

3

Invariant Trace Fields

The main algebraic invariants associated to a Kleinian group are its invariant trace field and invariant quaternion algebra. For a finite-covolume Kleinian group, its invariant trace field is shown in this chapter to be a number field (i.e., a finite extension of the rationals). This allows the invariants and the algebraic number-theoretic structure of such fields to be used in the study of these groups. This will be carried out in subsequent chapters. The invariant trace field is not, in general, the trace field itself but the trace field of a suitable subgroup of finite index. It is an invariant of the commensurability class of the group and that is established in this chapter. This invariance applies more generally to any finitely generated non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Likewise, the invariance, with respect to commensurability, of the associated quaternion algebra is also established. Given generators for the group, these invariants, the trace field and the quaternion algebra, can be readily computed and techniques are developed here to simplify these computations.

3.1 Trace Fields for Kleinian Groups of Finite Covolume

We begin with a basic definition:

Definition 3.1.1 *Let Γ be a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Let $\hat{\Gamma} = P^{-1}(\Gamma)$, where $P : \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{PSL}(2, \mathbb{C})$. Then the trace field of Γ ,*

denoted $\mathbb{Q}(\text{tr } \Gamma)$, is the field:

$$\mathbb{Q}(\text{tr } \hat{\gamma} : \hat{\gamma} \in \hat{\Gamma}).$$

Note that for any $\gamma \in \text{PSL}(2, \mathbb{C})$, the traces of any lifts to $\text{SL}(2, \mathbb{C})$ will only differ by \pm . Note also that in the above definition, we take traces in $\hat{\Gamma} = P^{-1}(\Gamma)$, so that we are not concerned with lifting the representation $\Gamma \rightarrow \text{PSL}(2, \mathbb{C})$ to a representation of Γ into $\text{SL}(2, \mathbb{C})$. When there is a lifting ρ of the representation, then, of course, $\mathbb{Q}(\text{tr } \rho(\gamma) : \gamma \in \Gamma) = \mathbb{Q}(\text{tr } \hat{\gamma} : \hat{\gamma} \in \hat{\Gamma})$. Thus we will frequently mildly abuse notation and simply write $\mathbb{Q}(\text{tr } \Gamma) = \mathbb{Q}(\pm \text{tr } \gamma : \gamma \in \Gamma)$. Of course, $\mathbb{Q}(\text{tr } \Gamma)$ is a conjugacy invariant.

The starting point for much of what follows in the book is the next result.

Theorem 3.1.2 *Let Γ be a Kleinian group of finite covolume. Then the field $\mathbb{Q}(\text{tr } \Gamma)$ is a finite extension of \mathbb{Q} .*

Later in this chapter, a number of useful identities on traces in $\text{SL}(2, \mathbb{C})$ will be given which are essential in calculations. For the moment, for the purposes of proving the above theorem, we prove one such identity which will also be used subsequently.

Lemma 3.1.3 *If $X \in \text{SL}(2, \mathbb{C})$, then $X^n = p_n(\text{tr } X)X - q_n(\text{tr } X)I$, where p_n and q_n are monic integral polynomials of degrees $n-1$ and $n-2$, respectively.*

Proof: The result follows from repeated use of

$$X^2 = (\text{tr } X)X - I, \tag{3.1}$$

from which we see that $p_n(x) = xp_{n-1}(x) - q_{n-1}(x)$ and $q_n(x) = p_{n-1}(x)$. \square

Corollary 3.1.4 *$\text{tr}(X^n)$ is a monic integral polynomial of degree n in $\text{tr}(X)$.*

Before commencing with the proof of Theorem 3.1.2, we prove a lemma, referring the reader to §1.6 for notation.

Lemma 3.1.5 *Let V be an algebraic variety defined over an algebraic number field k and let V have dimension 0. Then V is a single point and its coordinates are algebraic numbers.*

Proof: In this case, $\mathbb{C}(V) = \mathbb{C}$ since \mathbb{C} is algebraically closed. Hence $\mathbb{C}[V] = \mathbb{C}$. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in V$. The maximal ideal defined by $m_{\mathbf{x}} = \{f \in \mathbb{C}[V] \mid f(\mathbf{x}) = 0\}$ must be the trivial ideal $\{0\}$. Now the function f_i , obtained from the polynomial $X_i - x_i$, lies in $m_{\mathbf{x}}$ and so $X_i - x_i \in I(V)$. Thus $I(V)$ contains $X_1 - x_1, X_2 - x_2, \dots, X_n - x_n$ and so its vanishing set is

a single point. However, for the algebraically closed field $\bar{k} = \bar{\mathbb{Q}}$, $\bar{k}(V) = \bar{k}$. Thus as above, $X_i - x_i \in \bar{k}[X]$ lies in $I(V)$. Thus $x_i \in \bar{k}$. \square

We now commence with the proof of Theorem 3.1.2.

Proof: Since Γ is of finite covolume it is finitely presented. We lift Γ to $\mathrm{SL}(2, \mathbb{C})$ and, by abuse, continue to denote it by Γ . Now by Selberg's Lemma, Theorem 1.3.5, Γ contains a torsion-free subgroup Γ_1 of finite index. If all the traces in Γ_1 are algebraic, it follows from Corollary 3.1.4 that all traces in Γ are algebraic. It thus suffices to assume that Γ is torsion free.

As in §1.6, we form the algebraic subset $V(\Gamma)$ of $\mathrm{Hom}(\Gamma, \mathrm{SL}(2, \mathbb{C}))$. We will show that the dimension of $V(\Gamma)$ is 0. This will complete the proof, for by Lemma 3.1.5 the entries of the matrices A_i will be algebraic numbers. However, since Γ is finitely generated, all matrix entries in Γ will lie in a finite extension F of \mathbb{Q} , so that $\mathbb{Q}(\mathrm{tr} \Gamma) \subset F$ and the result follows.

Thus suppose by way of contradiction that the dimension of $V(\Gamma)$ is positive. Thus there are elements of $V(\Gamma) \subset \mathrm{Hom}(\Gamma, \mathrm{SL}(2, \mathbb{C}))$, arbitrarily close to the inclusion map, but distinct from it. By the Local Rigidity Theorem 1.6.2, these image groups must be finite-covolume Kleinian groups isomorphic to Γ . By Mostow's Rigidity Theorem 1.6.3, these groups are all conjugate in $\mathrm{Isom}(\mathbf{H}^3)$ to Γ . However the equations (1.14) imply that only four inner automorphisms of Γ , respecting this fix-point normalisation, are possible. This completes the proof. \square

Since Mostow Rigidity implies that the hyperbolic structure is a topological invariant of a finite-covolume hyperbolic 3-manifold, we have the following consequence:

Corollary 3.1.6 *Let $M = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold which has finite volume. Then $\mathbb{Q}(\mathrm{tr} \Gamma)$ is a topological invariant of M .*

There are several methods and techniques which simplify the calculation of these number fields described in this section in various types of examples. These will be given in later sections of this chapter once we have developed other related useful invariants of finite-covolume hyperbolic groups.

Exercise 3.1

1. If \mathbf{H}^3/Γ is the figure 8 knot complement, show directly (i.e., without using the Rigidity Theorems as in the proof of Theorem 3.1.2), that $V(\Gamma)$ has dimension 0.

2. Let $p_n(x)$ be the polynomials described in Lemma 3.1.3. If x is real and > 2 , so that $x = 2 \cosh \theta$, show that

$$p_n(x) = \frac{\sinh n\theta}{\sinh \theta} \quad \text{for } n \geq 1.$$

3.2 Quaternion Algebras for Subgroups of $\mathrm{SL}(2, \mathbb{C})$

Throughout this section, Γ is a non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$. Here we associate to Γ a quaternion algebra over $\mathbb{Q}(\mathrm{tr} \Gamma)$. Let

$$A_0\Gamma = \{\Sigma a_i \gamma_i \mid a_i \in \mathbb{Q}(\mathrm{tr} \Gamma), \gamma_i \in \Gamma\} \quad (3.2)$$

where only finitely many of the a_i are non-zero.

Theorem 3.2.1 *$A_0\Gamma$ is a quaternion algebra over $\mathbb{Q}(\mathrm{tr} \Gamma)$.*

Proof: It is clear that $A_0\Gamma$ is an algebra and so, by Theorem 2.1.8, we need to show that $A_0\Gamma$ is four-dimensional, central and simple over $\mathbb{Q}(\mathrm{tr} \Gamma)$.

