Calculated Security? Mathematical Modelling of Conflict and Cooperation

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The pioneering work of Lewis Fry Richardson on modelling the arms race raised expectations that mathematics can contribute to peace and conflict resolution. Based upon Richardson's model, various extensions are discussed, with a focus on time-discrete nonlinear models showing chaotic behavior. As a general framework for the modelling of conflict and cooperation in international security a multi-actor dynamic game is introduced, with mathematical conditions for unstable interaction, potentially leading to violent conflict, arms race and war. On the other hand, the approach provides a basis for the evolution of cooperation and coalition formation. In more detail, the case of instabilities in the offense-defense competition is discussed and the role of uncertainties in complex international relations.

1 Conflict Modelling in a Complex World

While the simple two-player arms race became the paradigm of conflict studies during the Cold War [Rapoport 1960], many of the world's current conflicts are more adequately described by complex multi-actor dynamic games. Decisions and interactions are shaped by a variety of actors and factors, which could provoke instability, rapidly changing security conditions and the outbreak of violence, making conflict-resolution more difficult. Not only the arsenals of armament are relevant for security, but also economic and technological interconnections as well as social and ecological factors, on global and regional levels. Security and sustainability are increasingly linked.

Understanding the emergence of collective behavior and the evolution of cooperation is a dynamic field of current interdisciplinary research, and knowledge transfer between the natural and social sciences can be highly fruitful. Mathematical modelling and its computer-based implementation can contribute to a deeper understanding and the development of new instruments for conflict-resolution and cooperation, disarmament, security and peace-building. Different mathemati-

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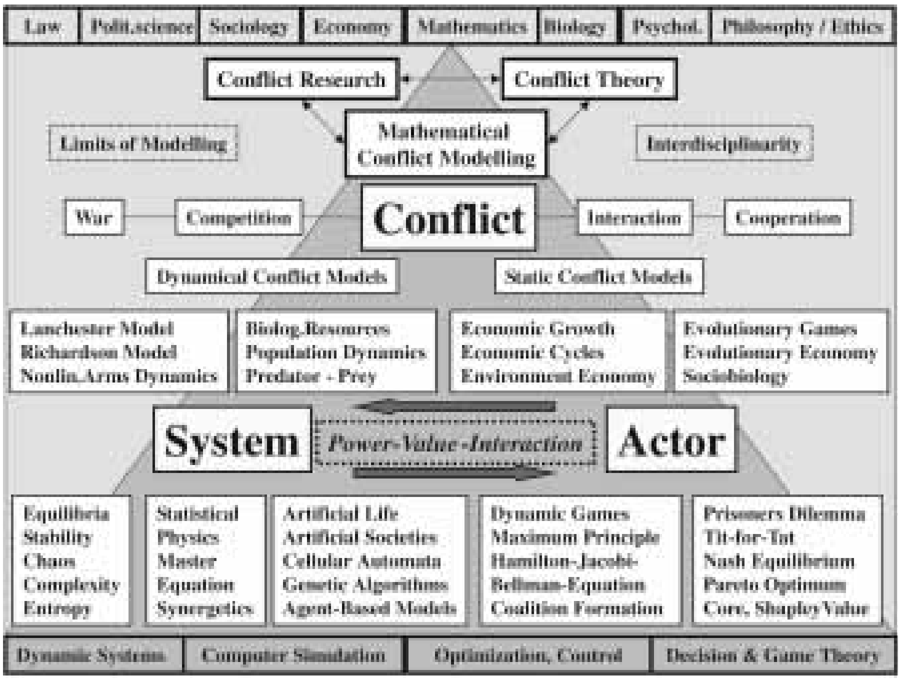

Figure 1. Landscape of mathematical conflict models.

cal methods have been applied to this field (on some see [Neuneck 1995]; a survey is given in Figure 1). The theory of *dynamical systems* determines equilibria and stability of differential and difference equations, as well as chaos, self-organization and phase transitions [Krabs 1998; for applications see Braun et al. 1983]. *Control theory* helps to determine optimal paths in dynamical systems. *Game theory*, based on the epochal work of von Neumann and Morgenstern 1944, analyzes rational choice among different options depending on decisions by other players.

Dynamic games deal with repeated game situations in which players interactively adapt their behavior to their environment, according to their own incentives, preferences and expected outcomes as well as the information sets and decisions taken by other players.

Differential games extend Pontryagin's maximum principle to the optimal control of dynamical systems by a few number of players, seeking to optimize their individual payoff functions for a given time-period (see for instance Dockner et al. 2000). The *repeated prisoners' dilemma* has been used to analyze the evolution of cooperation in experimental games [Axelrod 1984, 1997]. Increasing attention is focused on the link between *cooperative dynamic games* and *coalition formation*. *Evolutionary games* analyze the competition among populations of game-strategies depending on their fitness in the replica equation (for a survey see [Hofbauer/Sigmund 1998]). In economic *oligopoly theory* the competition of firms can be represented by the adaptation towards reaction functions according to Nash-Cournot strategies, the so-called "tatonnement process" [Simaan/Cruz 1976, Szidarovsky/

Li 2000]. Applications of dynamic-game models comprise a large number of natural and social systems, ranging from warfare to the environment-economy interaction (see for instance [Olsder 1995, Carraro/Filar 1995]).

Models of *artificial life and artificial societies* use computer simulation to analyze complex interaction between a great number of actors which follow given action rules and stimulus-response patterns in virtual environments [see Epstein/Axtell 1997, Gaylord/D'Andria 1998]. *Nonlinear dynamical systems* used in physics (Master equation, Boltzmann equation, Synergetics) can be partly transferred to socio-economic interactions, collective phenomena and self-organization [Helbing 1995, Weidlich 2000]. Improved computational capabilities facilitate the simulation of multi-actor systems in highly complex environments. There is a methodological gap between models for a few number of actors which pursue optimizing game strategies and models for a large number of actors which interact according to deterministic or stochastic rules of behavior. In this article, some of the approaches are combined by developing and investigating a mathematical framework for a dynamic game with multiple actors which in discrete time act upon a system by use of their power resources according to feedback strategies to achieve value goals. The actors' dynamic feedback, adjustment and learning strategies are represented by reaction functions which generate dynamical systems. Mathematical conditions are identified under which the dynamic system is stable or can be stabilized by parametric control of the interaction matrix. The combination of cooperative and dynamic game theory with computer-based multi-actor modelling provides a basis to further develop instruments of decision-support that can be used in international relations and economy as well as in other fields.

