Preface

This book is an introduction to the algebraic, algorithmic, and analytic aspects of the Galois theory of homogeneous linear differential equations. Although the Galois theory has its origins in the 19th Century and was put on a firm footing by Kolchin in the middle of the 20th Century, it has experienced a burst of activity in the last 30 years. In this book we present many of the recent results and new approaches to this classical field. We have attempted to make this subject accessible to anyone with a background in algebra and analysis at the level of a first-year graduate student. Our hope is that this book will prepare and entice the reader to delve further.

Here we will describe the contents of this book. Various researchers are responsible for the results described here. We will not attempt to give proper attributions here but refer the reader to each of the individual chapters for appropriate bibliographic references.

The Galois theory of linear differential equations (which we shall refer to simply as differential Galois theory) is the analog for linear differential equations of the classical Galois theory for polynomial equations. The natural analog of a field in our context is the notion of a differential field. This is a field k together with a derivation $\partial : k \to k$, that is, an additive map that satisfies $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in k$ (we will usually denote ∂a for $a \in k$ as a'). Except for those in Chap. 13, all differential fields will be of characteristic zero. A linear differential equation is an equation of the form $\partial Y = AY$ where A is an $n \times n$ matrix with entries in k, although sometimes we shall also consider scalar linear differential equations $L(y) = \partial^n y + a_{n-1}\partial^{n-1} y + \cdots + a_0 y = 0$ (these objects are in general equivalent, as we show in Chap. 2). One has the notion of a "splitting field," the Picard-Vessiot extension, which contains "all" solutions of $L(y) = 0$ and in this case has the additional structure of being a differential field. The differential Galois group is the group of field automorphisms of the Picard-Vessiot field fixing the base field and commuting with the derivation. Although defined abstractly, this group can be easily represented as a group of matrices and has the structure of a linear algebraic group, that is, it is a group of invertible matrices defined by the vanishing of a set of polynomials in the entries of these matrices. There is a Galois correspondence identifying differential subfields with linear algebraic subgroups of the Galois group. Corresponding to the notion of solvability by radicals for polynomial equations is the notion of solvability in terms of integrals, exponentials, and algebraics, that is, solvable in terms of liouvillian functions, and one can characterize this in terms of the differential Galois group as well.

Chapter 1 presents these basic facts. The main tools come from the elementary algebraic geometry of varieties over fields that are not necessarily algebraically closed and the theory of linear algebraic groups. In Appendix A we develop the results necessary for the Picard-Vessiot theory.

In Chap. 2, we introduce the ring $k[\partial]$ of differential operators over a differential field k, that is, the (in general, noncommutative) ring of polynomials in the symbol ∂ where multiplication is defined by $\partial a = a' + a\partial$ for all $a \in k$. For any differential equation $\partial Y = AY$ over k one can define a corresponding $k[\partial]$-module in much the same way that one can associate an $F[X]$-module with any linear transformation of a vector space over a field F. If $\partial Y = A_1 Y$ and $\partial Y = A_2 Y$ are differential equations over k and M_1 and M_2 are their associated $k[\partial]$-modules, then $M_1 \simeq M_2$ as $k[\partial]$-modules if and only if there is an invertible matrix Z with entries in k such that $Z^{-1}(\partial - A_1)Z = \partial - A_2$, i.e., $A_2 = Z^{-1}A_1 Z - Z^{-1}Z'$. We say two equations are *equivalent over k* if such a relation holds. We show that equivalent equations have the same Galois groups and so can define the Galois group of a $k[\partial]$-module. This chapter is devoted to further studying the elementary properties of modules over $k[\partial]$ and their relationship to linear differential equations. Furthermore, the tannakian equivalence between differential modules and representations of the differential Galois group is presented.

In Chap. 3, we study differential equations over the field of fractions $k = \mathbf{C}((z))$ of the ring of formal power series $\mathbf{C}[[z]]$ over the field of complex numbers, provided with the usual differentiation $\frac{d}{dz}$. The main result is to classify $k[\partial]$-modules over this ring or, equivalently, show that any differential equation $\partial Y = AY$ can be put in a normal form over an algebraic extension of k (an analog of the Jordan Normal Form of complex matrices). In particular, we show that any equation $\partial Y = AY$ is equivalent (over a field of the form $\mathbf{C}((t))$, $t^m = z$ for some integer $m > 0$) to an equation $\partial Y = BY$ where B is a block diagonal matrix where each block B_i is of the form $B_i = q_i I + C_i$ where $q_i \in t^{-1}\mathbf{C}[t^{-1}]$ and C_i is a constant matrix. We give a proof (and formal meaning) of the classical fact that any such equation has a solution matrix of the form $Z = Hz^L e^Q$, where H is an invertible matrix with entries in $\mathbf{C}((t))$, L is a constant matrix (i.e., with coefficients in $\mathbf{C}$) z^L means $e^{\log(z)L}$, and Q is a diagonal matrix whose entries are polynomials in t^{-1} without a constant term. A differential equation of this type is called *quasisplit* (because of its block form over a finite extension of $\mathbf{C}((z))$). Using this, we are able to explicitly give a universal Picard-Vessiot extension containing solutions for all such equations. We also show that the Galois group of the above equation $\partial Y = AY$ over $\mathbf{C}((z))$ is the smallest linear algebraic group containing a certain commutative group of diagonalizable matrices (the *exponential torus*) and one more element (the *formal monodromy*) and these can be explicitly calculated from its normal form. In this chapter we also begin the study of differential equations over $\mathbf{C}(\{z\})$, the field of fractions of the ring of convergent power series $\mathbf{C}\{z\}$. If A has entries in $\mathbf{C}(\{z\})$, we show that the equation $\partial Y = AY$ is equivalent over $\mathbf{C}((z))$ to a unique (up to equivalence over $\mathbf{C}(\{z\})$) equation with entries in $\mathbf{C}(\{z\})$, that is *quasisplit*. This latter fact is key to understanding the analytic behavior of solutions of these equations and will be used repeatedly in succeeding chapters. In Chaps. 2 and 3 we also use the language of tannakian categories to describe some of these results. This theory is explained in Appendix B. This appendix also contains a proof of the general result that the category of $k[\partial]$-modules for a differential field k forms a tannakian category and explains how one can deduce from this the fact that the Galois groups of the associated equations are linear algebraic groups. In general, we shall use tannakian categories throughout the book to deduce facts about categories of special $k[\partial]$-modules, i.e., deduce facts about the Galois groups of restricted classes of differential equations.

In Chap. 4, we consider the "direct" problem, which is to calculate explicitly for a given differential equation or differential module its Picard-Vessiot ring and its differential Galois group. A complete answer for a given differential equation should, in principle, provide all the algebraic information about the differential equation. Of course this can only be achieved for special base fields k, such as $\overline{\mathbf{Q}}(z)$, $\partial z = 1$ (where $\overline{\mathbf{Q}}$ is the algebraic closure of the field of rational numbers). The direct problem requires factoring many differential operators L over k. A right-hand factor $\partial - u$ of L (over k or over an algebraic extension of k) corresponds to a special solution f of $L(f) = 0$, which can be rational, exponential, or liouvillian. Some of the ideas involved here were already present in Beke's classical work on factoring differential equations. The "inverse" problem, namely to construct a differential equation over k with a prescribed differential Galois group G and action of G on the solution space, is treated for a *connected* linear algebraic group in Chap. 11. In the opposite case that G is a finite group (and with base field $\overline{\mathbf{Q}}(z)$) an effective algorithm is presented together with examples for equations of order 2 and 3. We note that some of the algorithms presented in this chapter are efficient and others are only the theoretical basis for an efficient algorithm.

Starting with Chap. 5, we turn to questions that are, in general, of a more analytic nature. Let $\partial Y = AY$ be a differential equation where A has entries in $\mathbf{C}(z)$, where $\mathbf{C}$ is the field of complex numbers and $\partial z = 1$. A point $c \in \mathbf{C}$ is said to be a *singular point* of the equation $\partial Y = AY$ if some entry of A is not analytic at c (this notion can be extended to the point at infinity on the Riemann sphere $\mathbf{P}$ as well). At any point p on the manifold $\mathbf{P}\backslash\{\text{the singular points}\}$, standard existence theorems imply that there exists an invertible matrix Z of functions, analytic in a neighborhood of p, such that $\partial Z = A\,Z$. Furthermore, one can analytically continue such a matrix of functions along any closed path γ, yielding a new matrix Z_γ which must be of the form $Z_\gamma = ZA_\gamma$ for some $A_\gamma \in \mathrm{GL}_n(\mathbf{C})$. The map $\gamma \mapsto A_\gamma$ induces a homomorphism, called the monodromy homomorphism, from the fundamental group $\pi_1(\mathbf{P}\backslash\{\text{the singular points}\}, c)$ into $\mathrm{GL}_n(\mathbf{C})$. As explained in Chap. 5, when all the singular points of $\partial Y = AY$ are regular singular points (that is, all solutions have at most polynomial growth in sectors at the singular point), the smallest linear algebraic group containing the image of this homomorphism is the Galois group of the equation. In Chaps. 5 and 6 we consider the inverse problem: Given points $\{p_0, \ldots, p_n\} \subset \mathbf{P}^1$ and a representation $\pi_1(\mathbf{P}\backslash\{p_1, \ldots, p_n\}, p_0) \to \mathrm{GL}_n(\mathbf{C})$, does there exist a differential equation with regular singular points having this monodromy representation? This is one form of Hilbert's 21st Problem and we describe its positive solution. We discuss refined versions of this problem that demand the existence of an equation of a more restricted form, as well as the existence of scalar linear differential equations having prescribed monodromy. Chapter 5 gives an elementary introduction to this problem concluding with an outline of the solution depending on basic facts concerning sheaves and vector bundles. In Appendix C, we give an exposition of the necessary results from sheaf theory needed in this and later sections. Chapter 6 contains deeper results concerning Hilbert's 21st problem and uses the machinery of connections on vector bundles, material that is developed in Appendix C and this chapter.

In Chap. 7, we study the analytic meaning of the formal description of solutions of a differential equation that we gave in Chap. 3. Let $w \in \mathbf{C}(\{z\})^n$ and let A be a matrix with entries in $\mathbf{C}(\{z\})$. We begin this chapter by giving analytic meaning to formal solutions $\hat{v} \in \mathbf{C}((z))^n$ of equations of the form $(\partial - A)\hat{v} = w$. We consider open sectors $S = S(a, b, \rho) = \{z \mid z \neq 0, \arg(z) \in (a, b) \text{ and } |z| < \rho(\arg(z))\}$, where $\rho(x)$ is a continuous positive function of a real variable and $a \leq b$ are real numbers and functions f analytic in S and define what it means for a formal series $\sum a_i z^i \in \mathbf{C}((z))$ to be the asymptotic expansion of f in S. We show that for any formal solution $\hat{v} \in \mathbf{C}((z))^n$ of $(\partial - A)\hat{v} = w$ and any sector $S = S(a, b, \rho)$ *with* $|a - b|$ *sufficiently small and suitable* ρ, there is a vector of functions v analytic in S satisfying $(\partial - A)v = w$ such that each entry of v has the corresponding entry in $\hat{v}$ as its asymptotic expansion. The vector v is referred to as an *asymptotic lift* of $\hat{v}$. In general, there will be many asymptotic lifts of $\hat{v}$ and the rest of the chapter is devoted to describing conditions that guarantee uniqueness. This leads us to the study of *Gevrey functions* and *Gevrey asymptotics*. Roughly stated, the main result, the multisummation theorem, allows us to associate, in a functorial way, to any formal solution $\hat{v}$ of $(\partial - A)\hat{v} = w$ and all but a finite number (mod 2π) of directions d, a unique asymptotic lift in an open sector $S(d - \epsilon, d + \epsilon, \rho)$ for suitable ϵ and ρ. The exceptional values of d are called the *singular directions* and are related to the so-called *Stokes phenomenon*. They play a crucial role in the succeeding chapters where we give an analytic description of the Galois group as well as a classification of meromorphic differential equations. Sheaves and their cohomology are the natural way to take analytic results valid in small neighborhoods and describe their extension to larger domains and we use these tools in this chapter. The necessary facts are described in Appendix C.

