Discovery of the Maximum Principle in Optimal Control

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Discovery of the Maximum Principle for the needs of optimal control and its subsequent development give a classical example of a theory, which initially emerged as an effective device for solving a purely engineering problem not amenable by existing methods, and eventually developed into a mathematical theory of major significance.

I present a short history of the discovery of the Maximum Principle in Optimal Control by L. S. Pontryagin and his associates.

1 Formulation of the Time–Optimal Problem

In 1970, at the World Congress in Nice, Prof. Pontryagin gave a plenary talk on differential games, which was motivated by pursuit-evasion strategies of aircrafts for a very simplified model of behavior. During the after-talk discussions, A. Grothendieck put a rhetorical question to Pontryagin. He said that though the listeners witnessed a beautiful piece of mathematics, still he would like to know whether the speaker feels himself morally responsible for supporting military trends in the society. Pontryagin's answer was quite definite and blunt. He was convinced, he said, that, on an intellectual level, any intellectual problems could be discussed openly in a developed society, and if we would follow to the logical end Prof. Grothendieck's recommendation, we should be prohibited from speaking openly about some topics of abstract Algebra, since Cryptography, which has much deeper correlations with military problems than the differential game considerations he spoke about, is completely based on the theory of finite fields.

Lev Semenovich Pontryagin was one of the leading figures in 20th century algebraic topology and topological algebra, but in mid-1950s he abandoned topology, never to return to it, and completely devoted himself to purely engineering problems of mathematics. He organized at the Steklov Mathematical Institute a seminar on applied problems of mathematics, often inviting theoretical engineers as speakers, since he considered a professional command over the engineering part of the problem under investigation to be mandatory for an adequate mathematical development.

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The activity in the seminar culminated very soon in the formulation of two major mathematical problems. One of them developed into the general theory of singularly perturbed systems of ordinary differential equations. The second problem brought the discovery of the Maximum Principle and the emergence of optimal control theory.

Pontryagin was led to the formulation of the general time-optimal problem by an attempt to solve a concrete fifth-order system of ordinary differential equations with three control parameters related to optimal maneuvers of an aircraft, which was proposed to him by two Air Force colonels during their visit to the Steklov Institute in the early spring of 1955. Two of the control parameters entered the equations linearly and were bounded, hence from the beginning it was clear that they could not be found by classical methods, as solutions of the Euler equations. The problem was highly specific, and very soon Pontryagin realized that some general guidelines were needed in order to tackle the problem. I remember he even said half-jokingly, "we must invent a new calculus of variations." As a result, the following general time-optimal problem was formulated.

Consider a controlled object represented in the n-dimensional state space $\mathbb{R}^n$ of points

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$$

by a system of n autonomous differential equations with r control parameters

$$u = \begin{pmatrix} u^1 \\ \vdots \\ u^r \end{pmatrix} \in U \subset \mathbb{R}^r,$$

$$\frac{dx}{dt} = f(x, u), \quad f = \begin{pmatrix} f^1 \\ \vdots \\ f^n \end{pmatrix}. \tag{1}$$

Initially it was supposed that the control vector u attains its values from an open set $U \subset \mathbb{R}^r$. Necessity of a closed U, the most important case for applied problems, was evident from the very beginning, though could be handled only later. To denote control parameters, the letter "u" was chosen, as the first letter of the Russian word "control" – "управление".

Formulation of the problem

Given initial and terminal states x_0, $x_1 \in \mathbb{R}^n$, find a control function $u(t) \in U$ $\forall t \in [t_0, t_1]$, such that it minimizes the transition time of the state point x, moving *from x_0 to x_1 according to the non autonomous equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} x = f(x(t), u(t)) \cdot$$

Thus, we come to the time--optimal control $u(t)$ and the corresponding time-optimal trajectory $x(t)$, $t_0 \leq t \leq t_1$, which satisfies the boundary value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = f(x(t), u(t)), \quad x(t_0) = x_0, \quad x(t_1) = x_1$$

and minimizes the transition time,

$$t_1 - t_0 = \min.$$

It should be noticed that the general optimal problem with an arbitrary integral-type functional is easily reduced to the formulated time-optimal problem, so that by solving the time-optimal problem with fixed boundary conditions we actually overcome all essential difficulties inherent in the general case.

