + 46x^2 - 40x + 8.$$

Thus it follows from above, that M is Haken. In addition, since 5_2 is two-bridge, it is known that there is no closed embedded essential surface in its complement. It follows that $(10, 1)$ is a boundary slope for 5_2 , which means that there is an incompressible surface in the complement of 5_2 whose boundary consists of curves parallel to the $(10, 1)$ curves on the boundary torus.

Remark The phenomena discussed in the preceding example fits into the following general theorem of Cooper and Long, which is proved using the A -polynomial, which will not be discussed here.

Theorem 5.2.9 *Let N be a compact 3-manifold with boundary a torus. Suppose that α is an essential simple closed curve on the boundary torus which is not a boundary slope, and let $N(\alpha)$ denote the result of Dehn surgery along α . Let ρ be any irreducible representation of $\pi_1(N(\alpha))$ into $\text{SL}(2, \mathbb{C})$, such that ρ is non-trivial when restricted to the peripheral subgroup. Let ξ be the eigenvalue of the core curve γ of the attached solid torus. Then ξ is an algebraic unit.*

5.2.3 Free Product with Amalgamation

With a little more technology, one can prove a stronger algebraic result on the group Γ in Theorem 5.2.2. This technology involves using some results on \mathcal{P} -adic Lie groups.

As described in the proof of Theorem 5.2.2, $\hat{\Gamma}$ injects into $\mathrm{SL}(2, K)$, where K is a \mathcal{P} -adic field, such that the image G has non-integral traces. A stronger version of Serre's splitting theorem states the following:

Theorem 5.2.10 *If $G \subset \mathrm{SL}(2, K)$ where K is a \mathcal{P} -adic field, and G is dense in $\mathrm{SL}(2, K)$, then G splits as a free product with amalgamation.*

Suppose that K as above is such that $\mathbb{Q}_p \subset K$ and $\ell = \mathbb{Q}_p(\{\mathrm{tr} g : g \in G\})$. As in Chapter 3, let

$$A = \left\{ \sum a_i g_i : a_i \in \ell, g_i \in G \right\}.$$

Now Γ contains infinitely many loxodromic elements x_i such that, for $i \neq j$, $\mathrm{tr}[x_i, x_j] \neq 2$. This then implies that the images of I, x_i, x_j and $x_i x_j$ in G are linearly independent over ℓ so that A is a quaternion algebra over ℓ .

By Corollary 2.6.4, there are two possibilities for A . If A is a division algebra, the valuation ring \mathcal{O} in A , defined in Corollary 2.6.2, is the unique maximal order in A (see Exercise 2.6, No. 1 and §6.4). Furthermore, from the definition of \mathcal{O} , it is clear that $\mathcal{O}^1 = A^1$ so that $G \subset A^1$ would have all traces being integers. Thus we conclude that $A \cong M_2(\ell)$.

By conjugating in $\mathrm{GL}(2, \ell)$ using the Skolem Noether Theorem, we can assume that $A = M_2(\ell)$ and so $G \subset \mathrm{SL}(2, \ell)$. Now $\mathrm{SL}(2, \ell)$ is a \mathcal{P} -adic Lie group and we can form \bar{G} , the closure of G . The subgroup G cannot be discrete. Otherwise, let G_1 be a torsion-free subgroup of finite index. Then G_1 acts on the tree of $\mathrm{SL}(2, \ell)$, whose vertex stabilisers are compact. Thus being torsion free, G_1 would act freely on the tree and so be free. Thus G , and hence Γ , would be virtually free, which is not possible for a finite-covolume group. Since \bar{G} is then not discrete, the theory of \mathcal{P} -adic Lie groups ensures that \bar{G} has a unique structure as a \mathcal{P} -adic Lie group. The theory further characterises \bar{G} as containing an open subgroup H which is a uniform pro- p group. It is not necessary to expand on the definition of uniform here, but it suffices to note that, as a profinite group, H is compact and its open subgroups form a basis of the neighbourhoods of the identity. By its action on the tree of $\mathrm{SL}(2, \ell)$, H will have a fixed point and so can be conjugated to an open subgroup of $\mathrm{SL}(2, R_\ell)$. It is straightforward to see that such open subgroups have, as a basis, the principal congruence subgroups Γ_i (see Exercise 5.2, No. 2). Thus, by conjugation, we can assume that $\bar{G} \supset \Gamma_j$ for all $j \geq i$ and an element with non-integral trace. The groups Γ_j are normal in $\mathrm{SL}(2, R_\ell)$ and a further conjugation by an element in $\mathrm{SL}(2, R_\ell)$ allows us to assume that \bar{G} contains Γ_j for $j \geq i$ and an element $g = \begin{pmatrix} \pi^n & 0 \\ 0 & \pi^{-n} \end{pmatrix}$ for some $n \neq 0$, since it has non-integral traces.

Now $\mathrm{SL}(2, \ell)$ is generated by the subgroups

$$U = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \ell \right\}, \quad L = \left\{ \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \mid \alpha \in \ell \right\}. \quad (5.1)$$

(See Exercise 5.2, No. 5.) Let $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in U$, so that $\alpha = \pi^t u$, where u is a unit. Choose m such that $2mn + t \geq i$. Then $g^m \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} g^{-m} \in \Gamma_i$. Applying a similar argument to elements of L , this yields $G = \mathrm{SL}(2, \ell)$. Thus from Theorem 5.2.10, we obtain the following extension to Theorem 5.2.2.

Theorem 5.2.11 *Let Γ be as in Theorem 5.2.2. Then Γ splits as a free product with amalgamation.*

Exercise 5.2

1. Let Γ and Γ' be commensurable groups contained in $\mathrm{SL}(2, \bar{\mathbb{Q}})$. Show that Γ has integral traces if and only if Γ' has (see §3.1).
2. Let K be a \mathcal{P} -adic field with ring of integers R . Show that the principal congruence subgroups Γ_i form a basis for the open subgroups of $\mathrm{SL}(2, R)$.
3. Prove that the tree \mathcal{T} described in §5.2.1 has valency $N\mathcal{P} + 1$.
4. Show that there exist hyperbolic Haken manifolds whose trace field has arbitrarily large degree over \mathbb{Q} .
5. Prove that the subgroups U and L defined at (5.1), generate $\mathrm{SL}(2, \ell)$.
6. Let Γ be the group generated by reflections in the faces of the prism obtained by truncating the infinite-volume tetrahedron with Coxeter symbol shown in Figure 5.1 ($m \geq 7$) by a face orthogonal to faces 2, 3 and 4. Let Γ^+ be the polyhedral subgroup. Show, for $m = 6p$ where p is a prime ≥ 5 , that Γ^+ is a free product with amalgamation. (See §4.7.3, in particular Exercise 4.7, No. 4. See also §10.4).

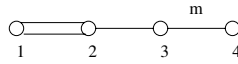


FIGURE 5.1.

5.3 Geodesics and Totally Geodesic Surfaces

The aim of this section is to prove several theorems relating the geometry of geodesics, and totally geodesic surfaces in finite-volume hyperbolic 3-manifolds, to the invariant trace field and quaternion algebra. We remind the reader that for Kleinian groups of finite covolume, the invariant trace field is always a finite non-real extension of \mathbb{Q} .

5.3.1 Manifolds with No Geodesic Surfaces

Theorem 5.3.1 *Let Γ be a Kleinian group of finite covolume which satisfies the following conditions:*

- (a) $k\Gamma$ contains no proper subfield other than \mathbb{Q} .
- (b) $A\Gamma$ is ramified at at least one infinite place of $k\Gamma$.

Then Γ contains no hyperbolic elements.

Proof: Note that Γ contains a hyperbolic element if and only if $\Gamma^{(2)}$ contains a hyperbolic element. Let us suppose that $\gamma \in \Gamma^{(2)}$ is hyperbolic, and let $t = \text{tr}(\gamma)$. By assumption, $t \in k\Gamma \cap \mathbb{R} = \mathbb{Q}$ and $|t| > 2$.

Now $A\Gamma$ is ramified at an infinite place v of $k\Gamma$, which is necessarily real. Let $\sigma : k\Gamma \rightarrow \mathbb{R}$ be the Galois embedding of $k\Gamma$ associated to v , and let $\psi : A\Gamma \rightarrow \mathcal{H}$ extend σ , where \mathcal{H} denotes Hamilton's quaternions. Thus

$$\psi(\Gamma^{(2)}) \subset \psi(A\Gamma^1) \subset \mathcal{H}^1.$$

Since $t \in \mathbb{Q}$,

$$t = \sigma(t) = \psi(\gamma + \bar{\gamma}) = \psi(\gamma) + \overline{\psi(\gamma)} = \text{tr } \psi(\gamma).$$

Since $\text{tr } \mathcal{H}^1 \subset [-2, 2]$ we obtain a contradiction. \square

We record the most important geometric corollary of this. This follows from the discussion in §1.2.

