Subgrid Phenomena and Numerical Schemes

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Abstract. In recent times, several attempts have been made to recover some information from the subgrid scales and transfer them to the computational scales. Many stabilizing techniques can also be considered as part of this effort. We discuss here a framework in which some of these attempts can be set and analyzed.

1 Introduction

In the numerical simulation of a certain number of problems, there are physical effects that take place on a scale which is much smaller than the smallest one representable on the computational grid, but have a strong impact on the larger scales, and, therefore, cannot be neglected without jeopardizing the overall quality of the final solution.

In other cases, the discrete scheme lacks the necessary stability properties because it does not treat in a proper way the smallest scales allowed by the computational grid. As a consequence, some "smallest scale mode" appears as abnormally amplified in the final numerical results. Most types of numerical instabilities are produced in this way, such as the checkerboard pressure mode for nearly incompressible materials, or the fine-grid spurious oscillations in convection-dominated flows. See for instance [19] and the references therein for a classical overview of several types of these and other instabilities of this nature.

In the last decade it has become clear that several attempts to recover stability, in these cases, could be interpreted as a way of improving the simulation of the effects of the smallest scales on the larger ones. By doing that, the small scales can be *seen* by the numerical scheme and therefore be kept under control.

These two situations are quite different, in nature and scale. Nevertheless it is not unreasonable to hope that some techniques that have been developed for dealing with the latter class of phenomena might be adapted to deal with the former one. In this sense, one of the most promising technique seems to be the use of Residual-Free Bubbles (see e.g. [10], [18].) In the following sections, we are going to summarize the general idea behind it, trying to underline its potential and its limitations. In Section 2 we present the continuous problems in an abstract setting, and provide examples of applications, related to advection dominated flows, composite materials, and viscous incompressible flows. For application of these concepts to other problems we refer, for instance, to [13], [14], [16], [18], [24]. In Section 3 we introduce the basic features of the RFB method. Starting from a given discretization (that might possibly be unstable), we discuss the suitable *bubble space* that can be added to the original finite element space. Increasing the space with bubbles leads to the *augmented problem*, usually infinite dimensional, which, in the end, will have to be solved in some suitable approximate way. In Section 4 we give an idea of how error estimates can be deduced for the augmented problem. In Section 5 we discuss the related computational aspects, and we present several strategies that can be used to deal with the augmented problem, in order to minimize the computational cost. We shall see in particular that several other methods that are known in the literature can actually be seen as variants of the RFB procedure, in which one or another of the above strategies is employed. This includes, for advection dominated problems, the classical SUPG methods (as it was already well known, see, e.g., [4]) as well as the older Petrov-Galerkin methods based on suitable operator dependent choices of test and trial functions [25]. For composite materials, this includes both the multiscale methods of [22], [23], and the upscaling methods of [1], [2]. Finally, in Section 6 we draw some conclusions.

2 The Continuous Problem

We consider the following continuous problem

$$\begin{cases} \text{find } u \in V \text{ such that} \\ \mathcal{L}(u,v) = \langle f, v \rangle \qquad \forall\, v \in V, \end{cases} \tag{2.1}$$

where V is a Hilbert space, and V' its dual space, $\mathcal{L}(u,v)$ is a continuous bilinear form on $V \times V$, and $f \in V'$ is the forcing term. We assume that, for all $f \in V'$, problem (2.1) has a unique solution. Various problems arising from physical applications can be written in the variational form (2.1), according to different choices of the space V and the bilinear form $\mathcal{L}$. Typical choices for V, when V is a space of scalar functions, are the following: if $\mathcal{O} \subset \mathbb{R}^d$, $(d = 1,2,3)$ denotes a generic domain, V could be, for instance, $L^2(\mathcal{O})$, $H^1(\mathcal{O})$, $H^1_0(\mathcal{O})$, $H^2(\mathcal{O})$ or $L^2_0(\mathcal{O})$, the last one being the space of L^2-functions having zero mean value. In the case where V is a space of vector valued functions, a first choice could be to take the Cartesian product of the previous scalar spaces. Other typical choices for V can be:

$$H(\mathrm{div};\mathcal{O}) := \{\boldsymbol{\tau} \in (L^2(\mathcal{O}))^d \text{ such that } \nabla \cdot \boldsymbol{\tau} \in L^2(\mathcal{O})\},$$
$$H_0(\mathrm{div};\mathcal{O}) := \{\boldsymbol{\tau} \in H(\mathrm{div};\mathcal{O}) \text{ such that } \boldsymbol{\tau} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O}\},$$

or also, for a generic domain $\mathcal{O} \subset \mathbb{R}^3$,

$$H(\mathbf{curl};\mathcal{O}) := \{\boldsymbol{\tau} \in (L^2(\mathcal{O}))^3 \text{ such that } \nabla \wedge \boldsymbol{\tau} \in (L^2(\mathcal{O}))^3\}$$
$$H_0(\mathbf{curl};\mathcal{O}) := \{\boldsymbol{\tau} \in H(\mathbf{curl};\mathcal{O}) \text{ such that } \boldsymbol{\tau} \wedge \mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}.$$

Product spaces are also frequently used: for instance, $H(\mathrm{div};\mathcal{O}) \times L^2(\mathcal{O})$, or $(H_0^1(\mathcal{O}))^d \times L_0^2(\mathcal{O})$, etc. Next, we provide some classical examples of problems and we indicate the corresponding space V, the bilinear form $\mathcal{L}$, and the variational formulation.