2 Weather, Arms and Mathematics – Richardson's World

Lewis Fry Richardson (1881–1953), a british physicist, psychologist and pacifist, applied mathematics to better understand arms race and war. He also made important contributions to weather forecasting, but the main drawback of his mathematical technique was the time necessary to produce a weather forecast. It usually took three months to predict the weather of the following day, but with the electronic computers after World War II, Richardson's method of weather prediction became more practical.[1]

Less known during his lifetime remained Richardson's work on mathematical conflict modelling [Hess 1995]. He tried to predict and prevent war by finding general laws, common to all nations. Being concerned that the arms race between the major powers in Europe during the 1930s could lead to another major war, he derived a set of differential equations that would describe the arms buildup, using empirical data from the First World War to fit the curves [Richardson 1960ab]. He was aware of the strengths as well as the weaknesses of applying mathematics to social phenomena:

[1] On Richardson's life, work and publications see Ashford 1985, 1993.

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Figure 2. Lewis Fry Richardson (1881–1953).

> To have to translate one's own verbal statements into mathematical formulae compels one carefully to scrutinize the ideas therein expressed. Next the possession of formulae makes it much easier to deduce the consequences. In this way absurd implications, which might have passed unnoticed in a verbal statement, are brought clearly into view and stimulate one to amend the formula. An additional advantage of a mathematical model is its brevity, which greatly diminishes the labor of memorizing the idea expressed. If the statements of an individual become the subject of a controversy, this definiteness and brevity lead to a speeding up of discussions over disputable points, so that obscurities can be cleared away, errors refuted, and truth found and expressed more quickly. [...] Mathematical expressions have, however, their special tendencies to pervert thought: the definiteness may be spurious, existing in the equations but not in the phenomena to be described; and the brevity may be due to the omission of the more important things, simply because they cannot be mathematized. [Richardson 1960a, p. xvii]

The basic philosophy in Richardson's model is the assumption that for two arbitrary countries each country increases its own armament level x_i proportional to the armament of an opponent x_j and reduces it proportional to its own armament (i, $j = 1, 2$). These three terms drive the increase or decrease of armament in a linear and additive way:

$$\dot{x}_1 = \frac{dx_1}{dt} = k_1 x_2 - a_1 x_1 + g_1 ,$$

$$\dot{x}_2 = \frac{dx_2}{dt} = k_2 x_1 - a_2 x_2 + g_2 .$$

The *defense coefficients* k_i measure the degree to which the ith country reacts to the opponent's armament (military threat) x_j, *the fatigue coefficients* $a_i > 0$ take into account the effect of economic constraints, reducing own armament proportional to x_i . The *grievance term* g_i measures political and strategic objectives and perceptions, the "*outward attitude of threatening or cooperation*" [Richardson 1960a, p. 13]. $g_i > 0$ represents aggressive intentions, promoting an arms buildup, while $g_i < 0$ corresponds to good-will and the tendency to cooperate for an arms builddown.

The equilibrium conditions $\dot{x}_1 = \dot{x}_2 = 0$, where the armament levels of both sides do not change, leads to straight lines in the (x_1, x_2)-space of armament levels, whose intersection point (x_1^*, x_2^*) is the so-called "*balance of power*"

$$x_i^* = \frac{k_i g_j + a_j g_i}{a_1 a_2 - k_1 k_2}, \quad (i, j = 1,2),$$

provided that $a_1 a_2 - k_1 k_2 \neq 0$. For $g_1 = g_2 = 0$ the only equilibrium for all coefficients is the point (0, 0). Stability of the equilibria is determined by the eigenvalues λ of the matrix A of coefficients a_i and k_i which are the solutions of the characteristic equation $|A - \lambda I| = 0$. Asymptotic stability is guaranteed for all eigenvalues having negative real parts.[2] Stability can be characterized by the stability index

$$\sigma \equiv \frac{a_1 a_2}{k_1 k_2} > 1$$

for positive a_i, k_i. If the product of the defense coefficients k_i exceeds the product of the fatigue coefficients a_i ($\sigma < 1$) the arms race escalates (unstable arms race), while for $\sigma > 1$ the armament level approaches the equilibrium asymptotically.

Richardson extended the equations to several nations for the arms races 1909–1914 and 1933–1939, using military expenditures as armament variables x_i. These calculations supported his view that "*foreign policy had then a rather machine-like quality*" [Richardson 1960a, p. 33] and led him to conclude that increasing armaments could lead to war breakout, while a constant level of armament corresponds to a steady state without war.

3 Critique and Extensions of the Richardson Model

Richardson's model initiated a flood of publications on the armament dynamics and a debate about its applicability to real-world phenomena which raised several critical issues. Describing countries as structureless entities by one single "catch all variable" x_i was seen as questionable. Expenditures are a weak indicator of military threat and its security impact which depends on the systems procured.

[2] The eigenvalues λ describe how the trajectories approach to or escape from the equilibrium once they are slightly displaced. The equilibrium is stable if nearby solutions stay nearby for all future times. If all eigenvalues have negative real part (Re $\lambda < 0$), all nearby solutions will be attracted by the equilibrium (asymptotic stability), requiring trace$(A) < 0$ and det$(A) > 0$. For Re $\lambda > 0$ the equilibrium solution is unstable and the trajectory goes off.

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The variables and coefficients of the Richardson model are difficult to measure. Expenditures are not easily available, verifiable and reliable for all countries, and difficult to compare for different countries.

An arms race does not only have quantitative aspects, but also qualitative aspects, which are important for new strategies and doctrines. Not only objective and measurable quantities are relevant, but also subjective and irrational factors, like expectations and anticipations, instincts and traditions of political leaders. The Richardson model describes "politics without personalities", where state authorities are black boxes and decisions are hidden in the budget. The Richardson coefficients are fixed parameters, decoupled from strategy and security interests. The armament dynamics is reduced to a mechanistic and deterministic interaction, where by the choice of the initial conditions $x_i(0)$ and the coefficients a_i, k_i, g_i the future is determined, leaving no room for political decisions or control. In reality, the coefficients are time dependent and influenced by strategic objectives.

Both sides are assumed to have complete knowledge of the armament levels and react instantaneously. In reality, each side has limited information about the other side's strength, and worst-case assumptions provoke higher reactions towards arms buildup. Additionally, time lags exist for information gathering, decision making and weapons acquisition. Another critical time constant is the long lead time between technological innovations and deployed weapons systems.

The linearity and simplicity of the Richardson equations leads to a few types of system behavior (oscillations, asymptotic decay, exponential increase). Reactions of real systems may be disproportionate and highly nonlinear, showing a variety of qualitatively different modes of behavior. To describe the decisionmaking on the armament dynamics, time-discrete difference equations may be more adequate than continuous differential equations. The arms buildup is not only an action-reaction process, driven by the mutual stimulation of the opponents' armament (positive k_i), but is also stimulated by a bureaucratic and budgetary eigendynamics (negative a_i), guided by a competition of domestic influences and interests. Mutual stimulation and self-stimulation are limiting cases, which are coupled and can enforce each other.