The first and the most important step toward the final solution was made by Pontryagin right after the formulation of the problem, during three days, or better to say, during three consecutive sleepless nights. He suffered from severe insomnia and very often used to do math in bed all night long. As a result, he completely disrupted his sleep in his later years and systematically took barbiturates in great quantities.

Thanks to his wonderful geometric insight, he derived from very simple duality considerations about the first order variational equation the initial version of necessary conditions, introducing an auxiliary covector-function $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ subject to the adjoint system of differential equations,

$$\left.\begin{array}{l} \dfrac{\mathrm{d}\psi_i}{\mathrm{d}t} = \displaystyle\sum_{\alpha=1}^{n} \psi_\alpha \frac{\partial f^\alpha}{\partial x^i}(x(t), u(t)), \quad i = 1, \ldots, n, \\ \Updownarrow \\ \dfrac{\mathrm{d}\psi}{\mathrm{d}t} = -\psi \dfrac{\partial f}{\partial x}(x(t), u(t)). \end{array}\right\} \tag{2}$$

This was the first appearance in optimal control theory of the adjoint system, which turned out to be of crucial importance for the whole subject. Actually, Pontryagin constructed for the first time, for the needs of optimization, what is usually called the *Hamiltonian lift* of the initial family of vector fields on the state space of the problem into its cotangent bundle, the phase space of the problem, cf. n° 5.

2 Initial Formulation of Necessary Conditions

Initial formulation of necessary conditions, reported by Pontryagin at the seminar right after they were derived, is expressed in formulas

$$\left.\begin{aligned}
&\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), u(t)), \ \ x(t_0) = x_0,\, x(t_1) = x_1, && (3.1)\\
&\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = -\psi(t)\frac{\partial f}{\partial x}(x(t), u(t)), && (3.2)\\
&\psi(t)\frac{\partial f(x(t), u(t))}{\partial u^j} = 0, && (3.3)\\
&u(t) \in U, \forall t \in [t_0, t_1],\, j = 1, \dots, r.
\end{aligned}\right\} \qquad (3)$$

They assert that if $x(t)$, $u(t)$, $t_0 \le t \le t_1$, is an optimal solution, then there exists a nonzero covector-function $\psi(t)$ such that $\psi(t)$, $x(t)$, $u(t)$, $t_0 \le t \le t_1$, is a solution of the system of differential equations (3.1)–(3.2), and along the solution, for every t, j = 1,... , r, "finite" equations (3.3) are satisfied.

This formulation supposes that the set U of admissible values of control is open, though, as I already mentioned above, from the very beginning it was clear that the ultimate result should be applicable to closed sets as well.

I shall describe now Pontryagin's very simple and straightforward geometric arguments which directly lead to equations (3).

Consider an arbitrary admissible variation of the optimal control $u(t)$,

$$\delta u(t) = \begin{pmatrix} \delta u^1(t) \\ \vdots \\ \delta u^r(t) \end{pmatrix}, \ u(t) + \delta u(t) \in U, \ \ t_0 \le t \le t_1 .$$

If we ignore quadratic and higher order terms with respect to δu in the Taylor expansion of the corresponding trajectory variation, we obtain the *first (linear) variation*

$$\delta x(t) = \begin{pmatrix} \delta x^1(t) \\ \vdots \\ \delta x^n(t) \end{pmatrix}, \ t_0 \le t \le t_1 ,$$

of the optimal trajectory, which satisfies the standard *linear variational equation*

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\delta x = \frac{\partial f}{\partial x}(x(t), u(t))\,\delta x + \frac{\partial f}{\partial u}(x(t), u(t))\,\delta u(t),\\ \delta x(t_0) = 0, i = 1\ldots, n.\end{aligned}\right\} \tag{4}$$