In Chap. 8 we give an analytic description of the differential Galois group of a differential equation $\partial Y = AY$ over $\mathbf{C}(\{z\})$ where A has entries in $\mathbf{C}(\{z\})$. In Chap. 3, we show that any such equation is equivalent to a unique *quasisplit* equation $\partial Y = BY$

with the entries of B in $\mathbf{C}(\{z\})$ as well, that is there exists an invertible matrix $\hat{F}$ with entries in $\mathbf{C}((z))$ such that $\hat{F}^{-1}(\partial - A)\hat{F} = \partial - B$. The Galois groups of $\partial Y = BY$ over $\mathbf{C}(\{z\})$ and $\mathbf{C}((z))$ coincide and are generated (as linear algebraic groups) by the associated exponential torus and formal monodromy. The differential Galois group G' over $\mathbf{C}(\{z\})$ of $\partial Y = BY$ is a subgroup of the differential Galois group of $\partial Y = AY$ over $\mathbf{C}(\{z\})$. To see what else is needed to generate this latter differential Galois group we note that the matrix $\hat{F}$ also satisfies a differential equation $\hat{F}' = A\hat{F} - \hat{F}B$ over $\mathbf{C}(\{z\})$ and so the results of Chap. 7 can be applied to $\hat{F}$. Asymptotic lifts of $\hat{F}$ can be used to yield isomorphisms of solution spaces of $\partial Y = AY$ in overlapping sectors, and using this we describe how, for each singular direction d of $\hat{F}' = A\hat{F} - \hat{F}B$, one can define an element St_d (called the *Stokes map in the direction* d) of the Galois group G of $\partial Y = AY$ over $\mathbf{C}(\{z\})$. Furthermore, it is shown that G is the smallest linear algebraic group containing the Stokes maps $\{St_d\}$ and G'. Various other properties of the Stokes maps are described in this chapter.

In Chap. 9, we consider the meromorphic classification of differential equations over $\mathbf{C}(\{z\})$. If one fixes a quasisplit equation $\partial Y = BY$, one can consider pairs $(\partial - A, \hat{F})$, where A has entries in $\mathbf{C}(\{z\})$, $\hat{F} \in \mathrm{GL}_n(\mathbf{C}((z)))$ and $\hat{F}^{-1}(\partial - A)\hat{F} = \partial - B$. Two pairs $(\partial - A_1, \hat{F}_1)$ and $(\partial - A_2, \hat{F}_2)$ are called *equivalent* if there is a $G \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$ such that $G(\partial - A_1)G^{-1} = \partial - A_2$ and $\hat{F}_2 = \hat{F}_1 G$. In this chapter, it is shown that the set E of equivalence classes of these pairs is in bijective correspondence with the first cohomology set of a certain sheaf of nonabelian groups on the unit circle, the *Stokes sheaf*. We describe how one can, furthermore, characterize those sets of matrices that can occur as Stokes maps for some equivalence class. This allows us to give the above cohomology set the structure of an affine space. These results will be further used in Chaps. 10 and 11 to characterize those groups that occur as differential Galois groups over $\mathbf{C}(\{z\})$.

In Chap. 10, we consider certain differential fields k and certain classes of differential equations over k and explicitly describe the *universal Picard-Vessiot ring* and its group of differential automorphisms over k, the *universal differential Galois group*, for these classes. For the special case $k = \mathbf{C}((z))$ this universal Picard-Vessiot ring is described in Chap. 3. Roughly speaking, a universal Picard-Vessiot ring is the smallest ring such that any differential equation $\partial Y = AY$ (with A an $n \times n$ matrix) in the given class has a set of n independent solutions with entries from this ring. The group of differential automorphisms over k will be an affine group scheme, and for any equation in the given class its Galois group will be a quotient of this group scheme. The necessary information concerning affine group schemes is presented in Appendix B. In Chap. 10, we calculate the universal Picard-Vessiot extension for the class of regular differential equations over $\mathbf{C}((z))$, the class of arbitrary differential equations over $\mathbf{C}((z))$ and the class of meromorphic differential equations over $\mathbf{C}(\{z\})$.

In Chap. 11, we consider the problem of, given a differential field k, determining which linear algebraic groups can occur as differential Galois groups for linear differential equations over k. In terms of the previous chapter, this is the, a priori, easier problem of determining the linear algebraic groups that are quotients of the universal Galois group. We begin by characterizing those groups that are differential Galois groups over $\mathbf{C}((z))$. We then give an analytic proof of the fact that any linear algebraic group occurs as a differential Galois group of a differential equation $\partial Y = AY$ over $\mathbf{C}(z)$, and describe the minimal number and type of singularities of such an equation that are necessary to realize a given group. We end by discussing an algebraic (and constructive) proof of this result for connected linear algebraic groups and give explicit details when the group is semisimple.

In Chap. 12, we consider the problem of finding a fine moduli space for the equivalence classes E of differential equations considered in Chap. 9. In that chapter, we describe how E has a natural structure as an affine space. Nonetheless, it can be shown that there does not exist a universal family of equations parameterized by E. To remedy this situation, we show the classical result that for any meromorphic differential equation $\partial Y = AY$, there is a differential equation $\partial Y = BY$ where B has coefficients in $\mathbf{C}(z)$ (i.e., a differential equation on the Riemann sphere) having singular points at 0 and ∞ such that the singular point at infinity is regular and such that the equation is equivalent to the original equation when both are considered as differential equations over $\mathbf{C}(\{z\})$. Furthermore, this latter equation can be identified with a (meromorphic) connection on a free vector bundle over the Riemann sphere. In this chapter we show that, loosely speaking, there exists a fine moduli space for connections on a fixed free vector bundle over the Riemann sphere having a regular singularity at infinity and an irregular singularity at the origin together with an extra piece of data (corresponding to fixing the formal structure of the singularity at the origin).

In Chap. 13, the differential field K has characteristic $p > 0$. A perfect field (i.e., $K = K^p$) of characteristic $p > 0$ has only the zero derivation. Thus we have to assume that $K \neq K^p$. In fact, we will consider fields K such that $[K : K^p] = p$. A nonzero derivation on K is then unique up to a multiplicative factor. This seems to be a good analog of the most important differential fields $\mathbf{C}(z)$, $\mathbf{C}(\{z\})$, $\mathbf{C}((z))$ in characteristic zero. Linear differential equations over a differential field of characteristic $p > 0$ have attracted, for various reasons, a lot of attention. Some references are [90, 139, 152, 153, 162, 205, 217, 227, 229, 8, 226]. One reason is Grothendieck's conjecture on p-curvatures, which states that the differential Galois group of a linear differential equation in characteristic zero is finite if and only if the p-curvature of the reduction of the equation modulo p is zero for almost all p. N. Katz has extended this conjecture to one that states that the Lie algebra of the differential Galois group of a linear differential equation in characteristic zero is determined by the collection of its p-curvatures (for almost all p). In this chapter we will classify a differential module over K essentially by the Jordan normal form of its p-curvature. Algorithmic considerations make this procedure effective. A glimpse at order two equations gives an indication of how this classification could be used for linear differential equations in characteristic 0. A more or less obvious observation is that these linear differential equations in positive characteristic behave very differently from what might be expected from the characteristic zero case. A different class of differential equations in positive characteristic, namely the iterative differential equations, is introduced. The chapter ends with a survey on iterative differential modules.

Appendix A contains the tools from the theory of affine varieties and linear algebraic groups that are needed, particularly in Chap. 1. Appendix B contains a description of the formalism of tannakian categories that are used throughout the book. Appendix C describes the results from the theory of sheaves and sheaf cohomology that are used in the analytic sections of the book. Finally, Appendix D discusses systems of linear partial differential equations and the extent to which the results of this book are known to generalize to this situation.

Conspicuously missing from this book are discussions of the arithmetic theory of linear differential equations as well as the Galois theory of nonlinear differential equations. A few references are [162, 197, 199, 222, 223, 293, 294, 295, 296]. We have also not described the recent applications of differential Galois theory to Hamiltonian mechanics for which we refer to [11] and [213]. For an extended historical treatment of linear differential equations and group theory in the 19th century see [113].

Notation and Terminology. We shall use the letters **C**, **N**, **Q**, **R**, and **Z** to denote the complex numbers, the non-negative integers, the rational numbers, the real numbers, and the integers, respectively. Authors of any book concerning functions of a complex variable are confronted with the problem of how to use the terms analytic and holomorphic. We consider these terms synonymous and use them interchangeably but with an eye to avoiding such infelicities as “analytic differential” and “holomorphic continuation”.

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2 Differential Operators and Differential Modules

2.1 The Ring $\mathcal{D} = k[\partial]$ of Differential Operators

In this chapter k is a differential field such that its subfield of constants C is different from k and has characteristic 0. The skew (i.e., noncommutative) ring $\mathcal{D} := k[\partial]$ consists of all expressions $L := a_n\partial^n + \cdots + a_1\partial + a_0$ with $n \in \mathbf{Z}$, $n \geq 0$ and all $a_i \in k$. These elements L are called *differential operators*. The degree of L $\deg L$ above is m if $a_m \neq 0$ and $a_i = 0$ for $i > m$. In the case $L = 0$ we define the degree to be $-\infty$. The addition in $\mathcal{D}$ is obvious. The multiplication in $\mathcal{D}$ is completely determined by the prescribed rule $\partial a = a\partial + a'$. Since there exists an element $a \in k$ with $a' \neq 0$, the ring $\mathcal{D}$ is not commutative. One calls $\mathcal{D}$ *the ring of linear differential operators with coefficients in* k.

A differential operator $L = a_n\partial^n + \cdots + a_1\partial + a_0$ acts on k and on differential extensions of k, with the interpretation $\partial(y) := y'$. Thus the equation $L(y) = 0$ has the same meaning as the scalar differential equation $a_n y^{(n)} + \cdots + a_1 y^{(1)} + a_0 y = 0$. In connection with this one sometimes uses the expression *order of* L, instead of the degree of L.

The ring of differential operators shares many properties with the ordinary polynomial ring in one variable over k.

Lemma 2.1 *For $L_1, L_2 \in \mathcal{D}$ with $L_1 \neq 0$, there are unique differential operators $Q, R \in \mathcal{D}$ such that $L_2 = QL_1 + R$ and* $\deg R < \deg L_1$.

The proof is not different from the usual division with remainder for the ordinary polynomial ring over k. The version where left and right are interchanged is equally valid. An interesting way to interchange left and right is provided by the "involution" $i : L \mapsto L^*$ of $\mathcal{D}$ defined by the formula $i(\sum a_i\partial^i) = \sum(-1)^i\partial^i a_i$. The operator L^* is often called the *formal adjoint* of L.

Exercise 2.2 The term "involution" means that i is an additive bijection, $i^2 = id$ and $i(L_1L_2) = i(L_2)i(L_1)$ for all $L_1, L_2 \in \mathcal{D}$. Prove that i, as defined above, has these properties. Hint: Let $k[\partial]^\star$ denote the additive group $k[\partial]$ made into a ring by the opposite multiplication given by the formula $L_1 \star L_2 = L_2L_1$. Show that $k[\partial]^\star$ is also a skew polynomial ring over the field k and with variable $-\partial$. Observe that $(-\partial) \star a = a \star (-\partial) + a'$. $\square$

Corollary 2.3 *For any left ideal $I \subset k[\partial]$ there exists an $L_1 \in k[\partial]$ such that $I = k[\partial]L_1$. Similarly, for any right ideal $J \subset k[\partial]$ there exists an $L_2 \in k[\partial]$ such that $J = L_2k[\partial]$.*

From these results one can define the *least common left multiple*, $\mathrm{LCLM}(L_1, L_2)$, of $L_1, L_2 \in k[\partial]$ as the unique monic generator of $k[\partial]L_1 \cap k[\partial]L_2$ and the *greatest common left divisor*, $\mathrm{GCLD}(L_1, L_2)$, of $L_1, L_2 \in k[\partial]$ as the unique monic generator of $L_1k[\partial] + L_2k[\partial]$. The *least common right multiple* of $L_1, L_2 \in k[\partial]$, $\mathrm{LCRM}(L_1, L_2)$ and the *greatest common right divisor* of $L_1, L_2 \in k[\partial]$, $\mathrm{GCRD}(L_1, L_2)$ can be defined similarly. We note that a modified version of the Euclidean algorithm can be used to find the $\mathrm{GCLD}(L_1, L_2)$ and the $\mathrm{GCRD}(L_1, L_2)$.

