8 Parabolic Equations

In this chapter we study both the pure initial value problem and the mixed initial-boundary value problem for the model heat equation, using Fourier techniques as well as energy arguments. In Sect. 8.1 we analyze the solution of the pure initial value problem for the homogeneous heat equation by means of a representation in terms of the Gauss kernel, and use it to investigate properties of the solution. In the remainder of the chapter we consider the initial-boundary value problem in a bounded spatial domain. In Sect. 8.2 we solve the homogeneous equation by means of eigenfunction expansions, and apply Duhamel's principle to find a solution of the inhomogeneous equation. In Sect. 8.3 we introduce the variational formulation of the problem and give examples of the use of energy arguments, and in Sect. 8.4 we show and apply the maximum principle.

8.1 The Pure Initial Value Problem

We begin our study of parabolic equations by considering the pure initial value problem (or the Cauchy problem) for the heat equation, which is to find $u(x,t)$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= 0, \qquad \text{in } \mathbf{R}^d \times \mathbf{R}_+, \\ u(\cdot,0) &= v, \qquad \text{in } \mathbf{R}^d. \end{aligned} \tag{8.1}$$

We shall employ the Fourier transform of u with respect to x, cf. App. A.3,

$$\hat{u}(\xi,t) = \mathcal{F}u(\cdot,t)(\xi) = \int_{\mathbf{R}^d} u(x,t)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x, \quad \text{for } \xi \in \mathbf{R}^d.$$

If u and its derivatives are small enough for large $|x|$, then we have

$$\big(\mathcal{F}\Delta u(\cdot,t)\big)(\xi) = \int_{\mathbf{R}^d} \Delta u(x,t)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x = -|\xi|^2\hat{u}(\xi,t)$$

and, with $u_t = \partial u/\partial t$,

$$\big(\mathcal{F}u_t(\cdot,t)\big)(\xi) = \frac{d\hat{u}}{dt}(\xi,t).$$

Hence we conclude from (8.1) that $\hat{u}$ satisfies

$$\frac{d\hat{u}}{dt} = -|\xi|^2\hat{u}, \qquad \text{for } \xi \in \mathbf{R}^d,\ t > 0,$$
$$\hat{u}(\xi, 0) = \hat{v}(\xi), \qquad \text{for } \xi \in \mathbf{R}^d.$$

This is a simple initial value problem for a first order linear ordinary differential equation, with ξ as a parameter, and its solution is

$$\hat{u}(\xi, t) = \hat{v}(\xi)\mathrm{e}^{-t|\xi|^2}. \tag{8.2}$$

Recalling that $w(x) = \mathrm{e}^{-|x|^2}$ has the Fourier transform

$$\hat{w}(\xi) = \pi^{d/2}\mathrm{e}^{-|\xi|^2/4}$$

(cf. Problem A.19), we conclude from (A.34) that $\mathrm{e}^{-t|\xi|^2}$ is the Fourier transform of the *Gauss kernel*

$$U(x, t) = (4\pi t)^{-d/2}\mathrm{e}^{-|x|^2/4t},$$

and hence we obtain formally from (8.2) that

$$u(x, t) = \big(U(\cdot, t) * v\big)(x) = (4\pi t)^{-d/2}\int_{\mathbf{R}^d} v(y)\mathrm{e}^{-|x-y|^2/4t}\,\mathrm{d}y. \tag{8.3}$$

The function $U(x, t)$ is a *fundamental solution* of the initial value problem. We shall now show that the function defined in (8.3) is, in fact, a solution of (8.1) under a weak assumption on the initial function. Note that $U(x, t)$ and $u(x, t)$ in (8.3) are only defined for $t > 0$.

Theorem 8.1. *If v is a bounded continuous function on $\mathbf{R}^d$, then the function $u(x, t)$ defined by* (8.3) *is a solution of the heat equation for $t > 0$, and tends to the initial data v as t tends to* 0.

Proof. We first note that for $t > 0$ we may differentiate the integral in (8.3) with respect to x and t under the integral sign, and show directly that this function satisfies the heat equation in (8.1). To see that $u(x, t)$ tends to the desired initial values as $t \to 0$ we let $x_0 \in \mathbf{R}^d$ be arbitrary and show that

$$u(x, t) \to v(x_0), \quad \text{as } \ (x, t) \to (x_0, 0).$$

In fact, using the transformation $\eta = (y - x)/\sqrt{4t}$, and the formula

$$\pi^{-d/2}\int_{\mathbf{R}^d} \mathrm{e}^{-|x|^2}\,\mathrm{d}x = 1, \tag{8.4}$$

we may write

$$\begin{aligned} u(x,t) - v(x_0) &= (4\pi t)^{-d/2} \int_{\mathbf{R}^d} v(y) \mathrm{e}^{-|x-y|^2/4t} \, \mathrm{d}y - v(x_0) \\ &= \pi^{-d/2} \int_{\mathbf{R}^d} \big(v(x + \sqrt{4t}\eta) - v(x_0) \big) \, \mathrm{e}^{-|\eta|^2} \, \mathrm{d}\eta. \end{aligned}$$