Corollary 5.3.2 *Let $M = \mathbf{H}^3/\Gamma$ be a finite-volume hyperbolic 3-manifold for which Γ satisfies the conditions of Theorem 5.3.1. Then M contains no immersed totally geodesic surface.*

We also give the group theoretic version of this.

Corollary 5.3.3 *Let Γ be a Kleinian group of finite covolume which satisfies the conditions of Theorem 5.3.1. Then Γ contains no non-elementary Fuchsian subgroups (i.e., no non-elementary subgroups leaving a disc or half-plane invariant).*

As will follow from our later discussions on arithmetic Kleinian groups in §9.5, many Kleinian groups satisfy the conditions of Theorem 5.3.1. In §4.8.3, the Weeks manifold was shown to satisfy these conditions, as does the manifold constructed in Exercise 4.8, No. 5.

5.3.2 Embedding Geodesic Surfaces

In §5.2, we considered conditions which gave rise to embedded surfaces in hyperbolic 3-manifolds. On the other hand, the corollaries of the preceding subsection give obstructions to the existence of immersions of totally geodesic surfaces. Connecting these results, we have the following result due to Long:

Theorem 5.3.4 *Let M be a closed hyperbolic 3-manifold containing a totally geodesic immersion of a closed surface. Then there is a finite covering of M which contains an embedded closed orientable totally geodesic surface.*

To prove this theorem, we first recall the notion of subgroup separability.

Definition 5.3.5 *Let G be a group and H a finitely generated subgroup. Then G is said to be H -subgroup separable if given any element $g \in G \setminus H$, there is a finite index subgroup K of G with $H < K$ and $g \notin K$.*

G is subgroup separable, if it is H -subgroup separable for all such H .

To prove the theorem, we first establish the following:

Lemma 5.3.6 *Let \mathcal{C} be a circle or straight line in $\mathbb{C} \cup \infty$ and $M = \mathbf{H}^3/\Gamma$, a closed hyperbolic 3-manifold. Let*

$$\text{Stab}(\mathcal{C}, \Gamma) = \{\gamma \in \Gamma : \gamma\mathcal{C} = \mathcal{C}\}.$$

Then $\text{Stab}(\mathcal{C}, \Gamma)$ is separable in Γ .

Proof: Let H denote $\text{Stab}(\mathcal{C}, \Gamma)$. We may assume without loss of generality that $H \neq 1$ —because 1 is separable since Γ is residually finite. Note that H is either a Fuchsian group or a \mathbb{Z}_2 -extension of a Fuchsian group. To prove the lemma, we need to show that, given $g \notin H$, there is a finite index subgroup of G containing H but not g . By conjugating, if necessary, we can assume that H stabilises the real line. If \mathbf{c} is the complex conjugate map, then \mathbf{c} extends to $\text{SL}(2, \mathbb{C})$ and is well-defined on $\text{PSL}(2, \mathbb{C})$. The stabiliser of \mathbb{R} in $\text{PSL}(2, \mathbb{C})$ is then characterised as those elements γ such that $\mathbf{c}(\gamma) = \gamma$. Let $\hat{\Gamma}$ be generated by matrices g_1, g_2, \dots, g_t . Let R be the subring of \mathbb{C} generated by all the entries of the matrices g_i , their complex conjugates and 1. Then R is a finitely generated integral domain with 1 so that for any non-zero element there is a maximal ideal which does not contain that element. Note that Γ and $\mathbf{c}(\Gamma)$ embed in $\text{PSL}(2, R)$.

If $\gamma \in \Gamma \setminus H$, then $\gamma = P(g)$, where $g = (g_{ij})$ with at least one element from each set $\{\mathbf{c}(g_{ij}) - g_{ij}\}$ and $\{\mathbf{c}(g_{ij}) + g_{ij}\}$, $i, j = 1, 2$, being non-zero. Call these x and y and choose a maximal ideal \mathcal{M} such that $xy \notin \mathcal{M}$. Let

$$\rho : \text{PSL}(2, R) \rightarrow \text{PSL}(2, R/\mathcal{M}) \times \text{PSL}(2, R/\mathcal{M})$$

be the homomorphism defined by $\rho(\gamma) = (\pi(\gamma), \pi(\mathbf{c}(\gamma)))$, where π is induced by the natural projection $R \rightarrow R/\mathcal{M}$. The image group is finite since R/\mathcal{M} is a finite field. By construction, the image of γ is a pair of distinct elements in $\text{PSL}(2, R/\mathcal{M})$, whereas the image of H lies in the diagonal. This proves the lemma. \square

The connection between the group theory and topology is given by the following lemma (which holds in greater generality than stated here).

Lemma 5.3.7 *Let $M = \mathbf{H}^3/\Gamma$ be a finite-volume hyperbolic 3-manifold and $f : S \hookrightarrow M$ be an incompressible immersion of a closed surface. Let $H = f_*(\pi_1(S)) \subset \Gamma$. If Γ is H -subgroup separable, there is a finite covering M_0 of M to which f lifts so that $f(S)$ is an embedded surface in M_0 .*

Proof: Let p denote the cover $\mathbf{H}^3 \rightarrow M$. Since S is compact, standard covering space arguments imply that there is a compact set $D \subset \mathbf{H}^3$ with $p(D) = f(S)$. Since Γ acts discontinuously on \mathbf{H}^3 , there are a finite number of elements $\gamma_1, \dots, \gamma_n \in \Gamma$ with $\gamma_i D \cap D \neq \emptyset$. Since Γ is H -subgroup separable, there is a finite index subgroup K in Γ containing H , but none of the γ_i 's. The covering \mathbf{H}^3/K is the covering required in the statement. \square

Proof: (of Theorem 5.3.4) Let $i : S \rightarrow M$ be a totally geodesic immersion of a closed surface. Let Γ be the covering group of M in $\mathrm{PSL}(2, \mathbb{C})$ and let $H = i_*(\pi_1(S))$. Then H is Fuchsian and preserves some circle or straight line \mathcal{C} in $\mathbb{C} \cup \infty$. Thus by Lemma 5.3.6, the group $\mathrm{Stab}(\mathcal{C}, \Gamma)$ is separable in Γ . Let K denote a finite index subgroup achieving this (recall the definition). By Lemma 5.3.7, since $\mathrm{Stab}(\mathcal{C}, \Gamma)$ is separable, the covering M_K of M determined by K will contain an embedded orientable totally geodesic surface, as required.

If $\mathrm{Stab}(\mathcal{C}, \Gamma)$ is not Fuchsian, we obtain, in the same way, a closed non-orientable hyperbolic surface S' embedded in the cover M_K of M corresponding to K . Now pass to the index 2 orientable double cover S'' of S' . Now construct a double cover of M_K by taking two copies of $M_K \setminus S'$ and doubling to obtain a covering of M_K and, hence, M in which the orientable totally geodesic surface S'' embeds. \square

Theorem 5.3.4 answers a special case of the conjecture due to Waldhausen and Thurston that every closed hyperbolic 3-manifold has a finite cover which is Haken. Indeed more is conjectured: that every closed hyperbolic 3-manifold has a finite cover with positive first betti number. In the totally geodesic case as described in Theorem 5.3.4, the separability can be used to promote the embedded surface to an embedded non-separating orientable surface in a finite cover, as the reader may wish to prove. In general, although the evidence is overwhelmingly for a positive answer to both of these conjectures, at present, there are no general methods for approaching a solution. The reader should consult the Further Reading section.

5.3.3 The Non-cocompact Case

Note that *any* finite-covolume Kleinian group satisfying the conditions of Theorem 5.3.1 is necessarily cocompact since $A\Gamma$ must be a division algebra (see Theorem 3.3.8). We will next address the non-cocompact case. First recall that, in §4.9, it was noted that some Bianchi groups contain

cocompact Fuchsian subgroups and, indeed, it will be shown in §9.6 that this is true of all Bianchi groups. In contrast, we have the following result:

Theorem 5.3.8 *Let Γ be a non-cocompact Kleinian group which has finite covolume and satisfies the following two conditions:*

- $k = \mathbb{Q}(\text{tr } \Gamma)$ is of odd degree over \mathbb{Q} and contains no proper subfield other than \mathbb{Q} .
- Γ has integral traces.

Then Γ contains no cocompact Fuchsian subgroups.