Although an arms race may provoke crisis unstable situations, it does not necessarily lead to war, either because both sides want to avoid warlike situations by building more or less weapons or because one side reaches economic, technological or political upper limits of armament. In the latter case, which excludes a permanent and unlimited arms race, the economic damage to a society by additional armament exceeds its security gain.

Such critical remarks do not only apply to the Richardson model, but to any arms race model. Several extensions and adaptations have been proposed to improve the deficiencies with regard to the following points: using the number of weapons or their lethality as a measure of the armament level; derivation of the Richardson coefficients from strategic considerations; nonlinear (economic) constraints; the application of optimal control theory and differential games; time-discretization and difference equations; the influence of the bureaucratic process; stochastic Markov processes; decision rules.

Michael D. Intriligator developed a framework for the strategic armament dynamics, discussing "*decision rules with regard to weapons procurement and their implications for stability and war initiation in the context of a dynamic model of an arms race*" [see Intriligator 1975]. Critizing the simple action-reaction scheme, Intriligator included strategic considerations, institutional aspects and decision rules [Intriligator/Brito 1985, pp. 133–134]:

> Modelling the arms race as either a mechanistic or an optimizing process, however, fails to account for the institutions of defense decision making. To the extent that these institutions are large, complex, bureaucratic organizations, as indeed they are, they tend to rely on neither passive mechanical responses nor on explicit optimization rules but rather on rules of thumb or heuristic decision rules with regard to weapons procurement.

Both sides build or destroy weapons (missiles) x_i based on a model of a strategic missile duel according to decision rules $\dot{x}_i = F_i(x_i, x_j)$ for countries $i, j = 1, 2$, matching the strategic objectives. If the acquisition of missiles is proportional to the gap between desired levels x_i^* and actual levels x_i of missiles, the decision rule has the linear form $\dot{x}_i = k_i(x_i^* - x_i)$ where x_i^* depends on the missile duel model and the strategic objectives. $k_i > 0$ is the adjustment or *reaction coefficient*. For $x_i < x_i^*$ more weapons will be built in order to reach x_i^*; for $x_i > x_i^*$, country i will reduce its missiles (e.g., for cost-saving reasons). The equilibrium lines $x_i^* = x_i$, where the armament is constant, separate the regions of deterrence and war initiation (where one side loses second strike capability) in the (x_1, x_2)-space. The resulting dynamical equations for these decision rules are linear Richardson type equations, whose coefficients can be derived from strategic considerations. A specific type of decision rules for military expenditures, combining mutual stimulation and self-stimulation, has been used by Lambelet and Luterbacher 1979.

4 Chaos and Predictability in the Arms Race

While mathematical models such as the Richardson equations help to structure the field of conflict research and international security by identifying fundamental relationships among actors and basic system variables, they are often rather simple in dealing with the complexity of reality. Here the "fog of war" (as Clausewitz named it) comes into play, that is, the lack of knowledge and a chaotic sequence of events that occurs in real conflicts.

Because of the linearity and continuity of the Richardson equations, an analytic treatment is quite simple. During the 1980s new mathematical concepts have been developed, such as the theory of complexity, chaos and nonlinear dynamics, which may be applicable to conflicts and arms race phenomena. Contrary to the well-ordered world of Newtonian mechanics, symbolized by the predictable swinging pendulum or the regular movement of the celestial bodies, is the unpredictability of everyday-life experience. Since the 1970s the natural sciences have begun to systematically explore critical phenomena, such as self organization, discontinuous phase transitions, and catastrophe theory. Methods and results were transferred to the social sciences. Chaos became the paradigm for the turbulent transformation of the international system after 1989.

The concept of deterministic chaos was developed to study simple nonlinear phenomena with complicated dynamics, in which it is "practically impossible to predict the long-term behavior of these systems, because in practice one can only fix their initial conditions with finite accuracy, and errors increase exponentially fast."[Schuster 1988, pp. 3–4]

Many properties of chaos have been studied for the time-discrete logistic mapping $x_{t+1} = f_r(x_t) = rx_t(1 - x_t)$, where $r \in [0, 4]$ is the reaction coefficient (order parameter), whose value determines the transition from predictability to chaos. For small r the variable x_t converges against the stable *fixed point* $x^* = 0$, independent of the initial conditions. For $r > 3$ the single fixed point bifurcates into *periodic cycles*. For large times a chaotic region is reached. An *attractor* is a set of system states on which the time evolution accumulates over longer periods. For *dissipative* systems, where phase-space volumes on the attractor are separated exponentially, one speaks of *strange attractors*.

The concept of chaos as a model for arms race and war outbreak was introduced by [Saperstein 1984, 1986], to show that even simple nonlinear deterministic arms race models may lead to the breakdown of predictability, which is defined in this context as [Saperstein 1984, p. 303]

> a situation in which small perturbations of initial conditions, such as malfunctions of early-warning radar systems or irrational acts of individuals disobeying orders, lead to large unforeseen changes in the solutions to the dynamical equations of the model. There is no way then to predict the effect of the actions of any participant analyst, planner, statesman or general with any certainty.

Saperstein used a pair of nonlinear difference equations with quadratic mappings for two variables, denoting the fractions of the available resources which two countries pay annually. The problem of chaotic dynamics in arms race models was further investigated by Grossmann/Mayer-Kress 1989. The nonlinear difference equations have the following Richardson-like form but with discrete time and a damping of the fraction of expenditure x_t and y_t in year t at the upper cost limits x_m and y_m :

$$\Delta x_t = x_{t+1} - x_t = (x_m - x_t)(-k_{11}(x_t - x_s) + k_{12} \cdot y_t),$$

$$\Delta y_t = y_{t+1} - y_t = (y_m - y_t)(-k_{22}(y_t - y_s) + k_{21} \cdot x_t),$$

where $\Delta x_t, \Delta y_t$ is the change in the armament expenditure between the years t and $t + 1$, and $k_{ii}, k_{ij} > 0$ are the reaction coefficients corresponding to the defense and fatigue coefficients of Richardson. In addition, x_s, y_s represent self-establishing armament levels, comparable to Richardson's grievance terms and Intriligator's decision rules.

Factors provoking chaotic behavior are multinational interactions (more than two actors), overshooting or underestimation, hectic responses, delay in informa-

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tion processing or discretization. There is an important distinction between chaos and instability:

> In particular, regarding the arms race and international relations, it is wrong to identify the general onset of bounded chaos with the outbreak of a war or another global crisis. The really dangerous case is instability. By this we mean that the $x_\alpha(t)$ leave their previously bounded range rapidly and dramatically. [Grossmann/Mayer-Kress 1989, p. 702]

In the asymmetric case

$$\sigma = \frac{k_{11}k_{22}}{k_{12}k_{21}} > 1$$

is a condition for stability of a fixed point, which corresponds to Richardson's stability condition.