Let $M = \|v\|_{\mathcal{C}} = \|v\|_{\mathcal{C}(\mathbf{R}^d)}$ and let δ be so small that

$$|v(z) - v(x_0)| < \epsilon, \quad \text{if } |z - x_0| < \delta. \tag{8.5}$$

For any $\omega > 0$, we have

$$\begin{aligned} |u(x,t) - v(x_0)| \le 2M\pi^{-d/2} &\int_{|y|>\omega} \mathrm{e}^{-|y|^2} \, \mathrm{d}y \\ + \pi^{-d/2} &\int_{|y|<\omega} \big| v(x + \sqrt{4t}y) - v(x_0) \big| \, \mathrm{e}^{-|y|^2} \, \mathrm{d}y = I + II. \end{aligned}$$

We now fix ω so large that $I < \epsilon$, which is possible in view of (8.4). Then, with ω fixed, we obtain, using (8.5) and (8.4),

$$II \le \sup_{|y|<\omega} |v(x + \sqrt{4t}y) - v(x_0)| < \epsilon, \quad \text{if } |x - x_0| + \sqrt{4t}\omega < \delta.$$

Hence, for these x, t we have

$$|u(x,t) - v(x_0)| < 2\epsilon,$$

which completes the proof. □

Theorem 8.1 thus shows that the initial value problem (8.1) admits a solution, and is therefore an existence theorem. We shall show that this solution depends continuously on the initial data v.

We write (8.3) in the form

$$u(x,t) = (E(t)v)(x) = (4\pi t)^{-d/2} \int_{\mathbf{R}^d} v(y) \mathrm{e}^{-|x-y|^2/4t} \, \mathrm{d}y, \tag{8.6}$$

where we may think of $E(t)$ as defining a linear operator, the solution operator of (8.1), which takes the given initial data into the solution at time t.

Note that by (8.4)

$$|u(x,t)| \le (4\pi t)^{-d/2} \int_{\mathbf{R}^d} \mathrm{e}^{-|x-y|^2/4t} \, \mathrm{d}y \, \|v\|_{\mathcal{C}} = \|v\|_{\mathcal{C}},$$

so that

$$\|u(\cdot,t)\|_{\mathcal{C}} \le \|v\|_{\mathcal{C}}, \quad \text{for } t > 0.$$

This shows that the operator $E(t)$ is bounded with respect to the maximum-norm, with operator norm 1, which is the first part of the following result.

Theorem 8.2. *The solution operator* $E(t)$ *defined by* (8.6) *is bounded in* $\mathcal{C}$, *and*

$$\|E(t)v\|_{\mathcal{C}} \le \|v\|_{\mathcal{C}}, \quad \text{for } t \ge 0. \tag{8.7}$$

If v_1 *and* v_2 *are two bounded continuous functions on* $\mathbf{R}^d$ *and* u_1 *and* u_2 *are the corresponding solutions of the initial value problem* (8.1), *then*

$$\|u_1(t) - u_2(t)\|_{\mathcal{C}} \le \|v_1 - v_2\|_{\mathcal{C}}, \quad \text{for } t \ge 0. \tag{8.8}$$

Proof. It remains only to show the second part of the theorem. But, since $E(t)$ is a linear operator,

$$u_1(t) - u_2(t) = E(t)v_1 - E(t)v_2 = E(t)(v_1 - v_2),$$

and hence (8.8) follows at once from (8.7). □

Bu using a maximum principle we shall prove in Sect. 8.4 the corresponding uniqueness result, i.e., that there exists at most one bounded solution of (8.1) and thus (8.3) is the only one.

Together the existence, uniqueness, and continuous dependence properties make the problem (8.1) a *well posed problem.* In particular, the continuous dependence property is important in applications. It shows that a small change in the data of the problem has only a small effect on the solution.

Not all problems which admit solutions has this continuous dependence property. Consider for example the initial value problem

$$\begin{aligned} &u_t + u_{xx} = 0, && \text{in } \mathbf{R} \times \mathbf{R}_+, \\ &u(x,0) = v_n(x) = n^{-1}\sin(nx), && \text{for } x \in \mathbf{R}, \end{aligned} \tag{8.9}$$

which has the solution

$$u_n(x,t) = n^{-1}\mathrm{e}^{n^2 t}\sin(nx).$$

Here

$$\|v_n\|_{\mathcal{C}} = n^{-1} \to 0, \quad \text{as } n \to \infty,$$

whereas, for any $t > 0$,

$$\|u_n(t)\|_{\mathcal{C}} = n^{-1}\mathrm{e}^{n^2 t} \to \infty, \quad \text{as } n \to \infty.$$

Hence, although the initial value v_n is close to 0, the solution is not for $t > 0$.