Exercises 2.4 *The ring $k[\partial]$*
1. Show that for any nonzero operators $L_1, L_2 \in k[\partial]$, with $\deg(L_1) = n_1$, $\deg(L_2) = n_2$ we have the fact that $\deg(L_1L_2 - L_2L_1) < n_1 + n_2$. Show that $k[\partial]$ has no two-sided ideals other than (0) and $k[\partial]$.

2. Let M be a $\mathcal{D} = k[\partial]$-submodule of the free left module $F := \mathcal{D}^n$. Show that F has a free basis $e_1, \dots, e_n$ over $\mathcal{D}$ such that M is generated by elements $a_1e_1, \dots, a_ne_n$ for suitable $a_1, \dots, a_n \in \mathcal{D}$. Conclude that M is also a free $\mathcal{D}$-module. Hints:
(a) For any element $f = (f_1, \dots, f_n) \in F$ there is a free basis $e_1, \dots, e_n$ of F such that $f = ce_n$, with $c \in \mathcal{D}$ such that $\mathcal{D}c = \mathcal{D}f_1 + \cdots + \mathcal{D}f_n$.
(b) Choose $m = (b_1, \dots, b_n) \in M$ such that the degree of the $c \in \mathcal{D}$ with $\mathcal{D}c = \mathcal{D}b_1 + \cdots + \mathcal{D}b_n$ is minimal. Choose a new basis, called $e_1, \dots, e_n$ of F, such that $m = ce_n$. Prove that M is the direct sum of $M \cap (\mathcal{D}e_1 \oplus \cdots \oplus \mathcal{D}e_{n-1})$ and $\mathcal{D}ce_n$.
(c) Use induction to finish the proof.

3. Let $L_1, L_2 \in k[\partial]$ with $\deg(L_1) = n_1,\ \deg(L_2) = n_2$. Let K be a differential extension of k having the same constants C as k and let $\mathrm{Soln}_K(L_i)$ denote the C-space of solutions of $L_i(y) = 0$ in K. Assume that $\dim_C(\mathrm{Soln}_K(L_2)) = n_2$. Show that:
(a) Suppose that every solution in K of $L_2(y) = 0$ is a solution of $L_1(y) = 0$. Then there exists a $Q \in k[\partial]$ such that $L_1 = QL_2$.
(b) Suppose that L_1 divides L_2 on the right, then $\mathrm{Soln}_K(L_1) \subset \mathrm{Soln}_K(L_2)$ and $\dim_C(\mathrm{Soln}_K(L_1)) = n_1$. □

Lemma 2.5 Finitely generated left $k[\partial]$-modules.
Every finitely generated left $k[\partial]$-module is isomorphic to a finite direct sum $\oplus M_i$, where each M_i is isomorphic to either $k[\partial]$ or $k[\partial]/k[\partial]L$ for some $L \in k[\partial]$ with $\deg L > 0$.

Proof. Let M be a finitely generated left $k[\partial]$-module. Then there is a surjective homomorphism $\phi : k[\partial]^n \to M$ of $k[\partial]$-modules. The kernel of ϕ is a submodule of the free module $k[\partial]^n$. Exercises 2.4.2 applied to $\ker(\phi)$ yields the required direct sum decomposition of M. □

Observation 2.6 *A differential module M over k is the same object as a left $k[\partial]$-module such that* $\dim_k M < \infty$.

Exercise 2.7 Let $y' = Ay$ be a matrix differential equation over k of dimension n with corresponding differential module M. Show that the following properties are equivalent:
(1) There is a fundamental matrix F for $y' = Ay$ with coefficients in k.
(2) $\dim_C \ker(\partial, M) = n$.
(3) M is a direct sum of copies of $\mathbf{1}_k$, where $\mathbf{1}_k$ denotes the 1-dimensional differential module ke with $\partial e = 0$.

A differential module M over k is called *trivial* if the equivalent properties (2) and (3) hold for M. Assume now that C is algebraically closed. Prove that M is a trivial differential module if and only if the differential Galois group of M is $\{1\}$. □

Intermezzo on multilinear algebra.
Let F be any field. For vector spaces of finite dimension over F there are "constructions of linear algebra" that are used very often in connection with differential modules. Apart from the well-known "constructions" *direct sum* $V_1 \oplus V_2$ *of two vector spaces*, *subspace* $W \subset V$, *quotient space* V/W, *dual space* V^* *of* V, there are the less elementary constructions:

The *tensor product* $V \otimes_F W$ (or simply $V \otimes W$) of two vector spaces. Although we have already used this construction many times, we recall its categorical definition. A bilinear map $b : V \times W \to Z$ (with Z any vector space over F) is a map $(v, w) \to b(v, w) \in Z$ that is linear in v and w separately. The tensor product $(t, V \otimes W)$ is defined by $t : V \times W \to V \otimes W$ is a bilinear map such that there exists for each bilinear map $b : V \times W \to Z$ a unique linear map $\ell : V \otimes W \to Z$ with $\ell \circ t = b$. The elements $t(v, w)$ are denoted by $v \otimes w$. It is easily seen that bases $\{v_1, \ldots, v_n\}$ of V and $\{w_1, \ldots, w_m\}$ of W give rise to a basis $\{v_i \otimes w_j\}_{i=1,\ldots,n;\, j=1,\ldots,m}$ of $V \otimes W$. The tensor product of several vector spaces $V_1 \otimes \cdots \otimes V_s$ can be defined in a similar way by multilinear maps. A basis of this tensor product can be obtained in a similar way from bases for every V_i.

The *vector space of the homomorphisms* $\mathrm{Hom}(V, W)$ consist of the F-linear maps $\ell : V \to W$. Its structure as an F-vector space is given by $(\ell_1 + \ell_2)(v) := \ell_1(v) + \ell_2(v)$ and $(f\ell)(v) := f\ell(v)$. There is a natural isomorphism $\alpha : V^* \otimes W \to \mathrm{Hom}(V, W)$, given by the formula $\alpha(\ell \otimes w)(v) := \ell(v) \cdot w$.

The *symmetric powers* $\mathrm{sym}^d V$ of a vector space V. Consider the d-fold tensor product $V \otimes \cdots \otimes V$ and its subspace W generated by the vectors $(v_1 \otimes \cdots \otimes v_d) - (v_{\pi(1)} \otimes \cdots \otimes v_{\pi(d)})$, with $v_1, \ldots, v_d \in V$ and $\pi \in S_d$, the group of all permutations on $\{1, \ldots, d\}$. Then $\mathrm{sym}^d V$ is defined as the quotient space $(V \otimes \cdots \otimes V)/W$. The notation for the elements of $\mathrm{sym}^d V$ is often the same as for the elements of $V \otimes \cdots \otimes V$, namely finite sums of expressions $v_1 \otimes \cdots \otimes v_d$. For the symmetric powers, one has (by definition) $v_1 \otimes \cdots \otimes v_d = v_{\pi(1)} \otimes \cdots \otimes v_{\pi(d)}$ for any $\pi \in S_d$. Sometimes one omits the tensor product in the notation for the elements in the symmetric powers. Thus $v_1 v_2 \cdots v_d$ is an element of $\mathrm{sym}^d V$. Let $\{v_1, \ldots, v_n\}$ be a basis of V, then $\{v_1^{a_1} v_2^{a_2} \cdots v_n^{a_n} |$ all $a_i \geq 0$ and $\sum a_i = d\}$ is a basis of $\mathrm{sym}^d V$. One extends this definition by $\mathrm{sym}^1 V = V$ and $\mathrm{sym}^0 V = F$.

The *exterior powers* $\Lambda^d V$. One considers again the tensor product $V \otimes \cdots \otimes V$ of d copies of V. Let W be the subspace of this tensor product generated by the expressions $v_1 \otimes \cdots \otimes v_d$, where there are (at least) two indices $i \neq j$ with $v_i = v_j$. Then $\Lambda^d V$ is defined as the quotient space $(V \otimes \cdots \otimes V)/W$. The image of the element $v_1 \otimes \cdots \otimes v_d$ in $\Lambda^d V$ is denoted by $v_1 \wedge \cdots \wedge v_d$. If $\{v_1, \ldots, v_n\}$ is a basis of V, then the collection $\{v_{i_1} \wedge \cdots \wedge v_{i_d} | \ 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$ is a basis of $\Lambda^d V$. In particular, $\Lambda^d V = 0$ if $d > n$ and $\Lambda^n V \cong F$. This isomorphism is made explicit by choosing a basis of V and mapping $w_1 \wedge \cdots \wedge w_n$ to the determinant in F of the matrix with columns the expressions of the w_i as linear combinations of the given basis. One extends the definition by $\Lambda^1 V = V$ and $\Lambda^0 V = F$. We note that for $1 \leq d \leq n$ one has $w_1 \wedge \cdots \wedge w_d \neq 0$ if and only if $w_1, \ldots, w_d$ are linearly independent over F.

Both the symmetric powers and the exterior powers can also be defined in a categorical way using symmetric multilinear maps and alternating multilinear maps.

Definition 2.8 *Cyclic vector.*
Let M be a differential module over k. An element $e \in M$ is called a *cyclic vector* if M is generated over k by the elements $e, \partial e, \partial^2 e, \ldots$. □

The following proposition extends Lemma 2.5.

Proposition 2.9 *Every finitely generated left $k[\partial]$-module has the form $k[\partial]^n$ or $k[\partial]^n \oplus k[\partial]/k[\partial]L$ with $n \geq 0$ and $L \in k[\partial]$.*

Proof. The only thing that we have to show is that a differential module M of dimension n over k is isomorphic to $k[\partial]/k[\partial]L$ for some L. This translates into the existence of an element $e \in M$ such that M is generated by $e, \partial e, \ldots, \partial^{n-1} e$. In other words, e is a cyclic vector for M.

Any $k[\partial]$-linear map $\phi : k[\partial] \to M$ is determined by $e := \phi(1) \in M$, where $1 \in k[\partial]$ is the obvious element. The map ϕ is surjective if and only if e is a cyclic element. If the map is surjective, then its kernel is a left ideal in $k[\partial]$ and has the form $k[\partial]L$. Thus $k[\partial]/k[\partial]L \cong M$. On the other hand, an isomorphism $k[\partial]/k[\partial]L \cong M$ induces a surjective $k[\partial]$-linear map $k[\partial] \to M$. The proof of the existence of a cyclic vector for M is reproduced from N. Katz's paper [155].

Choose an element $h \in k$ with $h' \neq 0$ and define $\delta = \frac{h'}{h}\partial$. Then $k[\partial] = \mathcal{D}$ is also equal to $k[\delta]$. Furthermore, $\delta h = h\delta + h$ and $\delta h^k = h^k \delta + k h^k$ for all $k \in \mathbf{Z}$. Take an $e \in M$. Then $\mathcal{D}e$ is the subspace of M generated over k by $e, \delta e, \delta^2 e, \ldots$.