A nonlinear time-discrete model, using decision rules for weapons procurement, was derived in 1988 by Saperstein and Mayer-Kress to simulate the implications of ballistic missile defense systems (SDI) on the arms race between the USA and the former USSR. Two questions have been raised in this context [Saperstein 1988b, p. 41]:

- Can we reasonably predict that there will be a transition from the present offense-dominated strategic nuclear confrontation to a defensive security posture?
- Can we be reasonably assured that predictions about the transition from offense-dominated strategic security postures will actually be fulfilled? Or will the resulting world be chaotic, so sensitive to small changes (small threats) as to be 'crisis unstable'?

To answer these questions, a nonlinear model was formulated, describing the dynamical interaction between the number of intercontinental ballistic missiles (ICBMs), anti-ballistic missile (ABM) satellites, anti-ABM systems (such as anti-satellite weapons, ASAT). Each procurement step depends on the anticipated role of the weapons in a simple two-strike model of nuclear war and on the immediately preceding steps of the opponent. The "*weapons builders recursion relations*" are coupled dynamical difference equations for the deployment of the three weapon types, having the form of deterministic decision rules with nonlinear constraints due to maximum costs. The time evolution (arms race) is simulated numerically under various parameter conditions (scenarios), starting from a set of initial conditions. In most cases, the arms race remains predictable and offense-dominated. But if the production rates are increased by a factor of 10, a chaotic transition occurs [Saperstein 1988b, p. 43]:

- When the model does not turn chaotic the present offense dominance does not evolve to the desired defensive configuration.
- Conversely, when the situation evolves from offense to defense, the model turns chaotic. The model seems to indicate that if SDI evolves to the point where the defense can overwhelm the offense, the result is a crisis-unstable international system that cannot be mathematically distinguished from a system of war.

5 A Dynamic Multi-Actor Conflict Model

The Richardson and other arms race models can be embedded into a larger framework of conflict modelling based on a multi-actor dynamic game of power-value interaction, called the VCX model [Scheffran 2001]. It describes the interaction among multiple actors which influence their system environments by using power resources to pursue their objectives. For players acting upon the same system, conflicts may occur if actions and objectives are incompatible. With such a general framework it is possible to study the complexity and instability of multi-actor constellations in more detail. In particular, it is possible to express problems of war and peace, of arms race and disarmament as well as environmental conflicts and international security problems within the model framework.

5.1 The representative actor

In order to understand the interaction among multiple actors, it is important first to analyze the actions of an individual or representative actor (see Figure 3) which acts in a system environment (context or situation), represented by relevant *system variables*, defining a system state x, which in general is a vector but will be treated as a scalar in this section. The system state is observed and evaluated by the actor with a problem-specific *value function* $V(x)$, and compared to a target value V^*. The actor decides on investing its *power resources* (costs) C into a particular *action* $a(x, C)$ that changes the system state by $\Delta x = g(x, a)$ to approach the state-dependent value function $V(x)$ towards the target value V^* in repeated action cycles. The flow of power resources is allocated with a priority p to several action alternatives.

In several cases (such as buying goods on a market for a fixed price) the action a and the induced system change is not state-dependent and can thus be completely controlled by the power resources. Then by use of C an actor changes the system state $\Delta x = g(a) = C / c$ (with unit cost or price c), resulting in a value gain $V = v \cdot \Delta x = f \cdot C$ where v is the value per unit of system changes and $f = v/c$ is the efficiency of resource input with regard to the value output (also called benefit-cost ratio). If an actor aims at achieving a particular value gain $V = V^*$, this leads to the required power resources $C^* = V^*/f$.

If at a given time t the actual resource flow $C(t)$ differs from the required one $C^*(t)$, this gap can be bridged by an adaptation process among power resources according to "decision rules":

$$\Delta C(t) = C(t+1) - C(t) = \alpha(C^*(t) - C(t)) = (\alpha / f)(V^*(t) - V(t))$$

where α is the reaction strength. The underlying decision rule is to increase power resource input as long as $V < V^*$ and to decrease it if the goal is exceeded ($V > V^*$). In the satisfactory state $V = V^*$ the resource input remains constant. The dynamical system describes a single actor who pursues a goal by use of power resources.

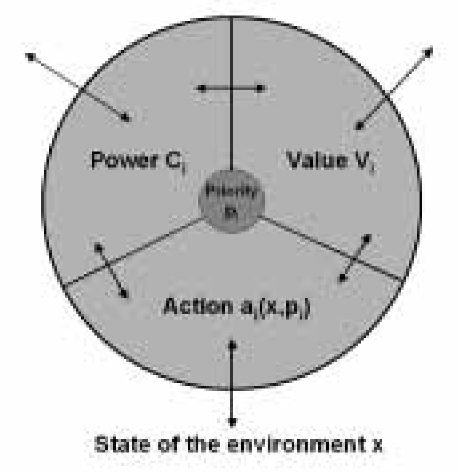

Figure 3.
The representative actor and its basic variables.

The actor has to take into consideration that its resource flow should not exceed the upper limit C^+ of available power resources, such that $0 \le C \le C^+$. In case of $C^+ < C^*$ the resources are insufficient to achieve the objective. In this case the actor can try to improve the action efficiency f which implies that the same objective V^* can be achieved at lower cost C^*. This may be done by switching to another action 2 with higher efficiency $f^2 > f^1$ than the first one. This implies that a higher share p of the power resources is allocated to this new option, at the cost of a lower share $1 - p$ for the old option. Then the overall efficiency of the mixed action is $f(p) = (1-p)f^1 + p \cdot f^2$.

5.2 From action to interaction

If several representative actors act upon the same system, an interaction evolves which is controlled by n actors (in the following called players) according to their strategies. During the interaction process each player P_i ($i = 1,..., n$) invests its power resources C_i into m system variables x^k ($k = 1,..., m$), and evaluates the outcome of the combined actions according to its own value criteria V_i to derive new actions in the next time step. The players can adapt the amount C_i and allocation of power resources with regard p_i^k to variables x^k, leading to an induced change $\Delta x_i^k = (p_i^k / c_i^k) C_i$, where the condition

$$\sum_{k=1}^{m} p_i^k = 1$$

is to be satisfied. Important is the learning capability of the players to adapt the resource allocation to achieve the value target V_i^*. In the linear case the value change ($i = 1,..., n$)

$$V_i = \sum_j f_{ij}(p) C_j + V_i^U$$

depends on the power resources of all players, weighted by the mutual efficiencies f_{ij} which describe the value impact of players on each other per resource unit. V_i^U includes all other value gains which can be important but are neglected in the following analysis. Now the target conditions $V_i = V_i^*$ cannot be achieved by one player alone but depends on the actions and powers of others according to (i = 1,..., n)

$$C_i^* = \frac{V_i^* - \sum_{j \neq i} f_{ij} C_j}{f_{ii}} .$$

These target conditions correspond to reaction curves (which in general are multidimensional hyperplanes) which lead to an adaptation $\Delta C_i = \alpha_i (C_i^* - C_i)$ in the space of power resources (costs) and thus a multi-player dynamical system. The intersection of the reaction curves is the cost equilibrium vector $\hat{C} = F^{-1} V^*$ (balance of power) where det F is the determinant of the interaction matrix $F = (f_{ij})_{i,j=1,\ldots,n}$. The equilibrium coordinates can be determined by Cramer's Rule, which for two players is $\hat{C}_i = (f_{jj} V_i^* - f_{ij} V_j^*) / \det F$.