Since Γ is non-elementary, it contains a pair of loxodromic elements, say g and h , such that $\langle g, h \rangle$ is irreducible, and so the vectors I, g, h and gh in $M_2(\mathbb{C})$ are linearly independent by Lemma 1.2.4. Now $A_0\Gamma \mathbb{C}$ is a ring and, by the above, of dimension at least 4 over \mathbb{C} . Thus $A_0\Gamma \mathbb{C} = M_2(\mathbb{C})$. Note also that $A_0\Gamma$ is central for if a lies in the centre of $A_0\Gamma$, then it lies in the centre of $M_2(\mathbb{C})$. Thus a is a multiple of the identity. It will now be shown that $A_0\Gamma$ is four dimensional over $\mathbb{Q}(\mathrm{tr} \Gamma)$.

Let T denote the trace form on $M_2(\mathbb{C})$ so that

$$T(a, b) = \mathrm{tr}(ab) \quad (3.3)$$

is a non-degenerate symmetric bilinear form (see Exercise 2.3, No. 1). A dual basis of $M_2(\mathbb{C})$, $\{I^*, g^*, h^*, (gh)^*\}$, is therefore well-defined. Since this spans, if $\gamma \in \Gamma$, then

$$\gamma = x_0 I^* + x_1 g^* + x_2 h^* + x_3 (gh)^*, \quad x_i \in \mathbb{C}. \quad (3.4)$$

If $\gamma_i \in \{I, g, h, gh\}$, then

$$T(\gamma, \gamma_i) = \mathrm{tr}(\gamma \gamma_i) = x_j, \quad \text{for some } j \in \{0, 1, 2, 3\}. \quad (3.5)$$

Hence as $\gamma \gamma_i \in \Gamma$, $\mathrm{tr} \gamma \gamma_i \in \mathbb{Q}(\mathrm{tr} \Gamma)$, and so we deduce from (3.5) that $x_0, \dots, x_3 \in \mathbb{Q}(\mathrm{tr} \Gamma)$. Thus

$$\mathbb{Q}(\mathrm{tr} \Gamma)[I, g, h, gh] \subset A_0\Gamma \subset \mathbb{Q}(\mathrm{tr} \Gamma)[I^*, g^*, h^*, (gh)^*].$$

Thus $A_0\Gamma$ is four dimensional over $\mathbb{Q}(\mathrm{tr} \Gamma)$.

Finally, we show that $A_0\Gamma$ is simple. For if J is a non-zero two-sided ideal, then $J \mathbb{C}$ is a non-zero two-sided ideal in $M_2(\mathbb{C})$. Thus $J \mathbb{C} = M_2(\mathbb{C})$ and J has dimension 4 over \mathbb{C} . Hence it must have dimension at least 4 over $\mathbb{Q}(\mathrm{tr} \Gamma)$ so that $J = A_0\Gamma$. \square

Note that multiplication in $A_0(\Gamma)$ is just the restriction of matrix multiplication in $M_2(\mathbb{C})$. Thus since the pure quaternions, and hence the reduced trace and norm, are determined by the multiplication (see §2.1), the reduced trace and norm in $A_0(\Gamma)$ coincide with the usual matrix trace and determinant.

Corollary 3.2.2 *If Γ is a non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$ and $g, h \in \Gamma$ are any pair of loxodromic elements such that $\langle g, h \rangle$ is irreducible, then $A_0\Gamma = \mathbb{Q}(\mathrm{tr} \Gamma)[I, g, h, gh]$.*

Corollary 3.2.3 *Let the subgroup Γ of $\mathrm{SL}(2, \mathbb{C})$ contain two elements g and h such that $\langle g, h \rangle$ is irreducible. Then $A_0\Gamma$ is a quaternion algebra over $\mathbb{Q}(\mathrm{tr} \Gamma)$ and*

$$A_0\Gamma = \mathbb{Q}(\mathrm{tr} \Gamma)[I, g, h, gh].$$

Proof: Note that in Theorem 3.2.1, the assumption that the group Γ is non-elementary was only used to exhibit elements g and h such that $\{I, g, h, gh\}$ is a linearly independent set over \mathbb{C} . Given any such pair of elements in Γ , like those guaranteed by the conditions given in this corollary, the same conclusion follows. \square

By normalising the elements g and h described in these corollaries, a fairly explicit representation of $A_0\Gamma$ can be obtained. Thus, assuming that g is not parabolic, conjugate so that

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad h = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}, \quad c \neq 0. \quad (3.6)$$

If $k = \mathbb{Q}(\mathrm{tr} \Gamma)$, then the eigenvalue λ satisfies a quadratic over k and so $K = k(\lambda)$ is an extension of degree 1 or 2 over k . Since $a + d$ and $\lambda a + \lambda^{-1}d \in k$, it follows that a, d and $c = ad - 1$ all lie in $k(\lambda)$. Thus after conjugation, $A_0\Gamma \subset M_2(k(\lambda))$.

Corollary 3.2.4 *With Γ, g, h , and λ as described above, Γ is conjugate to a subgroup of $\mathrm{SL}(2, k(\lambda))$.*

It should be noted that since g satisfies the same minimum polynomial as λ , the field $k(\lambda)$ embeds in $A_0\Gamma$. The above is thus a direct exhibition of the result that $k(\lambda)$ splits the algebra $A_0\Gamma$ as given in Corollary 2.1.9. For more details, see Exercise 3.2, No. 2.

These corollaries and various refinements of them will be frequently used in the determination of the quaternion algebras.

The following particular case of the above corollary is worth noting.

Corollary 3.2.5 *If Γ is a non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$ such that $\mathbb{Q}(\mathrm{tr} \Gamma)$ is a subset of \mathbb{R} , then Γ is conjugate to a subgroup of $\mathrm{SL}(2, \mathbb{R})$.*

Proof: If we choose g to be loxodromic, then as it has real trace, g will be hyperbolic. Thus $\lambda \in \mathbb{R}$ and the result follows from Corollary 3.2.4. \square

Exercise 3.2

1. Let $A_0\Gamma$ be as described at (3.2). Assume, in addition, that all traces in Γ are algebraic integers. Define

$$\mathcal{O}\Gamma = \left\{ \sum a_i \gamma_i \mid a_i \in R_{\mathbb{Q}(\text{tr } \Gamma)}, \gamma_i \in \Gamma \right\}. \quad (3.7)$$

Show that $\mathcal{O}\Gamma$ is an order in $A_0\Gamma$.

2. Suppose in the normalisation given at (3.6) that $\lambda \notin k$. Prove that

(a) a, d are $k(\lambda) \mid k$ conjugates and so $c \in k$.

(b)

$$A_0(\Gamma) = \left\{ \begin{pmatrix} x & y \\ cy' & x' \end{pmatrix} \mid x, y \in k(\lambda) \right\},$$

where x' and y' are the $k(\lambda) \mid k$ conjugates of x and y , respectively.

(c) Hence show that $A_0(\Gamma) = \left(\frac{\beta^2 \cdot c}{k} \right)$, where $\beta \in k(\lambda)$ is such that $k(\beta) = k(\lambda)$ and $\beta^2 \in k$.

3. If $\Gamma = \text{SL}(2, \mathbb{Z})$, show that

$$A_0(\Gamma) = M_2(\mathbb{Q}) \cong \left\{ \begin{pmatrix} a & b \\ -b' & a' \end{pmatrix} \mid a, b \in \mathbb{Q}(\sqrt{5}) \right\}$$

where a' and b' are the $\mathbb{Q}(\sqrt{5}) \mid \mathbb{Q}$ conjugates of a and b .

4. If Γ is the $(4, 4, 4)$ -triangle group, show that $k = \mathbb{Q}(\text{tr } \Gamma) = \mathbb{Q}(\sqrt{2})$ and $A_0(\Gamma) = \left(\frac{-1, 1 + \sqrt{2}}{k} \right)$. Deduce that $A_0(\Gamma)$ is a division algebra.

5. If Γ_1, Γ are non-elementary Kleinian groups and $\Gamma_1 \subset \Gamma$, show that $A_0(\Gamma) \cong A_0(\Gamma_1) \otimes_{\mathbb{Q}(\text{tr } \Gamma_1)} \mathbb{Q}(\text{tr } \Gamma)$.

6. The binary tetrahedral group G is a central extension of a group of order 2 by the tetrahedral group A_4 . Show that G can be embedded in $\text{SL}(2, \mathbb{C})$ as an irreducible subgroup. Determine $\mathbb{Q}(\text{tr } G)$ and $A_0(G)$. (See Exercise 2.3, No.7.)