Example 2.1. Advection-dominated scalar equations:

$$-\varepsilon\Delta u + \mathbf{c}\cdot\nabla u = f \quad \text{in } \Omega; \quad u = 0 \ \text{ on } \partial\Omega$$
$$V := H_0^1(\Omega); \quad \mathcal{L}(u,v) := \int_\Omega \varepsilon\nabla u\cdot\nabla v\,dx + \int_\Omega \mathbf{c}\cdot\nabla u\, v\,dx; \quad \langle f,v\rangle := \int_\Omega fv\,dx$$
$$\mathcal{L}(u,v) = \langle f,v\rangle \qquad \forall\, v\in V.$$

Example 2.2. Linear elliptic problems with composite materials:

$$-\nabla\cdot(\alpha(x)\nabla u) = f \quad \text{in } \Omega; \quad u = 0 \ \text{ on } \partial\Omega$$
$$V := H_0^1(\Omega); \quad \mathcal{L}(u,v) := \int_\Omega \alpha(x)\nabla u\cdot\nabla v\,dx; \qquad \langle f,v\rangle := \int_\Omega fv\,dx$$
$$\mathcal{L}(u,v) = \langle f,v\rangle \qquad \forall\, v\in V$$

(where $\alpha(x) \geq \alpha_0 > 0$ might have a very fine structure).

Example 2.3. Composite materials in mixed form, i.e., the same problem of the previous example, but now with:

$$\boldsymbol{\sigma} = -\alpha\nabla\psi \quad \text{in } \Omega; \qquad \nabla\cdot\boldsymbol{\sigma} = f \quad \text{in } \Omega; \qquad \psi = 0 \ \text{ on } \partial\Omega$$
$$V := \boldsymbol{\Sigma}\times\Phi; \qquad \boldsymbol{\Sigma} := H(\mathrm{div};\Omega); \qquad \Phi := L^2(\Omega)$$
$$a_0(\boldsymbol{\sigma},\boldsymbol{\tau}) := \int_\Omega \alpha^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx, \quad b(\boldsymbol{\tau},\varphi) := \int_\Omega \nabla\cdot\boldsymbol{\tau}\,\varphi\,dx$$
$$\mathcal{L}((\boldsymbol{\sigma},\psi),(\boldsymbol{\tau},\varphi)) := a_0(\boldsymbol{\sigma},\boldsymbol{\tau}) - b(\boldsymbol{\tau},\psi) + b(\boldsymbol{\sigma},\varphi); \quad \langle f,(\boldsymbol{\tau},\varphi)\rangle := \int_\Omega f\varphi\,dx$$
$$\mathcal{L}((\boldsymbol{\sigma},\psi),(\boldsymbol{\tau},\varphi)) = \langle f,(\boldsymbol{\tau},\varphi)\rangle \qquad \forall\,(\boldsymbol{\tau},\varphi)\in V.$$

Example 2.4. Stokes problem for viscous incompressible fluids:

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega; \qquad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega; \qquad \mathbf{u} = 0 \ \text{ on } \partial\Omega$$
$$V := \mathbf{U}\times Q; \quad \mathbf{U} := (H_0^1(\Omega))^d; \quad Q := L_0^2(\Omega)$$
$$a_1(\mathbf{u},\mathbf{v}) := \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,dx \quad b(\mathbf{v},q) := \int_\Omega \nabla\cdot\mathbf{v}\,q\,dx$$
$$\mathcal{L}((\mathbf{u},p),(\mathbf{v},q)) := a_1(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) + b(\mathbf{u},q); \qquad \langle f,(\mathbf{v},q)\rangle := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx$$
$$\mathcal{L}((\mathbf{u},p),(\mathbf{v},q)) = \langle f,(\mathbf{v},q)\rangle \qquad \forall\,(\mathbf{v},q)\in V.$$

3 From the Discrete Problem to the Augmented Problem

Let $\mathcal{T}_h$ be a decomposition of the computational domain Ω, with the usual nondegeneracy conditions [12], and let $V_h \subset V$ be a finite element space. The original discrete problem is then:

$$\begin{cases} \text{find } u_h\in V_h \text{ such that} \\ \mathcal{L}(u_h,v_h) = \langle f,v_h\rangle \qquad \forall\, v_h\in V_h. \end{cases} \tag{3.1}$$

Note that we do not assume that (3.1) has a unique solution. Indeed, the stabilization that we are going to introduce can, in some cases, take care of problems originally ill-posed. Our aim is, essentially, to solve eventually a final linear system having as many equations as the number of degrees of freedom of V_h. Apart from that, we are ready to pay some extra price, in order to have a better method. In some cases, the total amount of additional work will be small. In other cases, it can be huge. However, we want to be able to perform the extra work independently in each element so that we can do it, as a pre-processor, *in parallel*. This implies that we are ready to add as many degrees of freedom as we want at the interior of each element. For that, to V and $\mathcal{T}_h$ we associate the *maximal space of bubbles*

$$B(V;\mathcal{T}_h) = \prod_K B_V(K), \qquad \text{with } B_V(K) = \{v \in V : \ \operatorname{supp}(v) \subseteq \overline{K}\}.$$

Let us give some examples of the dependence of $B_V(K)$ on V.