The differential equation in (8.9) is the heat equation with the sign for the time derivative reversed, i.e., the *backward heat equation.* The result therefore means that the problem of determining an earlier distribution of heat in a body from the present one is *ill posed.*

We have already noted above that the representation of $u(x,t)$ in terms of v in (8.3) allows differentiation with respect to both x and t under the integral sign for $t > 0$, even without regularity assumptions on v. In fact, this differentiation can be carried out an arbitrary number of times so that u is infinitely differentiable, $u \in \mathcal{C}^\infty$, for $t > 0$. Using the multi-index notation from (1.8), one finds easily

$$\begin{aligned}|D_t^j D^\alpha U(x,t)| &\leq t^{-j-|\alpha|/2-d/2} P(|x|/\sqrt{4t}) \mathrm{e}^{-|x|^2/4t} \\ &\leq C t^{-j-|\alpha|/2-d/2} \mathrm{e}^{-|x|^2/8t},\end{aligned}$$

where $P(y)$ is a polynomial in y, and where we have used the fact that for any polynomial P there is a C such that

$$|P(y)\mathrm{e}^{-y^2}| \leq C\mathrm{e}^{-y^2/2}, \quad \text{for } y > 0.$$

Hence

$$\begin{aligned}\sup_{x\in\mathbf{R}^d} |D_t^j D^\alpha u(x,t)| &\leq C t^{-j-|\alpha|/2-d/2} \sup_{x\in\mathbf{R}^d} \int_{\mathbf{R}^d} |v(y)| \mathrm{e}^{-|x-y|^2/8t} \,\mathrm{d}y \\ &\leq C t^{-j-|\alpha|/2} \sup_{y\in\mathbf{R}^d} |v(y)|,\end{aligned}$$

or

$$\|D_t^j D^\alpha E(t) v\|_{\mathcal{C}} \leq C t^{-j-|\alpha|/2} \|v\|_{\mathcal{C}}, \quad \text{for } t > 0,$$

which shows that the operator $E(t)$ has a *smoothing property*: The solution of (8.1) is smooth for $t > 0$ even if v is nonsmooth. However, the bounds for the derivatives then grow as t tends to zero.

On the other hand, if the initial values are smooth then the derivatives of the solution are bounded uniformly down to $t = 0$: We have from (8.6), after the change of variables $z = x - y$,

$$\begin{aligned}\big(D^\alpha E(t)v\big)(x) = D_x^\alpha u(x,t) &= (4\pi t)^{-d/2} D_x^\alpha \int_{\mathbf{R}^d} v(x-z) \mathrm{e}^{-|z|^2/4t} \, dz \\ &= (4\pi t)^{-d/2} \int_{\mathbf{R}^d} D_x^\alpha v(x-z) \mathrm{e}^{-|z|^2/4t} \, dz = (E(t) D^\alpha v)(x),\end{aligned}$$

and hence, by (8.1) and (8.7),

$$\|D_t^j D^\alpha E(t) v\|_{\mathcal{C}} = \|\Delta^j D^\alpha E(t) v\|_{\mathcal{C}} = \|E(t) \Delta^j D^\alpha v\|_{\mathcal{C}} \leq \|\Delta^j D^\alpha v\|_{\mathcal{C}}.$$

It can be shown that the solution of the initial value problem for the inhomogeneous heat equation,

$$\begin{aligned} u_t - \Delta u &= f, \quad && \text{in } \mathbf{R}^d \times \mathbf{R}_+, \\ u(\cdot, 0) &= v, && \text{in } \mathbf{R}^d, \end{aligned}$$

where $f = f(x,t)$ is given, may be represented in the form

$$\begin{aligned} u(x,t) &= \int_{\mathbf{R}^d} v(y)U(x-y,t)\,\mathrm{d}y + \int_0^t \int_{\mathbf{R}^d} f(y,s)U(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= E(t)v + \int_0^t E(t-s)f(\cdot,s)\,\mathrm{d}s, \end{aligned}$$

provided, e.g., that v, f, and ∇f are continuous and bounded.

8.2 Solution of the Initial-Boundary Value Problem by Eigenfunction Expansion

We shall first consider the mixed initial-boundary value problem for the homogeneous heat equation: Find $u(x,t)$ such that

$$\begin{aligned} &u_t - \Delta u = 0 &&\text{in } \Omega \times \mathbf{R}_+, \\ &u = 0, &&\text{on } \Gamma \times \mathbf{R}_+, \\ &u(\cdot,0) = v &&\text{in } \Omega, \end{aligned} \tag{8.10}$$

where Ω is a bounded domain in $\mathbf{R}^d$ with smooth boundary Γ, $u_t = \partial u/\partial t$, and v is a given function in $L_2 = L_2(\Omega)$. We shall now solve this problem by using eigenfunction expansions. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm in $L_2 = L_2(\Omega)$, respectively.

We recall from Chapt. 6 that there exists an orthonormal basis $\{\varphi_i\}_{i=1}^\infty$ in L_2 of smooth eigenfunctions φ_i and corresponding eigenvalues $\{\lambda_i\}_{i=1}^\infty$ satisfying

$$-\Delta\varphi_i = \lambda_i\varphi_i \quad \text{in } \Omega, \quad \text{with } \varphi_i = 0 \quad \text{on } \Gamma, \tag{8.11}$$

or, equivalently, with our usual notation

$$a(\varphi_i, v) = \int_\Omega \nabla\varphi_i \cdot \nabla v\,\mathrm{d}x = \lambda_i(\varphi_i, v), \quad \forall v \in H_0^1,$$