3.3 Invariant Trace Fields and Quaternion Algebras

Although the trace field is an invariant of a Kleinian group, it is *not*, in general, an invariant of the commensurability class of that group in $\text{PSL}(2, \mathbb{C})$. As we shall show in this section, there is a field which is an invariant of the commensurability class, but first we give an example to show that the trace field is not that invariant field.

Example 3.3.1 Let Γ be the subgroup of $\mathrm{PSL}(2, \mathbb{C})$ generated by the images of A and B , where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}.$$

Here $\omega = (-1 + \sqrt{-3})/2$ so that the ring of integers O_3 in the field $\mathbb{Q}(\sqrt{-3})$ is $\mathbb{Z}[\omega]$. Clearly all the entries of the matrices in Γ lie in O_3 . Thus Γ is discrete and $\mathbb{Q}(\mathrm{tr} \Gamma) = \mathbb{Q}(\sqrt{-3})$. If $X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, then one easily sees that the image of X normalises Γ and its square is the identity. Thus $\Gamma_1 = \langle \Gamma, PX \rangle$ contains Γ as a subgroup of index 2. Now Γ_1 contains the image of $XBA = \begin{pmatrix} i & i \\ i\omega & -i+i\omega \end{pmatrix}$ so that i lies in the trace field of Γ_1 .

It should also be remarked that Γ is, in addition, of finite covolume, as it is a subgroup of index 12 in the arithmetic group $\mathrm{PSL}(2, O_3)$. (See §1.4.3.)

Now let Γ be a finitely generated non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$. We will next construct a subgroup of finite index in Γ whose trace field is an invariant of the commensurability class of the group.

Definition 3.3.2 Let $\Gamma^{(2)} = \langle \gamma^2 \mid \gamma \in \Gamma \rangle$.

Lemma 3.3.3 $\Gamma^{(2)}$ is a finite index normal subgroup of Γ whose quotient is an elementary abelian 2-group.

Proof: $\Gamma^{(2)}$ is obviously normal in Γ and such that all elements in the quotient have order 2. Since Γ is finitely generated, it follows that $\Gamma/\Gamma^{(2)}$ is a finite elementary abelian 2-group. \square

With this, we now prove one of the main results:

Theorem 3.3.4 Let Γ be a finitely generated non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$. The field $\mathbb{Q}(\mathrm{tr} \Gamma^{(2)})$ is an invariant of the commensurability class of Γ .

Proof: It will be shown that if Γ_1 has finite index in Γ , then $\mathbb{Q}(\mathrm{tr} \Gamma^{(2)}) \subset \mathbb{Q}(\mathrm{tr} \Gamma_1)$. With this, the theorem will follow. To see this, suppose Δ is commensurable with Γ . Hence by Lemma 3.3.3, $\Gamma^{(2)}$ and $\Delta^{(2)}$ are commensurable and so $\Gamma^{(2)} \cap \Delta^{(2)}$ has finite index in both Γ and Δ . Thus assuming the above claim we have the following inclusions:

- $\mathbb{Q}(\mathrm{tr} \Gamma^{(2)}) \subset \mathbb{Q}(\mathrm{tr} \Gamma^{(2)} \cap \Delta^{(2)})$
- $\mathbb{Q}(\mathrm{tr} \Delta^{(2)}) \subset \mathbb{Q}(\mathrm{tr} \Gamma^{(2)} \cap \Delta^{(2)})$

By definition, $\mathbb{Q}(\mathrm{tr} \Gamma^{(2)} \cap \Delta^{(2)}) \subset \mathbb{Q}(\mathrm{tr} \Gamma^{(2)})$ and so the above inclusions are all equalities. In particular, $\mathbb{Q}(\mathrm{tr} \Gamma^{(2)}) = \mathbb{Q}(\mathrm{tr} \Delta^{(2)})$, as required.

To establish the claim, first note that we can assume, in addition, that Γ_1 is a normal subgroup of finite index in Γ because, if C is the core of

Γ_1 in Γ (i.e., the intersection of all conjugates of Γ_1 under Γ), then C is normal of finite index in Γ . Since $\mathbb{Q}(\text{tr } C) \subset \mathbb{Q}(\text{tr } \Gamma_1)$, it suffices to show that $\mathbb{Q}(\text{tr } \Gamma^{(2)}) \subset \mathbb{Q}(\text{tr } C)$.

Recalling (3.2), let

$$A_0\Gamma_1 = \{\Sigma a_i \gamma_i \mid a_i \in \mathbb{Q}(\text{tr } \Gamma_1), \gamma_i \in \Gamma_1\}.$$

We next claim that given any $g \in \Gamma$, $g^2 \in A_0\Gamma_1$. Notice that since Γ_1 is normal in Γ , any such g induces by conjugation an automorphism of Γ_1 and hence an automorphism ϕ_g of $A_0\Gamma_1$. By Theorem 3.2.1, $A_0\Gamma_1$ is a quaternion algebra over $\mathbb{Q}(\text{tr } \Gamma_1)$, and so ϕ_g is inner by the Skolem Noether Theorem (see Corollary 2.9.9). Thus there exists $a \in (A_0\Gamma_1)^*$ such that

$$\phi_g(x) = axa^{-1} \quad (3.8)$$

for all $x \in A_0\Gamma_1$. Thus in $A_0\Gamma\mathbb{C} = M_2(\mathbb{C})$, $g^{-1}a$ commutes with every element and so $g^{-1}a = yI$ for some $y \in \mathbb{C}$. Consequently,

$$y^2 = \det(g^{-1}a) = \det(g^{-1})\det(a) = \det(a). \quad (3.9)$$

Now $(\det a)I = a^2 - \text{tr}(a)a \in A_0(\Gamma_1)$ so that $y^2 \in \mathbb{Q}(\text{tr } \Gamma_1)$. Hence, $g^2 = y^{-2}a^2 \in A_0\Gamma_1$, as claimed. Since g was chosen arbitrarily from Γ , $\Gamma^{(2)} \subset A_0(\Gamma)$ and, hence, $\mathbb{Q}(\text{tr } \Gamma^{(2)}) \subset \mathbb{Q}(\text{tr } \Gamma_1)$. \square

Corollary 3.3.5 *If Γ is a finitely generated non-elementary subgroup of $\text{SL}(2, \mathbb{C})$, then the quaternion algebra $A_0\Gamma^{(2)}$ is an invariant of the commensurability class of Γ .*

Proof: If Γ and Δ are commensurable, then $\mathbb{Q}(\text{tr } \Gamma^{(2)}) = \mathbb{Q}(\text{tr } \Delta^{(2)})$. Now choose an irreducible pair of loxodromic elements in $\Gamma^{(2)} \cap \Delta^{(2)}$. Then by Corollary 3.2.2, the quaternion algebras $A_0\Gamma^{(2)}$ and $A_0\Delta^{(2)}$ are equal. \square

Of course, the field $\mathbb{Q}(\text{tr } \Gamma^{(2)})$ is also an invariant of the wide commensurability class of Γ , where Γ and Δ are in the same wide commensurability class if there exists $t \in \text{SL}(2, \mathbb{C})$ such that $t\Gamma t^{-1}$ and Δ are commensurable (see Definition 1.3.4). Also, the quaternion algebras $A\Gamma^{(2)}$ and $A\Delta^{(2)}$ will be isomorphic since conjugation by t will define an isomorphism, acting like the identity on the centre, from the quaternion algebra $A\Gamma^{(2)}$ to the quaternion algebra $A\Delta^{(2)}$.

Definition 3.3.6 *Let Γ be a finitely generated non-elementary subgroup of $\text{PSL}(2, \mathbb{C})$. The field $\mathbb{Q}(\text{tr } \Gamma^{(2)})$ will henceforth be denoted by $k\Gamma$ and referred to as the invariant trace field of Γ . Likewise, the quaternion algebra $A_0\Gamma^{(2)}$ over $\mathbb{Q}(\text{tr } \Gamma^{(2)})$ will be denoted by $A\Gamma$ and referred to as the invariant quaternion algebra of Γ .*

The cases of particular interest here occur when Γ has finite covolume.

Theorem 3.3.7 *If Γ is a Kleinian group of finite covolume, then its invariant trace field is a finite non-real extension of \mathbb{Q} .*

Proof: That $k\Gamma$ is a finite extension of \mathbb{Q} follows from Theorem 3.1.2. Suppose that $k\Gamma$ is a real field. By Corollary 3.2.5, $\Gamma^{(2)}$ is conjugate to a subgroup of $\mathrm{SL}(2, \mathbb{R})$. However, $\Gamma^{(2)}$ cannot then have finite covolume. \square

We also note the fundamental relationship between the basic structure of quaternion algebras and the topology of the quotient space.