- if $V = H^1_0(\Omega)$ then $B_V(K) = H^1_0(K)$
- if $V = H^1(\Omega)$ then $B_V(K) = \{v \in H^1(K),\, v = 0 \textit{ on } \partial K \cap \Omega\}$
- if $V = L^2(\Omega)$ then $B_V(K) = L^2(K)$
- if $V = L^2_0(\Omega)$ then $B_V(K) = L^2_0(K)$
- if $V = H^2_0(\Omega)$ then $B_V(K) = H^2_0(K)$
- if $V = H_0(\mathrm{div};\Omega)$ then $B_V(K) = H_0(\mathrm{div};K)$
- if $V = H(\mathrm{div};\Omega)$ then $B_V(K) = \{\boldsymbol{\tau} \in H(\mathrm{div};K),\, \boldsymbol{\tau}\cdot\mathbf{n} = 0 \textit{ on } \partial K \cap \Omega\}$

Similar definitions and properties hold for the spaces $H(\mathbf{curl};\mathcal{O})$, but we are not going to use them here.

Let us now turn to the choice of the local bubble space $B_h(K)$. If possible, we would like to augment the space V_h by adding, in each element K, the whole $B_V(K)$. This would change V_h into $V_h + B(V;\mathcal{T}_h)$. However, some conditions are needed, as we shall see below. This might forbid, in some cases, taking the whole $B_V(K)$ in the augmentation process: some components of $B_V(K)$ have to be discarded. This will become more clear in the examples below. At this very abstract and general level, we assume that, in each $K \in \mathcal{T}_h$, we choose a subspace $B_h(K) \subseteq B_V(K)$ and, for the moment, "the bigger the better". A first condition that we require is that, for every $g \in V'$, the auxiliary problem

$$\begin{cases} \text{find } w_{B,K} \in B_h(K) \text{ such that} \\ \mathcal{L}(w_{B,K}, v) = \langle g, v\rangle \qquad \forall\, v \in B_h(K) \end{cases} \tag{3.2}$$

has a unique solution. We point out that the choice "the bigger the better" for $B_h(K)$ is made (so far) in order to understand the full potential of the method. As we shall see, in practice we will need to solve (3.2) a few times in each K. This implies that a finite dimensional choice for $B_h(K)$ will be, in the end, necessary.

Having chosen $B_h(K)$, we can now write the *augmented problem*. For that, let

$$V_A := V_h + \Pi_K B_h(K). \tag{3.3}$$

Two requirements have to be fulfilled: first of all, in (3.3) we must have a direct sum, and, second, for every $f \in V'$, the augmented problem

$$\begin{cases} \text{find } u_A \in V_A \text{ such that} \\ \mathcal{L}(u_A, v_A) = \langle f, v_A \rangle \qquad \forall\, v_A \in V_A \end{cases} \tag{3.4}$$

must have a unique solution. To summarize, in the augmentation process three conditions have to be fulfilled:

1) the local problems (3.2) must have a unique solution;
2) in (3.3) we must have a direct sum;
3) the augmented problem (3.4) must have a unique solution.

These are then the requirements that can guide us in choosing $B_h(K)$ in the various cases.

Examples of choices of $B_h(K)$.

Example 3.1. Referring to Examples 2.1 and 2.2 of the previous section, suppose that V_h is made of continuous piecewise linear functions. In this case it is easy to check that the choice $B_h(K) = B_V(K) \equiv H^1_0(K)$ verifies all of the three conditions.

Example 3.2. Suppose now that, always referring to Examples 2.1 and 2.2, V_h is made of continuous piecewise cubic functions. The choice $B_h(K) = B_V(K)$ is not viable anymore, as clearly condition 2) is violated: V_h contains functions of $B_V(K)$. In situations like this we should then choose a different $B_h(K)$, but we could also *reduce* the original space V_h. This is actually the simplest strategy, and we are going to follow it. Here, for instance, we can just remove the cubic bubble from $V_{h|K}$ and take a reduced space, still denoted by V_h with an abuse of notation, as a space of any serendipity cubic element (see, for instance, the element described in [12], page 50). Or we might take V_h as the space of functions v_h that are polynomials of degree ≤ 3 at the interelement boundaries and verify $Lv_h = 0$ separately in each K. Notice that these two choices produce the same augmented space V_A, and hence the same solution u_A to (3.4).

Example 3.3. Let us consider the problem of Example 2.3, and assume that $V_h = \Sigma_h \times U_h$ is made by lowest order Raviart-Thomas elements (see for instance [3]). For this problem we have

$$B_V(K) = \{\boldsymbol{\tau} \in H(\mathrm{div}; K),\, \boldsymbol{\tau} \cdot \mathbf{n} = 0 \textit{ on } \partial K \cap \Omega\} \times L^2(K).$$

we notice now that taking $B_h(K) = B_V(K)$ would not guarantee that problem (3.2) has a unique solution. Indeed, for internal elements K, the inf-sup condition is not satisfied, since $\int_K \operatorname{div} \tau\, v\, dx = 0\ \forall\ v$ constant on K. Condition 2) would also be violated by the choice $B_h(K) = B_V(K)$: in fact, U_h being the space of piecewise constants, $U_{h|K}$ contains bubbles of $L^2(K)$. A possible remedy in this case is to take

$$B_h(K) = H_0(\mathrm{div}; K) \times L^2_0(K) \subset B_V(K).$$

With this choice V_h remains the same, and B_h is the space of all pairs $(\tau, v) \in V$ such that τ has zero normal component at the boundary of each element, and v has zero mean value in each element. The same choice for B_h would be suitable also in the case of higher order Raviart-Thomas spaces (or, say, for BDM spaces; see always [3]), but then V_h should lose all internal degrees of freedom, apart from the piecewise constant scalars.