Recall that $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_i \le \dots$, that $\lambda_i \to \infty$ as $i \to \infty$, and that (with Kronecker's symbol $\delta_{ij} = 1$ for $j = i$ and 0 otherwise)

$$a(\varphi_i, \varphi_j) = \lambda_i \delta_{ij}.$$

We now seek a solution to (8.10) of the form

$$u(x,t) = \sum_{i=1}^\infty \hat{u}_i(t)\varphi_i(x), \tag{8.12}$$

where the $\hat{u}_i : \mathbf{R}_+ \to \mathbf{R}$ are coefficients to be determined. Because this is a sum of products of functions of x and t this approach is also called the method of *separation of variables.* Inserting (8.12) into the differential equation in (8.10) and using (8.11) we obtain formally

$$\sum_{i=1}^{\infty} \big(\hat{u}_i'(t) + \lambda_i \hat{u}_i(t)\big)\varphi_i(x) = 0, \quad \text{for } x \in \Omega,\ t \in \mathbf{R}_+,$$

and hence, since the φ_i form a basis,

$$\hat{u}_i'(t) + \lambda_i \hat{u}_i(t) = 0, \quad \text{for } t \in \mathbf{R}_+, \quad i = 1, 2, \dots,$$

so that

$$\hat{u}_i(t) = \hat{u}_i(0)\mathrm{e}^{-\lambda_i t}.$$

Moreover, from the initial condition in (8.10) it follows that

$$u(\cdot, 0) = \sum_{i=1}^{\infty} \hat{u}_i(0)\varphi_i = v = \sum_{i=1}^{\infty} \hat{v}_i \varphi_i, \quad \text{where } \hat{v}_i = (v, \varphi_i) = \int_\Omega v\, \varphi_i \, \mathrm{d}x.$$

We thus see that, at least formally, the solution of (8.10) has to be

$$u(x,t) = \sum_{i=1}^{\infty} \hat{v}_i \mathrm{e}^{-\lambda_i t} \varphi_i(x), \tag{8.13}$$

where by Parseval's relation, with $\|\cdot\| = \|\cdot\|_{L_2}$,

$$\|u(\cdot,t)\|^2 = \sum_{i=1}^{\infty} \big(\hat{v}_j^k \mathrm{e}^{-\lambda_j t}\big)^2 \le \mathrm{e}^{-2\lambda_1 t} \sum_{i=1}^{\infty} \hat{v}_i^2 = \mathrm{e}^{-2\lambda_1 t}\|v\|^2 < \infty,$$

Thus $u(\cdot,t) \in L_2$ for $t \ge 0$, and its L_2-norm decreases exponentially as $t \to \infty$. Although this decay is important in some situations, for simplicity we shall refrain from keeping track of the behavior of $u(\cdot,t)$ for large t in the sequel and content ourselves with the conclusion that

$$\|u(\cdot,t)\| \le \|v\|, \quad \text{for } t \in \mathbf{R}_+.$$

We now show that for $t > 0$ the function $u(\cdot,t)$ defined in (8.13) is smooth and satisfies the differential equation and the boundary condition in (8.10) in the classical sense, and that the initial condition holds in the sense that

$$\|u(\cdot,t) - v\| \to 0, \quad \text{as } t \to 0. \tag{8.14}$$

We first note that for any $k \ge 0$ there is a constant C_k such that $s^k \mathrm{e}^{-s} \le C_k$ for $s \ge 0$, Using this with $k = 1$ we have

$$|u(\cdot,t)|_1^2 = \sum_{i=1}^{\infty} \lambda_i \big(\hat{v}_i \mathrm{e}^{-\lambda_i t}\big)^2 = t^{-1} \sum_{i=1}^{\infty} \hat{v}_i^2 (\lambda_i t)\ \mathrm{e}^{-2\lambda_i t} \le C_1 t^{-1}\|v\|^2,$$

so that

$$|u(\cdot,t)|_1 \le C t^{-1/2}\|v\|, \quad \text{for } t > 0. \tag{8.15}$$

Thus $u(\cdot,t) \in H_0^1$ for $t > 0$, by Theorem 6.4, and, in particular, $u(\cdot,t)$ satisfies the boundary condition in (8.10). Now, applying $(-\Delta)^k$ to each term in (8.13), we obtain since $-\Delta\varphi_i = \lambda_i\varphi_i$

$$(8.16)\qquad (-\Delta)^k u(x,t) = \sum_{i=1}^{\infty} \hat{v}_i \lambda_i^k \mathrm{e}^{-\lambda_i t}\varphi_i(x),$$

and hence, for $t > 0$,

$$\|\Delta^k u(\cdot,t)\|^2 = \sum_{i=1}^{\infty} \big(\hat{v}_i \lambda_i^k \mathrm{e}^{-\lambda_i t}\big)^2 \le C_k^2 t^{-2k} \sum_{i=1}^{\infty} \hat{v}_i^2 = C_k^2 t^{-2k}\|v\|^2 < \infty.$$

In the same way as in (8.15), we also have

$$|\Delta^k u(\cdot,t)|_1 \le C_k t^{-k-1/2}\|v\| < \infty, \quad \text{for } t > 0.$$

and thus $\Delta^k u(\cdot,t) = 0$ on Γ for any $k \ge 0$ when $t > 0$. We may also apply D_t^m to each term in (8.16), and since $D_t \mathrm{e}^{-\lambda_i t} = -\lambda_i \mathrm{e}^{-\lambda_i t}$, we obtain

$$|D_t^m \Delta^k u(\cdot,t)|_\delta \le C t^{-m-k-\delta/2}\|v\| < \infty, \quad \text{for } t > 0, \quad \delta = 0, 1.$$