Let $\mathcal{D}e$ have dimension m. If $m = n$ then we are finished. If $m < n$ then we will produce an element $\tilde{e} = e + \lambda h^k f$, where $\lambda \in \mathbf{Q}$ and $k \in \mathbf{Z}$ and $f \in M \setminus \mathcal{D}e$, such that $\dim \mathcal{D}\tilde{e} > m$. This will prove the existence of a cyclic vector. We will work in the exterior product $\Lambda^{m+1}M$ and consider the element

$$E := \tilde{e} \wedge \delta(\tilde{e}) \wedge \cdots \wedge \delta^m(\tilde{e}) \in \Lambda^{m+1}M.$$

The multilinearity of the $\wedge$ and the rule $\delta h^k = h^k\delta + kh^k$ lead to a decomposition of E of the form

$$E = \sum_{0 \le a \le m} (\lambda h^k)^a (\sum_{0 \le b} k^b \omega_{a,b}), \text{ with } \omega_{a,b} \in \Lambda^{m+1}M \text{ independent of } \lambda, k.$$

Suppose that E is zero for every choice of λ and k. Fix k. For every $\lambda \in \mathbf{Q}$ one finds a linear dependence of the $m+1$ terms $\sum_{0 \le b} k^b \omega_{a,b}$. One concludes that for every a the term $\sum_{0 \le b} k^b \omega_{a,b}$ is zero for all choices of $k \in \mathbf{Z}$. The same argument shows that each $\omega_{a,b} = 0$. However, one easily calculates that $\omega_{1,m} = e \wedge \delta(e) \wedge \cdots \wedge \delta^{m-1}(e) \wedge f$. This term is not zero by our choice of f. □

There are other proofs of the existence of a cyclic vector, relevant for algorithms. These proofs produce a set $S \subset M$ of small cardinality such that S contains a cyclic vector. We will give two of those statements. The first one is due to Kovacic [168] (with some similarities to Cope [72, 73]).

Lemma 2.10 *Let M be a differential module with k-basis $\{e_1, \ldots, e_n\}$ and let $\eta_1, \ldots, \eta_n \in k$ be linearly independent over C, the constants of k. Then there exist integers $0 \le c_{i,j} \le n$, $1 \le i, j \le n$, such that $m = \sum_{i=1}^n a_i e_i$ is a cyclic vector for M, where $a_i = \sum_{j=1}^n c_{i,j}\eta_j$. In particular, if $z \in k$, $z' \neq 0$, then $a_i = \sum_{j=1}^n c_{i,j} z^{j-1}$ is, for suitable $c_{i,j}$ as above, a cyclic vector.*

The second one is due to Katz [155].

Lemma 2.11 *Assume that k contains an element z such that $z' = 1$. Let M be a differential module with k-basis $\{e_0, \ldots, e_{n-1}\}$. There exists a set $S \subset C$ with at most $n(n-1)$ elements such that if $a \notin S$ the element*

$$\sum_{j=0}^{n-1} \frac{(z-a)^j}{j!} \sum_{p=0}^{j} (-1)^p \binom{j}{p} \partial^p(e_{j-p}) \text{ is a cyclic vector.}$$

We refer to the literature for the proofs of these and to [80, 144, 237, 6, 30, 31]. For a generalization of the cyclic vector construction to systems of nonlinear differential equations, see [70].

2.2 Constructions with Differential Modules

The constructions with vector spaces (direct sums, tensor products, symmetric powers, etc.) extend to several other categories. The first interesting case concerns a finite group G and a field F. The category has as objects the representations G in finite dimensional vector spaces over F. A representation (ρ, V) is a homomorphism $\rho : G \to \mathrm{GL}(V)$, where V is a finite dimensional vector space over F. The tensor product $(\rho_1, V_1) \otimes (\rho_2, V_2)$ is the representation (ρ_3, V_3) with $V_3 = V_1 \otimes_F V_2$ and ρ_3 given by the formula $\rho_3(v_1 \otimes v_2) = (\rho_1 v_1) \otimes (\rho_2 v_2)$. In a similar way one defines direct sums, quotient representations, symmetric powers and exterior powers of a representation.

A second interesting case concerns a linear algebraic group G over F. A representation (ρ, V) consists of a finite dimensional vector space V over F and a homomorphism of algebraic groups over F, $\rho : G \to \mathrm{GL}(V)$. The formulas for tensor products and other constructions are the same as for finite groups. This example (and its extension to affine group schemes) is explained in the appendices.

A third example concerns a Lie algebra L over F. A representation (ρ, V) consists of a finite dimensional vector space V over F and an F-linear map $\rho : L \to \mathrm{End}(V)$ satisfying the property $\rho([A, B]) = [\rho(A), \rho(B)]$. The tensor product (ρ_1, V_1) $\otimes(\rho_2, V_2) = (\rho_3, V_3)$ with again $V_3 = V_1 \otimes_F V_2$ and with ρ_3 given by the formula $\rho_3(v_1 \otimes v_2) = (\rho_1 v_1) \otimes v_2 + v_1 \otimes (\rho_2 v_2)$.

As we will see, the above examples are related with constructions with differential modules. The last example is rather close to the constructions with differential modules.
The category of all differential modules over k will be denoted by Diff_k. *Now we start the list of constructions of linear algebra for differential modules.*

The *direct sum* $(M_1, \partial_1) \oplus (M_2, \partial_2)$ is (M_3, ∂_3), where $M_3 = M_1 \oplus M_2$ and $\partial_3(m_1 \oplus m_2) = \partial_1(m_1) \oplus \partial_2(m_2)$.

A *(differential) submodule* N of (M, ∂) is a k-vector space $N \subset M$ such that $\partial(N) \subset N$. Then $N = (N, \partial|_N)$ is a differential module.

Let N be a submodule of (M, ∂). Then M/N, provided with the induced map ∂, given by $\partial(m + N) = \partial(m) + N$, is the *quotient differential module*.

The *tensor product* $(M_1, \partial_1) \otimes (M_2, \partial_2)$ is (M_3, ∂_3) with $M_3 = M_1 \otimes_k M_2$ and ∂_3 is given by the formula $\partial_3(m_1 \otimes m_2) = (\partial_1 m_1) \otimes m_2 + m_1 \otimes (\partial_2 m_2)$. We note that this is not at all the tensor product of two $k[\partial]$-modules. In fact, the tensor product of two left $k[\partial]$-modules does not exist since $k[\partial]$ is not commutative.

A *morphism* $\phi : (M_1, \partial_1) \to (M_2, \partial_2)$ is a k-linear map such that $\phi \circ \partial_1 = \partial_2 \circ \phi$. If we regard differential modules as special left $k[\partial]$-modules, then the above translates into ϕ is a $k[\partial]$-linear map. We will sometimes write $\mathrm{Hom}_{k[\partial]}(M_1, M_2)$ (omitting ∂_1 and ∂_2 in the notation) for the C-vector space of all morphisms. This object is *not* a differential module over k, but it is $\mathrm{Mor}(M_1, M_2)$ the C-linear vector space of the morphisms in the category Diff_k.

The *internal Hom*, $\mathrm{Hom}_k((M_1, \partial_1), (M_2, \partial_2))$ of two differential modules is the k-vector space $\mathrm{Hom}_k(M_1, M_2)$ of the k-linear maps form M_1 to M_2 provided with a ∂ given by the formula $(\partial\ell)(m_1) = \ell(\partial_1 m_1) - \partial_2(\ell(m_1))$. This formula leads to the observation that

$$\mathrm{Hom}_{k[\partial]}(M_1, M_2) \text{ is equal to } \{\ell \in \mathrm{Hom}_k(M_1, M_2) |\ \partial\ell = 0\}.$$

In particular, the C-vector space $\mathrm{Mor}(M_1, M_2) = \mathrm{Hom}_{k[\partial]}(M_1, M_2)$ has dimension at most $\dim_k M_1 \cdot \dim_k M_2$.

The *trivial differential module of dimension 1* over k is again denoted by $\mathbf{1}_k$ or $\mathbf{1}$. A special case of internal Hom is the *dual M^* of a differential module M* defined by $M^* = \mathrm{Hom}_k(M, \mathbf{1}_k)$.

Symmetric powers and exterior powers are derived from tensor products and the formation of quotients. Their structure can be made explicit. The exterior power $\Lambda^d M$, for instance, is the k-vector space $\Lambda^d_k M$ provided with the operation ∂ given by the formula $\partial(m_1 \wedge \cdots \wedge m_d) = \sum_{i=1}^d m_1 \wedge \cdots \wedge (\partial m_i) \wedge \cdots \wedge m_d$.

The next collection of exercises presents some of the many properties of the above constructions and their translations into the language of differential operators and matrix differential equations.

Exercises 2.12 *Properties of the constructions*
1. Show that the tensor product of differential modules as defined above is indeed a differential module.

2. Show that, for a differential module M over k, the natural map $M \to M^{**}$ is an isomorphism of differential modules.

3. Show that the differential modules $\mathrm{Hom}_k(M_1, M_2)$ and $M_1^* \otimes M_2$ are "naturally" isomorphic.

4. Show that the k-linear map $M^* \otimes M \to 1_k$, defined by $\ell \otimes m = \ell(m)$, is a morphism of differential modules. Conclude that $M^* \otimes M$ has a nontrivial submodule if $\dim_k M > 1$.

5. Suppose that M is a trivial differential module. Show that all the constructions of linear algebra applied to M again produce trivial differential modules. Hint: Show that M^* is trivial; show that the tensor product of two trivial modules is trivial; show that any submodule of a trivial module is trivial too.

6. Suppose that $M \cong k[\partial]/k[\partial]L$. Show that $M^* \cong k[\partial]/k[\partial]L^*$. Here $L \mapsto L^*$ is the involution defined in Exercise 2.2. Hint: Let L have degree n. Show that the element $e \in \mathrm{Hom}_k(k[\partial]/k[\partial]L, \mathbf{1}_k)$ given by $e(\sum_{i=0}^{n-1} b_i \partial^i) = b_{n-1}$ is a cyclic vector and that $L^* e = 0$.

7. *The differential module M_L associated to the differential operator L.*
Consider an operator $L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in k[\partial]$. As in Sect. 1.2, one associates to L a matrix differential equation $Y' = A_L Y$, where A_L is the companion matrix

$$A_L = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & \dots & \dots & -a_{n-1} \end{pmatrix}.$$

This matrix differential equation induces a differential module M_L and we call this the *differential module associated with the operator L*.
(a) Prove that the differential modules M_L and $(k[\partial]/k[\partial]L)^*$ are isomorphic.
(b) *Operators of the same type*.
Let $L_1, L_2 \in k[\partial]$ be monic of degree n. Prove that M_{L_1} and M_{L_2} are isomorphic if and only if there are elements $R, S \in k[\partial]$ of degree $< n$ such that $L_1 R = S L_2$ and $GCRD(R, L_2) = 1$.
Hint: Describe an isomorphism $\phi : k[\partial]/k[\partial]L_1 \to k[\partial]/k[\partial]L_2$ by an operator of degree $< n$ representing the element $\phi(1) \in k[\partial]/k[\partial]L_2$.
In the classical literature, operators L_1, L_2 such that $M_{L_1} \cong M_{L_2}$ are called *of the same type*. This concept appears in the 19th century literature (for references to this literature as well as more recent references, see [271]).
(c) Prove that every matrix differential equation is equivalent to an equation of the form $Y' = A_L Y$.

8. *The matrix differential of the dual M^*.*
Let M be a differential equation and let $y' = Ay$ be an associated matrix differential equation by the choice of a basis $\{e_1, \dots, e_n\}$. Find the matrix differential equation for M^* associated to the dual basis $\{e_1^*, \dots, e_n^*\}$ of M^*.

9. *Extensions of differential fields.*
Let $K \supset k$ be an extension of differential fields. For any differential module (M, ∂) over k one considers the K-vector space $K \otimes_k M$. One defines ∂ on $K \otimes_k M$ by $\partial(a \otimes m) = a' \otimes m + a \otimes (\partial m)$. Show that this definition makes sense and that $(K \otimes_k M, \partial)$ is a differential module over K. Prove that the formation $M \mapsto K \otimes_k M$ commutes with all constructions of linear algebra.