Example 3.4. Let us now examine the Stokes problem of Example 2.4, and assume that V_h is made of piecewise quadratic velocities in $(H^1_0(\Omega))^d$, and discontinuous piecewise linear pressures in $L^2_0(\Omega)$, a choice which is known not to be stable, but can be stabilized with the present technique. Actually, in this case one can see that $B_V(K) = (H^1_0(K))^d \times L^2_0(K)$. Taking $B_h(K) = B_V(K)$ would violate condition 2), but we can reduce the space V_h, taking it to be the space of quadratic velocities and *constant* pressures. It is easy to check that with this last choice we have a direct sum in (3.3). Moreover, problem (3.4) has a unique solution, because the inf-sup condition is now verified in V_A.

Example 3.5. Let us again consider the Stokes problem of Example 2.4, but now with $V_h = U_h \times Q_h$ made of piecewise linear continuous velocities in $(H^1_0(\Omega))^d$, and piecewise constant pressures in $L^2_0(\Omega)$. It is well known that for this choice the inf-sup condition does not hold. Moreover, if we augment V_h with bubble functions, in any way, the augmented problem (3.4) will **never** verify the inf-sup condition. To see that, augment the velocity space: $U_A = U_h + \Pi_K (H^1_0(K))^d$ as much as you can, and augment the pressure space: $Q_A = Q_h + \{0\}$ as little as you can. For every $v \in (H^1_0(K))^d$ and for every constant q in K, we clearly have $(\mathrm{div}\ v, q) = 0$. Hence, for $q \in Q_h$:

$$\sup_{v \in V_A} \frac{(\mathrm{div}\ v, q)}{\|v\|_1} = \sup_{v \in U_h} \frac{(\mathrm{div}\ v, q)}{\|v\|_1},$$

and we know that the last quantity cannot bound $\|q\|_0$ for all $q \in Q_h$. We clearly see that, in cases like this, our strategy is totally useless, and should not be applied.

4 An Example of Error Estimates

To give an idea of how to proceed to obtain error estimates, let us consider, as an example, a general singular perturbation problem where

$$\mathcal{L}(u, v) := \varepsilon a_1(u, v) + a_0(u, v)$$

with

$$a_1(v,v) \geq \alpha\|v\|_V^2 \quad \forall\, v \in V, \qquad a_1(u,v) \leq \|u\|_V\,\|v\|_V \quad \forall\, u,v \in V \tag{4.1}$$

$$a_0(v,v) \geq 0 \quad \forall\, v \in V, \qquad a_0(u,v) \leq M\,\|u\|_V\,\|v\|_H \quad \forall\, u,v \in V \tag{4.2}$$

where H is a space such that $V \subset H$ with continuous embedding. We set $e := u - u_A$ and $\eta := u - u_I$, u_I being some interpolant of u in V_h. Proceeding as usual we have

$$\varepsilon\alpha\|e\|_V^2 \leq \mathcal{L}(e,e) = \mathcal{L}(e,\eta) = \varepsilon a_1(e,\eta) + a_0(e,\eta), \tag{4.3}$$

and the term $a_0(e,\eta)$ is the source of all difficulties, since it does not contain ε as an explicit factor. In order to estimate it, let $\eta = \eta_B + \eta_H$ be any decomposition of η with $\eta_B \in B_h$ and $\eta_H \in H$. Notice that $\eta_B \in B_h \subset V_A$, so that, by Galerkin orthogonality,

$$\varepsilon a_1(e,\eta_B) = -a_0(e,\eta_B). \tag{4.4}$$

Using this and the bounds (4.1)-(4.2) we can proceed as in [9] and deduce:

$$\begin{aligned} a_0(e,\eta) &= a_0(e,\eta_B) + a_0(e,\eta_H) = -\varepsilon a_1(e,\eta_B) + a_0(e,\eta_H) \\ &\leq \varepsilon\|e\|_V\|\eta_B\|_V + M\|e\|_V\|\eta_H\|_H \\ &\leq \varepsilon^{1/2}\Big(\varepsilon^{1/2}\|e\|_V\|\eta_B\|_V + M\varepsilon^{-1/2}\|e\|_V\|\eta_H\|_H\Big) \\ &\leq \varepsilon^{1/2}(1+M)\|e\|_V\Big(\varepsilon^{1/2}\|\eta_B\|_V + \varepsilon^{-1/2}\|\eta_H\|_H\Big). \end{aligned} \tag{4.5}$$

Taking now the supremum over all possible decompositions $\eta = \eta_B + \eta_H$, and then over $\varepsilon > 0$ we obtain

$$\begin{aligned} &a_0(e,\eta) \\ &\leq \varepsilon^{1/2}(1+M)\|e\|_V \sup_{\varepsilon>0}\Big[\sup_{\eta_B+\eta_H=\eta}\Big(\varepsilon^{1/2}\|\eta_B\|_V + \varepsilon^{-1/2}\|\eta_H\|_H\Big)\Big]. \end{aligned} \tag{4.6}$$