Theorem 3.3.8 *If Γ is a non-elementary group which contains parabolic elements, then $A_0\Gamma = M_2(\mathbb{Q}(\mathrm{tr}\Gamma))$. In particular, if Γ is a Kleinian group such that \mathbf{H}^3/Γ has finite volume but is non-compact, then $A\Gamma = M_2(k\Gamma)$.*

Proof: If Γ has a parabolic element γ , then $\gamma - I$ is non-invertible in the quaternion algebra. Thus $A_0\Gamma$ cannot be a division algebra. The result then follows from Theorem 2.1.7. \square

Given Γ as a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ means that its trace field is naturally embedded in \mathbb{C} . Thus the invariant trace field is a subfield of \mathbb{C} and so is not just defined up to isomorphism, but is embedded in \mathbb{C} .

Only in the first section of this chapter do we use the fact that the trace field is a number field. The results elsewhere in this chapter apply to any finitely generated non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$ and so, in particular, apply to all finitely generated Fuchsian groups.

It should be noted that even in the cases where the Kleinian groups are of finite covolume, the invariant trace field and quaternion algebra are not complete commensurability invariants. There are many examples of non-commensurable manifolds with the same invariant trace field and, indeed, of cocompact and non-cocompact groups with the same invariant trace field. Examples will be given in the next chapter and more will emerge later, particularly in the discussion of arithmetic groups. There are also examples of non-commensurable manifolds with isomorphic quaternion algebras and these will be discussed later.

Let Γ be a finitely generated non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$ so that $\Gamma^{(2)}$ is a normal subgroup of finite index. Then, as in the proof of Theorem 3.3.4, conjugation by $g \in \Gamma$ induces an automorphism of $\Gamma^{(2)}$ and, hence, induces an automorphism of the quaternion algebra $A\Gamma$ which is necessarily inner. Thus using (3.8), the assignment $g \rightarrow a$ induces a homomorphism of Γ into $A\Gamma^*/(k\Gamma)^*$ and, hence, into $\mathrm{SO}((A\Gamma)_0, n)$ by Theorem 2.4.1. Thus any finite-covolume Kleinian group Γ in $\mathrm{PSL}(2, \mathbb{C})$ admits a faithful representation in the $k\Gamma$ points of a linear algebraic group defined over $k\Gamma$, where $k\Gamma$ is a number field.

Exercise 3.3

1. Let Γ be a Kleinian group of finite covolume. Show that there are only finitely many Kleinian groups Γ_1 such that $\Gamma_1^{(2)} = \Gamma^{(2)}$.
2. Show that if \mathbf{H}^3/Γ is a compact hyperbolic manifold whose volume is bounded by c , then $[\Gamma : \Gamma^{(2)}]$ is bounded by a function of c .
3. Let Γ be a Kleinian group such that every element of Γ leaves a fixed circle in the complex plane invariant. Prove that the invariant trace field $k\Gamma \subset \mathbb{R}$.
4. Let Ad denote the adjoint representation of SL_2 to $\text{GL}(\mathcal{L})$, where \mathcal{L} is the Lie algebra of SL_2 . Let Γ be a subgroup of finite covolume in $\text{SL}(2, \mathbb{C})$. Show that $k\Gamma = \mathbb{Q}(\{\text{tr Ad } \gamma : \gamma \in \Gamma\})$.
5. Let Γ be a Kleinian group of finite covolume. Let σ be a Galois embedding of $k\Gamma$ such that $\sigma(k\Gamma)$ is real and $A\Gamma$ is ramified at the real place corresponding to σ . Prove that if τ is a Galois embedding of $\mathbb{Q}(\text{tr } \Gamma)$ such that $\tau|_{k\Gamma} = \sigma$, then $\tau(\mathbb{Q}(\text{tr } \Gamma))$ is real. (See Exercise 2.9, No. 6.)
6. Show that, if Γ is the $(2, 3, 8)$ -Fuchsian triangle group, then $\mathbb{Q}(\text{tr } \Gamma) \neq k\Gamma$. Show that $A\Gamma$ does not split over $k\Gamma$. (See Exercise 3.2, No. 4.) Describe the linear algebraic group G defined over $k\Gamma$ such that Γ has a faithful representation in the $k\Gamma$ points of G . Deduce that Γ has a faithful representation in $\text{SO}(3, \mathbb{R})$.
7. Let Γ denote the orientation-preserving subgroup of index 2 in the Coxeter group generated by reflections in the faces of the (ideal) tetrahedron in \mathbf{H}^3 bounded by the planes $y = 0, x = \sqrt{3}y, x = (1 + \sqrt{5})/4$ and the unit hemisphere. Determine the invariant trace field and quaternion algebra of Γ . Let Δ denote the orientation-preserving subgroup of index 2 in the Coxeter group generated by reflections in the faces of a regular ideal dodecahedron in \mathbf{H}^3 with dihedral angles $\pi/3$. Find the invariant trace field and quaternion algebra of Δ .

3.4 Trace Relations

There are a number of identities between traces of matrices in $\text{SL}(2, \mathbb{C})$. These are particularly useful in the determination of generators of the trace fields, which is carried out in the next section. The most useful of these identities are listed below and many are established by straightforward calculation.

Trace is, of course, invariant on conjugacy classes so that

$$\text{tr } XY = \text{tr } ZXYZ^{-1} \quad \text{for } X, Y \in M_2(\mathbb{C}), Z \in \text{GL}(2, \mathbb{C}). \quad (3.10)$$

In particular,

$$\operatorname{tr} XY = \operatorname{tr} YX \quad \text{and} \quad \operatorname{tr} X_1 X_2 \cdots X_n = \operatorname{tr} X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(n)} \quad (3.11)$$

for any cyclic permutation σ of $1, 2, \dots, n$.

Recall that for $X \in \operatorname{SL}(2, \mathbb{C})$

$$X^2 = (\operatorname{tr} X)X - I, \quad (3.12)$$

from which we deduce

$$\operatorname{tr} X^2 = \operatorname{tr}^2 X - 2 \quad (3.13)$$

and other identities for higher powers of X , as given in Lemma 3.1.3.

The other basic identities for elements $X, Y \in \operatorname{SL}(2, \mathbb{C})$ are

$$\operatorname{tr} XY = (\operatorname{tr} X)(\operatorname{tr} Y) - \operatorname{tr} XY^{-1}, \quad \operatorname{tr} X = \operatorname{tr} X^{-1}. \quad (3.14)$$

By repeated application of these relations, the following identities, which will be useful in the next two sections, are readily obtained.

$$\operatorname{tr} [X, Y] = \operatorname{tr}^2 X + \operatorname{tr}^2 Y + \operatorname{tr}^2 XY - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} XY - 2 \quad (3.15)$$

$$\operatorname{tr} XYXZ = \operatorname{tr} XY \operatorname{tr} XZ - \operatorname{tr} YZ^{-1} \quad (3.16)$$

$$\operatorname{tr} XYX^{-1}Z = \operatorname{tr} XY \operatorname{tr} X^{-1}Z - \operatorname{tr} X^2 YZ^{-1} \quad (3.17)$$

$$\operatorname{tr} X^2 YZ = \operatorname{tr} X \operatorname{tr} XYZ - \operatorname{tr} YZ \quad (3.18)$$

$$\operatorname{tr} XYZ + \operatorname{tr} YXZ + \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} Z = \operatorname{tr} X \operatorname{tr} YZ + \operatorname{tr} Y \operatorname{tr} XZ + \operatorname{tr} Z \operatorname{tr} XY \quad (3.19)$$

For this last identity, we argue as follows:

$$\begin{aligned} \operatorname{tr} XYZ &= \operatorname{tr} X \operatorname{tr} YZ - \operatorname{tr} XZ^{-1}Y^{-1} \\ &= \operatorname{tr} X \operatorname{tr} YZ - (\operatorname{tr} XZ^{-1} \operatorname{tr} Y - \operatorname{tr} XZ^{-1}Y) \\ &= \operatorname{tr} X \operatorname{tr} YZ - \operatorname{tr} Y(\operatorname{tr} X \operatorname{tr} Z - \operatorname{tr} XZ) + (\operatorname{tr} YX \operatorname{tr} Z - \operatorname{tr} YXZ). \end{aligned}$$