10. *The characterization of the "internal hom".*
For the reader familiar with representable functors this exercise, which shows that the "internal hom" is derived from the tensor product, might be interesting. Consider two differential modules M_1, M_2. Associate to this the contravariant functor $\mathcal{F}$ from Diff_k to the category of sets given by the formula $\mathcal{F}(T) = \mathrm{Hom}_{k[\partial]}(T \otimes M_1, M_2)$. Show that $\mathcal{F}$ is a representable functor and that it is represented by $\mathrm{Hom}_k(M_1, M_2)$. Compare also the definition of tannakian category given in the appendices. □

Now we continue Exercise 2.12.7 and the set of morphisms between two differential modules in terms of differential operators. An operator $L \in k[\partial]$ is said to be *reducible over k* if L has a nontrivial right-hand factor. Otherwise L is called *irreducible*. Suppose that L is reducible, say $L = L_1L_2$. Then there is an obvious exact sequence of differential modules

$$0 \to \mathcal{D}/\mathcal{D}L_1 \stackrel{.L_2}{\to} \mathcal{D}/\mathcal{D}L_1L_2 \to \mathcal{D}/\mathcal{D}L_2 \to 0,$$

where the first nontrivial arrow is multiplication on the right by L_2 and the second nontrivial arrow is the quotient map. In particular, the monic right-hand factors of L correspond bijectively to the quotient modules of $\mathcal{D}/\mathcal{D}L$ (and at the same time to the submodules of $\mathcal{D}/\mathcal{D}L$).

Proposition 2.13 *For* $L_1, L_2 \in k[\partial]$*, one defines* $\mathcal{E}(L_1, L_2)$ *to consist of the* $R \in k[\partial]$ *with* $\deg R < \deg L_2$*, such that there exists an* $S \in k[\partial]$ *with* $L_1R = SL_2$.
(1) *There is a natural* C*-linear bijection between* $\mathcal{E}(L_1, L_2)$ *and* $\mathrm{Hom}_{k[\partial]}(k[\partial]/k[\partial]L_1, k[\partial]/k[\partial]L_2)$.
(2) $\mathcal{E}(L, L)$ *or* $\mathcal{E}(L)$ *is called the* (right) eigenring *of* L*. This eigenring* $\mathcal{E}(L)$ *is a finite dimensional* C*-subalgebra of* $\mathrm{End}_k(k[\partial]/k[\partial]L)$*, which contains* $C.id$*. If* L *is irreducible and* C *is algebraically closed, then* $\mathcal{E}(L) = C.id$.

Proof. (1) A $k[\partial]$-linear map $\phi : k[\partial]/k[\partial]L_1 \to k[\partial]/k[\partial]L_2$ lifts uniquely to a $k[\partial]$-linear map $\psi : k[\partial] \to k[\partial]$ such that $R := \psi(1)$ has degree $< \deg L_2$. Furthermore, $\psi(k[\partial]L_1) \subset k[\partial]L_2$. Hence, $\psi(L_1) = L_1R \in k[\partial]L_2$ and $L_1R = SL_2$ for some $S \in k[\partial]$. On the other hand, an R and S with the stated properties determine a unique ψ that induces a $k[\partial]$-linear map $\phi : k[\partial]/k[\partial]L_1 \to k[\partial]/k[\partial]L_2$.
(2) The first statement is obvious. The kernel of any element of $\mathcal{E}(L)$ is a submodule of $k[\partial]/k[\partial]L$. If L is irreducible, then any nonzero element of $\mathcal{E}(L)$ is injective and therefore also bijective. Hence $\mathcal{E}(L)$ is a division ring. Since C is algebraically closed, one has $\mathcal{E}(L) = C$. □

Exercise 2.14 *The Eigenring.*
The eigenring provides a method to obtain factors of a reducible operator, see [136, 271] and Sect. 4.2. However, even if C is algebraically closed, a reducible operator L may satisfy $\mathcal{E}(L) = C.id$. In this case no factorization is found. The aim of this exercise is to provide an example.

1. The field C of the constants of k is supposed to be algebraically closed. Let $M = k[\partial]/k[\partial]L$ be a differential module over k of dimension 2. Prove that $\mathcal{E}(L) \neq C.id$ if and only if M has submodules N_1, N_2 of dimension 1 such that N_2 and M/N_1 are isomorphic. Hint: $\mathcal{E}(L) \neq C.id$ implies that there is a morphism $\phi : M \to M$ such that $N_1 := \ker(\phi)$ and $N_2 := \mathrm{im}(\phi)$ have dimension 1.

2. Take $k = C(z)$, $z' = 1$ and $L = (\partial + 1 + z^{-1})(\partial - 1)$. Show that $M := k[\partial]/k[\partial]L$ has only one submodule N of dimension 1 and that N and M/N are not isomorphic. Hint: The submodules of dimension 1 correspond to right-hand factors $\partial - u$ of L,

with $u \in k$. Perform Kovacic's algorithm to obtain the possibilities for u. This works as follows (see also Chap. 4). Derive the equation $u^2 + z^{-1}u + u' - (1 + z^{-1}) = 0$. Expand a potential solution u at $z = 0$ and $z = \infty$ as a Laurent series and show that u has no poles at $z = 0$ and $z = \infty$. At any point $c \in C^*$, the Laurent series of u has the form $\frac{\epsilon}{z-c} + \cdots$ with $\epsilon = 0, 1$. Calculate that $u = 1$ is the only possibility. □

We end this section with a discussion of the "solution space" of a differential module. To do this we shall need a *universal differential extension field* of a field k. This is defined formally (and made explicit in certain cases) in Sect. 3.2 but for our purposes it is enough to require this to be a field $\mathcal{F} \supset k$ with the same field of constants of k such that any matrix differential equation $Y' = AY$ over k has a solution in $\mathrm{GL}_n(\mathcal{F})$. Such a field can be constructed as a direct limit of all Picard-Vessiot extensions of k and we shall fix one and denote it by $\mathcal{F}$. We note that Kolchin [162] uses the term universal extension to denote a field containing solutions of ALL differential equations but our restricted notion is sufficient for our purposes.

Definition 2.15 Let M be a differential module over k with algebraically closed constants C and $\mathcal{F}$ a universal differential extension of k. The *covariant solution space* of M is the C-vector space $\ker(\partial, \mathcal{F} \otimes_k M)$. The *contravariant solution space* is the C-vector space $\mathrm{Hom}_{k[\partial]}(M, \mathcal{F})$. □

The terms "covariant" and "contravariant" reflect the following properties. Let $\phi : M_1 \to M_2$ be a morphism of differential modules. Then there are induced homomorphisms of C-vector spaces $\phi_* : \ker(\partial, \mathcal{F} \otimes_k M_1) \to \ker(\partial, \mathcal{F} \otimes_k M_2)$ and $\phi^* : \mathrm{Hom}_{k[\partial]}(M_2, \mathcal{F}) \to \mathrm{Hom}_{k[\partial]}(M_1, \mathcal{F})$. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of differential modules, then so is

$$0 \to \ker(\partial, \mathcal{F} \otimes_k M_1) \to \ker(\partial, \mathcal{F} \otimes_k M_2) \to \ker(\partial, \mathcal{F} \otimes_k M_3) \to 0.$$

This follows easily from the exactness of the sequence

$$0 \to \mathcal{F} \otimes_k M_1 \to \mathcal{F} \otimes_k M_2 \to \mathcal{F} \otimes_k M_3 \to 0$$

and the observation that $\dim_C \ker(\partial, \mathcal{F} \otimes_k M) = \dim_k M$ for any differential module M over k. The contravariant solution space also induces a contravariant exact functor from differential modules to finite dimensional C-vector spaces.

Lemma 2.16 *Let M be a differential modules with basis $e_1, \ldots, e_n$ and let $\partial e_i = -\sum_j a_{j,i} e_j$ and $A = (a_{i,j})$. Then*
1. $\ker(\partial, \mathcal{F} \otimes_k M) \simeq \{y \in \mathcal{F}^n \mid y' = Ay\}$.
2. *There are natural C-vector space isomorphisms*

$$\mathrm{Hom}_{k[\partial]}(M, \mathcal{F}) \simeq \mathrm{Hom}_{\mathcal{F}[\partial]}(\mathcal{F} \otimes_k M, \mathcal{F}) \simeq \mathrm{Hom}_C(\ker(\partial, \mathcal{F} \otimes_k M), C).$$

3. *Let $e \in M$ and let $L \in k[\partial]$ be its minimal monic annihilator. Let $W = \{y \in \mathcal{F} \mid L(y) = 0\}$. The map $\mathrm{Hom}_{k[\partial]}(M, \mathcal{F}) \to W \subset \mathcal{F}$, given by $\phi \mapsto \phi(e)$, is surjective.*

Proof. 1. The basis $e_1, \ldots, e_n$ yields an identification of $\mathcal{F} \otimes M$ with $\mathcal{F}^n$ and of ∂ with the operator $\frac{d}{dz} - A$ on $\mathcal{F}^n$.

2. Any $k[\partial]$-linear map $\phi : M \to \mathcal{F}$ extends to an $\mathcal{F}[\partial]$-linear map $\mathcal{F} \otimes_k M \to \mathcal{F}$. This gives the first isomorphism. Any ϕ in $\mathrm{Hom}_{\mathcal{F}[\partial]}(\mathcal{F} \otimes_k M, \mathcal{F})$ defines by restriction a C-linear map $\tilde{\phi} : \ker(\partial, \mathcal{F} \otimes_k M) \to C$. The map $\phi \mapsto \tilde{\phi}$ is a bijection since the natural map $\mathcal{F} \otimes_C \ker(\partial, \mathcal{F} \otimes_k M) \to \mathcal{F} \otimes_k M$ is an isomorphism.

3. The natural morphism $\mathrm{Hom}_{k[\partial]}(M, \mathcal{F}) \to \mathrm{Hom}_{k[\partial]}(k[\partial]e, \mathcal{F})$ is surjective, since these spaces are contravariant solution spaces and $k[\partial]e$ is a submodule of M. The map $\mathrm{Hom}_{k[\partial]}(k[\partial]e, \mathcal{F}) \to W$, given by $\phi \mapsto \phi(e)$, is bijective since the map $k[\partial]/k[\partial]L \to k[\partial]e$ (with $1 \mapsto e$) is bijective. $\square$

2.3 Constructions with Differential Operators

Differential operators do not form a category where one can perform constructions of linear algebra. However, in the literature tensor products, symmetric powers, etc., of differential operators are often used. In this section we will explain this somewhat confusing terminology and relate it with the constructions of linear algebra on differential modules.

A pair (M, e) of a differential module M and a cyclic vector $e \in M$ determines a monic differential operator L, namely the operator of smallest degree with $Le = 0$. Two such pairs (M_i, e_i), $i = 1, 2$ define the same monic operator if and only if there exists an isomorphism $\psi : M_1 \to M_2$ of differential modules such that $\psi e_1 = e_2$. Moreover, this ψ is unique. For a monic differential operator L one chooses a corresponding pair (M, e). On M and e one performs the construction of linear algebra. This yields a pair $(\mathrm{constr}(M), \mathrm{constr}(e))$. Now $\mathrm{constr}(L)$ is defined as the monic differential operator of minimal degree with $\mathrm{constr}(L)\mathrm{constr}(e) = 0$. This procedure extends to constructions involving several monic differential operators. There is one complicating factor, namely $\mathrm{constr}(e)$ is, in general, not a cyclic vector for $\mathrm{constr}(M)$.