By definition (see [7]) the double supremum is the norm of η in a suitable interpolation space, usually denoted by $[B_h, H]_{\frac{1}{2},\infty}$, that for brevity we shall denote by F. Hence, (4.6) becomes

$$a_0(e,\eta) \leq \varepsilon^{1/2}(1+M)\|e\|_V\|\eta\|_F. \tag{4.7}$$

Inserting (4.7) in (4.3) gives

$$\varepsilon\alpha\|e\|_V^2 \leq \varepsilon a_1(e,\eta) + a_0(e,\eta) \leq \varepsilon^{1/2}\|e\|_V(\varepsilon^{1/2}\|\eta\|_V + (1+M)\|\eta\|_F),$$

and finally

$$\varepsilon^{1/2}\alpha\|u - u_A\|_V \leq \varepsilon^{1/2}\|u - u_I\|_V + (1+M)\|u - u_I\|_F. \tag{4.8}$$

Notice that an estimate for $\varepsilon^{1/2}\|u-u_A\|_V$ is not as bad as we are used to. For instance, with an argument similar to the one used before, using (4.4)-(4.5), from (4.8) we can see that

$$\begin{aligned}\|A_0(u-u_A)\|_{F'} &:= \sup_{\varphi} \frac{a_0(u-u_A,\varphi)}{\|\varphi\|_F} \\ &= \sup_{\varphi} \frac{a_0(u-u_A,\varphi_B)+a_0(u-u_A,\varphi_H)}{\|\varphi\|_F} \\ &= \sup_{\varphi} \frac{-\varepsilon a_1(u-u_A,\varphi_B)+a_0(u-u_A,\varphi_H)}{\|\varphi\|_F} \\ &\leq (1+M)\varepsilon^{1/2}\|u-u_A\|_V \sup_{\varphi} \frac{\varepsilon^{1/2}\|\varphi_B\|_V+\varepsilon^{-1/2}\|\varphi_H\|_H}{\|\varphi\|_F} \\ &\leq (1+M)\varepsilon^{1/2}\|u-u_A\|_V \leq C(\varepsilon^{1/2}\|u-u_I\|_V+\|u-u_I\|_F),\end{aligned}$$

which is a typical estimate that can be obtained with stabilized methods (see, e.g., [22], [27]). We refer to [6], [9], [28] for the error analysis for residual-free bubbles methods for advection dominated problems.

5 Computational Aspects

Let us now examine the structure of the abstract augmented problem (3.4). Since we constructed the space V_A as a direct sum:

$$V_A := \Pi_K B_h(K) \oplus V_h$$

we then have the unique splittings: $u_A = u_B + u_h$, $v_A = v_B + v_h$. The augmented problem can then be written as

$$\begin{cases} \text{find } u_A = u_B + u_h \in V_A \text{ such that} \\ \mathcal{L}(u_B+u_h, v_B+v_h) = \langle f, v_B+v_h\rangle \qquad \forall\, v_B \in B_h,\ \forall\, v_h \in V_h. \end{cases} \tag{5.1}$$

The associated system will therefore have the form:

$$\begin{pmatrix} L_{B,B} & L_{B,h} \\ L_{h,B} & L_{h,h} \end{pmatrix} \begin{pmatrix} u_B \\ u_h \end{pmatrix} = \begin{pmatrix} f_B \\ f_h \end{pmatrix} \qquad \text{with } L_{B,B} \text{ block diagonal.}$$

There are different strategies for solving the (still infinite dimensional) problem (5.1). All of them are based on the (approximate) solution of the problems

$$\begin{cases} \text{find } w_B^i \in B_h \text{ such that} \\ \mathcal{L}(w_B^i, v_B) = \mathcal{L}(v_i, v_B) \equiv \langle L v_i, v_B\rangle \qquad \forall\, v_B \in B_h, \end{cases} \tag{5.2}$$

where the $\{v_i\}$'s are a basis for V_h, plus, if necessary, the solution of the problem

$$\begin{cases} \text{find } w_B^f \in B_h \text{ such that} \\ \mathcal{L}(w_B^f, v_B) = \langle f, v_B\rangle \qquad \forall\, v_B \in B_h. \end{cases} \tag{5.3}$$

As we shall see, what is actually needed, for all strategies, is the computation (for $i, j = 1, ..., \dim(V_h)$) of the quantities

$$S_{j,i} := \mathcal{L}(w_B^i, v_j) \equiv \langle w_B^i, L^* v_j \rangle, \quad \text{and} \quad T_j := \mathcal{L}(w_B^f, v_j) \equiv \langle w_B^f, L^* v_j \rangle, \quad (5.4)$$

where L^* is the adjoint operator of L. In turn, the computation of the solution of the problems (5.2) amounts to solving, in each K, the local bubble problem

$$\begin{cases} \text{find } w_{B,K}^i \in B_h(K) \text{ such that} \\ \mathcal{L}(w_{B,K}^i, b) = \langle L v_i, b \rangle \qquad \forall\, b \in B_h(K). \end{cases} \quad (5.5)$$

The same is obviously true for (5.3). Moreover, f can often be approximated, in each K, by elements of $LV_{h|K}$, so that the solution of (5.3) can be easily obtained from the solutions of the problems (5.2).

A careful inspection of the local problems (5.5) suggests several observations that are computationally relevant.

- For each v_i, the computation of w_B^i can be done in parallel.
- In each element K, the dimension of span$\{Lv_{i|K}\}$ will be small. In general, it will be less than or equal to the number of degrees of freedom of V_h in K.
- Finally, as we already pointed out, only the quantities $S_{j,i} = \langle w_B^i, L^* v_j \rangle$ are actually needed. Hence, only some averages of w_B^i will be used, and therefore a rough approximation might often be sufficient.
- The same considerations clearly hold for the contributions T_j to the right-hand side.