Finally, we take combinations of this last identity:

$$\begin{aligned} \operatorname{tr} XYZW + \operatorname{tr} YXZW &= \operatorname{tr} X \operatorname{tr} YZW + \operatorname{tr} Y \operatorname{tr} XZW + \operatorname{tr} ZW \operatorname{tr} XY \\ &\quad - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} ZW, \\ \operatorname{tr} WXYZ + \operatorname{tr} XWYZ &= \operatorname{tr} W \operatorname{tr} XYZ + \operatorname{tr} X \operatorname{tr} WYZ + \operatorname{tr} WX \operatorname{tr} YZ \\ &\quad - \operatorname{tr} W \operatorname{tr} X \operatorname{tr} YZ, \\ \operatorname{tr} XZWY + \operatorname{tr} ZXWY &= \operatorname{tr} X \operatorname{tr} ZWY + \operatorname{tr} Z \operatorname{tr} XWY + \operatorname{tr} XZ \operatorname{tr} WY \\ &\quad - \operatorname{tr} X \operatorname{tr} Z \operatorname{tr} WY. \end{aligned}$$

By subtracting the last one from the sum of the first two and using the earlier identities we obtain

$$\begin{aligned}
2\operatorname{tr} XYZW &= \operatorname{tr} X \operatorname{tr} YZW + \operatorname{tr} Y \operatorname{tr} ZWX + \operatorname{tr} Z \operatorname{tr} WXY \\
&+ \operatorname{tr} W \operatorname{tr} XYZ + \operatorname{tr} XY \operatorname{tr} ZW - \operatorname{tr} XZ \operatorname{tr} YW \\
&+ \operatorname{tr} XW \operatorname{tr} YZ - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} ZW - \operatorname{tr} Y \operatorname{tr} Z \operatorname{tr} XW \\
&- \operatorname{tr} X \operatorname{tr} W \operatorname{tr} YZ - \operatorname{tr} Z \operatorname{tr} W \operatorname{tr} XY + \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} Z \operatorname{tr} W.
\end{aligned} \tag{3.20}$$

We now establish trace relations among triple products of matrices, which will subsequently be useful. The method used in establishing these is less straightforward than the simple calculations used to establish the identities so far.

In the quaternion algebra $A = M_2(\mathbb{C})$, the pure quaternions A_0 are the matrices of trace 0 and the norm form induces a bilinear form B on A_0 given by

$$B(X, Y) = \frac{-1}{2}(XY + YX) = \frac{-1}{2}\operatorname{tr} XY. \tag{3.21}$$

Thus, for $X, Y, Z \in A_0$, $\operatorname{tr} XYZ = \operatorname{tr} ((\operatorname{tr} XY)I - YX)Z = -\operatorname{tr} YXZ$. Thus if we define F on A_0^3 by $F(X, Y, Z) = \operatorname{tr} XYZ$, then F is an alternating trilinear form. Thus if X', Y' and Z' also lie in A_0 , then

$$\operatorname{tr} XYZ \operatorname{tr} X'Y'Z' = c \det \begin{pmatrix} B(X, X') & B(X, Y') & B(X, Z') \\ B(Y, X') & B(Y, Y') & B(Y, Z') \\ B(Z, X') & B(Z, Y') & B(Z, Z') \end{pmatrix}$$

for some constant c . Using (3.21), and choosing suitable matrices [e.g., $X = X' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y = Y' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = Z' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$], we obtain $c = 4$ and

$$\operatorname{tr} XYZ \operatorname{tr} X'Y'Z' = \frac{-1}{2} \det \begin{pmatrix} \operatorname{tr} XX' & \operatorname{tr} XY' & \operatorname{tr} XZ' \\ \operatorname{tr} YX' & \operatorname{tr} YY' & \operatorname{tr} YZ' \\ \operatorname{tr} ZX' & \operatorname{tr} ZY' & \operatorname{tr} ZZ' \end{pmatrix}. \tag{3.22}$$

Now if we take any matrices X, Y, Z, X', Y' and Z' in $M_2(\mathbb{C})$, then their projections in A_0 are of the form $X_1 = X - 1/2(\operatorname{tr} X)I$ and so satisfy (3.22). Rearranging then gives that for any matrices X, Y, Z, X', Y' and Z' in $M_2(\mathbb{C})$,

$$\operatorname{tr} XYZ \operatorname{tr} X'Y'Z' + P' \operatorname{tr} XYZ + P \operatorname{tr} X'Y'Z' + Q = 0 \tag{3.23}$$

where P', P and Q are rational polynomials in the traces of these six matrices and their products taken in pairs.

Now choose $X = X', Y = Y'$ and $Z = Z'$, where $X, Y, Z \in \operatorname{SL}(2, \mathbb{C})$. A tedious calculation using (3.22) shows that $\operatorname{tr} XYZ$ satisfies the following

quadratic polynomial:

$$\begin{aligned}
 x^2 &= (\operatorname{tr} XY \operatorname{tr} Z + \operatorname{tr} YZ \operatorname{tr} X + \operatorname{tr} ZX \operatorname{tr} Y - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} Z)x \\
 &+ \operatorname{tr} XY \operatorname{tr} YZ \operatorname{tr} ZX + (\operatorname{tr}^2 XY + \operatorname{tr}^2 YZ + \operatorname{tr}^2 ZX) \\
 &- (\operatorname{tr} X \operatorname{tr} Y \operatorname{tr} XY + \operatorname{tr} Y \operatorname{tr} Z \operatorname{tr} YZ + \operatorname{tr} Z \operatorname{tr} X \operatorname{tr} ZX) \\
 &+ (\operatorname{tr}^2 X + \operatorname{tr}^2 Y + \operatorname{tr}^2 Z) - 4.
 \end{aligned} \tag{3.24}$$

Notice, in particular, that the coefficients here are *integral* polynomials in the traces of the three matrices involved and their products taken in pairs.

Exercise 3.4

1. Establish (3.15).
2. Prove that $\operatorname{tr} X^2 Y^2 = \operatorname{tr}^2 XY - \operatorname{tr} [X, Y]$.
3. Let $\phi : M_2(\mathbb{C}) \rightarrow \mathbb{C}$ be a \mathbb{C} -linear function such that ϕ is a conjugacy invariant and $\phi(I) = 2$. Prove that $\phi = \operatorname{tr}$.
4. For $X, Y \in \operatorname{SL}(2, \mathbb{C})$, let $\beta(X) = \operatorname{tr}^2 X - 4$ and $\gamma(X, Y) = \operatorname{tr} [X, Y] - 2$.
 - (a) Prove that $\operatorname{tr} X^n = (\operatorname{tr} X)^\epsilon q(\beta(X))$, where q is an integral polynomial with (i) $\epsilon = 0$ if n is even and (ii) $\epsilon = 1$ if n is odd.
 - (b) Prove that $\operatorname{tr} (X^n Y X^m Y^{-1}) = (\operatorname{tr} X)^\epsilon p(\gamma(X, Y), \beta(X))$, where p is an integral polynomial in two variables with (i) $\epsilon = 0$ if $n + m$ is even and (ii) $\epsilon = 1$ if $n + m$ is odd.
 - (c) Prove that $\operatorname{tr} (X^{n_1} Y X^{n_2} Y^{-1} \dots X^{n_{2r}} Y^{-1}) = (\operatorname{tr} X)^\epsilon p(\gamma(X, Y), \beta(X))$, where p is an integral polynomial in two variables with (i) $\epsilon = 0$ if $\sum n_i$ is even and (ii) $\epsilon = 1$ if $\sum n_i$ is odd.
 - (d) Determine $\gamma(X, (XYXY^{-1})^n)$ in terms of $\gamma(X, Y)$ and $\beta(X)$ for $n = 1$ and 2.
5. Establish (3.24).

3.5 Generators for Trace Fields

Let Γ be generated by $\gamma_1, \gamma_2, \dots, \gamma_n$. The aim is to show, first of all, that $\mathbb{Q}(\operatorname{tr} \Gamma)$ is generated over \mathbb{Q} by the traces of a small collection of elements in Γ . This will later be modified to obtain a small collection generating $\mathbb{Q}(\operatorname{tr} \Gamma^{(2)})$.

Let P denote the collection

$$\{\gamma_{j_1} \cdots \gamma_{j_t} \mid t \geq 1 \text{ and all } j_i \text{ are distinct}\}.$$

Let Q denote the collection

$$\{\gamma_{i_1} \cdots \gamma_{i_r} \mid r \geq 1 \text{ and } 1 \leq i_1 < \cdots < i_r \leq n\}.$$

Let R denote the collection

$$\{\gamma_i, \gamma_{j_1}\gamma_{j_2}, \gamma_{k_1}\gamma_{k_2}\gamma_{k_3} \mid 1 \leq i \leq n, 1 \leq j_1 < j_2 \leq n, 1 \leq k_1 < k_2 < k_3 \leq n\}.$$

We show successively that $\mathbb{Q}(\text{tr } \Gamma)$ is generated over \mathbb{Q} by the traces of the elements in P , then in Q , and finally in R .

