Preface

> "How can anything a computer produces have to do with chaos? I thought computers were based on *logic*."
>
> *Inspector Morse, in response to an explanation of the Mandelbrot fractal.*

This book deals with nonlinear Hamiltonian systems, depending on parameters. Such systems occur for example in the modeling of frictionless mechanics and optics. The general goal is to understand their dynamics in a qualitative, and if possible, also quantitative way. The dynamical behavior generally is expressed in terms of equilibria, periodic and quasi periodic solutions as well as corresponding homo– and heteroclinic connections between those. Such connections often are accompanied by chaotic dynamics. In many important cases, it is possible to reduce a skeleton of the dynamics to lower dimensions, sometimes leading to a Hamiltonian system in one degree of freedom. Such reduced systems allow a singularity theory or catastrophe theory approach which gives rise to transparent, in a sense polynomial, normal forms. Moreover the whole process of arriving at these normal forms is algorithmic. The purpose of this book is to develop computer-algebraic tools for the implementation of these algorithms, which involves Gröbner basis techniques. This set-up allows for many applications concerning resonances in coupled or driven oscillators, the n-body problem, the dynamics of the rigid body, etc. in Hamiltonian mechanics. A version of the spring–pendulum is used as a test case.

The present work appeared earlier as the PhD thesis [Lun99a] of Gerton Lunter, written under supervision of Henk Broer and Gert Vegter.

Background

General background of this work is the theory of nonlinear Dynamical Systems as it has evolved since Poincaré [Poi87]. A main aspect of this development is that the 19th century programme of the explicit analytic computation of individual evolutions (solutions) largely has broken down. Indeed, since then the emphasis has shifted to considering the whole phase space as it contains all possible evolutions, in simple cases expressed by devices like phase portraits.

More generally, increasing importance was given to the study of geometric objects like tori and stable or unstable manifolds, and their relative positions in phase space with respect to each other, periodic evolutions, and equilibria.

Another development was that the interest moved away from individual systems, and towards the study of dynamical properties that are persistent for small perturbations of the given system, compare the treatment of the term 'genericity' by Smale [Sma67] in the 1960s. Parallel to this development is the introduction of parameters into dynamical systems, so as to effect deformations or unfoldings of these in a systematic way, compare Thom [Tho72], also see Arnold [Arn88].

Summarizing one could say that during one century, the emphasis in dynamical systems shifted from the explicit analytic computation of a given evolution to the consideration of generic properties of families of dynamical systems deforming a given one, where the methods became more geometric and qualitative.

In a parallel development, during the past 50 years the electronic computer became increasingly important for studying concrete dynamical systems, possibly depending on parameters. On the one hand we saw visualization of phase portraits, chaos, invariant manifolds, bifurcation diagrams, etc., while on the other hand also the computation of underlying dynamical characteristics like normal form coefficients, dimensions, Lyapunov exponents, power spectra, etc., became possible. Here, apart from purely numerical computations, also symbolic computations and computer algebra plays a role.

The present book is focused on various forms of normal form computations, which are deeply involved with computer algebra. We restrict to the Hamiltonian context, where moreover the systems can be approximated by those admitting a reduction to one degree of freedom. Therefore, apart from the approximation aspect, we are back in the paradise of the 19th century. The one degree of freedom reduction supports the coherent dynamics of the original model, while it also provides a skeleton for the chaos.

By this combination of geometric and algebraic methods, the quantitative element resurfaces: the geometric descriptions can be traced in detail to the original physical model equations.

Formal normal forms, a perturbation problem

Several methods exist by which dynamical systems can be reduced to lower dimensions. One standard way is by restricting the system to an invariant manifold, like a center manifold. Factoring out symmetry is another way to reduce the phase space. In the Hamiltonian context this also is the classical approach, since symmetry by Noether's theorem [Arn89] is related to the existence of (first) integrals and therefore conservation laws. A textbook example of this is the Kepler problem, where rotational symmetry thus leads to the conservation of angular momentum and a cyclic variable, which allows reduction to one degree of freedom. In this example the symmetry group is the circle or a 1-torus. In many applications the symmetry group is a higher dimensional torus.

In the present Hamiltonian context we concentrate on reduction by symmetry. Within the class of general Hamiltonian systems, those that admit a symmetry group like an n-torus, are highly degenerate: in other words, having such a torus symmetry is not a generic property. Nevertheless, the theory of formal normal forms near certain equilibria or (quasi–) periodic orbits admits the local approximation of the given system by a symmetric one. This method of simplifying formal series goes back to Poincaré and Birkhoff [Poi28, Bir50] also see Gustafson [Gus66]. For general information, see e.g. Arnold [Arn89, Arn88].

To be somewhat more precise, if the original vector field is denoted by X, where for the moment parameter dependence is suppressed, then the present normal form theory asks for a canonical transformation Φ, such that

$$\Phi_*(X) = N + R.$$

Here N is the normalized, symmetric part, that describes the slow dynamics after factoring out the symmetry, while R is the small remainder term. In this way the study of the dynamical system is divided into two parts. The first is to understand the symmetric approximation N and the second to take the perturbation R into account. Since this book will be entirely devoted to the former of these two problems, we just give a few remarks about the latter, e.g., see Broer and Takens [BT89]. Indeed, in this setting the occurrence of separatrix splitting as associated to chaos, is a flat phenomenon. One could say that the approximation N contains the regular skeleton supporting the chaotic zones of instability.