5.1 First Strategy

Let us see in more detail how the whole procedure can be applied in practice. For this, consider problem (5.1) and note that u_B is the solution of

$$\mathcal{L}(u_B, v_B) = -\mathcal{L}(u_h, v_B) + \langle f, v_B \rangle \qquad \forall\, v_B \in B_h,$$

and can be seen as an (affine) function of u_h and f:

$$u_B = L_{B,B}^{-1}(f - Lu_h).$$

Substituting into (5.1), and taking now v_h as a test function, gives

$$\mathcal{L}(u_h, v_h) + \mathcal{L}(L_{B,B}^{-1}(f - Lu_h), v_h)) = \langle f, v_h \rangle \qquad \forall\, v_h \in V_h, \quad (5.6)$$

which is an equation in terms of u_h alone, where the additional term

$$\mathcal{L}(L_{B,B}^{-1}(f - Lu_h), v_h) \equiv \mathcal{L}(u_B, v_h) \quad (5.7)$$

represents the effect of the small scales onto the coarse ones. To see how to compute the additional term (5.7) let us write $u_h := \sum_i U_i v_i$ and take v_j as a test function. We have

$$\begin{aligned}\mathcal{L}(u_B, v_j) &= \mathcal{L}(L_{B,B}^{-1}(f - Lu_h), v_j) = \mathcal{L}(L_{B,B}^{-1} f, v_j) - \sum_i \mathcal{L}(L_{B,B}^{-1} L v_i, v_j) U_i \\ &= \mathcal{L}(w_B^f, v_j) - \sum_i \mathcal{L}(w_B^i, v_j) U_i = T_j - \sum_i S_{j,i} U_i,\end{aligned}$$

that clearly shows the use of the auxiliary terms T_j and $S_{j,i}$. Indeed, setting

$$K_{j,i} = \mathcal{L}(v_i, v_j), \qquad \text{and} \qquad F_j = \langle f, v_j \rangle, \tag{5.8}$$

we have from (5.6) that the U_i's can be obtained as the solution of the following linear system of equations:

$$\sum_i (K_{j,i} - S_{j,i})\, U_i = F_j - T_j \qquad j = 1, ..., \dim(V_h). \tag{5.9}$$

Example 5.1. To see how this strategy can be applied, let us go back to the advection-dominated equation, that we recall here:

$$\begin{gathered} -\varepsilon \Delta u + \mathbf{c} \cdot \nabla u = f \quad \text{in } \Omega; \quad u = 0 \ \text{ on } \partial\Omega, \\ V := H_0^1(\Omega); \ \ \mathcal{L}(u, v) := \int_\Omega \varepsilon \nabla u \cdot \nabla v \, dx + \int_\Omega \mathbf{c} \cdot \nabla u \, v \, dx. \end{gathered}$$

Assume that the original finite element space V_h is made of piecewise linear continuous functions. Assume moreover that both the source term f and the convective term $\mathbf{c}$ are piecewise constant. Then, it is easy to see that for all v_i the terms Lv_i and $L^* v_i$ are constant in each K. Consequently, all the w_B^i can be computed by solving a *single* problem in each K, that is

$$\begin{cases} \text{find } b_K \in H_0^1(K) \text{ such that} \\ \mathcal{L}(b_K, b) = \langle 1, b \rangle \qquad \forall\, b \in H_0^1(K). \end{cases} \tag{5.10}$$

With some computations, the problem becomes now (see, e.g., [4]):

$$\begin{cases} \text{find } u_h \in V_h \text{ such that, for all } v_h \in V_h : \\ \mathcal{L}(u_h, v_h) - \displaystyle\sum_K \frac{\int_K b_K \, dx}{|K|} \int_K (f - \mathbf{c} \cdot \nabla u_h) \mathbf{c} \cdot \nabla v_h \, dx = \langle f, v_h \rangle. \end{cases} \tag{5.11}$$

This coincides with the *SUPG* method with $\tau_K = \dfrac{\int_K b_K \, dx}{|K|}$ (see [11], [16]).

5.2 Alternative Computational Strategies

Another possibility is to change the space V_h: for every basis function $v_i \in V_h$, define

$$\widetilde{v}_i := v_i - w_B^i, \tag{5.12}$$

and remember that w_B^i was defined by

$$\mathcal{L}(w_B^i, v_B) = \mathcal{L}(v_i, v_B) \qquad \forall\, v_B \in B_h. \tag{5.13}$$

Therefore,

$$\mathcal{L}(\widetilde{v}_i, v_B) = 0 \qquad \forall\, v_B \in B_h. \tag{5.14}$$

Set now $\widetilde{V}_h = \mathrm{span}\{\widetilde{v}_i\}$, and notice that, again, $V_A = \widetilde{V}_h \oplus B_h$. Split u_A into $u_A = \widetilde{u}_h + \widetilde{u}_B$, with $\widetilde{u}_h$ in $\widetilde{V}_h$, and $\widetilde{u}_B$ in B_h. Then, thanks to (5.14), $\widetilde{u}_B$ is the solution of

$$\mathcal{L}(\widetilde{u}_B, v_B) \equiv \mathcal{L}(u_A, v_B) = \langle f, v_B\rangle \quad \forall\, v_B \in B_h. \tag{5.15}$$