The mapping $\{\delta u(t), t_0 \le t \le t_1\} \mapsto \delta x(t_1)$ is a linear operator from the space of variations $\delta u(t), t_0 \le t \le t_1$ into the state space $\mathbb{R}^n$. Since the set of admissible values of u is assumed open, the admissible variations $\delta u(t)$ are arbitrary (piecewise continuous) functions. Hence the set

$$L = x(t_1) + \{\delta x(t_1) | \delta x(t_0) = 0\} = x(t_1) + \Gamma \subset \mathbb{R}^n \tag{5}$$

is a plane through $x(t_1)$ in $\mathbb{R}^n$, Γ is the corresponding subspace of $\mathbb{R}^n$. Since $x(t)$, $t_0 \le t \le t_1$,is optimal, we obtain the relation

$$\dim L = \dim \Gamma \le n - 1 \ ,$$

which is easily derived from the implicit function theorem. Hence, there exists a (nonzero) covector $\chi = (\chi_1, \ldots, \chi_n)$ orthogonal to Γ,

$$\chi \delta x(t_1) = \sum_{\alpha=1}^{n} \chi_\alpha \delta x^\alpha (t_1) = 0 \ \forall \delta x(t_1) \in \Gamma.$$

This is the geometric condition, from which the equations (3) follow immediately, if we express $\delta x(t_1)$ through $\delta u(t)$, i.e., integrate the variational equation (4). For this purpose we introduce the fundamental matrix $\Phi(t)$ of the corresponding homogeneous equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x = \frac{\partial f}{\partial x}((x(t), u(t))\,\delta x,$$

and the inverse $\Psi(t) = \Phi^{-1}(t)$. They satisfy matrix differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi = \frac{\partial f}{\partial x}((x(t), u(t))\,\Phi, \ \frac{\mathrm{d}}{\mathrm{d}t}\Psi = -\Psi \frac{\partial f}{\partial x}((x(t), u(t)). \tag{6}$$

Solution of the nonhomogeneous equation (4) with the initial condition $\delta x(t_0) = 0$ is represented as

$$\delta x(t) = \Phi(t) \int_{t_0}^{t} \Psi(\tau) \frac{\partial f}{\partial u}(x(\tau), u(\tau))\,\delta u(\tau)\, d\tau, \ \ t \in [t_0, t_1],$$

hence

$$\chi \delta x(t_1) = \chi \Phi(t_1) \int_{t_0}^{t_1} \Psi(\tau) \frac{\partial f}{\partial u}(x(\tau), u(\tau))\,\delta u(\tau)\, d\tau = 0 \ \ \forall \delta u(\tau). \tag{7}$$

The n-dimensional covector

$$\psi(t) = (\psi_1(t), \ldots, \psi_n(t)) = \chi\,\Phi(t_1)\Psi(t), \;\; t_0 \le t \le t_1,$$

is nonzero and satisfies, according to (6), the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = -\psi(t)\frac{\partial f}{\partial x}(x(t), u(t)),$$

which coincides with the adjoint system (3.2). Finally, the equation (7) attains the form

$$\int_{t_0}^{t_1} \psi(\tau)\frac{\partial f}{\partial u}(x(\tau), u(\tau))\,\delta u(\tau)\,d\tau = 0.$$

Since the control variation $\delta u(t), t_0 \le t \le t_1$, is an arbitrary vector function, we obtain equations (3.3) and come to the optimality conditions (3) formulated above. They easily imply the Euler-Lagrange equations for the Lagrange problem of the classical calculus of variations.

3 The Second Variation

As soon as the equations (3) were obtained, Pontryagin recognized the decisive role of the covector-function $\psi(t)$ and the adjoint system (2) for the whole problem. He considered, in the generic case, r finite equations (3.3) as conditions, which eliminate r control parameters $u^1, \ldots, u^r$ from system (3), thus making it possible to solve uniquely the $2n$-th order system of differential equations (3.1)–(3.2) with a given initial condition $x(t_0) = x_0$ and an arbitrary (nonzero) initial condition for ψ. All such solutions were declared as *extremals of the problem*, from which the optimal solutions were to be derived.

Pontryagin's idea about a universal procedure of elimination of control parameters, which reduces the problem of determining extremals to solving ordinary differential equations with given boundary conditions, found its ultimate realization in the maximum principle, which was formulated by him several months later after his first report at the seminar and was supported by the subsequent advancements obtained meanwhile at the seminar.