Since the interaction efficiencies $f_{ij}(p)$ between the players depend on the vector of the allocation priorities $p = (p_1,..., p_n)$ of all players, each of them can control the dynamical system by changing their own p_i (which themselves can be vectors) to achieve the respective target equilibria and stabilize or destabilize it. If the action priorities p_i are changed towards hostile relations, the equilibrium moves towards higher costs, for a cooperative relation towards lower costs (see the stylized symmetric cases in Figure 4). In asymmetric cases one player can show cooperative attitudes to another player, which in return can show hostile attitudes, and vice versa. The players can also negotiate and cooperate on the right choice of their own p_i which can be expressed as a dynamic game problem. In addition they can form coalitions by pooling some of their resources and redistribute the gains (or losses) to the individual players.

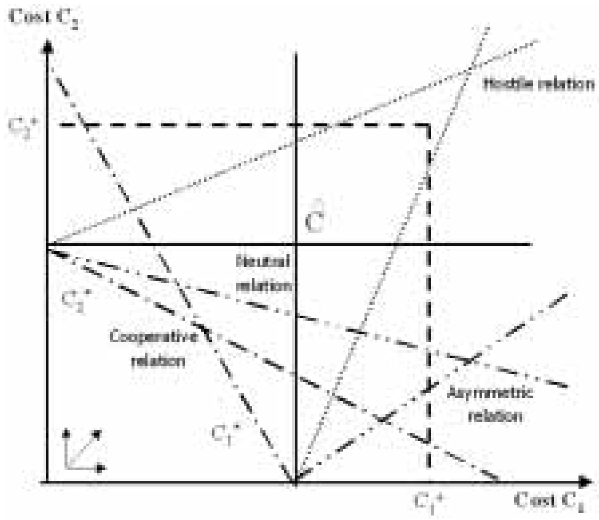

Figure 4.
Cost balance for neutral, hostile, friendly and asymmetric relations

Using the deviations from the equilibrium $y_i = C_i - \hat{C}_i$ as variables, we obtain a system of linear difference equations for the vector $y = (y_1,..., y_n)$

$$\Delta y(t) = y(t+1) - y(t) = -\bar{F} \cdot y(t)$$

where $\bar{F}$ is the modified interaction matrix with elements $\bar{f}_{ij} = \alpha_i f_{ij} / f_{ii}$. The dynamical system can be also expressed in the form $y(t + 1) = A\, y(t)$ with matrix $A = I - \bar{F}$ where I denotes the $n \times n$ unity matrix.

5.3 Stability conditions for interaction

From standard linear algebra it follows that as long as the interaction matrix F is stable, all players are able to achieve positive values, even for hostile relations with $f_{ij} < 0$. F also determines the stability of the resource dynamics $y(t+1) = (I - \bar{F})y(t)$, i.e., the conditions under which all eigenvalues are within the unit circle of the complex plane. For arbitrary f_{ij}, a set of sufficient stability conditions is [Scheffran 2001, Murata 1977]

$$\left|1 - \bar{f}_{ii}\right| + \sum_{j \neq i}^{n} \left|\bar{f}_{ij}\right| = \left|1 - \alpha_i\right| + \sum_{j \neq i}^{n} \left|\alpha_i f_{ij} / f_{ii}\right| < 1 \quad \text{for all } i = 1, 2, \ldots, n.$$

For $f_{ii} > 0$ and $0 < \alpha_i \leq 1$, this condition holds for

$$f_{ii} > \sum_{j \neq i} \left| f_{ij} \right|,$$

i.e., for each player positive self-efficiency should exceed the aggregate impact of cross-efficiencies by other players. For an increasing number of players, this condition is more difficult to achieve without cooperation. For $f_{ij} \leq 0$ $(j = 1,..., n)$, we obtain the condition

$$\sum_j f_{ij} > 0 .$$

The stability of the interaction matrix determines whether the players can achieve specific value goals unilaterally or need to form coalitions and cooperate. A coalition is called stable if all members of the coalition are satisfied and have no incentive to leave the coalition. If the interaction matrix of a coalition becomes unstable, some players are not satisfied and fail to achieve their target values, i.e., the coalition breaks apart into smaller coalitions for which the interaction matrix is stable. With an increasing number of players, the number of eigenvalues of $\bar{F}$ increases and thus the chance that some are outside the unit circle and the interaction would become unstable. This relationship has been extensively discussed in ecology and population dynamics as the “complexity-stability” tradeoff. For an initial selection of arbitrary players only those players survive which are powerful enough or are part of a stable coalition.

To clarify the escalating sequence of events it is assumed that a player P_i tries to leave the power equilibrium $\hat{C}$ by changing its power use ($\delta C_i > 0$) in order to achieve a unilateral value gain $\delta V_i = f_{ii}\,\delta C_i > 0$. If its action leads to value losses $\delta V_j = f_{ji}\,\delta C_i < 0$ for other players P_j, it provokes countermeasures $\delta C_j = -f_{ji} / f_{jj}\,\delta C$ to compensate for the loss and keep V_j constant, assuming $f_{ji} < 0$. Then the outcome of P_i's action and other players' reaction after the second time step is

$$\delta V_i = f_{ii}\,\delta C_i + \sum f_{ij}\,\delta C_j = f_{ii}(1-\sigma)\,\delta C_i$$

where

$$\sigma = \sum_j f_{ij} f_{ji} / (f_{jj}\, f_{ii}).$$

Thus, we have $\delta V_i < 0$ for $\sigma > 1$. Then, even though player P_i aims for an individual value gain, the overall social efficiency $f_{ii}^{\sigma} = f_{ii}(1-\sigma)$ due to the interaction with other players is negative and its value is worse than before. If this process continues, the interaction among the other players evolves, which may further contribute to instability (similar to an arms race), or stabilize the process by containing the first player's move. The latter can be induced externally by punishments or rewards, imposing some kind of "social regulation".