Hence $\widetilde{u}_B$ equals w_B^f, the solution of (5.3), and can be computed *before* knowing $\widetilde{u}_h$. Finally, $\widetilde{u}_h$ can be computed as the solution of

$$\mathcal{L}(\widetilde{u}_h, v_h) + \mathcal{L}(\widetilde{u}_B, v_h) = \langle f, v_h\rangle \quad \forall\, v_h \in V_h, \tag{5.16}$$

with the same number of unknowns and equations as the dimension of V_h. It is interesting to observe that the difference between this and the first strategy is mainly psychological. Indeed, setting $\widetilde{u}_h := \sum_i \widetilde{U}_i \widetilde{v}_i$, we have from (5.12), (5.8), and (5.4)

$$\begin{aligned}
&\mathcal{L}(\widetilde{u}_h, v_j) = \sum_i \mathcal{L}(\widetilde{v}_i, v_j)\widetilde{U}_i = \sum_i \mathcal{L}(v_i - w_B^i, v_j)\widetilde{U}_i = \sum_i (K_{j,i} - S_{j,i})\,\widetilde{U}_i,\\
&\mathcal{L}(\widetilde{u}_B, v_j) = \mathcal{L}(w_B^f, v_j) = T_j,
\end{aligned} \tag{5.17}$$

so that, inserting (5.17) into (5.16) we obtain

$$\sum_i (K_{j,i} - S_{j,i})\,\widetilde{U}_i = F_j - T_j \qquad j = 1, ..., \dim(V_h), \tag{5.18}$$

which is exactly (5.9).

A third possibility would be, assuming that the adjoint problem of (5.13) is uniquely solvable, to define $\hat{w}_B^i$ solution of

$$\mathcal{L}(v_B, \hat{w}_B^i) = \mathcal{L}(v_B, v_i) \qquad \forall\, v_B \in B_h, \tag{5.19}$$

and to associate to any v_i, basis function in V_h, the function

$$\hat{v}_i = v_i - \hat{w}_B^i. \tag{5.20}$$

Therefore, $\hat{v}_i$ is the solution of

$$\mathcal{L}(v_B, \hat{v}_i) \equiv \langle v_B, L^* \hat{v}_i \rangle = 0 \qquad \forall\, v_B \in B_h. \tag{5.21}$$

Set then $V_h^* = \mathrm{span}\{\hat{v}_i\}$, and notice that, in general, V_h^* will be different from $\widetilde{V}_h$, unless the bilinear form $\mathcal{L}$ is symmetric. We again have $V_A = V_h^* + B_h$, always with a direct sum. Take now in (5.1) for u_A the same splitting as before, that is, $u_A = \widetilde{u}_h + \widetilde{u}_B$, with $\widetilde{u}_h \in \widetilde{V}_h$, $\widetilde{u}_B \in B_h$, and for v_A take instead the splitting $v_A = \hat{v}_h + v_B$, with $\hat{v}_h \in V_h^*$, $v_B \in B_h$, always without changing the final solution u_A. Substituting in (5.1) shows that $\widetilde{u}_B$ is again the solution of (5.15). Hence, as before, $\widetilde{u}_B$ equals w_B^f, and can be computed before knowing $\widetilde{u}_h$. Finally, $\widetilde{u}_h$ can be computed as the solution of

$$\mathcal{L}(\widetilde{u}_h, \hat{v}_h) = \langle f, \hat{v}_h \rangle \quad \forall\, \hat{v}_h \in V_h^*. \tag{5.22}$$

The matrix associated with (5.22) is however given by

$$\mathcal{L}(\widetilde{v}_i, \hat{v}_j) = \mathcal{L}(\widetilde{v}_i, v_j - \hat{w}_B^j) = \mathcal{L}(\widetilde{v}_i, v_j) = K_{j,i} - S_{j,i} \tag{5.23}$$

(having used (5.20), (5.14), and (5.17)). On the other hand,

$$\langle f, \hat{v}_j \rangle = \langle f, v_j - \hat{w}_B^j \rangle = F_j - \langle f, \hat{w}_B^j \rangle, \tag{5.24}$$

and, using (5.3), (5.19), and (5.4),

$$\langle f, \hat{w}_B^j \rangle = \mathcal{L}(w_B^f, \hat{w}_B^j) = \mathcal{L}(w_B^f, v_j) = T_j. \tag{5.25}$$

We are therefore back to the system (5.18). It is somehow remarkable that the solution of (5.22) can be computed without actually computing the functions $\hat{v}_j$.

Remark 5.1. Although the above strategies, as we have seen, do coincide in practice, this is not often recognized in the literature. For instance, formulations (5.16) and (5.22), when applied to advection dominated problems coincide with the classical so-called Petrov-Galerkin methods in which suitable trial and test functions, depending on the operator, were used (see [25], and see, in Figure 5.1, the typical shape of the basis functions in $\widetilde{V}_h$ and V_h^*). The above computation shows that these methods coincide with SUPG when the choice of the stabilization parameter τ_K is made as in (5.11). On the other hand, when applied to problems related to composite materials, as in Example 2.2 (respectively, Example 2.3), the formulation (5.22) reproduces the multiscale methods of [22], [23] and the upscaling method of [1], [2], respectively.