After his talk in the seminar, Pontryagin suggested to V. Boltyanski and me, his former students and close collaborators at that time, to join him in his investigations of the problem. V. Boltyanski held a formal position at the Steklov Institute as Pontryagin's assistant, helping him in everyday computations and manuscript editing; I was a young member of the department of the Steklov Institute headed by Pontryagin.

Pontryagin's vision of the problem at this early stage of development could be described as follows.

Instead of considering the boundary value problem with fixed endpoints for the controlled system (3.1), we should only fix the initial point x_0, take an arbitrary initial value $\psi(t_0) = \psi_0 \neq 0$, and solve the system of $2n + r$ equations (3.1)–(3.3) with $2n + r$ unknowns x^i, ψ_j, u^k, proceeding along an arbitrary extremal through x_0. This should be possible, since the r control parameters u^k are, "in general", successfully eliminated by r conditions (3.3), hence only $2n$ unknown parameters are left, x_i, ψ_j, subject to the system of $2n$ differential equations (3.1)–(3.2) and the initial conditions $x(t_0) = x_0, \psi(t_0) = \psi_0$. Since the adjoint system (3.2) is linear in ψ, the function $\psi(t)$ is defined up to a nonzero constant factor, hence we can normalize the initial value $\psi(t_0)$, obtaining thus the $(n - 1)$-dimensional sphere of the initial values of ψ, which should generate an $(n - 1)$-parametric family of extremal trajectories of the problem emanating from the point x_0.

According to this initial picture, the final goal of the program consisted in the study of extremals $\psi(t), x(t), x(t_0) = x_0$, as solutions of the system (3.1)–(3.3), parametrized by the initial value $\psi_0 \neq 0$. Today we recognize in the given formulation the problem of controllability in its most rudimentary setting. Certainly at that stage, before the maximum principle was not even formulated, it was practically impossible to obtain in this direction any nontrivial results.

I was fascinated by Pontryagin's geometric approach and got an idea how to apply this picture to investigate the problem up to the second-order approximation. So, we decided to split our further advancements in two directions. Pontryagin, together with Boltyanski, pursued the problem in the controllability direction, I started to investigate the second variation of the problem. As it turned out, this latter direction led to the final formulation of the maximum principle.

Necessary conditions of optimality, expressed by equations (3), are derived from purely first-order approximation. They are independent of "general position" considerations. My second-order considerations demanded from the very beginning general position assumptions, which were overcome only in the final version of the proof by Boltyansky. The set of admissible values of the control parameters U was still assumed to be open.

Very briefly, the idea of the second-order approximation could be described in the following way.

Take an arbitrary "generic" solution of the optimal problem, $x(t), u(t), t_0 \le t \le t_1$, which means that the plane L in (5) is of maximal possible dimension,

$$\dim L = \dim \Gamma = n - 1,$$

and the trajectory $x(t)$ intersects L at $x(t_1)$ transversally (is not tangent to L). Hence L divides $\mathbb{R}^n$ in distinguishable half-spaces, $\mathbb{R}^n_-$ – before $x(t)$ intersects L, $\mathbb{R}^n_+$ – after the intersection. Every variation $\delta u(t)$ displaces the endpoint $x(t_1)$ in the first order into the hyperplane $L, x(t_1) + \delta x(t_1) \in L$. The real displacement $\Delta x(t_1)$ is certainly nonlinear in δu and, generally, stays off the hyperplane,

$$x(t_1) + \Delta x(t_1) \in \mathbb{R}^n_- \quad \text{or} \quad x(t_1) + \Delta x(t_1) \in \mathbb{R}^n_+ .$$

Denote from now on the first variation $\delta x(t)$ by $\delta_1\, x(t)$, and let $\mathbb{K}$ be the kernel of the linear operator from the space of control variations into the space of first variations of $x(t)$ for $t = t_1$; the operator is given by

$$\{\delta u(t), t_0 \le t \le t_1\} \mapsto \delta_1 x(t_1) = \Phi(t_1)\int_{t_0}^{t_1} \Psi(\tau)\left\| \frac{\partial f^i}{\partial u^j}(x(\tau), u(\tau))\right\| \delta u(\tau)\, d\tau \cdot$$