Recall from the theory of elliptic equations the regularity estimate (3.37),

$$\|w\|_s \le C\|\Delta w\|_{s-2}, \quad \forall w \in H^s \cap H_0^1, \quad \text{for } s \ge 2.$$

By repeated application of this we obtain, again for $\delta = 0$ or 1,

$$\|w\|_{2k+\delta} \le C\|\Delta^k w\|_\delta, \quad \forall w \in H^{2k+\delta}, \quad \text{if } \Delta^j w = 0 \text{ on } \Gamma \text{ for } j < k,$$

and we finally conclude that, for any nonnegative integers s and m,

$$(8.17)\qquad \|D_t^m u(\cdot,t)\|_s \le C t^{-m-s/2}\|v\|, \quad \text{for } t > 0.$$

It follows by Sobolev's inequality, Theorem A.5, that $D_t^m u(\cdot,t) \in \mathcal{C}^p$ for $t > 0$, for any $p \ge 0$.

Thus $u(x,t)$ is a smooth function of x and t for $t > 0$ even though we have not assumed the initial data v to be smooth, and $u(\cdot,t)$ therefore satisfies the heat equation in the classical sense. By above we also know that the boundary condition is satisfied, and finally we obtain (8.14) by showing that

$$\|u(\cdot,t) - v\|^2 = \sum_{i=1}^{\infty} \big(\mathrm{e}^{-\lambda_i t} - 1\big)^2 \hat{v}_i^2 \to 0, \quad \text{as } t \to 0.$$

To prove this we let $\epsilon > 0$ be arbitrarily small and choose N large enough that $\sum_{i=N+1}^{\infty} \hat{v}_i^2 < \epsilon$. Then

$$\|u(\cdot,t) - v\|^2 \le \sum_{i=1}^{N} \big(\mathrm{e}^{-\lambda_i t} - 1\big)^2 \hat{v}_i^2 + \epsilon.$$

Since each of the terms of the sum tends to zero as $t \to 0$, we conclude that

$$\|u(\cdot,t) - v\|^2 < 2\epsilon, \quad \text{for } t \text{ small enough.}$$

We collect these results in the following theorem.

Theorem 8.3. *For any $v \in L_2$ the function $u(x,t)$ defined by* (8.13) *is a classical solution of the heat equation in* (8.10)*, vanishes on Γ for $t > 0$, and satisfies the initial condition in the sense of* (8.14)*. Moreover, the smoothness estimate* (8.17) *holds.*

Since the factor t^{-k} on the right in (8.17) tends to infinity as t tends to zero, the smoothness of the solution is not guaranteed uniformly down to $t = 0$. If the initial function is smoother, then better results are possible in this regard. We have, for instance, the following result in H_0^1.

Theorem 8.4. *Assume that $v \in H_0^1$. Then the solution $u(x,t)$ of* (8.10) *determined in Theorem 8.3 satisfies*

$$|u(\cdot,t)|_1 \le |v|_1 \quad \textit{for } t \ge 0.$$

Proof. We have by (6.4)

$$|u(\cdot,t)|_1^2 = \sum_{i=1}^{\infty} \lambda_i \hat{v}_i^2 \mathrm{e}^{-2\lambda_i t} \le \sum_{i=1}^{\infty} \lambda_i \hat{v}_i^2 = |v|_1^2,$$

which shows our claim. □

We note that this result requires not only that the initial data are in H^1 but also that they vanish on Γ. This means that the initial data have to be compatible with the boundary data on $\Gamma \times \mathbf{R}_+$, which is obviously required for the solution to be continuous at $t = 0$. For higher order regularity further compatibility conditions are needed.

In the same way as in Sect. 8.1 we may think of the solution at time t as the result of a solution operator $E(t)$ acting on the initial data v, and thus write $u(t) = E(t)v$. By (8.13) this operator satisfies the stability estimate

$$\|E(t)v\| \le \|v\|, \quad \text{for } t > 0,$$

and the estimate (8.17) may be expressed as

$$\|D_t^m E(t)v(\cdot,t)\|_s \le Ct^{-m-s/2}\|v\|, \quad \text{for } t > 0, \quad m, s \ge 0, \tag{8.18}$$

which expresses a smoothing property of the solution operator.

The following simple example illustrates the above solution method.

Example 8.1. The solution of the spatially one-dimensional problem

$$\begin{aligned} & u_t - u_{xx} = 0, && \text{in } \Omega \times \mathbf{R}_+, \\ & u(0,\cdot) = u(\pi,\cdot) = 0, && \text{in } \mathbf{R}_+, \\ & u(\cdot,0) = v, && \text{in } \Omega, \end{aligned} \tag{8.19}$$

with $\Omega = (0,\pi)$ and $v \in L_2(\Omega)$, is given by

$$u(x,t) = \sum_{j=1}^{\infty} \hat{v}_j \mathrm{e}^{-j^2 t} \sin(jx), \quad \text{where } \hat{v}_j = \frac{2}{\pi} \int_0^{\pi} v(x) \sin(jx) \, \mathrm{d}x. \tag{8.20}$$