So far, we assumed that we were able to compute the solutions of the local bubble problems (5.2). As anticipated, these solutions cannot be computed exactly, but require some suitable approximation. Let us see, in the particular case of advection dominated problems, how this approximate solutions can be carried out in practice.

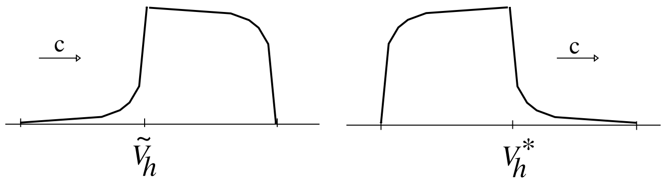

Fig. 5.1. Typical shape of the basis functions in $\widetilde{V}_h$ and V_h^*.

We recall that, in this case, solving (5.15) amounts in practice to compute, in each K, the "unitary bubble" b_K, solution of

$$-\varepsilon\Delta b_K + \mathbf{c}\cdot\nabla b_K = 1 \quad \text{in each } K. \tag{5.26}$$

Actually, what we really need is its mean value in each K (see (5.11)).

Several tricks can be used to compute $\int_K b_K\,dx$.

- A possibility is to solve by hand the pure convective problem, as advocated in [10]:

$$\begin{cases} \text{ find } \widetilde{b}_K \in H^1(K) \text{ such that} \\ \mathbf{c}\cdot\nabla\widetilde{b}_K = 1 \ \text{ in } K, \\ \widetilde{b}_K = 0 \ \text{ on } \partial K^- \,(= \text{inflow}) \end{cases}$$

  Notice that the integral of $\widetilde{b}_K$ on K is just the volume of a pyramid, as

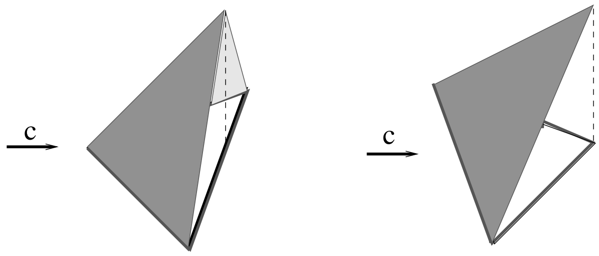

Fig. 5.2. Possible shapes of $\widetilde{b}_K$; here $\mathbf{c} = (1, 0)$.

  shown in Figure 5.2.
- Another possibility is to solve (5.26) on a subgrid with very few degrees of freedom, but well chosen (e.g., Pseudo RFB [8], Shishkin [17], etc, see Figure 5.3. Typically few nodes in the element boundary layer are needed.
- As an alternative, one could use subgrid artificial viscosity; that means solving, instead of (5.26), the problem

$$-(\varepsilon + \varepsilon_A)\Delta b_K + \mathbf{c}\cdot\nabla b_K = 1 \quad \text{in each } K$$

  on a very rough grid (typically, one node), where ε_A is a suitably chosen artificial viscosity, in general $\simeq h_K$ (see [20]). Unfortunately, the prob-

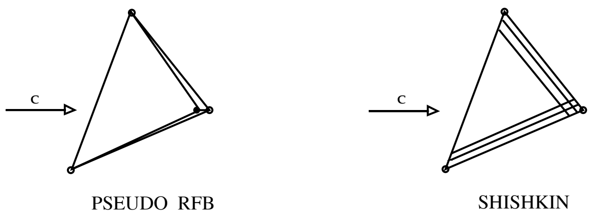

Fig. 5.3. Example of meshes.

lem of the optimal choice for ε_A is rather delicate. Indeed, using a one-dimensional space $B_h(K) = \text{span}\{\beta_K(x)\}$ results in an SUPG method with

$$\tau_K = \frac{(\int_K \beta \, dx)^2}{|K|(\varepsilon + \varepsilon_A) \int_K |\nabla \beta|^2 \, dx},$$

as shown in [5]. This implies that the bigger is ε_A the smaller is τ_K, that is, we add artificial viscosity for stabilizing and we decrease the stabilization parameter.

6 Conclusions

The Residual Free Bubble approach offers a unified framework for setting and analyzing several two-level and/or stabilized methods. It consists, essentially, in augmenting a given finite element space with spaces of functions having support in a single element. The necessary requirements for this augmentation process have been introduced and discussed for several examples. The disconnected nature of the bubble space allows us to eliminate the additional unknowns with an element by element procedure, that can be carried out in parallel. The elimination process involves, in general, the approximate solution of a partial differential equation in each element. We have seen however that in many cases a rough approximation can be sufficient.

The use of this type of approach for stabilizing unstable finite element formulations were already well known. Here we presented the method in a very general setting, and this allowed us to show that several other methods for stabilizing and, mostly, for dealing with subgrid phenomena, can actually be seen as a particular case of the RFB approach. This includes old methods like the Petrov Galerkin methods with special, operator dependent, trial and test functions for advection dominated problems, as well as more recent approaches like the multiscale method or the upscaling method for problems with composite materials.

Other developments and applications to different problems are surely worth further investigation, as well as some recent variants like the use of non-conforming bubbles, the possibility of adding edge-bubbles, or the connections with domain decomposition methods.

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http://www.springer.com/978-3-540-44319-3

Frontiers in Numerical Analysis
Durham 2002
Blowey, J.; Craig, A.; Shardlow, T. (Eds.)
2003, XIV, 354 p. 32 illus., 1 illus. in color., Softcover
ISBN: 978-3-540-44319-3