Define the *second variation* $\delta_2 x(t), t_0 \le t \le t_1$, of $x(t)$ as the solution of the linear nonhomogeneous equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta_2 x = \frac{\partial f}{\partial x}(x(t), u(t))\,\delta_2 x + Q(\delta u(\tau), \delta u(\tau)),\ \ \delta_2 x(t_0) = 0,$$

where Q is a vector-valued integral quadratic expression in $\delta u(t), t_0 \le t \le t_1$, which is easily computed if we take in the Taylor expansion of $\Delta x(t)$ only quadratic terms with respect to $\delta u(\tau)$. The obtained equation differs from (4) only by the nonhomogeneous part. The displacement of the endpoint $x(t_1)$ up to the second order is given by the vector $\delta_1 x(t_1) + \delta_2 x(t_1)$.

The key geometric fact for a generic optimal trajectory $x(t), t_0 \le t \le t_1$, *consists in the assertion that the second order displacement of its endpoint, considered on the kernel* $\mathbb{K}$, *belongs to the half-space* $\mathbb{R}^n_-$,

$$x(t_1) + \delta_1 x(t_1) + \delta_2 x(t_1) \in \mathbb{R}^n_- \ \Leftrightarrow\ x(t_1) + \delta_2 x(t_1) \in \mathbb{R}^n_- \ \ \forall \delta u \in \mathbb{K}.$$

Hence we come to the conclusion that, additionally to the system (3.1)–(3.3), the following integral quadratic form in δu is nonpositive, provided the covector $\psi(t_1)$, which is transversal to L, is correctly normalized (directed toward the half-space $\mathbb{R}^n_+$),

$$\psi(t_1)\,\delta_2 x(t_1) = \int_{t_0}^{t_1} \psi(\tau)\, Q(\delta u(\tau)), \delta u(\tau), \delta u(\tau))\, d\tau \le 0 \ \ \forall \delta u \in \mathbb{K}.$$

After some elaborate investigation of this integral quadratic form, I came to the conclusion that from its nonpositivity on $\mathbb{K}$ follows the nonpositivity of its singular part on $\mathbb{K}$, which easily implies the final form of the second order optimality condition, the pointwise (in $t \in [t_0, t_1]$) nonpositivity of the quadratic form in $v \in \mathbb{R}^r$

$$v^* \left\| \psi(t)\frac{\partial^2 f}{\partial u^i \partial u^j}(x(t), u(t)) \right\| v \le 0,\ \ \forall v \in \mathbb{R}^r, \forall t \in [t_0, t_1], \tag{8}$$

satisfied, together with the first-order conditions (3), by every generic optimal solution $x(t), u(t), t_0 \le t \le t_1$, .

4 Final Form of the Maximum Principle

Collecting all necessary conditions (3.1)–(3.3), (8) together, we immediately recognize that a certain stable combination of symbols reappears in all of them, the scalar function of three arguments ψ, x, u,

$$H(\psi, x, u) = \sum_{\alpha=1}^{n} \psi_\alpha f^\alpha(x, u) = \psi f(x, u). \tag{9}$$

It enables us to rewrite the system (3.1)–(3.2) as a Hamiltonian system (10.1) with the Hamiltonian function (9), together with additional conditions (3.3), (8), written as (10.2)–(10.3):

$$\left.\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \frac{\partial H}{\partial \psi}(\psi(t), x(t), u(t)), \\ \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} &= \frac{\partial H}{\partial x}(\psi(t), x(t), u(t)), \end{aligned}\right\} \tag{10.1}$$

$$\frac{\partial H}{\partial u^j}(\psi(t), x(t), u(t)) = 0, \ \ \forall t \in [t_0, t_1], \ \ j = 1, \dots, r, \tag{10.2}$$

$$v^* \left\| \frac{\partial^2 H}{\partial u^i \partial u^j}(\psi(t), x(t), u(t)) \right\| v \le 0, \ \ \forall v \in \mathbb{R}^r. \tag{10.3}$$

They assert that generic extremals are solutions of the Hamiltonian system (10.1), and, according to (10.2), their points are stationary points of the Hamiltonian (9) with respect to the control parameter u; furthermore, according to (10.3), along regular extremals, for which the form (10.3) is definite, the function H attains its local maximum with respect to u.