In this case the associated eigenvalue problem (8.11) reduces to (6.21) with $b = \pi$, and the result thus follows from Theorem 8.3 and the results obtained in Sect. 6.1, except that the eigenfunctions are not normalized here. Note that in (8.20) the coefficient $\hat{v}_j \mathrm{e}^{-j^2 t}$ of the eigenfunction $\sin(jx)$ is obtained by multiplying the corresponding coefficient $\hat{v}_j$ in the expansion of the initial function v by the factor $\mathrm{e}^{-j^2 t}$. If j is large, then $\sin(jx)$ is rapidly oscillating and the factor $\mathrm{e}^{-j^2 t}$ rapidly becomes very small as t increases from 0. Thus, the components of the solution $u(x,t)$ corresponding to the eigenfunctions $\sin(jx)$ with j large are strongly damped as t grows. This means that rapid variations or oscillations in the initial function v, such as, for instance, in the case of a discontinuity (jump), are smoothed out as t increases. This is thus a special case of the smoothing property of the solution operator discussed above, which is typical for parabolic problems.

The solution operator $E(t)$ introduced above is convenient to use in the study of the boundary value problem for the inhomogeneous equation,

$$\begin{aligned} & u_t - \Delta u = f, && \text{in } \Omega \times \mathbf{R}_+, \\ & u = 0, && \text{on } \Gamma \times \mathbf{R}_+, \\ & u(\cdot,0) = v, && \text{in } \Omega, \end{aligned} \tag{8.21}$$

In fact, as we shall see, the solution of this problem may be expressed as

$$u(t) = E(t)v + \int_0^t E(t-s) f(s) \, \mathrm{d}s, \tag{8.22}$$

This formula represents the solution of the inhomogeneous equation as a superposition of solutions of homogeneous equations, and is referred to as Duhamel's principle.

Clearly, since $E(t)$ is bounded in L_2-norm, the right hand side of (8.22) is well defined. The first term is the solution of (8.1) so that, since the second term vanishes for $t = 0$, in order to show that u is a solution of (8.21) we need to demonstrate that

$$D_tF(t) - \Delta F(t) = f(t), \quad \text{where } F(t) = \int_0^t E(t-s)f(s)\,\mathrm{d}s. \tag{8.23}$$

Formally, by differentiation of the integral, we have

$$D_tF(t) - \Delta F(t) = f(t) + \int_0^t D_tE(t-s)f(s)\,\mathrm{d}s - \int_0^t \Delta E(t-s)f(s)\,\mathrm{d}s, \tag{8.24}$$

and since $D_te(t-s) = \Delta E(t-s)$ the integrals should cancel. However, requiring only $f(s) \in L_2$ for $s \in (0,t)$, (8.18) indicates a singularity of order $O((t-s)^{-1})$ in the integrands, so that the integrals are not necessarily well defined. For this reason we now assume that $\|D_tf(t)\|$ is bounded for $t \in [0,T]$ with arbitrary $T > 0$, and write, after replacing $t-s$ by s in the last term,

$$F(t) = \int_0^t E(t-s)(f(s) - f(t))\,\mathrm{d}s + \int_0^t E(s)f(t)\,\mathrm{d}s.$$

By differentiation with respect to t we obtain

$$D_tF(t) = \int_0^t D_tE(t-s)(f(s) - f(t))\,\mathrm{d}s + E(t)f(t), \tag{8.25}$$

where the integrand is now bounded since $\|f(s) - f(t)\| \le C|s-t|$. Similarly, since $\Delta E(t-s) = D_tE(t-s)$,

$$\Delta F(t) = \int_0^t \Delta E(t-s)(f(s) - f(t))\,\mathrm{d}s + (E(t) - I)f(t). \tag{8.26}$$

Taking the difference between (8.25) and (8.26) shows (8.23).

Another way to deal with the singularities in the integrands in (8.24) would be to use regularity of $f(s)$ in the spatial variable, e.g., through the inequality $\|\Delta E(t-s)f(s)\| \le \|\Delta f(s)\|$. However, in addition to regularity of $f(s)$ this would require the unnatural boundary condition $f(s) = 0$ on Γ.

By (8.22) we obtain at once the stability estimate

$$\|u(t)\| = \|v\| + \int_0^t \|f(s)\|\,\mathrm{d}s. \tag{8.27}$$

In the standard way this may be used to show uniqueness of the solution of (8.21) as well as the continuous dependence of the solution on the data. For example, if u_1 and u_2 are solutions corresponding to the right-hand sides f_1 and f_2 and initial values v_1 and v_2, then we have

$$\|u_1(t) - u_2(t)\| \le \|v_1 - v_2\| + \int_0^t \|f_1(s) - f_2(s)\|\,\mathrm{d}s, \quad \text{for } t \in \mathbf{R}_+. \tag{8.28}$$

8.3 Variational Formulation. Energy Estimates

We shall now write the initial-boundary value problem (8.21) in variational, or weak, form, and use this to derive some estimates for its solution. Although we shall not pursue this here, variational methods may be used to prove existence and uniqueness of solutions of parabolic problems, which are considerably more general than (8.21), such as problems with time-dependent coefficients or non-selfadjoint elliptic operator, problems with inhomogeneous boundary conditions, and also some nonlinear problems. For such problems the method of eigenfunction expansion of the previous section is difficult or impossible to use. Moreover, the variational formulation is the basis for the finite element method for parabolic problems, which we shall study in Chapt. 10.