6 Striving for Security

6.1 The security-cost dynamics

In the following we apply the multi-actor framework to problems of international security. An important precondition for security is the absence of threat, i.e., the expectation that no intolerable damage is likely to occur. In mathematical terms this would imply that in a situation described by system variables $x(t)$ at time t, the expected security value $S^+(x, t) - S^-(x, t)$, combining gains and losses, is below a tolerable security threshold $S^*(x, t)$. Provided that the expected security loss S^- increases with potential damage $Y(x, t)$ and the likelihood $p(x, t)$, we obtain the net security function

$$S(x,t) = S^+(x,t) - p(x,t) \cdot Y(x,t) - S^*(x,t)$$

which is thus an indicator of relative security against threat. For $S > 0$ an actor feels relatively secure, for $S < 0$ relatively insecure. By use of power resources C (such as the defense budget) an actor can change its security state $\Delta S = f \cdot C$, where $f = s/c$ is the efficiency of the power resources with regard to the security impact. If an actor aims at achieving a particular security gain $\Delta S = V^*$, this translates into the required budget $C^* = V^* / f$. One obvious objective would be to reduce a negative security gap $S < 0$ through changes $\Delta S = -S / \tau = V^*$ where τ indicates how many time steps are required to bridge the security gap for constant ΔS. For $\tau = 1$ the gap shall be bridged in one single time step, for $\tau > 1$ the gap is expected to decay over an extended period. If at a given time t the actual budget $C(t)$ differs from the required one $C^*(t)$, the gap can be bridged by a budget adaptation process

$\Delta C_i(t) = \alpha_i(C_i^*(t) - C_i(t))$ where the reaction parameter $\alpha_i = \kappa_i C_i(C_i^+ - C_i)$ contains a logistic growth factor, reflecting the consideration that for a country with a limited budget $0 \le C_i \le C_i^+$ the dynamics is damped near the boundaries. κ_i is a constant cost reaction parameter (this model variant is called the SCX model, according to [Scheffran 1989] and [Jathe/Krabs/Scheffran 1997]).

6.2 Stability of the offense-defense competition

Throughout military history the offense-defense competition has been a major driver of the arms race, in which offenses increased the potential damage in war while defense tried to limit it. With the advent of nuclear-armed ballistic missiles any attempt to protect against this immense threat by defensive measures remained economically and technically unfeasible, depite enormous costs and efforts in programs such as the Strategic Defense Initiative (SDI) or the current Missile Defense programs of the Bush administration. One of the crucial issues is the impact of (missile) defense on strategic and international stability (for an earlier survey see [Scheffran 1989]).

The potential damage for a defender P_i grows with the offensive capability O_j of an attacker P_j, weighted with its offense efficiency o_j (representing the fraction of the existing offense units being efficiently launched) and destructive impact y_i (translating weapons into damage at the target), dimished by its own defense capability D_i, which is weighted with the defense efficiency d_i. In the simple scenario of first and second strike (which dominated Cold War thinking for a long time), each country perceives basically three options: being the first striker (index F), being the second striker (index S), no attack (index 0). The case of mutual attack (duel) in the same period is neglected here for reasons of simplicity. Assume that attacker P_i destroys some of P_j's offense capability before it can be launched in retaliatory attack. Then damage for the first and second striker is

$$Y_i^F = y_i^F[o_j(O_j - e_i(o_i^F O_i - d_j D_j)) - d_i^F D_i],$$

$$Y_i^S = y_i[o_j^F O_j - d_i D_i],$$

where o_j^F, d_j^F and y_i^F are the respective variables for the first striker while the index S for the second striker is neglected to simplify notation. e_i is the efficiency of the first striker's offense units in destroying the second striker's offense units, depending on factors such as the attacker's accuracy or the number of offense units per delivery vehicle. Now assume that each of the options is perceived with a likelihood $q_i^F, q_i = q_i^S, q_i^0$, which may also be interpreted as the weight that the decisionmakers in a country gives to the damage in a particular scenario. $q_i^k = 0$ implies that the decisionmakers completely neglect the damage in a scenario k. Then net security includes the weighted security impacts for all considered options

$$S_i = q_i^0 S_i^+ - q_i^F Y_i^F - q_i^S Y_i^S - S_i^*.$$

To norm the likelihoods one can make the additional assumption $q_i^0 = 1 - q_i^F - q_i$. To determine the impact of the military capabilities on security function S_i, we use the partial derivatives $s_{ij}^x = \partial S_i / \partial x_j$ where x_j can be either offensive or defensive capabilities of two countries P_i and P_j. For analytic treatment, we now use the special case that each country P_i ignores the option of being the first striker $(q_i^F = 0)$ but feels threatened by a first strike of an opponent P_j with likelihood $q_i > 0$. Then security $S_i = (1 - q_i)S_i^+ - q_i Y_i - S_i^*$ of each player only depends on the weighted damage Y_i of being attacked and the security gains of no attack.

A country can change its security situation by applying its power resources (military budget) C_i to increase or decrease the offensive and defensive capabilities according to $\Delta O_i = (p_i^o / c_i^o) C_i$, $\Delta D_i = (p_i^d / c_i^d) C_i$ where p_i^d, $p_i^o = 1 - p_i^d$ are the cost shares (action priorities) for the respective military capabilities. In this case, and for constant likelihoods,[3] the overall security efficiencies are $f_{ii} = p_i^d s_{ii}^d / c_i^d = q_i y_i d_i^c p_i^d$, $f_{ij} = p_j^o s_{ij}^o / c_j^o = -q_i y_i o_j^c (1 - p_j^a$ $(i, j = 1, \ldots, n)$ with $o_j^c = o_j / c_j^o$, $d_i^c = d_i / c_i^d$ and $p_j^o = 1 - p_j^d$. According to the analyis in the previous chapter, for $f_{ii} > 0$ and $f_{ij} < 0$ $(j \neq i)$, sufficient conditions for stability are $f_{ii} + f_{ij} > 0$ which leads to (for $c_i^d > 0$)

$$p_i^d > \frac{o_j^c}{d_i^c}(1 - p_j^d) \equiv p_i^{d*}.$$

This defines a defense share which country P_i has to exceed to ensure stability. Below this threshold the threat due to offensive weapons is intolerable and has to be compensated by increasing defenses. The situation is unstable which implies that there are incentives for an arms race. This threshold increases with the security-cost efficiency of the offense o_j^c and decreases with the efficiency of the defense d_i^c. There is an inverse coupling between the defense shares of both countries, which implies that the threshold for one country declines the lower the offense share for the other country is. For $o_j^c = d_i^c$ the lines connect the corners (0, 1) and (1, 0).

We use a simple example with relative units. Compared to a country P_1 another country P_2 has a higher offense efficiency $o_2^c = 2o_1^c = 2$, which exceeds the defense efficiencies $d_2^c = 2d_1^c = 1$ of both sides. Then the two stability conditions are $p_1^{d*} = 4(1 - p_2^d)$ and $p_2^{d*} = 1 - p_1^d$. The results are depicted in Figure 5.