We can unite two independent conditions (10.2)–(10.3) into one condition and write

$$H(\psi(t), x(t), u(t)) = \max_{v \in O_t} H(\psi(t), x(t), v), \tag{10.4}$$

where O_t is a neighborhood of $u(t)$. Furthermore, the equations (10.1)–(10.2) imply,

$$\frac{\mathrm{d}H}{\mathrm{d}t}(\psi(t), x(t), u(t)) = \frac{\partial H}{\partial \psi} \cdot \frac{\mathrm{d}\psi}{\mathrm{d}t} + \frac{\partial H}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial H}{\partial u} \cdot \frac{\mathrm{d}u}{\mathrm{d}t} \equiv 0.$$

It is also easy to show that $H(\psi(t), x(t), u(t))$, as a function of t, is continuous, even if the control function $u(t)$ has jumps. Hence, taking into account the generic character of the solution – the trajectory $x(t)$ is transversal to L at $x(t_1)$ – we obtain

$$H(\psi(t), x(t), u(t)) \equiv const = \psi(t_1) f(x(t_1), u(t_1)) > 0. \tag{11}$$

After the equations (10.1)–(10.3) were written, Pontryagin realized that the universal elimination method of the control parameters he was searching for, was found. He replaced the local maximum condition (10.4) by the global maximum over the whole set U , the "*Pontryagin maximum condition*" (12), which made any restrictive assumptions about the admissible set U superfluous,

$$H(\psi(t), x(t), u(t)) = \max_{u\in U} H(\psi(t), x(t), u) \equiv const \geq 0. \tag{12}$$

Thus, he came to the final formulation of the maximum principle, combining the Hamiltonian system (10.1) with the maximum condition (12) and dropping off any assumptions about genericity of the solutions or the nature of the admissible set U.

The Maximum Principle. *Suppose a controlled equation is given,*

$$\dot{x} = f(x, u), \quad x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \in \mathbb{R}^n, \quad u = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} \in U \subset \mathbb{R}^r,$$

where the admissible set U is arbitrary. We introduce the Hamiltonian function of the problem

$$H(\psi, x, u) = \psi f(x, u) = \sum_{\alpha=1}^{n} \psi_\alpha f^\alpha(x, u), \tag{13.1}$$

which depends on three arguments – the covector $\psi = (\psi_1, \dots, \psi_n)$ *and the vectors x, u. If* $u(t), t_0 \leq t \leq t_1,$ *is a time-optimal control,* $x(t), t_0 \leq t \leq t_1,$ *the corresponding time-optimal trajectory,*

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = f(x(t), u(t)), \ \ t_0 \leq t \leq t_1; \ \ t_1 - t_0 = \min,$$

then there exists a nonzero covector function $\psi(t)$ *such that the triple*

$$\psi(t), x(t), u(t), \ \ t_0 \leq t \leq t_1,$$

is a solution of the Hamiltonian system (13.2), and the maximum condition (13.3) holds,

$$\left.\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \frac{\partial H}{\partial \psi}(\psi(t), x(t), u(t)), \\ \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} &= -\frac{\partial H}{\partial x}(\psi(t), x(t), u(t)), \end{aligned}\right\} \tag{13.2}$$

$$\left.\begin{aligned} H(\psi(t), x(t), u(t)) = \max_{u\in U} H(\psi(t), x(t), u) \equiv const \geq 0, \\ \forall t \in [t_0, t_1] \end{aligned}\right\} \tag{13.3}$$

In this formulation, the maximum condition (13.3) could be viewed not only as a universal elimination method, but also as a generalization of the Legendre transformation from the state-space variables (x, u)tothe phase-space variables (ψ, x).