Proof: We argue by contradiction, and so assume that Γ contains a cocompact Fuchsian group F say. Since Γ has integral traces, F has integral traces, and by the first assumption, $\text{tr } F \subset \mathbb{Z}$. Next consider the quaternion algebra AF , defined over \mathbb{Q} , and $\mathcal{O}F$, as defined at (3.7) is an order of AF . We claim that AF is isomorphic to $M(2, \mathbb{Q})$. Assuming this and using the Skolem Noether Theorem, we can conjugate in $\text{GL}(2, \mathbb{C})$, so that $AF = M(2, \mathbb{Q})$. Now all maximal orders in $M_2(\mathbb{Q})$ are conjugate to $M_2(\mathbb{Z})$ (see Corollary 2.2.10). Thus by further conjugation, we can take $\mathcal{O}F$ to be a suborder of $M(2, \mathbb{Z})$. However, this means F is a subgroup of $\text{SL}(2, \mathbb{Z})$, which is a contradiction since F is assumed cocompact.

Thus it remains to establish the isomorphism between AF and $M_2(\mathbb{Q})$. If AF is not isomorphic to $M(2, \mathbb{Q})$, it is a division algebra over \mathbb{Q} and hence ramified at at least one finite place (see Theorem 2.7.3). Let $p \in \mathbb{Z}$ be the associated prime. Furthermore, a simple dimension count implies that $AF \otimes_{\mathbb{Q}} k = A\Gamma$, and since Γ is non-cocompact, $A\Gamma \cong M(2, k)$ by Theorem 3.3.8.

Let $\mathcal{P}_1, \dots, \mathcal{P}_g$ be the k -prime divisors of p , and consider the localization of AF . Since $A\Gamma$ is unramified at every place of k , we must have

$$(AF \otimes_{\mathbb{Q}} k)_{\mathcal{P}_i} \cong M(2, k_{\mathcal{P}_i})$$

for each $i = 1, \dots, g$. On the other hand, AF is ramified at p , so $AF \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra over \mathbb{Q}_p . Note that

$$(AF \otimes_{\mathbb{Q}} k) \otimes_k k_{\mathcal{P}_i} \cong (AF \otimes_{\mathbb{Q}} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} k_{\mathcal{P}_i}$$

for each $i = 1, \dots, g$. Now as noted, the left-hand side is simply $M(2, k_{\mathcal{P}_i})$. Thus $(AF \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ is split by the extension field $k_{\mathcal{P}_i}$. By assumption, the degree $[k : \mathbb{Q}]$ is odd, and since

$$[k : \mathbb{Q}] = \sum_{i=1}^g e_i [k_{\mathcal{P}_i} : \mathbb{Q}_p]$$

(see §0.3), at least one of the local degrees $[k_{\mathcal{P}_i} : \mathbb{Q}_p]$ is odd. However, by Exercise 2.3, No. 3, an odd-degree extension cannot split the division algebra over \mathbb{Q}_p . This contradiction completes the proof. \square

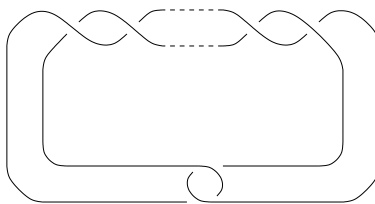


FIGURE 5.2.

From the remarks preceding this theorem and the fact, to be shown in Theorem 8.2.3, that all non-cocompact arithmetic Kleinian groups are commensurable with the Bianchi groups, examples having the properties given in Theorem 5.3.8 will necessarily be non-arithmetic.

Example 5.3.9 Twist Knots: Certain twist knots as shown in Figure 5.2 furnish examples which satisfy the conditions of Theorem 5.3.8 (see Theorem 1.5.6). These twist knots are two-bridge knots of the form $(p/p-2)$ (see §4.5). If we choose p to be of the form $4m+3$, then we obtain a symmetric sequence

$$\{e_1, e_2, \dots, e_{4m+2}\} = \{1, -1, 1, -1, \dots, -1, 1, 1, -1, 1, -1, \dots, -1, 1\}.$$

The polynomial described in §4.5, determining the trace field, then has degree $2m+1$, is monic and integral. If $2m+1$ is prime and the polynomial is irreducible, then the conditions of Theorem 5.3.8 hold. In the cases $m = 1, 2$, we obtain, respectively, the polynomials

$$1 + 2z - 3z^2 + z^3, \quad 1 + 3z - 13z^2 + 16z^3 - 7z^4 + z^5,$$

which are irreducible over \mathbb{Q} .

5.3.4 Simple Geodesics

We now turn our attention to relationships between the geometry of closed geodesics and the properties of the related invariant trace field and quaternion algebra. Let $M = \mathbf{H}^3/\Gamma$. A closed geodesic in M is called *simple* if it has no self-intersections. Otherwise, a closed geodesic is called *non-simple*. The following lemma (see Exercise 5.3, No.3) will prove useful.

Lemma 5.3.10 *Let $M = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold. Then M contains a non-simple closed geodesic if and only if there exists a primitive loxodromic element γ with axis A_γ and an element $\delta \in \Gamma$ such that $\delta A_\gamma \cap A_\gamma \neq \emptyset$ and $\delta A_\gamma \neq A_\gamma$.*

With this lemma, we can develop obstructions to the existence of non-simple closed geodesics in closed hyperbolic 3-manifolds. Note that (see

Exercise 5.3, No. 4) any finite-volume hyperbolic 3-manifold which contains an immersion of a totally geodesic surface contains a non-simple closed geodesic.

Let $M = \mathbf{H}^3/\Gamma$ be a closed hyperbolic 3-manifold and assume g is a non-simple closed geodesic in M . We begin with a few basic geometric observations. By definition, there exists a loxodromic element $\gamma \in \Gamma$ and a geodesic in \mathbf{H}^3 , namely the axis A of γ , such that under the canonical projection map to M , the image of A is freely homotopic to g . As g is non-simple, by Lemma 5.3.10 there is an element $\delta \in \Gamma$ such that $\delta A \neq A$ and $\delta A \cap A \neq \emptyset$. Then δA is the axis of the element $\eta = \delta\gamma\delta^{-1}$.

Let the fix points of γ be a_1 and a_2 ; these are just the endpoints in $\mathbf{C} \cup \infty$ of the geodesic A in \mathbf{H}^3 . Let the images of a_1 and a_2 under δ be b_1 and b_2 .

Lemma 5.3.11 *The points a_1, a_2, b_1 and b_2 lie on a circle in $\mathbf{C} \cup \infty$. The cross-ratio $[a_1, a_2, b_1, b_2]$ is a real number lying in the interval $(0, 1)$.*

Proof: By an element of $\text{PSL}(2, \mathbf{C})$, we can map $a_1 \rightarrow 0$, $b_1 \rightarrow 1$ and $b_2 \rightarrow \infty$. Assume that a_2 maps to w . Because $\delta A \neq A$ and $\delta A \cap A \neq \emptyset$, w must be a real number greater than 1. Since elements of $\text{PSL}(2, \mathbf{C})$ map circles to circles, this proves the first statement. The cross-ratio is also preserved by elements of $\text{PSL}(2, \mathbf{C})$. Therefore the cross-ratio we require is $[0, w, 1, \infty]$, which is simply $1/w$, hence real and lies in $(0, 1)$. \square

Expanding on the proof of Lemma 5.3.11, note that γ and η have the same trace since they are conjugate. The mapping described in the proof has the effect of conjugating Γ so that

$$\gamma = \begin{pmatrix} \lambda & 0 \\ r & \lambda^{-1} \end{pmatrix} \text{ and } \eta = \begin{pmatrix} \lambda & (\lambda^{-1} - \lambda) \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Let $t = (\lambda^{-1} - \lambda)$. With this notation, the fix point w of Lemma 5.3.11 is $-t/r$. Thus by Lemma 5.3.11, $-t/r$ is real and greater than 1.