For the variational formulation we multiply the heat equation in (8.21) by a smooth function $\varphi = \varphi(x)$, which vanishes on Γ and find, after integration over Ω and using Green's formula, that

$$(u_t, \varphi) + a(u, \varphi) = (f, \varphi), \quad \forall \varphi \in H_0^1, \quad t \in \mathbf{R}_+, \tag{8.29}$$

with our standard notation

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w \,\mathrm{d}x, \quad (v, w) = \int_\Omega v w \,\mathrm{d}x.$$

The variational problem may then be formulated: Find $u = u(x, t) \in H_0^1$, thus vanishing on Γ, for $t > 0$, such that (8.29) holds and such that

$$u(\cdot, 0) = v \quad \text{in } \Omega. \tag{8.30}$$

By taking the above steps in the opposite order it is easy to see that if u is a sufficiently smooth solution of this problem, it is also a solution of (8.21). In fact, by integration by parts in (8.29) we obtain

$$(u_t - \Delta u - f, \varphi) = 0, \quad \forall \varphi \in H_0^1, \quad t \in \mathbf{R}_+,$$

or, for any $t \in \mathbf{R}_+$,

$$\int_\Omega \rho(\cdot, t)\, \varphi \,\mathrm{d}x = 0, \quad \forall \varphi \in H_0^1, \quad \text{where } \rho = u_t - \Delta u - f.$$

We conclude, in the same way as for the stationary problem, that this is possible only if $\rho = 0$.

The following result shows some bounds in various natural norms for the solution of our above problem in terms of its data. We proceed formally and refrain from precise statements about the regularity requirements needed. We write $u(t)$ for $u(\cdot, t)$ and similarly for $f(t)$.

Theorem 8.5. *Let $u(t)$ satisfy* (8.29) *and* (8.30)*, vanish on Γ, and be appropriately smooth for $t \geq 0$. Then there is a constant C such that, for $t \geq 0$,*

$$\|u(t)\|^2 + \int_0^t |u(s)|_1^2 \, \mathrm{d}s \le \|v\|^2 + C \int_0^t \|f(s)\|^2 \, \mathrm{d}s \tag{8.31}$$

and

$$|u(t)|_1^2 + \int_0^t \|u_t(s)\|^2 \, \mathrm{d}s \le |v|_1^2 + \int_0^t \|f(s)\|^2 \, \mathrm{d}s. \tag{8.32}$$

Proof. Taking $\varphi = u$ in (8.29) we obtain

$$(u_t, u) + a(u, u) = (f, u), \quad \text{for } t > 0. \tag{8.33}$$

Here

$$(u_t, u) = \int_\Omega u_t u \, \mathrm{d}x = \int_\Omega \tfrac{1}{2}(u^2)_t \, \mathrm{d}x = \tfrac{1}{2} \frac{d}{dt} \|u\|^2.$$

Applying Poincaré's inequality, Theorem A.6, i.e.,

$$\|\varphi\| \le C|\varphi|_1 = C \; a(\varphi, \varphi)^{1/2}, \quad \text{for } \varphi \in H_0^1,$$

we have, using also the inequality $2ab \le a^2 + b^2$, that

$$|(f, u)| \le \|f\| \, \|u\| \le C\|f\| \, |u|_1 \le \tfrac{1}{2}|u|_1^2 + \tfrac{1}{2}C^2\|f\|^2.$$

We thus obtain from (8.33) that

$$\tfrac{1}{2} \frac{d}{dt}\|u\|^2 + |u|_1^2 \le \tfrac{1}{2}|u|_1^2 + \tfrac{1}{2}C^2\|f\|^2,$$

or, with a new C,

$$\frac{d}{dt}\|u\|^2 + |u|_1^2 \le C\|f\|^2.$$

By integration over $(0, t)$ this yields

$$\|u(t)\|^2 + \int_0^t |u(s)|_1^2 \, \mathrm{d}s \le \|v\|^2 + C \int_0^t \|f\|^2 \, \mathrm{d}s,$$

which is (8.31).

To prove (8.32) we now choose $\varphi = u_t$ in (8.29) and obtain

$$\|u_t\|^2 + a(u, u_t) = (f, u_t) \le \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|u_t\|^2.$$

Here

$$a(u, u_t) = \int_\Omega \nabla u \cdot \nabla u_t \, \mathrm{d}x = \int_\Omega \tfrac{1}{2}(|\nabla u|^2)_t \, \mathrm{d}x = \tfrac{1}{2} \frac{d}{dt}|u|_1^2,$$

so that we may conclude,

$$\|u_t\|^2 + \frac{d}{dt}|u|_1^2 \le \|f\|^2,$$

whence, by integration over $(0, t)$,

$$|u(t)|_1^2 + \int_0^t \|u_t\|^2 \, \mathrm{d}s \le |v|_1^2 + \int_0^t \|f\|^2 \, \mathrm{d}s,$$

which is (8.32). □

It follows in the standard way from (8.31) that if u_1 and u_2 are solutions corresponding to the right-hand sides f_1 and f_2 and initial values v_1 and v_2, then we have

$$\|u_1(t)-u_2(t)\|^2+\int_0^t |u_1-u_2|_1^2\,\mathrm{d}s \leq \|v_1-v_2\|^2+C\int_0^t \|f_1-f_2\|^2\,\mathrm{d}s, \quad \text{for } t \geq 0,$$

and a similar bound is obtained from (8.32). Note that these estimates also bound the error in H_0^1 and uses the L_2-norm in time rather than the L_1-norm employed in (8.28).