5 Symplectic Invariance of the Maximum Principle

It took approximately a year before a full proof of the maximum principle was found. It is not my intention here to go into the details of the history of this proof. I shall only mention that in full generality the proof of the PMP, as formulated above, became possible after V. Boltyansky's invention of the so-called "needle variations" of the control function. These variations are zero everywhere on the time-interval, except on several segments with a small total length, where they can attain arbitrary admissible values, and they have an important property of admitting an operation of convex combination, regardless of the shape of U. This was Boltyansky's major contribution to the subject.

I should like to make now some final remarks about the nature of the maximum principle and its significance for optimal control theory and calculus of variations.

The role of the Pontryagin Maximum Principle, discovered for the needs of optimal control, was twofold in mathematics. First, it created a really working device for solving important applied problems of optimization, not amenable by classical calculus of variations, and second, it gave a much broader perspective on necessary conditions of extremality in mathematics in general. Though formulated in 1955, the maximum principle was never changed, nor even slightly improved, since then. All first-order advancements were directed toward generalizations of the optimal problem itself, especially toward developing nonsmooth optimization, with corresponding first-order necessary conditions shaped after the maximum principle.

This could be explained by the very nature of the maximum principle. Despite its seemingly hard analytic form, it is deeply geometric and symplectic invariant already in its initial formulation. It prescribes a canonical way of deriving differential equations of extremals in the phase space of the problem, the cotangent bundle of the state-space manifold, on which the variational problem is formulated. Since the equations are obtained canonically, they are invariantly connected with the cotangent bundle, hence with the canonical symplectic structure of the bundle, and any information about the interrelations of the equations with the symplectic structure is a nontrivial information about the variational problem itself. We can even say that a real mathematical investigation of the variational problem starts after the equations of extremals are derived, and not after the formulation of the variational problem. The latter could be considered only as a "letter of intent", whereas the equations of extremals give us the "contract" itself, which should bring the mathematical dividends.

To reformulate the maximum principle in an explicitly invariant form, let me rewrite the initial control system in the "state-invariant" form on a smooth manifold M,

$$\left.\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,u),\, x \in M,\, u \in U, \\ &f_u : x \mapsto f(x,u) \in TM,\, x \in M,\, u \in U \end{aligned}\right\}$$

and consider the family of vector fields f_u as a family of scalar-valued functions H_u on the cotangent bundle $T^*M \xrightarrow{\pi} M$, which are linear on fibers,

$$f_u \approx H_u \in C^\infty(T^*M),\, H_u \text{ is linear on fibers.}$$

The family of scalar-valued functions H_u generates on T^*M a family of Hamiltonian vector fields $\vec{H}_u$ according to the standard relation

$$i_{\vec{H}_u}\omega = \mathrm{d}H_u,$$

where ω is the canonical symplectic form on T^*M. We thus obtain the Hamiltonian system of the maximum principle (13.2). The field $\vec{H}_u$ is the canonical lift into the cotangent bundle T^*M of the vector field f_u defined on the base manifold M, i.e., it generates on T^*M a fiber-preserving flow, mapping an arbitrary fiber T_x^*M linearly onto the fiber $T^*_{e^{tf_u}(x)}M$,

$$e^{t\vec{H}_u} : T_x^*M \to T^*_{e^{tf_u}(x)}M,\ x \in M.$$

The maximum principle asserts that the extremals of the problem are trajectories $\xi(t)$ of the nonstationary Hamiltonian vector field $\vec{H}_{u(t)}$ such that the maximum condition holds,

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = \vec{H}_{u(t)}(\xi(t)),\, H_{u(t)}(\xi(t)) = \max_{v \in U} H_v(\xi(t)),\ t_0 \le t \le t_1.$$

If the maximum condition

$$H_{u(t)}(\xi) = \max_{v \in U} H_v(\xi),\ \xi \in O \subset T^*M,$$

eliminates the parameter u from the family H_u in some region $O \subset T^*M,$ and as a result of this elimination, we obtain a *smooth* scalar-valued function (without parameters) H, the *master-Hamiltonian* of the problem, then the optimal problem considered on O is reduced to studying trajectories of a fixed Hamiltonian vector field $\vec{H}$:

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = \vec{H}(\xi(t)),\ \pi\xi(t_0) = x_0,\ H(\xi(t)) \equiv const \ge 0.$$

Regular problems of the calculus of variations are typical examples of this situation. Actually, this picture was envisaged by Pontryagin in his initial attempt to solve the problem.