Lemma 5.3.12 *With notation as above, $t^2, rt \in k\Gamma$ and, hence, so does $t/r = -[a_1, a_2, b_1, b_2]^{-1}$.*

Proof: Since $t^2 = \text{tr}^2 \gamma - 4 = \text{tr}^2 \eta - 4$, $t^2 \in k\Gamma$. Also, the element $\gamma\eta^{-1}$ is a commutator in Γ and so lies in $\Gamma^{(2)}$. Thus $rt = 2 - \text{tr}(\gamma\eta^{-1}) \in k\Gamma$. The last part follows since, by Lemma 5.3.11, $t, r \neq 0$. \square

Theorem 5.3.13 *If M has a non-simple closed geodesic, then $A\Gamma \cong \left(\frac{a, b}{k\Gamma}\right)$ for some $a \in k\Gamma$ and $b \in k\Gamma \cap \mathbb{R}$.*

Proof: Assume that M has a non-simple geodesic g . We shall compute the expression for $A\Gamma$ using the elements η and γ^{-1} described above, which generate an irreducible subgroup. Thus $A\Gamma \cong \left(\frac{a, b'}{k}\right)$, where $a = \text{tr}(\eta)^2 - 4$

and $b' = \text{tr}[\eta, \gamma^{-1}] - 2$ by Theorem 3.6.1. Now $a = t^2$ and $b' = r^2 t^2 + (rt)t^2 = (t^2(r/t))^2(1 + (t/r))$. Removing squares, we conclude that $A\Gamma \cong \left(\frac{a,b}{k\Gamma}\right)$, where $b = 1 + (t/r) \in k\Gamma \cap \mathbb{R}$. \square

Note that, if Γ is not cocompact, there are always elements a and b (equal to 1) satisfying the conditions of Theorem 5.3.13. This result can now be stated from the contrapositive viewpoint.

Corollary 5.3.14 *With the notation of Theorem 5.3.13, suppose that there are no elements $a \in k\Gamma$ and $b \in k\Gamma \cap \mathbb{R}$ such that $A\Gamma$ is isomorphic over $k\Gamma$ to the quaternion algebra $\left(\frac{a,b}{k\Gamma}\right)$. Then all of the closed geodesics of the closed hyperbolic 3-manifold $M = \mathbf{H}^3/\Gamma$ are simple.*

It will be shown in §9.7, that there exist number fields k with exactly one complex place and quaternion algebras over k such that there are no elements $a \in k, b \in k \cap \mathbb{R}$ as described in this corollary. The arithmetic groups Γ which arise from these, furnish examples of manifolds all of whose closed geodesics are simple.

Exercise 5.3

1. Let Γ be a finite-covolume Kleinian group such that $[k\Gamma : k\Gamma \cap \mathbb{R}] = n$ and $[k\Gamma \cap \mathbb{R} : \mathbb{Q}] = 2$. Show that if $A\Gamma$ is ramified at at least $n + 1$ real places, then Γ has no hyperbolic elements.
2. (a) Show that Theorem 5.3.4 holds when M has finite volume.
(b) Show that $\text{PSL}(2, \mathbb{Z})$ is separable in $\text{PSL}(2, O_d)$.
3. Prove Lemma 5.3.10.
4. Show that if a finite-volume hyperbolic manifold M contains an immersed totally geodesic non-boundary parallel surface, then it contains a non-simple closed geodesic.
5. Show that if Γ is as described in Theorem 5.3.8, then it can contain at most one wide commensurability class of non-cocompact finite-covolume Fuchsian groups. Show that the twist knot groups discussed in Example 5.3.9 do contain non-cocompact finite-covolume Fuchsian subgroups.

5.4 Further Hilbert Symbol Obstructions

As we have already seen, the Hilbert symbol appears naturally as an obstruction to certain geometric phenomena. In this section, we give further applications of the Hilbert Symbol in this role. As discussed in §1.2 and 1.3, if $Q = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-orbifold whose singular set contains at

least one vertex, then the vertex stabilizer is a finite group isomorphic to one of D_n , A_4 , S_4 or A_5 . We now discuss how the presence of a subgroup isomorphic to A_4 , S_4 and A_5 manifests itself in the Hilbert Symbol of the invariant quaternion algebra.

Let \mathcal{H} denote the Hamiltonian quaternions. Let σ denote the embedding $\sigma : \mathcal{H}^1 \rightarrow \mathrm{SL}(2, \mathbb{C})$ given by

$$\sigma(a_0 + a_1i + a_2j + a_3ij) = \begin{pmatrix} a_0 + a_1i & a_2 + a_3i \\ -a_2 + a_3i & a_0 - a_1i \end{pmatrix}$$

where \mathcal{H}^1 is the group of elements of norm 1.

If n denotes the norm on \mathcal{H} , then there is an epimorphism

$$\Phi : \mathcal{H}^1 \rightarrow \mathrm{SO}(3, \mathbb{R})$$

where $\mathrm{SO}(3, \mathbb{R})$ is represented as the orthogonal group of the quadratic subspace V of \mathcal{H} spanned by $\{i, j, ij\}$, (i.e., the pure quaternions), equipped with the restriction of the norm form, so that $n(x_1i + x_2j + x_3ij) = x_1^2 + x_2^2 + x_3^2$. The homomorphism Φ is defined by $\Phi(\alpha) = \phi_\alpha$, where

$$\phi_\alpha(\beta) = \alpha\beta\alpha^{-1}, \quad \alpha \in \mathcal{H}^1, \quad \beta \in V.$$

The kernel of Φ is $\{\pm 1\}$.

Let the tetrahedron in V have vertices

$$i + j + ij, \quad i - j - ij, \quad -i + j - ij, \quad -i - j + ij.$$

If $\alpha_1 = i$ and $\alpha_2 = (1 + i + j + ij)/2$, then ϕ_{α_1} is a rotation of order 2 about the axis through the edge mid-point i and ϕ_{α_2} is a rotation of order 3 about the axis through the vertex $i + j + ij$. Note that $\alpha_1^2 = \alpha_2^3 = -1$ and so we obtain a faithful representation of the binary tetrahedral group, BA_4 in \mathcal{H}^1 (see Exercise 2.3, No. 7). This is also true for the binary octahedral group and the binary icosahedral group (see Exercise 5.4, No. 1).

The group $P\sigma(BA_4) \cong A_4$ is said to be in standard form and we note that it fixes the point $(0, 0, 1)$ in \mathbf{H}^3 . If Γ is a Kleinian group containing a subgroup isomorphic to A_4 , then Γ can be conjugated so that A_4 is in standard form. Of course, if Γ contains an S_4 or an A_5 , it will contain a subgroup isomorphic to A_4 .

Lemma 5.4.1 *Let Γ be a Kleinian group of finite covolume with invariant quaternion algebra A and number field k . If Γ contains a subgroup isomorphic to A_4 , then*

$$A \cong \left(\frac{-1, -1}{k} \right). \quad (5.2)$$

In particular, the only finite primes at which A can be ramified are the dyadic primes.

Proof: Suppose that Γ contains a subgroup isomorphic to A_4 . Then since A_4 is generated by two elements of order 3, $A_4 = A_4^{(2)} \subset \Gamma^{(2)}$. Thus by conjugation, we can assume that $\sigma(BA_4) \subset \mathcal{G}$ where $P\mathcal{G} = \Gamma^{(2)}$, $\mathcal{G} \subset \mathrm{SL}(2, \mathbb{C})$. Now

$$A = \left\{ \sum a_i g_i : a_i \in k, \quad g_i \in \mathcal{G} \right\}.$$

Let

$$A_0 = \left(\frac{-1, -1}{\mathbb{Q}} \right).$$

Then

$$A_0 \cong \left\{ \sum a_i g_i : a_i \in \mathbb{Q}, \quad g_i \in \sigma(BA_4) \right\}$$

since $1, i, j, ij \in BA_4$. Now the quaternion algebra

$$\left\{ \sum a_i g_i : a_i \in k \quad g_i \in \sigma(BA_4) \right\}$$

lies in A , is isomorphic to $A_0 \otimes_{\mathbb{Q}} k$ and is four-dimensional. Thus

$$A \cong \left(\frac{-1, -1}{k} \right).$$

Finally, by Theorem 2.6.6, A splits over all \mathcal{P} -adic fields $k_{\mathcal{P}}$, where \mathcal{P} is non-dyadic. \square

Lemma 5.4.2 *Let Γ be a finite-covolume Kleinian group which contains a subgroup isomorphic to A_5 . If, furthermore, $[k\Gamma : \mathbb{Q}] = 4$, then $A\Gamma$ has no finite ramification.*

Proof: As above, let $k = k\Gamma$ and $A = A\Gamma$. Now A can, at worst, have dyadic finite ramification. Also, by Lemma 5.4.1, A is ramified at all real places of which there are either 0 or 2. Since Γ must contain an element of order 5, $\mathbb{Q}(\sqrt{5}) \subset k$. There is a unique prime \mathcal{P} in $\mathbb{Q}(\sqrt{5})$ such that $\mathcal{P} \mid 2$. So if \mathcal{P} ramifies or is inert in $k \mid \mathbb{Q}(\sqrt{5})$, then there will only be one dyadic prime in k at which A cannot be ramified for parity reasons. Suppose then that \mathcal{P} splits as $\mathcal{P}_1\mathcal{P}_2$ so that $k_{\mathcal{P}_1} \cong k_{\mathcal{P}_2} \cong \mathbb{Q}(\sqrt{5})_{\mathcal{P}}$. For parity reasons, the quaternion algebra $\left(\frac{-1, -1}{\mathbb{Q}(\sqrt{5})} \right)$ splits in the field $\mathbb{Q}(\sqrt{5})_{\mathcal{P}}$. Hence $\left(\frac{-1, -1}{k} \right)$ splits in $k_{\mathcal{P}_1}$ and $k_{\mathcal{P}_2}$, and A has no finite ramification. \square

Exercise 5.4

1. (a) Show that the binary octahedral group BS_4 has a faithful representation in \mathcal{H}^1 .