There is another interpretation of a monic differential operator L. Let, as before, $\mathcal{F} \supset k$ denote a fixed universal differential field. One can associate to L its solution space $\mathrm{Sol}(L) := \{y \in \mathcal{F} | L(y) = 0\}$. This space determines L. Indeed, assume that $L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0$. Then $\mathrm{Sol}(L)$ has dimension n over C. Let $y_1, \ldots, y_n$ be a basis of $\mathrm{Sol}(L)$. Then $a_{n-1}, \ldots, a_0$ satisfy the linear equations

$$y_i^{(n)} + a_{n-1}y_i^{(n-1)} + \cdots + a_1 y_i^{(1)} + a_0 y_i = 0 \text{ for } i = 1, \ldots, n.$$

The wronskian matrix of $y_1, \ldots, y_n$ has nonzero determinant and thus the equations determine $a_{n-1}, \ldots, a_0$. Let $\mathrm{Gal}(\mathcal{F}/k)$ denote the group of the differential automorphisms of $\mathcal{F}/k$, i.e., the automorphisms of the field $\mathcal{F}$ that are k-linear and commute with the differentiation on $\mathcal{F}$. For a Picard-Vessiot extension $K \supset k$ the group $\mathrm{Gal}(K/k)$ of differential automorphisms of K/k has the property that $K^{\mathrm{Gal}(K/k)} = k$. The universal differential extension $\mathcal{F}$ is the direct limit of all Picard-Vessiot field extensions of k. It follows from this that $\mathcal{F}^{\mathrm{Gal}(\mathcal{F}/k)} = k$. This leads to the following result.

Lemma 2.17 *Let $V \subset \mathcal{F}$ be a vector space over C of dimension n. There exists a (unique) monic differential operator $L \in k[\partial]$ with* $\mathrm{Sol}(L) = V$ *if and only if V is (set-wise) invariant under* $\mathrm{Gal}(\mathcal{F}/k)$.

Proof. As above, one observes that any V determines a unique monic differential operator $L \in \mathcal{F}[\partial]$ such that $V = \{y \in \mathcal{F} \mid L(y) = 0\}$. Then V is invariant under $\mathrm{Gal}(\mathcal{F}/k)$ if and only if L is invariant under $\mathrm{Gal}(\mathcal{F}/k)$. The latter is equivalent to $L \in k[\partial]$. □

We note that the lemma remains valid if $\mathcal{F}$ is replaced by a Picard-Vessiot field extension $K \supset k$ and $\mathrm{Gal}(\mathcal{F}/k)$ by $\mathrm{Gal}(K/k)$.

This leads to another way, omnipresent in the literature, of defining a construction of linear algebra to a monic differential operator L. One applies this construction to $\mathrm{Sol}(L)$ and finds a new subspace V of $\mathcal{F}$. This subspace is finite dimensional over C and invariant under G. By the above lemma this determines a new monic differential operator. This procedure extends to constructions with several monic differential operators.

The link between these two ways of making new operators is given by the contravariant solution space. Consider a monic differential operator L and a corresponding pair (M, e). By Definition 2.15 and Lemma 2.16, $\mathrm{Sol}(L)$ is the image of the contravariant solution space $\mathrm{Hom}_{k[\partial]}(M, \mathcal{F})$ of M under the map $\phi \mapsto \phi(e)$. We will make the above explicit for various constructions of linear algebra. Needless to say, this section is only concerned with the language of differential equations and does not contain new results.

Tensor Products. Let $(M_i, e_i),\ i = 1, 2$ denote two differential modules with cyclic vectors. The tensor product construction is $(M_1 \otimes M_2, e_1 \otimes e_2)$. In general, $e_1 \otimes e_2$ need not be a cyclic vector of $M_1 \otimes M_2$ (see Exercise 2.21). Our goal is to describe the contravariant solution space of $M_1 \otimes M_2$, the minimal monic annihilator of $e_1 \otimes e_2$ and its solution space in $\mathcal{F}$.

Lemma 2.18 *The canonical isomorphism*

$$\mathrm{Hom}_{k[\partial]}(M_1, \mathcal{F}) \otimes_C \mathrm{Hom}_{k[\partial]}(M_2, \mathcal{F}) \ \simeq\ \mathrm{Hom}_{k[\partial]}(M_1 \otimes M_2, \mathcal{F})$$

is described by $\phi_1 \otimes \phi_2 \mapsto \overline{\phi_1 \otimes \phi_2}$ *where* $\overline{\phi_1 \otimes \phi_2}(m_1 \otimes m_2) := \phi(m_1)\phi_2(m_2)$.

Proof. The canonical isomorphism $c : (\mathcal{F} \otimes_k M_1) \otimes_{\mathcal{F}} (\mathcal{F} \otimes_k M_2) \to \mathcal{F} \otimes_k (M_1 \otimes_k M_2)$ of differential modules over $\mathcal{F}$ is given by $(f_1 \otimes m_1) \otimes (f_2 \otimes m_2)$

$\mapsto f_1 f_2 \otimes m_1 \otimes m_2$. This c induces an isomorphism of the covariant solution spaces

$$\ker(\partial, \mathcal{F} \otimes_k M_1) \otimes_C \ker(\partial, \mathcal{F} \otimes_k M_2) \to \ker(\partial, \mathcal{F} \otimes_k (M_1 \otimes_k M_2)).$$

We write again c for this map. By taking duals as C-vector spaces and after replacing c by c^{-1} one obtains the required map $(c^{-1})^*$ (cf. Lemma 2.16). The formula for this map is easily verified. □

Corollary 2.19 *Let the monic differential operators L_i correspond to the pairs (M_i, e_i) for $i = 1, 2$. Let L be the monic operator of minimal degree such that $L(e_1 \otimes e_2) = 0$. Then the solution space of L in $\mathcal{F}$, i.e., $\{y \in \mathcal{F} | L(y) = 0\}$, is equal to the image of the contravariant solution space* $\mathrm{Hom}_{k[\partial]}(M_1 \otimes M_2, \mathcal{F})$ *under the map $\phi \mapsto \phi(e_1 \otimes e_2)$. In particular, L is the monic differential operator of minimal degree such that $L(y_1 y_2) = 0$ for all pairs $y_1, y_2 \in \mathcal{F}$ such that $L_1(y_1) = L_2(y_2) = 0$.*

Proof. Apply Lemma 2.16.3 to $e_1 \otimes e_2$. The image of the contravariant solution space of $M_1 \otimes M_2$ in $\mathcal{F}$ under the map $\phi \mapsto \phi(e_1 \otimes e_2)$ is generated as a vector space over C by the products $\phi_1(e_1)\phi_2(e_2)$, according to Lemma 2.18. □

It is hardly possible to compute the monic operator L of minimal degree satisfying $L(e_1 \otimes e_2) = 0$ by the previous corollary. Indeed, $\mathcal{F}$ is, in general, not explicit enough. The obvious way to find L consists of computing the elements $\partial^n(e_1 \otimes e_2)$ in $M_1 \otimes M_2$ and to find a linear relation over k between these elements. In the literature one finds the following definition (or an equivalent one) of the tensor product of two monic differential operators.

Definition 2.20 Let L_1 and L_2 be two differential operators. The minimal monic annihilating operator of $1 \otimes 1$ in $k[\partial]/k[\partial]L_1 \otimes k[\partial]/k[\partial]L_2$ is the *tensor product $L_1 \otimes L_2$ of L_1 and L_2.*

Exercise 2.21 Prove that $\partial^3 \otimes \partial^2 = \partial^4$. □

Similar definitions and results hold for tensor products with more than two factors.

Symmetric Powers. The d-th *symmetric power* $\mathrm{sym}^d M$ of a differential module is a quotient of the ordinary d-fold tensor product $M \otimes \cdots \otimes M$. The image of $m_1 \otimes m_2 \otimes \cdots \otimes m_d$ in this quotient will be written as $m_1 m_2 \cdots m_d$. In particular, m^d denotes the image of $m \otimes \cdots \otimes m$. This construction applied to (M, e) produces $(\mathrm{sym}^d M, e^d)$.

Lemma 2.22 *Let M be a differential module over k. The canonical isomorphism of contravariant solution spaces*

$$\mathrm{sym}^d(\mathrm{Hom}_{k[\partial]}(M, \mathcal{F})) \to \mathrm{Hom}_{k[\partial]}(\mathrm{sym}^d M, \mathcal{F})$$

is given by the formula $\phi_1\phi_2 \cdots \phi_d \mapsto \overline{\phi_1\phi_2 \cdots \phi_d}$,
where $\overline{\phi_1\phi_2 \cdots \phi_d}(m_1 m_2 \cdots m_d) := \phi(m_1)\phi_2(m_2) \cdots \phi_d(m_d)$.

The proof is similar to that of Lemma 2.18. The same holds for the next corollary.

Corollary 2.23 *Let L correspond to the pair (M, e). The image of the map $\phi \mapsto \phi(e^d)$ from $\mathrm{Hom}_{k[\partial]}(\mathrm{sym}^d M, \mathcal{F})$ to $\mathcal{F}$ is the C-vector space generated by $\{f_1 f_2 \cdots f_d \mid L(f_i) = 0\}$.*

Definition 2.24 Let L be a monic differential operator and let $e = 1$ be the generator of $k[\partial]/k[\partial]L$. The minimal monic annihilating operator of e^d in $\mathrm{sym}^d(k[\partial]/k[\partial]L)$ is the d-th symmetric power $\mathrm{Sym}^d(L)$ of L. □

Exercise 2.25
(1) Show that $\mathrm{Sym}^2(\partial^3) = \partial^5$.
(2) Show that $\mathrm{Sym}^d(L)$ has degree $d+1$ if L has degree 2. Hint: Use Proposition 4.26. □

Exterior Powers. One associates to a pair (M, e) (with e a cyclic vector) the pair $(\Lambda^d M, e \wedge \partial e \wedge \cdots \wedge \partial^{d-1} e)$.

Definition 2.26 Let L be a differential operator and let $e = 1$ be the generator of $k[\partial]/k[\partial]L$. The minimal monic annihilating operator of $e \wedge \partial e \wedge \ldots \wedge \partial^{d-1} e$ in $\Lambda^d(k[\partial]/k[\partial]L)$ is the d-th exterior power $\Lambda^d(L)$ of L. □

We denote by $\mathcal{S}_d$ the permutation group of d elements. Similar to the previous constructions one has the following lemma.

Lemma 2.27 *Let M be a differential module over k. The natural isomorphism of contravariant solution spaces*

$$\Lambda^d_C \mathrm{Hom}_{k[\partial]}(M, \mathcal{F}) \to \mathrm{Hom}_{k[\partial]}(\Lambda^d_k M, \mathcal{F})$$

is given by $\phi_1 \wedge \cdots \wedge \phi_d \mapsto \overline{\phi_1 \wedge \cdots \wedge \phi_d}$, where

$$\overline{\phi_1 \wedge \cdots \wedge \phi_d}(m_1 \wedge \cdots \wedge m_d) := \sum_{\pi \in \mathcal{S}_d} \mathrm{sgn}(\pi) \phi_1(m_{\pi(1)}) \phi_2(m_{\pi(2)}) \cdots \phi_d(m_{\pi(d)}).$$

Note that for $e \in M, \phi_1 \ldots, \phi_d \in \mathrm{Hom}_{k[\partial]}(M, \mathcal{F})$ and $y_i := \phi_i(e)$, we have

$$\begin{aligned}\overline{\phi_1 \wedge \cdots \wedge \phi_d}(e \wedge \partial e \wedge \cdots \wedge \partial^{d-1} e) &= \det \begin{pmatrix} y_1 & \cdots & y_d \\ y_1' & \ldots & y_d' \\ \vdots & \ldots & \vdots \\ y_1^{(d-1)} & \ldots & y_d^{(d-1)} \end{pmatrix} \\ &= wr(y_1, \ldots, y_d).\end{aligned}$$

One therefore has the following corollary.