It is remarkable that the described geometric approach permits us, practically in all interesting cases, including nonregular cases, to construct canonically from

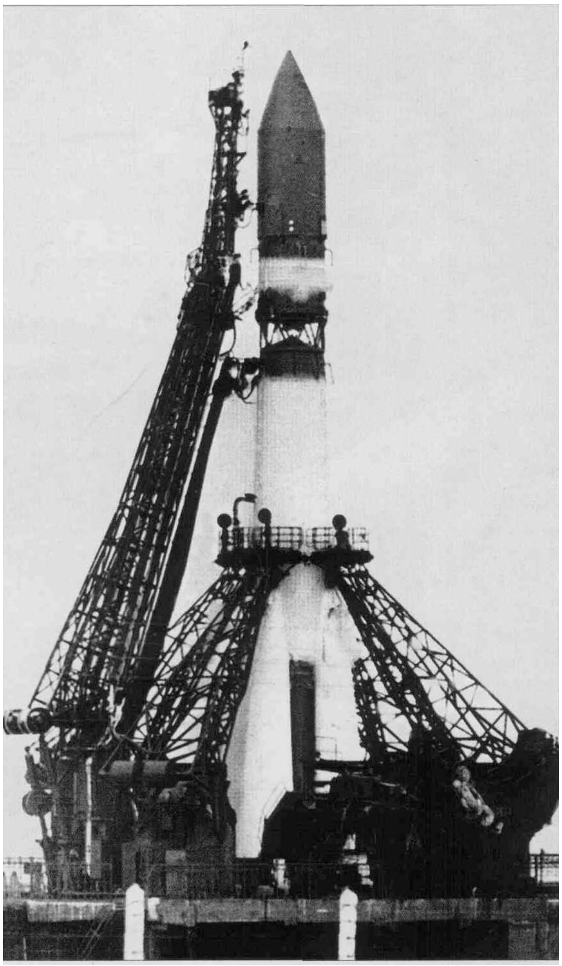

Figure 1. According to information received from Samara State Aerospace University, the launchings of Sputnik 1 and Sputnik 2 in 1957 were made without recourse to the Pontryagin's Maximum Principle. Later tasks – bringing down cosmonauts safely, and guaranteeing that intercontinental missiles surviving a first strike would not miss New York by more than the radius of efficiency of a hydrogen bomb – did use the Maximum Principle, which was in the public domain well before that. For Sputnik 1 and 2, ideas due to Pontryagin were used to find the correct weight of the space craft (same source). [Editors' note; photo: Samara State Aerospace University Museum]

the given optimal problem a nonlinear connection on T^*M, which produces new important infinitesimal invariants of the optimal problem that are nontrivial already in the regular case. In particular, we can obtain the curvature tensor of the optimal problem, cf. [4]. If we try to derive from here global invariants of the state manifold M, for example, try to express its Euler characteristic through the curvature of the optimal problem (a possible generalization of the Gauss-Bonnet-Chern formula), we should inevitably come to generalizations of some classical relations concerning characteristic classes due to Pontryagin and Chern, which were obtained in the special case when the standard Riemannian length on the manifold M is minimized. Thus, two major achievements of L. S. Pontryagin, based on completely different ideas and obtained in different periods of his activity, might be intimately related.

I have reached the end of the story, which started with two humble Air Force engineers who ignited the mathematical genius of L. S. Pontryagin almost fifty years ago.

Summing up, we should consider the discovery of the Maximum Principle and the subsequent development of optimal control as a classical example of a theory that initially emerged as an effective device for solving a concrete engineering problem not amenable by existing methods, and eventually developed into a mathematical theory of major significance.

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Among the first publications on the maximum principle I can indicate here [1]–[3]. In [4], construction of invariant connections is described.

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