An alternative way of stabilizing the security relationship is the option of mutually disarming the offense which in the model corresponds to negative unit costs $c_i^o < 0$. In this case we have $f_{ij} > 0$ which always guarantees $f_{ii} + f_{ij} > 0$ and thus stability. The equilibrium point (balance of power) is given by

$$C_i = \frac{p_j^d \left[q_j y_j d_j^c V_i^* - q_i y_i o_j^c V_j^* \right] + q_i y_i o_j^c V_j^*}{\det F}$$

where V_i^* is the given value target for player P_i (e.g., bridging a security gap). Assume $\det F = f_{11} f_{22} - f_{12} f_{21} > 0$, and a negative bracket [...]. Then the budget of

[3] The assumption of exogenously given likehoods can be revised, making the likelihood of being attacked dependent on the force structure. See further [Scheffran 1989].

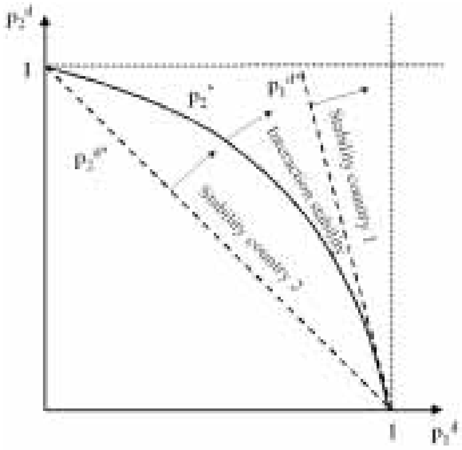

Figure 5.
Stability regions for defense shares p_1^d and p_2^d.

P_i can be reduced by increasing the defense share of $P_j(p_j^d \to 1)$; if the bracket is positive, costs can be reduced by decreasing the defense share $(p_j^d \to 0)$. The latter is the case in particular for disarming the offensive capability $(c_j^o < 0)$. The threshold between increasing and decreasing defense is given by

$$q_i = q_j \frac{y_j d_j V_i^*}{y_j o_j^c V_j^*} = q_i^*.$$

Thus for an attack likelihood $q_i < q_i^*$ there is a tendency to increase the offense share for cost reasons which depends on the perceived likelihood q_j of the opponent being attacked. The rule of thumb is that for a low perceived attack likelihood the defense share is also low. If both sides cooperate and trust each other, they don't believe in an attack by their partner and have thus less tendency to build defenses because they evaluate the other's offense as less threatening. Stability of the interaction matrix depends on the eigenvalues solving the characteristic equation:

$$\lambda_{1/2} = -(f_{11} + f_{22})/2 \pm \sqrt{(f_{11} + f_{22})^2/4 - \det F}\,.$$

Negative eigenvalues occur for $f_{11} + f_{22} > 0$ and the determinant of the interaction matrix

$$\det F = f_{11}f_{22} - f_{12}f_{21} = q_1 q_2 y_1 y_2 [d_1^c d_2^c p_1^d p_2^d - o_1^c o_2^c (1 - p_1^d)(1 - p_2^d)] > 0$$

which leads to the condition

$$p_2^d > \frac{1 - p_1^d}{1 + (do - 1)p_1^d} = p_2^*,$$

where

$$do = \frac{d_1^c d_2^c}{o_1^c o_2^c}.$$

Both the points $(p_1^d, p_2^d) = (0, 1)$ and (1, 0) satisfy the equality relation $p_2^d = p_2^*$, whereas for $p_1^d = 0.5$ we achieve the threshold

$$p_2^* = \frac{1}{1 + do} .$$

Our example yields $do = 1/4$ and a threshold $p_2^* = 4/5$ (see Figure 5).

The competition between offenses and defense can show a very complex and unstable dynamics which is difficult to control because of a wide range of uncertainties in the security-relevant variables [Scheffran 1989]. With an increasing number of countries whose offense-defense competition is closely linked, the potential for instability increases as the number of eigenvalues increases and thus the likelihood that the chosen parameter combination represents an unstable eigenvalue with positive real parts. With regard to the introduction of missile defense, there is the danger of a domino effect, i.e., the possibility, that Russia and China build more weapons because of the USA, that India follows China and Pakistan responds to India. Such a strong coupling can lead to an international escalation, with missile defense as a key trigger.

7 Chaos, Uncertainty, and Instability in International Security

The more variables and relations are included in a model to match the complexity of reality, the more does the combined uncertainty in model variables matter, leaving a wide range of possible outcomes. This also can play a role in the SCX conflict model outlined in the previous chapter, with the following set of equations for the variable changes of actors P_i ($i = 1,..., n$):

$$\Delta S_i = \sum_{j=1}^{n} f_{ij} C_j + U_i^S ,$$

$$\Delta C_i = -\kappa_i C_i (C_i^+ - C_i)(S_i + \tau_i \Delta S_i) ,$$

$$\Delta x_i^k = p_i^k C_i / c_i^k .$$

Here $k = 1, ..., m_i$ represents the index of system variables, κ_i is the cost reaction parameter, and U_i^S comprises all other security impacts (which are again ignored here). Each of the parameters can be time-dependent which is neglected to facilitate notation. Even in the bilateral case of two actors, chaos can emerge, as the following case shows. Here we start with anti-symmetric (antagonistic) security conditions $S_1(0) = - S_2(0) = 0.3$, which corresponds to a zero-sum situation. Both actors adapt their budgets towards achieving their satisfying security objectives $S_i = 0$, starting with an initial budget of $C_i(0) = C_i^* / 2$, which is half of the maxi-