(b) Taking the regular dodecahedron to have its vertices in V at

$$\pm i \pm j \pm ij, \quad \pm \tau i \pm \tau^{-1} j, \quad \pm \tau j \pm \tau^{-1} ij, \quad \pm \tau ij \pm \tau^{-1} i$$

where $\tau = (1 + \sqrt{5})/2$, show that the binary icosahedral group BA_5 has a faithful representation in \mathcal{H}^1 .

2. Let Γ be a cocompact tetrahedral group as described in §4.7.2.

(a) Show that the finite ramification of $A\Gamma$ is at most dyadic.

(b) Show that in all cases except one, $A\Gamma$ has no finite ramification. [To cut short lengthy calculations, see Theorem 10.4.1.]

5.5 Geometric Interpretation of the Invariant Trace Field

In this section, we give a geometric description of the invariant trace field $k\Gamma$ in the case $M = \mathbf{H}^3/\Gamma$ is a cusped hyperbolic manifold.

Let M be a finite-volume hyperbolic 3-manifold with a triangulation by ideal tetrahedra:

$$M = S_1 \cup S_2 \cup \cdots \cup S_n,$$

where each S_j is an ideal tetrahedron in \mathbf{H}^3 . As discussed in §1.7, the tetrahedron S_j is described up to isometry by a single complex number z_j with positive imaginary part (the *tetrahedral parameter* of S_j) such that the Euclidean triangle cut off at any vertex of S_j by a horosphere section is similar to the triangle in \mathbb{C} with vertices 0, 1 and z_j . Alternatively, z_j is the cross-ratio of the vertices of S_j (considered as points of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$). This tetrahedral parameter depends on a choice (an edge of S_j or an oriented ordering of its vertices); changing the choice replaces z_j by $1/(1 - z_j)$ or $1 - 1/z_j$. Denote the field $\mathbb{Q}(z_j : j = 1, \dots, n)$ by $k_\Delta M$ or $k_\Delta \Gamma$. A priori $k_\Delta \Gamma$ might depend on the choice of triangulation, but this is not the case.

Theorem 5.5.1 $k_\Delta \Gamma = k\Gamma$.

Proof: Denote $k_\Delta \Gamma$ by k_Δ for short. If we lift the triangulation of M to \mathbf{H}^3 , we get a tessellation of \mathbf{H}^3 by ideal tetrahedra. Let V be the set of vertices of these tetrahedra in the sphere at infinity. Let k_1 be the field generated by all cross-ratios of 4-tuples of points of V . Position V by an isometry of \mathbf{H}^3 (upper half-space model) so that three of its points are at 0, 1, and ∞ , and let k_2 be the field generated by the remaining points of V . This k_2 does not depend on which three points we put at 0, 1, ∞ ; in fact the following holds:

Lemma 5.5.2 $k_1 = k_2 = k_\Delta$.

Proof: $k_1 \subseteq k_2$ since k_1 is generated by cross-ratios of elements of k_2 while $k_2 \subseteq k_1$ because the cross-ratio of 0, 1, ∞ , and z is just z . The inclusion $k_\Delta \subseteq k_1$ is straightforward (see Exercise 5.5, No. 1). Finally, put three vertices of one tetrahedron of our tessellation at 0, 1, and ∞ , and then $k_2 \subseteq k_\Delta$ is a simple deduction on noting that, for any field l , if three

vertices and the tetrahedral parameter of an ideal tetrahedron $S \subseteq \mathbf{H}^3$ are in $l \cup \{\infty\}$, then so is the fourth vertex. \square

Now suppose we have positioned V as above. Any element $\gamma \in \Gamma$ maps 0, 1, and ∞ to points w_1 , w_2 , and w_3 of $V \subseteq k \cup \{\infty\}$. Thus γ is given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose entries satisfy

$$\begin{aligned} b - dw_1 &= 0, \\ a + b - cw_2 - dw_2 &= 0, \\ a - cw_3 &= 0. \end{aligned}$$

We can solve this for a , b , c and d in k_Δ and then γ^2 is represented by the element

$$\frac{1}{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \in \mathrm{PSL}(2, k_\Delta).$$

By definition, $k\Gamma = \mathbb{Q}(\mathrm{tr} \Gamma^{(2)})$, so we see that $k\Gamma \subseteq k_\Delta$.

For the reverse inclusion, we shall use Theorem 4.2.3, which says that Γ may be conjugated to lie in $\mathrm{PSL}(2, \mathbb{Q}(\mathrm{tr} \Gamma))$. Given this, the points of V , which are the fixed points of parabolic elements of Γ , lie in $\mathbb{Q}(\mathrm{tr} \Gamma)$, since the fixed point of a parabolic element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $(a - d)/2c$. Thus, by Lemma 5.5.2 $k_\Delta \subseteq \mathbb{Q}(\mathrm{tr} \Gamma)$. On the other hand, k_Δ is clearly an invariant of the commensurability class of Γ , so we can apply this to $\Gamma^{(2)}$ to see $k_\Delta \subseteq k\Gamma$. \square

Using the tetrahedral parameters to determine the invariant trace field as in Theorem 5.5.1 is a simple tool to apply once the data (i.e., the tetrahedral parameters), are known. When a cusped hyperbolic manifold is triangulated by ideal tetrahedra, the gluing pattern of the tetrahedra dictates the gluing conditions around each edge, which are equations in the tetrahedral parameters. Furthermore, for the structure to yield a complete hyperbolic structure, it is necessary and sufficient that the geometric structure of the cusps must be a Euclidean structure which yields the holonomy condition pinning down the precise values of the tetrahedral parameters. This process was set out by Thurston and for the figure 8 knot complement, given in §4.4.2 as a union of two ideal tetrahedra, it leads to the fact that the tetrahedra in that case are regular (see §1.7), thus determining the tetrahedral parameter field to be $\mathbb{Q}(\sqrt{-3})$. A further example, the complement of the knot 5_2 , is considered next.

Conversely, starting with a fixed small number of ideal tetrahedra, the number of possible gluing patterns which yield a manifold, is finite and can be expressed as gluing consistency equations on the tetrahedral parameters which must further satisfy the holonomy conditions at the cusps. In this way, SnapPea (a program of Jeff Weeks) created a census of cusped manifolds obtained from small numbers of ideal tetrahedra and the data so obtained lends itself readily to the calculation of the invariant trace field.

In fact, an exact version of SnapPea, called Snap, has been created by Coulson, Goodman, Hodgson and Neumann, and here the number fields can be read off very easily. (See further discussion below). Tables of such are presented in the Appendix to this book.

Example 5.5.3 Here we illustrate the above discussion using the complement of the knot 5_2 , whose invariant trace field as a two-bridge knot complement we have already calculated in §4.5.

Regard the knot as lying essentially in the plane P given by $z = 0$ in \mathbb{R}^3 . The complement can then be regarded as the union of two polyhedra with their faces identified and vertices deleted. To describe the two polyhedra, they consist of two balls filling the upper half-space $z \geq 0$ and the lower half-space $z \leq 0$. At each crossing, we adjoin small oriented 1-cells as shown in Figure 5.3 with end points on the knot K . Let two such cells be equivalent if one can be obtained from the other by sliding along the knot. Now take regions in the plane bounded by the knot and three or more 1-cells, as the two cell faces of the polyhedra, one for upper half-space and the other for lower half-space, with appropriate gluing given by the equivalence of 1-cells. This yields two polyhedra as depicted in Figure 5.4. If we further subdivide the polyhedron on the left as shown in Figure 5.5 to split C into two cells C_1 and C_2 , the 1-cell on the polyhedron in the lower half-space, shown on the right, is determined by the identifications already specified. This results in two tetrahedra in the upper half-space and two in the lower half-space, but one of these has ‘degenerated’ into a triangle. Identifying D and D' , we obtain the polyhedron shown in Figure 5.6. The upshot is that we obtain the knot complement as a union of three tetrahedra and we can now calculate the gluing consistency equations. Thus using the notation

$$z_1 = z, \quad z_2 = \frac{z-1}{z}, \quad z_3 = \frac{1}{1-z}$$

and similarly for u and w , the gluing consistency conditions require that the sum of the dihedral angles round an edge is 2π . These can be expressed

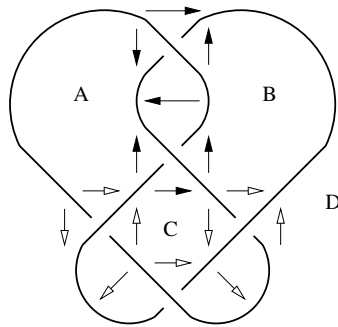


FIGURE 5.3.