For each $\gamma \in \Gamma$, define the length of γ [with respect to the generators $\gamma_1, \dots, \gamma_n$]

$$\ell(\gamma) = \min \left\{ \sum_{i=1}^s |\alpha_i| \mid \gamma = \gamma_{k_1}^{\alpha_1} \cdots \gamma_{k_s}^{\alpha_s} \right\}$$

where the minimum is taken over all representations of γ in terms of the given generators.

Lemma 3.5.1 *Let $\gamma \in \Gamma$. Then $\text{tr } \gamma$ is an integer polynomial in $\{\text{tr } \delta \mid \delta \in P\}$.*

Proof: We proceed by induction on the length of γ . From (3.13) and (3.14), the result is clearly true if $\ell(\gamma) = 1$ or 2. So suppose $\ell(\gamma) \geq 3$ and the result holds for all elements of length less than $\ell(\gamma)$. If $\gamma \notin P$, then either $k_i = k_j$ for distinct i and j or some $\alpha_i \neq 1$. If $k_i = k_j$, then γ , after conjugation, has the form $XYXZ$ or $XYX^{-1}Z$, and the result follows by induction from (3.16) to (3.18). In the same way, if some $|\alpha_i| \geq 2$, the result follows from (3.18). If some $\alpha_i = -1$, so that γ has the form $X\gamma_{k_i}^{-1}Y$, then

$$\text{tr } X\gamma_{k_i}^{-1}Y = \text{tr } YX\gamma_{k_i}^{-1} = \text{tr } YX \text{tr } \gamma_{k_i} - \text{tr } YX\gamma_{k_i}.$$

By repeated application of this and induction, the result follows. \square

Lemma 3.5.2 *Let $\gamma \in \Gamma$. Then $\text{tr } \gamma$ is an integer polynomial in $\{\text{tr } \delta \mid \delta \in Q\}$.*

Proof: For each permutation τ of S_n , define

$$\tau^*(Q) = \{\gamma_{\tau(i_1)} \cdots \gamma_{\tau(i_r)} \mid 1 \leq i_1 < \cdots < i_r \leq n\}$$

so that $P = \cup_{\tau \in S_n} \tau^*(Q)$. Each τ is a product of transpositions of the form $(i \ i+1)$ and we define the length of τ to be the minimum number of such transpositions required. We need to show that if $\gamma \in \tau^*(Q)$, then γ is an integer polynomial in $\{\text{tr } \delta \mid \delta \in Q\}$. Proceed by induction on the length of τ . The result is trivial if the length is 0, so let $\tau = \tau'\sigma$, where $\sigma = (i \ i+1)$ and the length of $\tau' < \text{length of } \tau$. Then using (3.19) and repeated use of (3.13), we obtain that $\gamma \in \tau^*(Q)$ has trace an integer polynomial in $\{\text{tr } \delta \mid \delta \in \tau'^*(Q)\}$. The result now follows by induction. \square

This last result suffices for many of the calculations which will appear.

It certainly suffices where the group Γ can be generated by two or three elements.

If $\Gamma = \langle g, h \rangle$, then

$$\mathbb{Q}(\text{tr } \Gamma) = \mathbb{Q}(\text{tr } g, \text{tr } h, \text{tr } gh). \quad (3.25)$$

If $\Gamma = \langle f, g, h \rangle$, then

$$\mathbb{Q}(\text{tr } \Gamma) = \mathbb{Q}(\text{tr } f, \text{tr } g, \text{tr } h, \text{tr } fg, \text{tr } fh, \text{tr } gh, \text{tr } fgh). \quad (3.26)$$

Remarks: Further use will be made of these results when we come to consider arithmetic Kleinian groups. Note, for this reason, that in all of the above results of this section, we could replace \mathbb{Q} by \mathbb{Z} .

Lemma 3.5.3 *Let $\gamma \in \Gamma$. Then $\text{tr } \gamma$ is a rational polynomial in $\{\text{tr } \delta \mid \delta \in R\}$.*

Proof: This follows immediately from Lemma 3.5.2 and the identity (3.20). \square

Given Γ , a non-elementary subgroup of $\text{SL}(2, \mathbb{C})$, we now want to determine the invariant trace field $k\Gamma = \mathbb{Q}(\text{tr } \Gamma^{(2)})$. From a presentation of Γ , a set of generators for $\Gamma^{(2)}$ can be obtained via, say, the Reidemeister-Schreier rewriting process. The above results can then be applied to $\Gamma^{(2)}$. However, note that if F is a free group on n generators, then $F^{(2)}$ has $2^n(n-1)+1$ generators, so that, in general, the number of generators of $\Gamma^{(2)}$ may increase exponentially with the number of generators of Γ . We now give an elementary result which gives a considerable saving in this direction.

Definition 3.5.4 *Let Γ be a non-elementary subgroup of $\text{SL}(2, \mathbb{C})$, with generators $\gamma_1, \gamma_2, \dots, \gamma_n$. Define Γ^{SQ} , with respect to this set of generators, by*

$$\Gamma^{SQ} = \langle \gamma_1^2, \gamma_2^2, \dots, \gamma_n^2 \rangle. \quad (3.27)$$

Lemma 3.5.5 *With Γ as above and $\text{tr } \gamma_i \neq 0$ for $i = 1, 2, \dots, n$, then $k\Gamma = \mathbb{Q}(\text{tr } \Gamma^{SQ})$.*

Proof: Clearly $\Gamma^{SQ} \subset \Gamma^{(2)}$ so that $\mathbb{Q}(\text{tr } \Gamma^{SQ}) \subset k\Gamma$. Now from (3.12), if $\text{tr } \gamma \neq 0$, then $\gamma = (\text{tr } \gamma)^{-1}(\gamma^2 + I)$ in $M_2(\mathbb{C})$. Thus, let $\gamma \in \Gamma^{(2)}$ so that $\gamma = \delta_1^2 \delta_2^2 \cdots \delta_r^2$ with $\delta_i \in \Gamma$. Now $\delta_i = \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_{r_i}}$. Thus

$$\begin{aligned} \delta_i &= \prod_{j=1}^{r_i} (\text{tr } \gamma_{i_j})^{-1} \prod_{j=1}^{r_i} (\gamma_{i_j}^2 + I), \\ \delta_i^2 &= \prod_{j=1}^{r_i} (\text{tr }^2 \gamma_{i_j})^{-1} \left(\prod_{j=1}^{r_i} (\gamma_{i_j}^2 + I) \right)^2. \end{aligned}$$

It follows that $\text{tr } \gamma \in \mathbb{Q}(\text{tr } \Gamma^{SQ})$. \square

Note that, when Γ is finitely generated, in contrast to $\Gamma^{(2)}$, Γ^{SQ} may well be of infinite index in Γ . However, under the conditions given, Γ^{SQ} has the same number of generators as Γ .

With this, we can obtain another description of $k\Gamma$ in terms of traces which is applicable to methods of characterising arithmetic Kleinian and Fuchsian groups.