Corollary 2.28 *Let e be a cyclic vector for M with minimal annihilating operator L. Let $W \subset \mathcal{F}$ be the C-span of $\{wr(y_1, \dots, y_d) \mid L(y_i) = 0\}$. Then the map $\phi \mapsto \phi(e \wedge \partial e \wedge \dots \wedge \partial^{d-1} e)$ defines a surjection of $\mathrm{Hom}_{k[\partial]}(\Lambda^d M, \mathcal{F})$ onto W and W is the solution space of the minimal annihilating operator of $e \wedge \partial e \wedge \dots \wedge \partial^{d-1} e$.*

The calculation of the d-th exterior power of L is similar to the calculations in the previous two constructions. Let $v = e \wedge \partial e \wedge \cdots \wedge \partial^{d-1} e$. Differentiate v $\binom{n}{d}$ times and use L to replace occurrences of $\partial^j,\ j \geq n$ with linear combinations of $\partial e^i,\ i < n$. This yields a system of $\binom{n}{d} + 1$ equations

$$\partial^i v = \sum_{\substack{J = (j_1, \dots, j_d) \\ 0 \leq j_1 < \cdots < j_d \leq n-1}} a_{i,J}\, \partial^{j_1} e \wedge \cdots \wedge \partial^{j_d} e \tag{2.1}$$

in the $\binom{n}{d}$ quantities $\partial^{j_1} e \wedge \cdots \wedge \partial^{j_d} e$ with $a_{i,J} \in k$. These equations are linearly dependent and a linear relation among the first t of these (with t as small as possible) yields the exterior power.

We illustrate this with one example. (A more detailed analysis and simplification of the process to calculate the associated equations is given in [58, 60].)

Example 2.29 Let $L = \partial^3 + a_2\partial^2 + a_1\partial + a_0,\ a_i \in k$ and $M = k[\partial]/k[\partial]L$. Letting $e = 1$, we have that $\Lambda^2 M$ has a basis $\{\partial^i \wedge \partial^j \mid 1 \leq i < j \leq 2\}$. We have

$$\begin{aligned} v &= e \wedge \partial e, \\ \partial v &= e \wedge d^2 e, \\ \partial^2 v &= e \wedge (-a_2\partial^2 e - a_1 \partial e -_0 e) + \partial e \wedge \partial^2 e. \end{aligned}$$

Therefore $(\partial^2 + a_2\partial + a_1)v = \partial e \wedge \partial^2 e$ and so $\partial(\partial^2 + a_2\partial + a_1)v = \partial e \wedge (-a_2\partial^2 e - a_1\partial e - a_0 e)$. This implies that the minimal annihilating operator of v is $(\partial + a_2) \cdot (\partial^2 + a_2\partial + a_1) - a_0$. □

It is no accident that the order of the $(n-1)$st exterior power of an operator of order n is also n. The following exercise outlines a justification.

Exercise 2.30 *Exterior powers and adjoint operators*
Let $L = \partial^n + a_{n-1}\partial^{(n-1)} + \cdots + a_0$ with $a_i \in k$. Let K be a Picard-Vessiot extension of k associated with L and let $\{y_1, \dots, y_n\}$ be a fundamental set of solutions of $L(y) = 0$. The set $\{u_1, \dots, u_n\}$, where $u_i = wr(y_1, \dots, \hat{y}_i, \dots, y_n\}$, spans the solution space of $\Lambda^{n-1}(L)$. The aim of this exercise is to show that the set $\{u_1, \dots, u_n\}$ is linearly independent and so $\Lambda^{n-1}(L)$ always has order n. We, furthermore, show that the operators $\Lambda^{n-1}(L)$ and L^* (the adjoint of L, see Exercise 2.1) are related in a special way (cf. [255], pages 167–171).

1. Show that $v_i = u_i / wr(y_1, \dots, y_n)$ satisfies $L^*(v_i) = 0$. Hint: Let A_L be the companion matrix of L and $W = Wr(y_1, \dots, y_n)$. Since $W' = A_L W$, we have the

fact that $U = (W^{-1})^T$ satisfies $U' = -A_L^T U$. Let $(f_0, \dots, f_{n-1})^T$ be a column of U. Note that $f_{n-1} = v_i$ for some i. One has (cf. Exercise 2.1),

$$\begin{aligned} -f'_{n-1} + a_{n-1} f_{n-1} &= f_{n-2} \\ -f'_i + a_i f_{n-1} &= f_{i-1} \quad 1 \le i \le n-2 \\ -f'_0 + a_0 f_{n-1} &= 0, \end{aligned}$$

and so

$$\begin{aligned} -f'_{n-1} + a_{n-1} f_{n-1} &= f_{n-2} \\ (-1)^2 f''_{n-1} - a_{n-1} f'_{n-1} + a_{n-2} f_{n-1} &= f_{n-3} \\ &\vdots\ \vdots\ \vdots \\ (-1)^n f^{(n)}_{n-1} + (-1)^{n-1}(a_{n-1} f_{n-1})^{(n-1)} + \ldots + a_0 f_{n-1} &= 0. \end{aligned}$$

This last equation implies that $0 = L^*(f_n) = L^*(v_i)$.

2. Show that $wr(v_1, \dots, v_n) \neq 0$. Therefore the map $z \mapsto z/wr(y_1, \dots, y_n)$ is an isomorphism of the solution space of $\Lambda^{n-1}(L)$ onto the solution space of L^* and, in particular, the order of $\Lambda^{n-1}(L)$ is always n. Hint: Standard facts about determinants imply that $\sum_{i=1}^n v_i y_i^j = 0$ for $j = 0, 1, \dots, n-2$ and $\sum_{i=1}^n v_i y_i^{(n-1)} = 1$. Use these equations and their derivatives to show that $Wr(v_1, \dots, v_n) Wr(y_1, \dots, y_n) = 1$. □

Exercise 2.31 Show that $\Lambda^2(\partial^4) = \partial^5$. Therefore the d-th exterior power of an operator of order n can have order less than $\binom{n}{d}$. Hint: Show that the solution space of $\Lambda^2(\partial^4)$ is the space of polynomials of degree at most 4. □

We note that in the classical literature (cf. [255], p. 167), the d-th exterior power of an operator is referred to as the $(n-d)$th *associated operator*.

In connection with Chap. 4, a generalization of $\Lambda^d(L)$ is of interest. This generalization is present in the algorithms developed by Tsarev, Grigoriev et al. that refine Beke's algorithm for finding factors of a differential operator. Let $\mathcal{I} = (i_1, \dots, i_d)$, $0 \le i_1 < \ldots < i_d \le n-1$. Let $e = 1$ in $k[\partial]/k[\partial]L$. We define the d-th *exterior power of L with respect to* $\mathcal{I}$, denoted by $\Lambda^d_{\mathcal{I}}(L)$, to be the minimal annihilating operator of $\partial^{i_1} e \wedge \cdots \wedge \partial^{i_d} e$ in $\Lambda^d(k[\partial]/k[\partial]L)$. One sees as above that the solution space of $\Lambda^d_{\mathcal{I}}(L)$ is generated by $\{w_{\mathcal{I}}(y_1, \dots, y_d) \mid L(y_i) = 0\}$, where $w_{\mathcal{I}}(y_1, \dots, y_d)$ is the determinant of the $d \times d$ matrix formed from the rows $i_1 + 1, \dots i_d + 1$ of the $n \times d$ matrix

$$\begin{pmatrix} y_1 & y_2 & \dots & y_d \\ y'_1 & y'_2 & \dots & y'_d \\ \vdots & \vdots & \dots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_d^{(n-1)} \end{pmatrix}.$$

This operator is calculated by differentiating the element $v = \partial^{i_1} e \wedge \cdots \wedge \partial^{i_d} e$ as above. The following lemma is useful.

Lemma 2.32 *Let k and L be as above and assume that $\Lambda^d(L)$ has order $\nu = \binom{n}{d}$. For any $\mathcal{I}$ as above, there exist $b_{\mathcal{I},0}, \ldots, b_{\mathcal{I},\nu-1} \in k$ such that*

$$w_{\mathcal{I}}(y_1, \ldots, y_d) = \sum_{j=0}^{\nu-1} b_{\mathcal{I},j} wr(y_1, \ldots, y_d)^{(i)}$$

for any solutions $y_1, \ldots, y_d$ of $L(y) = 0$.

Proof. If $\Lambda^d(L)$ has order ν, then this implies that the system of equations (2.1) has rank ν. Furthermore, $\partial^{i_1} e \wedge \cdots \wedge \partial^{i_d} e$ appears as one of the terms in this system. Therefore we can solve for $\partial^{i_1} e \wedge \cdots \wedge \partial^{i_d} e$ as a linear function $\sum_{i=0}^{\nu-1} b_{\mathcal{I},i} \partial^i v$ of $v = e \wedge \partial e \wedge \cdots \wedge \partial^{d-1} e$ and its derivatives up to order $\nu - 1$. This gives the desired equation. □

We close this section by noting that MAPLE V contains commands in its DEtools package to calculate tensor products, symmetric powers, and exterior powers of operators.

2.4 Differential Modules and Representations

Throughout this section k will denote a differential field with algebraically closed subfield of constants C.

We recall that Diff_k denotes the category of all differential modules over k. Fix a differential module M over k. For integers $m, n \geq 0$ one defines the differential module $M_n^m = M \otimes \cdots M \otimes M^* \otimes \cdots \otimes M^*$, i.e., the tensor product of n copies of M and m copies of the dual M^* of M. For $m = n = 0$ the expression M_0^0 is assumed to mean $\mathbf{1} = \mathbf{1}_k$, the trivial 1-dimensional module over k. A *subquotient* of a differential module N is a differential module of the form N_1/N_2 with $N_2 \subset N_1 \subset N$ submodules. The subcategory $\{\{M\}\}$ of Diff_k is defined by: The objects of this category are the subquotients of finite direct sums of the M_n^m. For objects A, B of $\{\{M\}\}$ one defines $\mathrm{Hom}(A, B)$ to be $\mathrm{Hom}_{k[\partial]}(A, B)$. Thus $\mathrm{Hom}(A, B)$ has the same meaning in $\{\{M\}\}$ and in Diff_k. This is usually expressed as "$\{\{M\}\}$ is a full subcategory of Diff_k." It is easily seen that $\{\{M\}\}$ is the smallest full subcategory of Diff_k that contains M and is closed under all operations of linear algebra (i.e., direct sums, tensor products, duals, subquotients).

For a linear algebraic group G over C one considers the category Repr_G that consists of the representations of G on finite dimensional vector spaces over C (see the beginning of Sect. 2.2 and the appendices). A finite dimensional vector space W over C together with a representation $\rho : G \to \mathrm{GL}(W)$ is also called a G-module.

In the category Repr_G one can also perform all operations of linear algebra (i.e., direct sums, tensor products, duals, subquotients). The strong connection between the differential module M and its differential Galois group G is given in the following theorem.

Theorem 2.33 *Let M be a differential module over k and let G denote its differential Galois group. There is a C-linear equivalence of categories*

$$S : \{\{M\}\} \to \mathrm{Repr}_G,$$

which is compatible with all constructions of linear algebra.

Proof. We start by *explaining the terminology*. First of all, S is a functor. This means that S associates to every object A of the first category an object $S(A)$ of the second category. Likewise, S associates to every morphism $f \in \mathrm{Hom}(A, B)$ of the first category a morphism $S(f) \in \mathrm{Hom}(S(A), S(B))$ of the second category. The following rules should be satisfied:
$S(1_A) = 1_{S(A)}$ and $S(f \circ g) = S(f) \circ S(g)$.
The term C-linear means that the map from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(S(A), S(B))$, given by $f \mapsto S(f)$, is C-linear. The term "equivalence" means that the map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(S(A), S(B))$ is bijective and that there exists for every object B of the second category an object A of the first category such that $S(A)$ is isomorphic to B. The compatibility of S with, say, tensor products means that there are isomorphisms $i_{A,B} : S(A \otimes B) \to S(A) \otimes S(B)$. These isomorphisms should be "natural" in the sense that for any morphisms $f : A \to A',\ g : B \to B'$ the following diagram is commutative.

$$\begin{array}{ccc} S(A \otimes B) & \xrightarrow{S(f\otimes g)} & S(A' \otimes B') \\ \downarrow{\scriptstyle i_{A,B}} & & \downarrow{\scriptstyle i_{A',B'}} \\ S(A) \otimes S(B) & \xrightarrow{S(f)\otimes S(g)} & S(A') \otimes S(B'). \end{array}$$

Thus, the compatibility with the constructions of linear algebra means that S maps a construction in the first category to one object in the second category that is in a "natural" way isomorphic to the same construction in the second category. For the S that we will construct almost all these properties will be obvious.