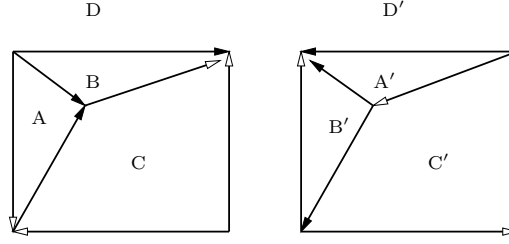


FIGURE 5.4.

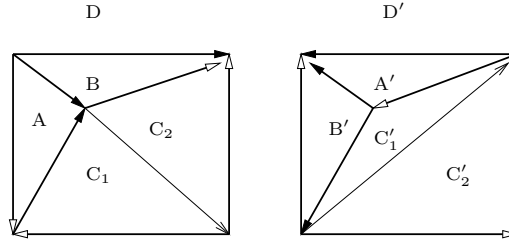


FIGURE 5.5.

by the logarithmic gluing equations requiring that the sum of the logarithmic parameters be $2\pi i$. Exponentiating gives the multiplicative gluing equations, which can be read off directly from Figure 5.7, one for each edge. These equations are

$$u_1 u_3 z_1 z_2 w_1 w_2 = 1, \quad \frac{u}{1-u}(z-1)w(w-1) = 1, \quad (5.3)$$

$$u_1 u_2 z_2 z_3 w_3 w_2 = 1, \quad (u-1)\frac{-1}{z}\frac{1}{(1-w)^2} = 1, \quad (5.4)$$

$$u_2 u_3 z_1 z_3 w_2 = 1, \quad \frac{-1}{u}\frac{z}{1-z}\frac{w-1}{w} = 1. \quad (5.5)$$

As a 1-cusped manifold, the link of the vertex is made up of 12 triangles arranged as in Figure 5.8. For a complete hyperbolic 3-manifold, the cusp must have a horospherical torus cross section. This can be determined from the holonomy of the similarity structure on the boundary torus which can be read off from Figure 5.8 as

$$\begin{aligned} H'(x) &= w_1 u_3 u_2 z_2 w_3 w_2 u_3 w_1 z_2 z_1 u_3 z_3 z_2 u_2 w_3 w_2 z_3 z_2, \\ H'(y) &= u_3 w_1 w_2. \end{aligned} \quad (5.6)$$

One then determines from these equations that w is a solution to

$$x^3 - x + 1 = 0$$

with positive imaginary part, $u = w$ and $z = 1/(1-u)$, thus providing a solution to the gluing equations with positive imaginary parts.

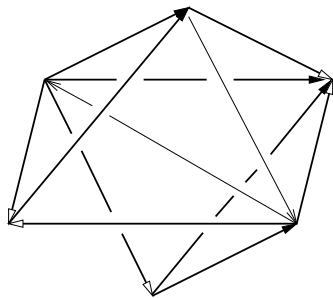


FIGURE 5.6.

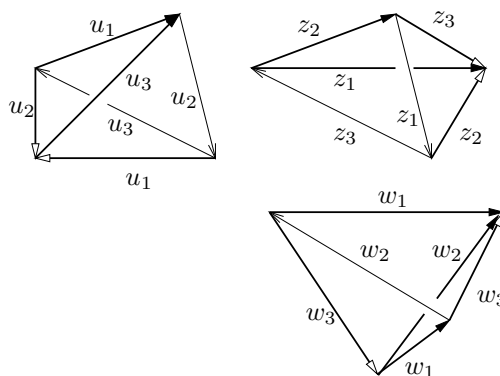


FIGURE 5.7.

This whole process, and more, has been automated. Starting with a knot or link complement, SnapPea will produce numerical values for the tetrahedral parameters. Then Snap, combining this with the number theory package Pari, yields polynomials satisfied by these tetrahedral parameters (provided the degree is not too large). This then yields the arithmetic data and, in particular, the invariant trace field. This applies not only to knot and link complements but also to other cusped manifolds which can be

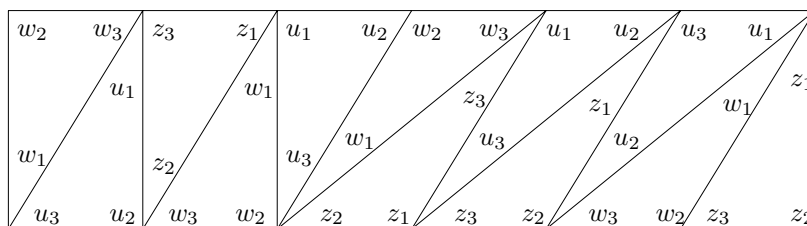


FIGURE 5.8.

constructed from ideal tetrahedra in such a way that the gluing conditions referred to earlier are satisfied. A census of such manifolds is available with the above packages. All of this has been extended to include manifolds and orbifolds obtained by Dehn filling cusped manifolds. Again the determination of the associated parameters yields the arithmetic data, describing, in particular, the invariant trace field and the invariant quaternion algebra. Note also, with reference to §5.2, that Snap also yields information about integral traces. For future reference, it also indicates whether or not the manifold or orbifold is arithmetic. Once again, these packages provide censuses of closed manifolds and their details.

Returning to the cusped case and the example 5_2 , we note that for a complete structure to be guaranteed, the method described with reference to the knot 5_2 , and these packages, not only produce a polynomial satisfied by a tetrahedral parameter but also the specific root of that polynomial which gives the appropriate tetrahedral parameter. Thus the invariant trace field is identified, not just up to isomorphism, but as a subfield of \mathbb{C} . In the case of the knot 5_2 , where the invariant trace field has just one complex place, this yields nothing new. However, recall that, for the knot 6_1 , discussed in Example B in §4.5, the invariant trace field has degree 4 over \mathbb{Q} , two complex places and discriminant 257. Obtaining a tetrahedral decomposition of the complement of 6_1 gives the associated tetrahedral parameters as elements of \mathbb{C} and hence identifies the invariant trace field as a subfield of \mathbb{C} . Following the procedure set out for 5_2 , the complement of 6_1 yields initially two pentagonal regions which, on further subdivision, shows that the complement of 6_1 is a union of four ideal tetrahedra in the upper half-space and one in the lower half-space together with a degenerate quadrangle. Computing the gluing and holonomy equations from this then exhibits the tetrahedral parameters. Alternatively, the package Snap will return, for the knot 6_1 , the identifying polynomial and the specific root. For the record, the polynomial is

$$x^4 + x^2 - x + 1$$

and the root, with positive imaginary part is approximately $0.547423 + 0.585652i$. For many other examples, see Appendix 13.4.

Exercise 5.5

1. Prove that $k_\Delta \subseteq k_1$ (in the notation of Lemma 5.5.2).
2. Generalize Lemma 5.5.2 to the following setting. Say a subset $\mathcal{V} \subset \mathbb{C} \cup \{\infty\}$ is defined over a subfield $k \subset \mathbb{C}$ if there is an element of $\mathrm{PSL}(2, \mathbb{C})$ transforming \mathcal{V} into a subset of $k \cup \{\infty\}$. Show that the following are equivalent:

- \mathcal{V} is defined over k .

- All cross-ratios of 4-tuples of points of \mathcal{V} are in $k \cup \{\infty\}$.
 - If, after transforming \mathcal{V} by an element of $\mathrm{PSL}(2, \mathbb{C})$, three of its points lie in $k \cup \{\infty\}$, then they all are.
3. In the notation of the previous question, suppose \mathcal{V} is defined over k , $|\mathcal{V}| \geq 3$ and $\Gamma \subset \mathrm{PGL}(2, \mathbb{C})$ is non-elementary and satisfies; $\Gamma\mathcal{V} = \mathcal{V}$. Prove that Γ may be conjugated into $\mathrm{PGL}(2, k)$.