Lemma 3.5.6 *Let Γ be a finitely generated non-elementary subgroup of $\text{SL}(2, \mathbb{C})$. Let $k = \mathbb{Q}(\{\text{tr } \gamma^2 : \gamma \in \Gamma\})$. Then $k = k\Gamma$.*

Proof: Note that $\text{tr } \gamma^2 = \text{tr }^2 \gamma - 2$ so that $k \subset k\Gamma$. Now choose a set of generators $\gamma_1, \dots, \gamma_n$ of Γ such that $\text{tr } \gamma_i \neq 0, \text{tr } \gamma_i^2 \gamma_j^2 \neq 0$ for all i and j . Thus by Lemmas 3.5.3 and 3.5.5, it suffices to show that $\text{tr } \gamma_i^2 \gamma_j^2, \text{tr } \gamma_i^2 \gamma_j^2 \gamma_k^2 \in k$ for all i, j and k . This follows by a manipulation of trace identities:

$$\text{tr } \gamma_i^2 \gamma_j^2 = \text{tr } \gamma_i \text{tr } \gamma_i \gamma_j \gamma_j - \text{tr } \gamma_j.$$

Squaring both sides gives that $\text{tr } \gamma_i \text{tr } \gamma_j \text{tr } \gamma_i \gamma_j \in k$ and hence so does $\text{tr } \gamma_i^2 \gamma_j^2$.

$$\text{tr } \gamma_i^2 \gamma_j^2 \gamma_k^{-1} = \text{tr } \gamma_i^2 \gamma_j^2 \text{tr } \gamma_k - \text{tr } \gamma_i^2 \gamma_j^2 \gamma_k.$$

Squaring both sides then gives that $\text{tr } \gamma_k \text{tr } \gamma_i^2 \gamma_j^2 \gamma_k \in k$ since $\text{tr } \gamma_i^2 \gamma_j^2 \neq 0$. The result follows since

$$\text{tr } \gamma_i^2 \gamma_j^2 \gamma_k^2 = \text{tr } \gamma_k \text{tr } \gamma_i^2 \gamma_j^2 \gamma_k - \text{tr } \gamma_i^2 \gamma_j^2.$$

\square

The cases where Γ has two generators deserve special attention, as there are numerous interesting examples of these. So suppose that $\Gamma = \langle g, h \rangle$ is a non-elementary group. Note that both g and h cannot have order 2. Suppose initially that neither has, so that $\text{tr } g, \text{tr } h \neq 0$. Thus by the Lemma 3.5.5 and (3.25), $k\Gamma = \mathbb{Q}(\text{tr } g^2, \text{tr } h^2, \text{tr } g^2 h^2)$. Now

$$\text{tr } g^2 h^2 = \text{tr } g \text{tr } h \text{tr } gh - \text{tr }^2 g - \text{tr }^2 h + 2. \quad (3.28)$$

Hence, from (3.25) and (3.13), we have the following:

Lemma 3.5.7 *Let $\Gamma = \langle g, h \rangle$, with $\text{tr } g, \text{tr } h \neq 0$, be a non-elementary subgroup of $\text{SL}(2, \mathbb{C})$. Then*

$$k\Gamma = \mathbb{Q}(\text{tr }^2 g, \text{tr }^2 h, \text{tr } g \text{tr } h \text{tr } gh). \quad (3.29)$$

Now suppose that $\text{tr } h = 0$ so that h has order 2. Then $\Gamma_1 = \langle g, hg^{-1}h^{-1} \rangle$ is a subgroup of index 2 in Γ and so $k\Gamma_1 = k\Gamma$ by Theorem 3.3.4. The following result is then an immediate consequence of Lemma 3.5.7.

Lemma 3.5.8 *Let $\Gamma = \langle g, h \rangle$, with $\text{tr } h = 0$, be a non-elementary subgroup of $\text{SL}(2, \mathbb{C})$. Then*

$$k\Gamma = \mathbb{Q}(\text{tr}^2 g, \text{tr}[g, h]). \quad (3.30)$$

We note that the conjugacy class of an irreducible Kleinian group $\Gamma = \langle g, h \rangle$ is determined by the three complex parameters

$$\beta(g) = \text{tr}^2 g - 4, \quad \beta(h) = \text{tr}^2 h - 4, \quad \gamma(g, h) = \text{tr}[g, h] - 2. \quad (3.31)$$

It is of interest to note how these relate to the invariant trace field when Γ is non-elementary. In the case where $\text{tr } h = 0$, it is immediate from Lemma 3.5.8 that

$$k\Gamma = \mathbb{Q}(\gamma(g, h), \beta(g)). \quad (3.32)$$

When $\text{tr } g, \text{tr } h \neq 0$, then from (3.15), one sees that $\text{tr } g \text{tr } h \text{tr } gh$ satisfies the monic quadratic polynomial

$$x^2 - (\beta(g) + 4)(\beta(h) + 4)x - (\beta(g) + 4)(\beta(h) + 4)(\gamma(g, h) - \beta(g) - \beta(h) - 4) = 0.$$

Thus from Lemma 3.5.7,

$$[k\Gamma : \mathbb{Q}(\gamma(g, h), \beta(g), \beta(h))] \leq 2. \quad (3.33)$$

Now consider the case where Γ has three generators.

Lemma 3.5.9 *Let $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$, with $\text{tr } \gamma_i \neq 0$ for $i = 1, 2$ and 3. Then $k\Gamma$ is generated over \mathbb{Q} by $\{\text{tr}^2 \gamma_i, 1 \leq i \leq 3; \text{tr } \gamma_i \gamma_j \text{tr } \gamma_i \text{tr } \gamma_j, 1 \leq i < j \leq 3; \text{tr } \gamma_1 \gamma_2 \gamma_3 \text{tr } \gamma_1 \text{tr } \gamma_2 \text{tr } \gamma_3\}$.*

Proof: From Lemma 3.5.5 and (3.26), $k\Gamma$ is generated over \mathbb{Q} by the traces of seven elements. Then using (3.28) and

$$\gamma_1^2 \gamma_2^2 \gamma_3^2 = \prod_{i=1}^3 ((\text{tr } \gamma_i) \gamma_i - I),$$

it is immediate that these seven traces can be replaced by the seven expressions given in the statement of this lemma. \square

There are many examples in the next chapter which illustrate the application of the results in this section.

Exercise 3.5

1. Show that the invariant trace field of a Fuchsian (ℓ, m, n) -triangle group is a totally real field. Suppose $\ell = 2$ and N is the least common multiple of m and n . Show that the invariant trace field has degree $\phi(N)/2$ or $\phi(N)/4$ over \mathbb{Q} according as $(m, n) > 2$ or not.

2. If $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle$, find the integer polynomial in $\{\text{tr } \delta \mid \delta \in Q\}$ for $\text{tr}(\gamma_1 \gamma_3^{-1} \gamma_2 \gamma_1 \gamma_4)$.
3. Show that for a standard set of $2g$ generators in a compact surface group Γ of genus g , Γ^{SQ} has infinite index in Γ .
4. If $\Gamma = \langle x, y, z \rangle$, show that $[\mathbb{Q}(\text{tr } \Gamma) : K] \leq 2$, where K is generated over \mathbb{Q} by the traces of the elements x, y and z and their products taken in pairs.
5. Let $\Gamma = \langle \gamma_1, \gamma_2, \dots, \gamma_n \rangle$, with $\text{tr } \gamma_i \neq 0$ for all i . Let

$$K = \mathbb{Q}(\{\text{tr }^2 \gamma_i, 1 \leq i \leq n : \text{tr } \gamma_j \text{tr } \gamma_k \text{tr } \gamma_j \gamma_k, 1 \leq j < k \leq n\}).$$

Show that $k\Gamma = K(\text{tr } \gamma_i \gamma_j \gamma_k \text{tr } \gamma_i \text{tr } \gamma_j \text{tr } \gamma_k)$ for one such triple product which does not lie in K . (See (3.23).)

6. Let Γ be a finitely presented non-elementary subgroup of $\text{SL}(2, \mathbb{C})$ with generators $\gamma_1, \gamma_2, \dots, \gamma_n$. Then the set $\text{Hom}(\Gamma)$ of homomorphisms $\rho : \Gamma \rightarrow \text{SL}(2, \mathbb{C})$ is an algebraic set in \mathbb{C}^{4n} defined over \mathbb{Q} as in §1.6. Let $X(\Gamma)$ denote the set of characters χ_ρ of such representations given by $\chi_\rho(\gamma) = \text{tr } \rho(\gamma)$. For each $g \in \Gamma$, $\tau_g : \text{Hom}(\Gamma) \rightarrow \mathbb{C}$ defined by $\tau_g(\rho) = \chi_\rho(g)$ is a regular function. Show that the ring T generated by all such functions is finitely generated. Let $\delta_1, \delta_2, \dots, \delta_m$ be such that $\{\tau_{\delta_i} : 1 \leq i \leq m\}$ generate T and define $t : \text{Hom}(\Gamma) \rightarrow \mathbb{C}^m$ by $t(\rho) = (\tau_{\delta_1}(\rho), \tau_{\delta_2}(\rho), \dots, \tau_{\delta_m}(\rho))$. Show that $X(\Gamma)$ can be identified with $t(\text{Hom}(\Gamma))$ and in this way becomes an algebraic set: the character variety of Γ .