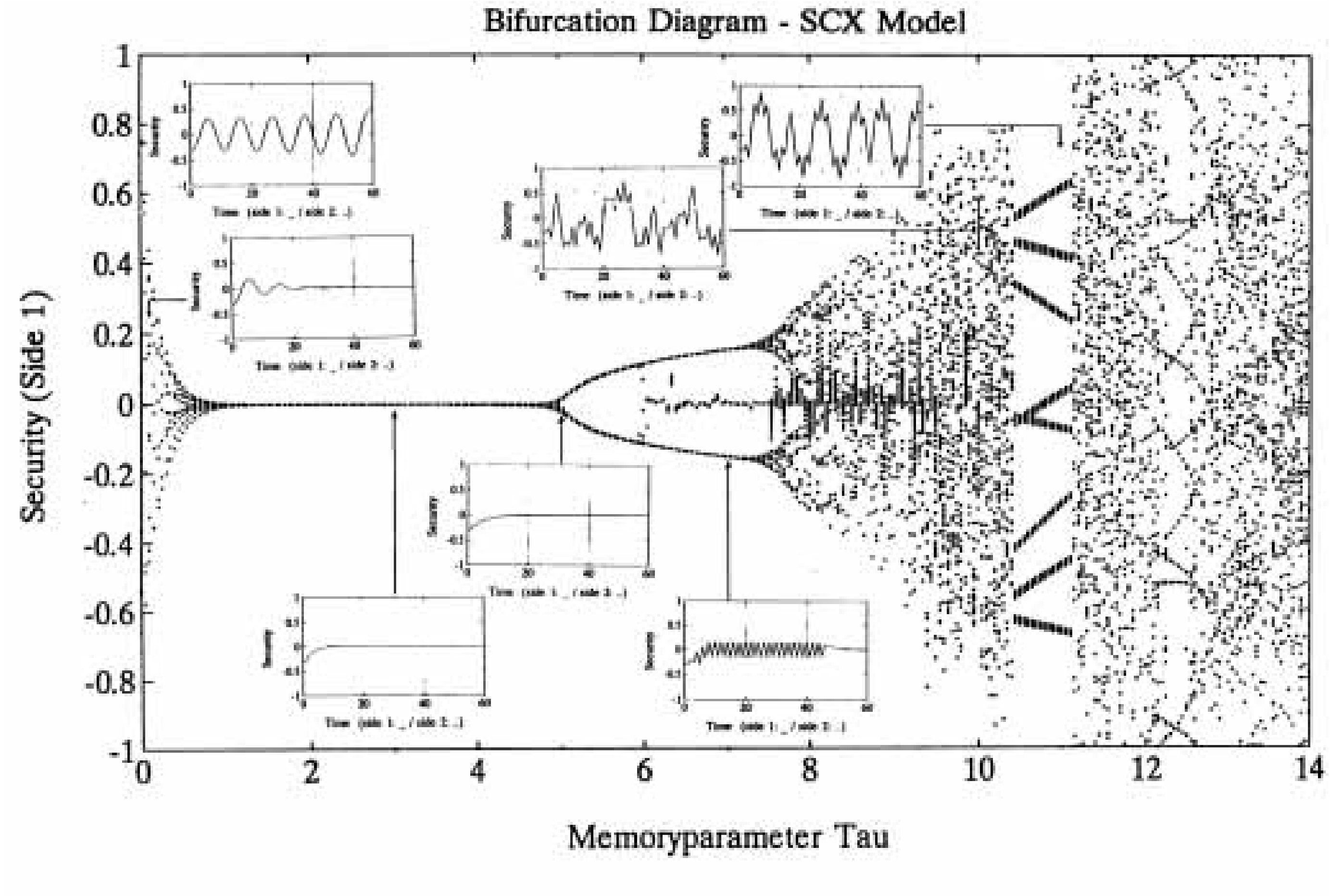

Figure 6. The security bifurcation diagram for two antagonistic actors.

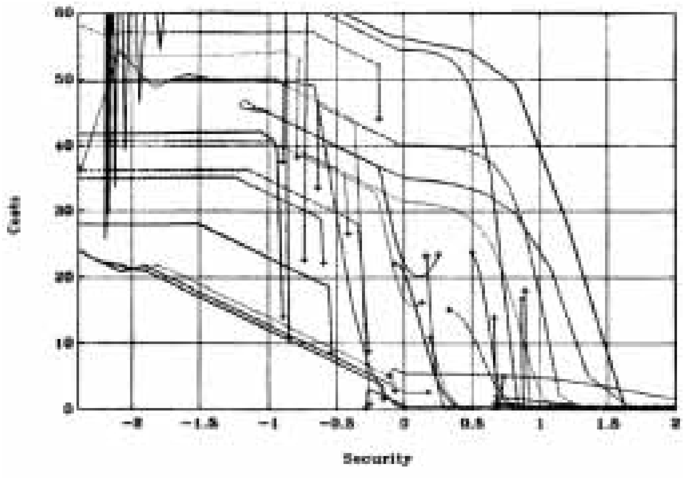

Figure 7. Dichotomy between winners and losers for a large number of actors.

mum budget $C_i^+ = 60$ cost units. Relations are hostile, represented by efficiencies $f_{ii} = -f_{ij} = 0.01$, which are chosen to be at the threshold between stability and instability. The cost reaction parameter is $\kappa_i = 0.04$. If we use the memory parameter τ_i as an order parameter and depict the security states to which the trajectory is attracted after some time, then we get a bifurcation diagram (Figure 6). For very low τ values, security oscillates around $S_1 = 0$, for increasing τ the trajectory approaches this state asymptotically (the second player shows inverse behavior). At some τ value, the first bifurcation occurs by splitting the equilibrium point into two, leading to a whole cascade. For high $\tau > 11$ the security situation is completely unpredictable because the overreaction of the actors moves the dynamics into any possible security state [Jathe/Scheffran 1995ab].

Now consider a large number of actors (in this case 30) whose interaction efficiencies f_{ij} as well as the initial conditions for security and costs (marked by +) are chosen stochastically around the previous baseline case. This implies that for some players the interactions are stable, for others unstable. Figure 7 clearly shows that a dichotomy emerges between two groups of actors. Some actors steadily win from the dynamics, forming a winning coalition, others lose by moving towards high insecurity and high costs. This polarization resembles some aspects that can be observed in real-world conflicts, such as the North-South conflict. Here also the asymmetry in maximum power resources C_i^+ plays an important role, which implies that a strong player (such as the US) can compensate unstable relations with much weaker players by use of their overwhelming force (dominance). However, this can be countered by an increased destruction efficiency f_{ij} , e.g., by a terror attack against vulnerable spots of the strong player which require only small efforts.

One way to avoid a polarization into blocks and minimize violent conflicts between them is to change the interaction efficiencies f_{ij} towards positive values, e.g., by strengthening of cooperation beyond the blocks and the formation of coalitions in order to achieve common aims. This can involve non-military security dimensions, such as environmental, economic, cultural and human rights dimensions. The model framework seems appropriate, in particular in the environmental sciences, to analyze conflicts between human behavior and ecological limits. Here systemic models (from ecology) and actor-based models (economic-social relations) are directly linked. If central control mechanisms are neither realistic nor desirable, the question arises under which conditions a spontaneous, self-organized transition to ecologically compatible modes of behavior would emerge.

One example is the cooperation between industrialized and developing countries in greenhouse gas emission reductions, e.g., via Joint Implementation and the Clean Development Mechanism of the Kyoto Protocol (see [Scheffran/Pickl 2000] and [Ipsen/Rösch/Scheffran 2001]). Another example deals with cooperative and sustainable management of fish resources, minimizing the potential for fish scarcity and severe conflict [Scheffran 2000]. In both fields the VCX model framework has been applied to identify conditions for cooperation, a task that is to be continued in the future.

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