5.6 Constructing Invariant Trace Fields

For a finite-covolume Kleinian group, the invariant trace field is a finite non-real extension of \mathbb{Q} by Theorem 3.3.7. Through the examples discussed in Chapter 4, we produced an array of fields which are the invariant trace fields of finite-covolume Kleinian groups. These do not yield, however, any clear picture of the nature of those fields which can arise, and this is, indeed, a wide open question. Via the Bianchi groups, we note that every quadratic imaginary field can arise and, more generally by the construction of arithmetic Kleinian groups, to be considered in subsequent chapters (see, in particular, Definition 8.2.1), any number field with exactly one complex place can also arise. In this section, we show how it is possible to build on known examples using free products with amalgamation and HNN extensions. We observe in passing that the answer to the corresponding question for finite-covolume Fuchsian groups is: all fields with at least one real place. Indeed for a torsion-free Fuchsian group, the set of all such groups with representations in a fixed field is dense in the Teichmüller space (cf. §4.9).

The main result in this section is the following:

Theorem 5.6.1 *Let Γ be a finitely generated Kleinian group expressed as a free product with amalgamation or HNN-extension $\Gamma_0 *_H \Gamma_1$ or $\Gamma_0 *_H$, where H is a non-elementary Kleinian group (where all groups are assumed finitely generated). Then $k\Gamma = k\Gamma_0 \cdot k\Gamma_1$ or $k\Gamma_0$, respectively, where \cdot denotes the compositum of the two fields.*

Proof: We deal with the HNN-extension case first. By definition of the HNN-extension, Γ is generated by Γ_0 and a stable letter t say, where $tH_0t^{-1} = H_1$ and $H_i \cong H$ for $i = 0, 1$. Furthermore, this stable letter is of infinite order, and by changing the generating set, if necessary we can assume that Γ_0 is finitely generated by elements of infinite order.

Now, by Lemma 3.5.5, $k\Gamma$ coincides with the trace field of the group $\Gamma_1 = \langle t^2, \Gamma_0^{(2)} \rangle$. We claim that this latter field K is simply $k\Gamma_0$. Certainly K contains $k\Gamma_0$.

To establish this claim, we argue as follows. Since H_0 is non-elementary, there exist $g_0, h_0 \in H_0^{(2)}$ such that $\langle g_0, h_0 \rangle$ is irreducible. Then, if $g_1 =$

tg_0t^{-1} and $h_1 = th_0t^{-1}$, the subgroup $\langle g_1, h_1 \rangle$ is also irreducible and

$$A\Gamma_0 = k\Gamma_0[1, g_0, h_0, g_0h_0] = k\Gamma_0[1, g_1, h_1, g_1h_1].$$

Conjugation by t induces an automorphism θ of $A\Gamma_0$ and so by the Skolem Noether Theorem, there exists $y \in A\Gamma_0^*$ such that $\theta(a) = yay^{-1}$ for all $a \in AH_0$. By the argument of the proof of Theorem 3.3.4, we deduce that t differs from y by a non-zero element in $k\Gamma_0$. Again, as in the case of Theorem 3.3.4, squaring and taking traces, we deduce that $t^2 \in A\Gamma_0^1$. Hence, $\langle t^2, \Gamma_0^{(2)} \rangle$ is contained in $A\Gamma_0^1$, and so traces lie in $k\Gamma_0$ as is required.

The proof in the free product with amalgamation case is similar. The structure theory of free products with amalgamation means that we have $\Gamma = \langle \Gamma_0, \Gamma_1 \rangle$ and $\Gamma_0 \cap \Gamma_1 = H$. By Lemma 3.5.5, we need to show that the trace field K of the group $\langle \Gamma_0^{(2)}, \Gamma_1^{(2)} \rangle$ coincides with $k\Gamma_0 \cdot k\Gamma_1$. One inclusion is obvious; thus it remains to establish that $K \subset k\Gamma_0 \cdot k\Gamma_1$.

Since H is non-elementary, by tensoring over kH , we see that $H^{(2)}$ contains a $k\Gamma_0$ -basis for $A\Gamma_0$ and $k\Gamma_1$ -basis for $A\Gamma_1$. The key observation in this case is the following:

$$A\Gamma_0 \otimes_{k\Gamma_0} k\Gamma_0 \cdot k\Gamma_1 \cong A\Gamma_1 \otimes_{k\Gamma_1} k\Gamma_0 \cdot k\Gamma_1 := A.$$

This follows since

$$AH \otimes_{kH} k\Gamma_0 \cdot k\Gamma_1 \cong (AH \otimes_{kH} k\Gamma_i) \otimes_{k\Gamma_i} k\Gamma_0 \cdot k\Gamma_1 \cong A\Gamma_i \otimes_{k\Gamma_i} k\Gamma_0 \cdot k\Gamma_1.$$

From this, we see that $\Gamma_0^{(2)}$ and $\Gamma_1^{(2)}$ are subgroups of A^1 , and hence the field K is a subfield of the field of definition of A , namely $k\Gamma_0 \cdot k\Gamma_1$. The proof is now complete. \square

Application

Let M be a hyperbolic 3-manifold, By a *mutation* of M we mean cutting M along an embedded incompressible surface Σ and regluing via an isometry τ of Σ giving a new manifold M^τ . The manifold M^τ is called a *mutant* of M . Mutants are well-known to be hard to differentiate. In this setting, Theorem 5.6.1 yields the first application:

Corollary 5.6.2 *Mutation preserves the invariant trace field.*

For discussion of mutation invariants, see Further Reading.

Example 5.6.3 The classical setting of mutation is when M is a hyperbolic knot or link complement in S^3 , and Σ is a Conway sphere (i.e., an incompressible four-punctured sphere with meridional boundary components). Figure 5.9 shows the Kinoshita-Terasaka and Conway mutant pair.

The main application of Theorem 5.6.1 is in building certain invariant trace fields.

Theorem 5.6.4 *Let $K = \mathbb{Q}(\sqrt{-d_1}, \dots, \sqrt{-d_r})$, where the positive integers d_1, \dots, d_r are square-free. Then K is the invariant trace field of a finite-volume hyperbolic 3-manifold.*

The proof of this relies on the fact that a twice-punctured disc in a hyperbolic 3-manifold has a unique hyperbolic structure. More precisely, we have (recall our convention that all immersions map boundary to boundary) the following lemma:

Lemma 5.6.5 *Let $M = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold and $f : D \hookrightarrow M$ an incompressible twice-punctured disc in M . Then, $f_*(\pi_1(D)) \subset \Gamma$ is conjugate in $\mathrm{PSL}(2, \mathbb{C})$ to the group $F(2)$, the level 2 congruence subgroup of $\mathrm{PSL}(2, \mathbb{Z})$.*

Proof: Since $f(D)$ is a hyperbolic twice-punctured disc in M , the fundamental group viewed as a subgroup F of $\mathrm{PSL}(2, \mathbb{C})$ is generated by a pair of parabolic elements, a and b say, whose product is also parabolic. Now by conjugating in $\mathrm{PSL}(2, \mathbb{C})$, we may assume

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}.$$

Since ab is also parabolic we must have that $\mathrm{tr} ab = \pm 2$. The case of $\mathrm{tr} ab = 2$ is easily ruled out, and we deduce that $r = -2$, which gives the level 2 congruence subgroup as required. \square

The Bianchi groups $\mathrm{PSL}(2, O_d)$ are a collection of finite-covolume Kleinian groups whose invariant trace fields are $\mathbb{Q}(\sqrt{-d})$.

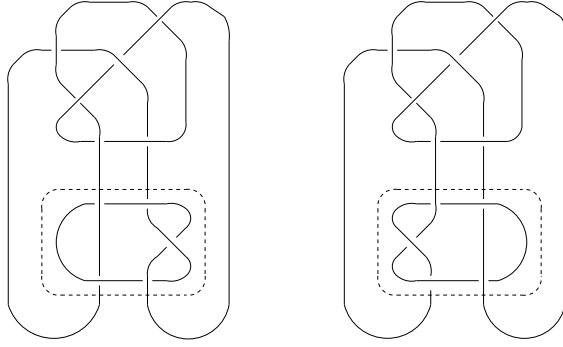


FIGURE 5.9.