3.6 Generators for Invariant Quaternion Algebras

Recall from Corollary 3.2.3 that $A\Gamma$ is the algebra $k\Gamma[I, g, h, gh]$, where $\langle g, h \rangle$ is an irreducible subgroup of $\Gamma^{(2)}$. The quaternion algebra can be conveniently described by its Hilbert symbol and for this, we require a standard basis of $A\Gamma$ (i.e., a basis of the form $\{1, i, j, ij\}$, where $i^2, j^2 \in k\Gamma^*$ and $ij = -ji$). Now $A\Gamma \cdot \mathbb{C} = M_2(\mathbb{C})$ (see Theorem 3.2.1), so that the pure quaternions form the subspace $sl(2, \mathbb{C})$, which, as described in §2.3, is a quadratic space with the restriction of the norm or determinant form. Let the associated symmetric bilinear form be B so that for $C, D \in sl(2, \mathbb{C})$,

$$B(C, D) = \frac{-1}{2}(CD + DC) = \frac{-1}{2}\text{tr } CD. \quad (3.34)$$

Thus C and D are mutually orthogonal if and only if $CD = -DC$. Hence, $\{i, j, ij\}$ must form an orthogonal basis of $sl(2, \mathbb{C})$ with respect to the bilinear form B .

Thus given g and h as above, let $t_0 = \text{tr } g$, $t_1 = \text{tr } h$ and $t_2 = \text{tr } gh$. Set $g' = g - (t_0/2)I$ and $h' = h - (t_1/2)I$, so that $g', h' \in sl(2, \mathbb{C})$. Also

$g'^2 = (t_0^2 - 4)/4$, $h'^2 = (t_1^2 - 4)/4$. Thus provided g and h are not parabolic, $g'^2, h'^2 \in k\Gamma^*$. Assuming that g is not parabolic, set

$$h'' = h' - \frac{B(g', h')}{B(g', g')} g'$$

so that $h'' \in \mathfrak{sl}(2, \mathbb{C})$ and is orthogonal to g' . Now

$$h''^2 = -\frac{t_0^2 + t_1^2 + t_2^2 - t_0 t_1 t_2 - 4}{t_0^2 - 4} = -\frac{\text{tr}[g, h] - 2}{t_0^2 - 4}. \quad (3.35)$$

Note that since $\langle g, h \rangle$ is irreducible, the numerator is non-zero. Thus removing squares (see Lemma 2.1.2), we have that

$$A\Gamma = \left(\frac{\text{tr}^2 g - 4, -(\text{tr}^2 g - 4)(\text{tr}[g, h] - 2)}{k\Gamma} \right) = \left(\frac{\text{tr}^2 g - 4, \text{tr}[g, h] - 2}{k\Gamma} \right). \quad (3.36)$$

See §2.1 for the last equality. We have thus established the following:

Theorem 3.6.1 *If g and h are elements of the non-elementary group $\Gamma^{(2)}$ such that $\langle g, h \rangle$ is irreducible and such that g is not parabolic, then*

$$A\Gamma = \left(\frac{\text{tr}^2 g - 4, \text{tr}[g, h] - 2}{k\Gamma} \right). \quad (3.37)$$

Now, it is convenient to describe the Hilbert symbol in terms of the elements of Γ rather than those of $\Gamma^{(2)}$.

Theorem 3.6.2 *If g and h are elements of the non-elementary group Γ such that $\langle g, h \rangle$ is irreducible, g and h do not have order 2 in $\text{PSL}(2, \mathbb{C})$ and g is not parabolic, then*

$$A\Gamma = \left(\frac{\text{tr}^2 g(\text{tr}^2 g - 4), \text{tr}^2 g \text{tr}^2 h(\text{tr}[g, h] - 2)}{k\Gamma} \right). \quad (3.38)$$

Proof: The elements g^2 and h^2 satisfy the conditions stated in the previous theorem so we can apply the method used in the proof of that theorem. Thus in (3.35), replacing t_0 by $t_0^2 - 2$, t_1 by $t_1^2 - 2$ and t_2 by $t_0 t_1 t_2 - t_0^2 - t_1^2 + 2$ (see (3.28)) gives

$$\frac{-t_0^2 t_1^2 (\text{tr}[g, h] - 2)}{t_0^2 (t_0^2 - 4)}.$$

Since $\text{tr}^2 g^2 - 4 = t_0^2 (t_0^2 - 4)$, the result follows. \square

Now if g is not parabolic and g and h generate a non-elementary subgroup, then g and h cannot both be of order 2. If neither has order 2, then $\langle g, h \rangle$ is irreducible and we can apply the above result. If h has order 2, then $\langle g, hgh^{-1} \rangle$ cannot be reducible and we can apply the above result to these elements.

Corollary 3.6.3 *Let g and h generate a non-elementary subgroup of Γ and be such that g is neither parabolic nor of order 2 in $\mathrm{PSL}(2, \mathbb{C})$ and h has order 2. Then*

$$A\Gamma = \left(\frac{\mathrm{tr}^2 g (\mathrm{tr}^2 g - 4), (\mathrm{tr} [g, h] - 2)(\mathrm{tr} [g, h] - \mathrm{tr}^2 g + 2)}{k\Gamma} \right). \quad (3.39)$$

Notice that if $\Gamma = \langle g, h \rangle$ in the above corollary, then the invariant trace field and the quaternion algebra are described in terms of the defining parameters given at (3.31).

Corollary 3.6.4 *Let $\Gamma = \langle g, h \rangle$ be a non-elementary subgroup where h has order 2 and g is not parabolic. Then*

$$A\Gamma \cong \left(\frac{(\beta(g) + 4)\beta(g), \gamma(g, h)(\gamma(g, h) - \beta(g))}{\mathbb{Q}(\beta(g), \gamma(g, h))} \right).$$

Exercise 3.6

1. Establish (3.35).
2. Let Γ, g and h be as in Theorem 3.6.2 with σ a real embedding of $k\Gamma$. Prove that $A\Gamma$ is ramified at the real place corresponding to σ if and only if $\sigma(\mathrm{tr}^2 g) < 4$ and $\sigma(\mathrm{tr} [g, h]) < 2$. (See Exercise 3.3, No. 5.)
3. Embed the group A_5 of symmetries of a regular icosahedron in $\mathrm{PSL}(2, \mathbb{C})$ and let G denote its lift to $\mathrm{SL}(2, \mathbb{C})$. Determine kG and AG (cf. Exercise 3.2, No. 6).
4. Let Γ be a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{C})$ which is generated by three elements γ_1, γ_2 and γ_3 of order 2. Let $t_1 = \mathrm{tr} \gamma_2 \gamma_3$, $t_2 = \mathrm{tr} \gamma_3 \gamma_1$, $t_3 = \mathrm{tr} \gamma_1 \gamma_2$ and $u = \mathrm{tr} \gamma_1 \gamma_2 \gamma_3$. Prove that, after a suitable permutation of γ_1, γ_2 and γ_3 ,

$$k\Gamma = \mathbb{Q}(t_2^2, t_3^2, t_1 t_2 t_3),$$

$$A\Gamma = \left(\frac{t_3^2(t_3^2 - 4), t_2^2 t_3^2(u^2 - 4)}{k\Gamma} \right).$$

3.7 Further Reading

The important Theorem 3.1.2 that the trace field is a number field for a Kleinian group of finite covolume is to be found in Thurston (1979) and also in Macbeath (1983). The connections between the matrix entries in finitely generated subgroups of $\mathrm{GL}(2, \mathbb{C})$ and the structure of the related groups was investigated in Bass (1980) and quaternion algebras constructed from the subgroups were employed in this. In the context of characterising arithmetic Fuchsian groups among all Fuchsian groups, Takeuchi, in the same

way, used the construction of quaternion algebras from Fuchsian groups in Takeuchi (1975). This was extended to Kleinian groups in Maclachlan and Reid (1987). In Reid (1990), the invariance up to commensurability of the invariant trace field was established (cf. Macbeath (1983)). For discrete subgroups of semi-simple Lie groups, fields of definition were investigated in Vinberg (1971) and the invariance of the invariant trace field described here can be deduced from these results (see §10.3). The invariance of the invariant quaternion algebra can be found in Neumann and Reid (1992a). The trace identities and the dependence of all traces in a finitely generated group on the simple sets described in Lemmas 3.5.1 to 3.5.3 are mainly well known and have been used in various contexts (Helling et al. (1995)). The energy-saving Lemma 3.5.5 appears in Hilden et al. (1992c). The simple formulas in terms of traces used to obtain the Hilbert symbols for quaternion algebras given in §3.6 arose mainly in the context of investigations into arithmetic Fuchsian and Kleinian groups (e.g., Takeuchi (1977b), Hilden et al. (1992c)). The dependence of a two generator group up to conjugacy on the parameters discussed in (3.31) is given in Gehring and Martin (1989).



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