Remarks

- The expression $\Phi_*(X) = N + R$ is reminiscent to the division algorithm. Below this fact will be elaborated further in the context of Gröbner basis techniques [CLO92].
- Concerning regularity, our context mainly is assumed to be C^∞. However, if one restricts to the real analytic C^ω case, the remainder term R can be estimated in an appropriate exponential way. Compare Neishtadt [Nei84, BRS96, BR01].

Singularity and Catastrophe Theory, polynomial normal forms

In many examples factoring out the symmetry leads to Hamiltonian systems in one degree of freedom where the dynamics largely is determined by the level curves of the corresponding Hamilton functions. The presence of parameters in the original problem so brings us in the setting of families of real functions in dimension 2. Since the reduced phase space is 2-dimensional, in a further simplification process we may abstain from symplecticity of the transformations, since this only affects the time parametrization of the dynamics and not the level curves of the functions. Now singularity and catastrophe theory normal forms can be computed, which to a large extent are polynomial. For a general reference, e.g., see Thom [Tho72], Bröcker and Lander [BL75].

Regarding the reduction to one degree of freedom Hamiltonian systems, we have selected two different approaches. The first of these is the *planar reduction method*, compare Broer and Vegter *et al.* [BV92, BCKV93, BCKV95], comparable to the classical Keplerian reduction. It is to be noted that the formal integrals obtained after the formal normalization show up as distinguished parameters in the singularity theory. The second approach concerns the *energy–momentum map*, compare Duistermaat [Dui84], Van der Meer [Mee85], Cushman and Bates [CB97]. Also see [BHLV98]. Both methods, after a formal normal form, reduce to one degree of freedom Hamiltonian systems after which singularity theory is used. As said before, here further simplifying transformation are applied. In the former case this leads to so-called *right equivalences* and in the latter to *left–right equivalences*.

The most interesting cases contain rather strong resonances, which also gives discrete symmetries in the normal form. Furthermore, certain discrete symmetries are considered that are *a priori* to the original physical problem, such as time reversibility.

For general background on the use of singularity and catastrophe theory to dynamical systems, frequently using contact equivalence, see Golubitsky, Schaeffer, Stewart and Marsden [GS85, GSS88, GMSD95]. In these references, as well as in Wassermann [Was75], also distinguished parameters play a role. However, in view of the special nature of the distinguished parameters, which are nonnegative action variables, a new unfolding theory was developed in [BCKV93]. For general background also see Damon [Dam84, Dam88, Dam95] and Montaldi [Mon91].

Algorithms, setting of the problem

An elementary observation is that, without the help of computers, the computations mentioned so far can not be extended to the level that is of interest for serious applications. A good example of this in the dissipative setting is given by Marsden and McCracken [MM76]. Fortunately the proofs involved here are highly constructive and lead to algorithms that can easily be implemented on computers.

The algorithm of formal normal forms already has been widely implemented on computers. In the Netherlands e.g. the Dynamical Systems Laboratory at the CWI Amsterdam has been active in this, unifying the process and making it more sophisticated. The singularity theory normal form also is algorithmic in nature. See Kas and Schlessinger [KS72]. Here, however, implementation largely was lacking, while the complication of practical computations renders the use of computers essential. As indicated before, this program is deeply involved with computer algebra, in particular with Gröbner basis techniques [CLO92]. It turns out that the methods of planar reduction and the energy–momentum map may be formalized in a unified way with help of *standard bases*. A recent reference in this direction of applying computational algebra to dynamical systems is Gatermann [Gat00].

One of the key ideas is to keep track of all normalizing transformations, which makes it possible to translate the mathematical conclusions back into the original physical context of the model. This task is effectively carried out for a mechanical example. In that sense, key aspects of the qualitative model are made quantitative. For a general reference on the application of computer algebraic methods in perturbation theory see Rand and Armbruster [RA87].

The scope and beyond

Extension of the research at hand can be pursued in the following directions. As introductory reduction algorithms also the Liapunov–Schmidt or the Moser–Weinstein method, as well as the restriction to center manifolds, may be taken into consideration. After this again a singularity and catastrophe theory approach seems feasible. Again compare [GMSD95]. Moreover, combinations of these approaches are of importance.

Another option is to incorporate the work of Hummel [Hum79], who investigates periodic points of diffeomorphisms by contact equivalence. Also here discrete symmetries are essential. For earlier, theoretical results compare Takens [Tak74a] and see [GS85, GSS88, GMSD95, BGV02].

Also at the level of concrete applications and case studies many extensions are possible, beyond the present case study of a spring pendulum and the resonances considered here. We just mention mechanical examples like driven or coupled oscillators (compare, e.g., [TRVN00]), the rigid body, etc. For earlier work in this direction compare, [Han95a, Han95b, Hov92, BHvN98, BHvNV99]. Also see [Lun99a, Lun99b]. Moreover, the research at hand forms the beginning of a theoretical basis for the future development of a coherent set (package or library) of computer programs, suitable for further use.

A further development that may be of great importance is the combination of computer algebraic and numerical means, compare Simó [Sim89]. In this way it becomes possible to be more efficient in dealing with the constants, parameters and transformations, only keeping track of essential things. Another aspect is the computation and visualization of invariant manifolds, e.g., compare Vegter *et al.*, [HOV95, BOV97].

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