Preface

A number of important results in combinatorics, discrete geometry, and theoretical computer science have been proved by surprising applications of algebraic topology. Lovász's striking proof of Kneser's conjecture from 1978 is among the first and most prominent examples, dealing with a problem about finite sets with no apparent relation to topology.

During the last two decades, topological methods in combinatorics have become more elaborate. On the one hand, advanced parts of algebraic topology have been successfully applied. On the other hand, many of the earlier results can now be proved using only fairly elementary topological notions and tools, and while the first topological proofs, like that of Lovász, are masterpieces of imagination and involve clever problem-specific constructions, reasonably general recipes exist at present. For some types of problems, they suggest how the desired result can be derived from the nonexistence of a certain map ("test map") between two topological spaces (the "configuration space" and the "target space"). Several standard approaches then become available for proving the nonexistence of such a map. Still, the number of different combinatorial results established topologically remains relatively small.

This book aims at making elementary topological methods more easily accessible to nonspecialists in topology. It covers a number of substantial combinatorial and geometric results, and at the same time, it introduces the required material from algebraic topology. Background in undergraduate mathematics is assumed, as well as a certain mathematical maturity, but no prior knowledge of algebraic topology. (But learning more algebraic topology from other sources is certainly encouraged; this text is no substitute for proper foundations in that subject.)

We concentrate on topological tools of one type, namely, the Borsuk–Ulam theorem and similar results. We develop a systematic theory as far as our restricted topological means suffice. Other directions of research in topological methods, often very beautiful and exciting ones, are surveyed in Björner [Bjö95].

History and notes on teaching. This text started with a course I taught in fall 1993 in Prague (a motivation for that course is mentioned in Section 6.8). Transcripts of the lectures made by the participants served as a basis of the first version. Some years later, a course partially based on that text was taught by Günter M. Ziegler in Berlin. He made a number of corrections and additions (in the present version, the treatment of Bier spheres in Section 5.6 is based on his writing, and Chapters 1, 2, and 4 bear extensive marks of his improvements). The present book is essentially a thoroughly rewritten version prepared during a predoctoral course I taught in Zürich in fall 2001, with a few things added later. Most of the material was covered in the course: Chapter 1 was assigned as introductory reading, and the other chapters were presented in approximately 25 hours of teaching, with some omissions throughout and only a sketchy presentation of the last chapter.

The material of this book should ultimately become a part of a more extensive project, a textbook of "topological combinatorics" with Anders Björner (the spiritual father of the project) and Günter M. Ziegler as coauthors. A substantial amount of additional text already exists, but it appears that finishing the whole project might still take some time. We thus chose to publish the present limited version, based on my lecture notes and revolving around the Borsuk–Ulam theorem, separately. Although Anders and Günter decided not to be "official" coauthors of this version, the text has certainly benefited immensely from discussions with them and from their insightful comments.

Sources. The 1994 version of this text was based on research papers, on a thorough survey of topological methods in combinatorics by Björner [Bjö95], and on a survey of combinatorial applications of the Borsuk–Ulam theorem by Bárány [Bár93]. The presentation in the current version owes much to the recent handbook chapter by Živaljević [Živ04] (an extended version of [Živ04] is [Živ96]). The continuation [Živ98] of that chapter deals with more advanced methods beyond the scope of this book.

For learning algebraic topology, many textbooks are available (although in this subject it is probably much better to attend good courses). The first steps can be made with Munkres [Mun00] (which includes preparation in general topology) or Stillwell [Sti93]. A very good and reliable basic textbook is Munkres [Mun84], and Hatcher [Hat01] is a vividly written modern book reaching quite advanced material in some directions.

Exercises. This book is accompanied by 114 exercises; many of them serve as highly compressed outlines of interesting results. Only some have actually been tried in class.

The exercises without a star have short solutions, and they should usually be doable by good students who understand the text, although they are not necessarily easy. All other exercises are marked with a star: the more laborious ones and/or those requiring a nonobvious idea. Even this rough classification is quite subjective and should not be taken very seriously.

Acknowledgments. Besides the already mentioned contributions of Günter M. Ziegler and Anders Björner, this book benefited greatly from the help of other people. For patient answers to my numerous questions I am much

indebted to Rade Živaljević and Imre Bárány. Special thanks go to Yuri Rabinovich for a particularly careful reading and a large number of inspiring remarks and well-deserved criticisms. I would like to thank Imre Bárány, Péter Csorba, Allen Hatcher, Tomáš Kaiser, Roy Meshulam, Karanbir Sarkaria, and Torsten Schönborn for reading preliminary versions and for very useful comments. The participants of the courses (in Prague and in Zürich) provided a stimulating teaching environment, as well as many valuable remarks. I also wish to thank everyone who participated in creating the friendly and supportive environments in which I have been working on the book. The end-of-proof symbol is based on a photo of the European badger ("borsuk" in Polish) by Steve Jackson, and it used with his kind permission.

Errors. If you find errors in the book, especially serious ones, I would appreciate it if you would let me know (email: `matousek@kam.mff.cuni.cz`). I plan to post a list of errors at `http://kam.mff.cuni.cz/~matousek`.

Prague, November 2002 Jiří Matoušek

On the second printing. This is a revised second printing of the book. Errors discovered in the first printing have been removed, few arguments have been clarified and streamlined, and some new pieces of information on developments in the period 2003–2007 have been inserted. Most notably, a brief treatment of the cohomological index and of the Hom complexes of graphs is now included.

For valuable comments and suggestions I'd like to thank José Raúl Gonzáles Alonso, Ben Braun, Péter Csorba, Ehud Friedgut, Dmitry Feichtner-Kozlov, Nati Linial, Mark de Longueville, Haran Pilpel, Mike Saks, Lars Schewe, Carsten Schultz, Gábor Simonyi, Gábor Tardos, Robert Vollmert, Uli Wagner, and Günter M. Ziegler.

Prague, August 2007 J. M.

http://www.springer.com/978-3-540-00362-5

Using the Borsuk-Ulam Theorem
Lectures on Topological Methods in Combinatorics and Geometry
Matousek, J.
2003, XII, 214 p., Softcover
ISBN: 978-3-540-00362-5