8.4 A Maximum Principle

We now consider the generalization of the mixed initial-boundary value problem of Sect. 8.2 which allows a source term and inhomogeneous boundary conditions, i.e., to find u on $\bar{\Omega} \times \bar{I}$ such that

$$(8.34) \qquad \begin{aligned} &u_t - \Delta u = f, && \text{in } \Omega \times I,\\ &u = g, && \text{on } \Gamma \times I,\\ &u(\cdot,0) = v, && \text{in } \Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbf{R}^d$ and $I = (0,T)$ is a finite interval in time. In order to show a maximum principle for this problem it is convenient to introduce the *parabolic boundary* of $\Omega \times I$ as the set $\Gamma_p = (\Gamma \times \bar{I}) \cup (\Omega \times \{t = 0\})$, i.e., the boundary of $\Omega \times I$ minus the interior of the top part of this boundary, $\Omega \times \{t = T\}$.

Theorem 8.6. *Let u be smooth and assume that $u_t - \Delta u \leq 0$ in $\Omega \times I$. Then u attains its maximum on the parabolic boundary Γ_p.*

Proof. If this were not true, then the maximum would be attained either at an interior point of $\Omega \times I$ or at a point of $\Omega \times \{t = T\}$, i.e., at a point $(\bar{x}, \bar{t}) \in \Omega \times (0, T]$, and we would have

$$u(\bar{x},\bar{t}) = \max_{\bar{\Omega}\times\bar{I}} u = M > m = \max_{\Gamma_p} u.$$

In such a case, for $\epsilon > 0$ sufficiently small, the function

$$w(x,t) = u(x,t) + \epsilon|x|^2$$

would also take its maximum at a point in $\Omega \times (0,T]$, since, for ϵ small,

$$\max_{\Gamma_p} w \leq m + \epsilon \max_{\Gamma_p} |x|^2 < M \leq \max_{\bar{\Omega}\times\bar{I}} w.$$

By our assumption we have since $\Delta(|x|^2) = 2d$ that

$$w_t - \Delta w = u_t - \Delta u - 2d\epsilon < 0, \quad \text{in } \Omega \times I. \tag{8.35}$$

On the other hand, at the point $(\tilde{x}, \tilde{t})$, where w takes its maximum, we have

$$-\Delta w(\tilde{x}, \tilde{t}) = -\sum_{i=1}^{d} w_{x_i x_i}(\bar{x}, \bar{t}) \geq 0,$$

and

$$w_t(\tilde{x}, \tilde{t}) = 0, \quad \text{if } \tilde{t} < T, \quad \text{or} \quad w_t(\tilde{x}, \tilde{t}) \geq 0, \quad \text{if } \tilde{t} = T,$$

so that in both cases

$$w_t(\tilde{x}, \tilde{t}) - \Delta w(\tilde{x}, \tilde{t}) \geq 0.$$

This is a contradiction to (8.35) and thus shows our claim. □

By considering the functions $\pm u$, it follows, in particular, that a solution of the homogeneous heat equation ($f = 0$) attains both its maximum and its minimum on Γ_p, so that in this case, with $\|w\|_{\mathcal{C}(\bar{M})} = \max_{x \in \bar{M}} |w(x)|$,

$$\|u\|_{\mathcal{C}(\bar{\Omega} \times \bar{I})} \leq \max\left\{\|g\|_{\mathcal{C}(\Gamma \times \bar{I})}, \|v\|_{\mathcal{C}(\bar{\Omega})}\right\}.$$

For the inhomogeneous equation one may show the following inequality, the proof of which we leave as an exercise, see Problem 8.7.

Theorem 8.7. *The solution of* (8.34) *satisfies*

$$\|u\|_{\mathcal{C}(\bar{\Omega} \times \bar{I})} \leq \max\left\{\|g\|_{\mathcal{C}(\Gamma \times \bar{I})}, \|v\|_{\mathcal{C}(\bar{\Omega})}\right\} + \frac{r^2}{2d}\|f\|_{\mathcal{C}(\bar{\Omega} \times \bar{I})},$$

where r is the radius of a ball containing Ω.

As usual such a result shows uniqueness and stability for the initial-boundary value problem.

We close this section by proving the uniqueness of a bounded solution to the pure initial value problem considered in Sect. 8.1.

Theorem 8.8. *The initial value problem* (8.1) *has at most one solution which is bounded in $\mathbf{R}^d \times [0, T]$, where T is arbitrary.*

Proof. If there were two solutions of (8.1), then their difference would be a solution with initial data zero. It suffices therefore to show that the only bounded solution u of

$$\begin{aligned} u_t &= \Delta u, && \text{in } \mathbf{R}^d \times I, \quad \text{where } I = (0, T), \\ u(\cdot, 0) &= 0, && \text{in } \mathbf{R}^d, \end{aligned}$$

is $u = 0$, or that and if (x_0, t_0) is an arbitrary point in $\mathbf{R}^d \times I$, and $\epsilon > 0$ is arbitrary, then $|u(x_0, t_0)| \leq \epsilon$. We introduce the auxiliary function

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