Lemma 5.6.6 *For all d , $\mathrm{PSL}(2, O_d)$ contains a torsion-free subgroup G_d such that \mathbf{H}^3/G_d contains an embedded totally geodesic twice-punctured disc.*

Proof: The groups of the complements of the Whitehead link and the chain link with four components are subgroups in the cases $d = 1$ and $d = 3$, respectively and as seen in Figure 5.10, these complements contain obvious twice-punctured discs. Being totally geodesic follows from Lemma 5.6.5. Let these manifolds be denoted M_1 and M_3 respectively. For $d \neq 1, 3$, we make use of a result of Fine and Frohman:

Theorem 5.6.7 *If $d \neq 1, 3$ then $\mathrm{PSL}(2, O_d)$ can be expressed as an HNN-extension with amalgamating subgroup $\mathrm{PSL}(2, \mathbb{Z})$.*

The theorem can be viewed topologically as asserting the orbifolds have embedded incompressible sub-2-orbifolds which are non-separating copies of $\mathbf{H}^2/\mathrm{PSL}(2, \mathbb{Z})$. We can pass to manifold covers with twice-punctured discs as follows. Let $\Gamma_d(2)$ denote the level 2 congruence subgroup in the Bianchi group $\mathrm{PSL}(2, O_d)$. Then as is easy to see, $\Gamma_d(2) \cap \mathrm{PSL}(2, \mathbb{Z}) = F(2)$ (in the notation above). Let $M_d(2)$ denote the manifold $\mathbf{H}^3/\Gamma_d(2)$. Then from our above remarks $M_d(2)$ contains an embedded twice-punctured disc. \square

Proof of Theorem 5.6.3: These twice-punctured discs can be used to cut-and-paste submanifolds of the manifolds constructed in Lemma 5.6.6. Thus given a field K as in the hypothesis, we proceed by induction. The details are left as an easy exercise using Theorem 5.6.1. \square

See §10.2 for other applications of this method.

Since the invariant trace field is preserved by mutation, one could further ask whether mutation preserves the property of having integral traces. In complete generality, Bass's Theorem says that we can amalgamate groups with integral traces together and create non-integral traces. In Figure 5.11 we give a pair of mutant links for which mutation destroys integral traces.

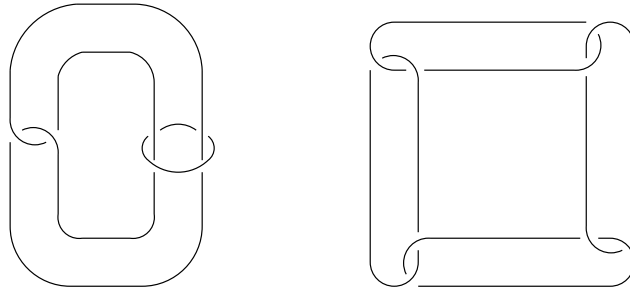


FIGURE 5.10.

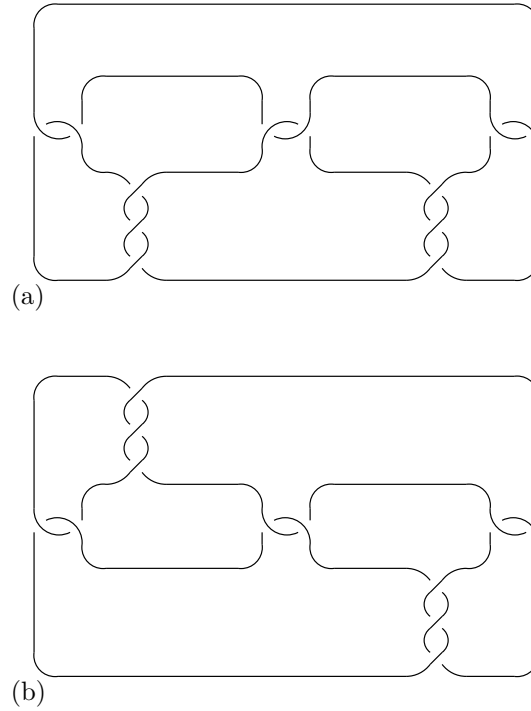


FIGURE 5.11.

The link in Figure 5.11(a) is commensurable with $\mathbf{H}^3/\mathrm{PSL}(2, O_3)$ and so has traces in O_3 . However, the mutant in Figure 5.11(b) has a non-integral trace.

These can be checked using SnapPea and Snap, for example.

Exercise 5.6

1. Complete the proof of Theorem 5.6.4.
2. Use the truncated tetrahedra with a super-ideal vertex, truncating a $(2, 3, n)$ -triangle group to obtain some further examples of invariant trace fields (see §4.7.3 and Exercise 5.2, No.6).
3. Use the knowledge of the trace fields of the Fibonacci groups to show that, given n , there exists a field with at least n real places which is the invariant trace field of a finite-covolume Kleinian group.

5.7 Further Reading

Versions of Theorem 5.1.2 and Lemma 5.1.3 appeared as results giving an intrinsic characterisation of finite-covolume Fuchsian and Kleinian groups as arithmetic in Takeuchi (1975) and Maclachlan and Reid (1987). The results given here were used in Gehring et al. (1997) to establish the existence of Kleinian groups with various extremal geometric properties. Special cases from Gehring et al. (1997) appear as Exercise 5.1, Nos 3 and 4 and Exercise 5.1, No. 5 is adapted from Maclachlan and Rosenberger (1983).

The proof of the Smith Conjecture (Morgan and Bass (1984)) had many components, one of which produced the relationship between traces, amalgam structures and embedded surfaces through the work of Bass (1980). The format employed in Theorem 5.2.2. results from methods of Culler et al. (1987). The tree associated to $SL(2, K)$ for K a \mathcal{P} -adic field is dealt with in Serre (1980).

Following Thurston (Thurston (1979)), most Dehn surgeries on a hyperbolic knot complement produce complete hyperbolic manifolds, but determining precisely those that do for any given knot is a detailed process which can be more or less ascertained by machine calculation in the form of the SnapPea program of Jeff Weeks (Weeks (2000)). Indeed, determining the actual invariant trace field after Dehn surgery and whether or not it has integral traces has also been mechanised as described in §5.5.

The work of Culler and Shalen contained in Culler and Shalen (1983) and Culler and Shalen (1984) was seminal in the development of understanding boundary slopes of hyperbolic cusped 3-manifolds from their representation and character varieties. Further developments came in Culler et al. (1987) and subsequently in the form of the A-polynomial in Cooper et al. (1994). The result referred to as Theorem 5.2.9 appears in Cooper and Long (1997) as a consequence of subtle properties of the A-polynomial.

The version of the splitting theorem 5.2.11 appears in Long and Reid (1998) following earlier versions in Long et al. (1996) and Maclachlan and Reid (1998). Tetrahedra in \mathbf{H}^3 with a super-ideal vertex which can be truncated as described in Exercise 5.2, No. 6 are enumerated in Vinberg (1985). The non-existence of totally geodesic surfaces in certain finite-covolume Kleinian groups as described in Theorems 5.3.1 and 5.3.8 were given in Reid (1991b). Separability and its use in connecting group theory and topology appears in Scott (1978) and Theorem 5.3.4 appears in Long (1987). For a variety of results and techniques used in proving that many classes of hyperbolic 3-manifolds have finite covers which are Haken, or have positive first Betti number, the reader should consult Millson (1976), Hempel (1986), Baker (1989), Li and Millson (1993), Cooper and Long (1999), Clozel (1987) and Shalen and Wagreich (1992). The twist knot examples are discussed in Reid (1991b) (see also Hoste and Shanahan (2001)), making use of earlier detailed descriptions in Riley (1974) and in Riley (1972). Theorem 5.3.13 and further consequences of this to be discussed later are

due to Chinburg and Reid (1993). The obstructions given in §5.4 appeared in Gehring et al. (1997) and borrowed from descriptions of the representations of regular solid groups in Vignéras (1980a).

The description of the invariant trace field of a cusped manifold via tetrahedral parameters was given and utilised in Neumann and Reid (1992a) and, as described in §5.5, has been combined with SnapPea to produce an exact version called Snap (Goodman (2001), Coulson et al. (2000)), by which the number fields can be quickly determined. For a given knot complement, the methodology of determining the decomposition into ideal tetrahedra was laid out in Thurston (1979) and expanded upon in Hatcher (1983) and in Menasco (1983).

The denseness of the representations of Fuchsian groups in Teichmüller space follows from work in Takeuchi (1971) and Maclachlan and Waterman (1985). An early version of Theorem 5.6.1 appears in Neumann and Reid (1991). The difficulty of distinguishing mutant pairs by any means is illustrated in the works of Lickorish and Millet (1987), Thistlethwaite (1984) and Ruberman (1987).

In his book, the structure of the Bianchi groups from a group presentational view point is discussed at length by Fine (1989), particularly concerning Fuchsian subgroups. Various amalgam and HNN descriptions of these groups are given there, including Theorem 5.6.7 which is taken from Fine and Frohman (1986).

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