For the definition of S we need the Picard-Vessiot field $K \supset k$ of M. The differential module $K \otimes_k M$ over K is trivial in the sense that there is a K-basis $e_1, \ldots, e_d$ of $K \otimes_k M$ such that $\partial e_i = 0$ for all i. In other words, the obvious map $K \otimes_C \ker(\partial, K \otimes_k M) \to K \otimes_k M$ is a bijection. Indeed, this is part of the definition of the Picard-Vessiot field. Also, every $K \otimes_k M_n^m$ is a trivial differential module over K. We conclude that for every object N of $\{\{M\}\}$ the differential module $K \otimes_k N$ is trivial. One defines S by $S(N) = \ker(\partial, K \otimes_k N)$. This object is a finite dimensional vector space over C. The action of G on $K \otimes_k N$ (induced by the action of G on K) commutes with ∂ and thus G acts on the kernel of ∂ on $K \otimes_k N$. From

Theorem 1.27 one easily deduces that the action of G on $\ker(\partial, K\otimes_k N)$ is algebraic. In other words, $S(N)$ is a representation of G on a finite dimensional vector space over C. Let $f : A \to B$ be a morphism in $\{\{M\}\}$. Then f extends to a K-linear map $1_K \otimes f : K\otimes_k A \to K\otimes_k B$, which commutes with ∂. Therefore, f induces a C-linear map $S(f) : S(A) \to S(B)$ which commutes with the G-actions.

We will omit the straightforward and tedious verification that S commutes with the constructions of linear algebra. It is not a banality to show that $\mathrm{Hom}(A, B) \to \mathrm{Hom}(S(A), S(B))$ is a bijection. Since $\mathrm{Hom}(A, B)$ is equal to $\ker(\partial, A^* \otimes B) = \mathrm{Hom}(\mathbf{1}_k, A^*\otimes B)$ we may assume that $A = \mathbf{1}_k$ and that B is arbitrary. Clearly $S(\mathbf{1}_k) = \mathbf{1}_G$, where the latter is the 1-dimensional trivial representation of G. Now $\mathrm{Hom}(\mathbf{1}_k, B)$ is equal to $\{b \in B|\ \partial(b) = 0\}$. Furthermore, $\mathrm{Hom}(\mathbf{1}_G, S(B))$ is equal to $\{v \in \ker(\partial, K\otimes_k B)|\ gv = v \text{ for all } g \in G\}$. Since $K^G = k$, one has $(K\otimes_k B)^G = B$. This implies that $\{b \in B|\ \partial b = 0\} \to \mathrm{Hom}(\mathbf{1}_G, S(B))$ is a bijection.

Finally, we have to show that any representation B of G is equivalent to the representation $S(A)$ for some $A \in \{\{M\}\}$. This follows from the following fact on representations of any linear algebraic group G (see [302] and the appendices):

Suppose that V is a faithful representation of G (i.e., $G \to \mathrm{GL}(V)$ is injective). Then every representation of G is a direct sum of subquotients of the representations $V\otimes\cdots\otimes V\otimes V^*\otimes\cdots\otimes V^*$.

In our situation we take for V the representation $S(M)$ that is by definition faithful. Since S commutes with the constructions of linear algebra, we have the fact that any representation of G is isomorphic to $S(N)$ for some N that is a direct sum of subquotients of the M_n^m. In other words, N is an object of $\{\{M\}\}$. □

Remarks 2.34

(1) In the terminology of tannakian categories, Theorem 2.33 states that the category $\{\{M\}\}$ is a neutral tannakian category and that G is the corresponding affine group scheme (see the appendices).

(2) The functor S has an "inverse". We will describe this inverse by constructing the differential module N corresponding to a given representation W. One considers the trivial differential module $K\otimes_C W$ over K with ∂ defined by $\partial(1\otimes w) = 0$ for every $w \in W$. The group G acts on $K\otimes_C W$ by $g(f\otimes w) = g(f)\otimes g(w)$ for every $g \in G$. Now one takes the G-invariants $N := (K\otimes_C W)^G$. This is a vector space over k. The operator ∂ maps N to N, since ∂ commutes with the action of G. One has now to show that N has finite dimension over k, that N is an object of $\{\{M\}\}$, and that $S(N)$ is isomorphic to W.

We know already that $W \cong S(A)$ for some object A in $\{\{M\}\}$. Let us write $W = S(A)$ for convenience. Then by the definition of S one has $K\otimes_C W = K\otimes_k A$ and the two objects have the same G-action and the same ∂. Then $(K\otimes_C W)^G = A$ and this finishes the proof.

(3) Let H be a closed normal subgroup of G. Choose a representation W of G such that the kernel of $G \to \mathrm{GL}(W)$ is H. Let N be an object of $\{\{M\}\}$ with

$S(N) = W$. The field K contains a Picard-Vessiot field L for N, since $K \otimes_k N$ is a trivial differential module over K. The action of the subgroup H on L is the identity since by construction the differential Galois group of N is G/H. Hence $L \subset K^H$. Equality holds by Galois correspondence, see Proposition 1.34 part 1. Thus, we have obtained a more natural proof of the statement in loc. cit. part 2, namely that K^H is the Picard-Vessiot field of some differential equation over k. □

Corollary 2.35 *Let $L \in k[\partial]$ be a monic differential operator of degree ≥ 1. Let K be the Picard-Vessiot field of $M := k[\partial]/k[\partial]L$ and G its differential Galois group. Put $V = \ker(\partial, K \otimes_k M)$ (This is the covariant solution space of M.) There are natural bijections between:*
(a) *The G-invariant subspaces of V.*
(b) *The submodules of M.*
(c) *The monic right-hand factors of L.*

The only thing to explain is the correspondence between (b) and (c). Let $e = 1$ be the cyclic element of M and let N be a submodule of M. There is a unique monic operator R of minimal degree such that $Re \in N$. This is a right-hand factor of L. Moreover, $M/N = \mathcal{D}/\mathcal{D}R$ (compare the exact sequence before Proposition 2.13). Of course R also determines a unique left-hand factor of L. We note that the above corollary can also be formulated for the contravariant solution space $\mathrm{Hom}_{k[\partial]}(M, K)$.

We recall that an operator $L \in k[\partial]$ is *reducible over k* if L has a nontrivial right-hand factor. Otherwise L is called *irreducible*. The same terminology is used for differential equations in matrix form or for differential modules or for representations of a linear algebraic group over C.

Exercise 2.36 Show that a matrix differential equation is reducible if and only if it is equivalent to an equation $Y' = BY,\ B \in M_n(k)$ where B has the form

$$B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix} .$$

□

Definition 2.37 A differential module M is called *completely reducible* or *semi-simple* if there exists for every submodule N of M a submodule N' such that $M = N \oplus N'$. □

The same terminology is used for differential operators and for representations of a linear algebraic group G over C. We note that the terminology is somewhat confusing because an irreducible module is, at the same time, completely reducible.

A G-module W and a G-submodule W_1 has a *complementary submodule* if there is a G-submodule W_2 of W such that $W = W_1 \oplus W_2$. Thus a (finite dimensional) G-module V is *completely reducible* if every G-submodule has a complementary submodule. This is equivalent to V being a direct sum of irreducible submodules (compare with Exercise 2.38 part 1).

The *unipotent radical* of a linear algebraic group G is the largest normal unipotent subgroup G_u of G (see [141] for definitions of these notions). The group G is called *reductive* if G_u is trivial. We note that for this terminology G is reductive if and only G^o is reductive.

When G is defined over an algebraically closed field of characteristic zero, it is known that G is reductive if and only if it has a *faithful* completely reducible G-module (cf. the Appendix of [32]). In this case, all G-modules will be completely reducible.

Exercise 2.38 *Completely reducible modules and reductive groups.*
(1) Show that M is completely reducible if and only if M is a direct sum of irreducible modules. Is this direct sum unique?

(2) Let M be a differential module. Show that M is completely reducible if and only if its differential Galois group is reductive. Hint: Use the above information on reductive groups.

(3) Let M be a completely reducible differential module. Prove that every object N of $\{\{M\}\}$ is completely reducible. Hint: Use the above information on reductive groups.

(4) Show that the tensor product $M_1 \otimes M_2$ of two completely reducible modules is again completely reducible. Hint: Apply (2) and (3) with $M := M_1 \oplus M_2$. We note that a direct proof (not using reductive groups) of this fact seems to be unknown. □

Exercise 2.39 *Completely reducible differential operators.*
(1) Let $R_1, \dots, R_s$ denote irreducible monic differential operators (of degree ≥ 1). Let L denote LCLM$(R_1, \dots, R_s)$, the least common left multiple of $R_1, \dots, R_s$. In other terms, L is the monic differential operator satisfying $k[\partial]L = \cap_{i=1}^{s} k[\partial]R_i$. This generalizes the LCLM of two differential operators, defined in Sect. 2.1. Show that the obvious map $k[\partial]/k[\partial]L \to k[\partial]/k[\partial]R_1 \oplus \cdots \oplus k[\partial]/k[\partial]R_s$ is injective. Conclude that L is completely reducible.

(2) Suppose that L is monic and completely reducible. Show that L is the LCLM of suitable (distinct) monic irreducible operators $R_1, \dots, R_s$. Hint: By definition $k[\partial]/k[\partial]L = M_1 \oplus \cdots \oplus M_s$, where each M_i is irreducible. The element $\bar{1} \in k[\partial]/k[\partial]L$ is written as $\bar{1} = m_1 + \cdots + m_s$ with each $m_i \in M_i$. Let L_i be the monic operator of smallest degree with $L_i m_i = 0$. Show that L_i is irreducible and that $L = \text{LCLM}(L_1, \dots, L_s)$.

(3) Let $k = C$ be a field of constants and let L be a linear operator in $C[\partial]$. We may write $L = p(\partial) = \prod p_i(\partial)^{n_i}$ where the p_i are distinct irreducible polynomials and $n_i \geq 0$. Show that L is completely reducible if and only if all the $n_i \leq 1$.

(4) Let $k = C(z)$. Show that the operator $L = \partial^2 + (1/z)\partial \in C(z)[\partial]$ is not completely reducible. Hint: The operator is reducible since $L = (\partial + (1/z))(\partial)$ and ∂ is the only first order right factor. $\square$

Proposition 2.40 *Let M be a completely reducible differential module. Then M can be written as a direct sum $M = M_1 \oplus \cdots \oplus M_r$, where each M_i is a direct sum of n_i copies of an irreducible module N_i. Moreover, $N_i \not\cong N_j$ for $i \neq j$. This unique decomposition is called the* isotypical decomposition of M. *Then the eigenring $\mathcal{E}(M)$ (i.e., the ring of the endomorphisms of M) is equal to the product $\prod_{i=1}^{r} \mathrm{M}_{n_i}(C)$ of matrix algebras over C.*

Proof. The first part of the proposition is rather obvious. For $i \neq j$, every morphism $N_i \to N_j$ is zero. Consider an endomorphism $f : M \to M$. Then $f(M_i) \subset M_i$ for every i. This shows already that the isotypical decomposition is unique. Furthermore, M_i is isomorphic to $N_i \otimes L_i$, where L_i is a trivial differential module over k of dimension n_i. One observes that $\mathcal{E}(N_i) = C.1_{N_i}$ follows from the irreducibility of N_i. From this one easily deduces that $\mathcal{E}(M_i) \cong \mathrm{M}_{n_i}(C)$. $\square$

We note that the above proposition is a special case of a result on semisimple modules over a suitable ring (compare [170], Chap. XVII.1, Proposition 1.2).

The Jordan-Hölder Theorem is also valid for differential modules. We recall its formulation. A tower of differential modules $M_1 \supset M_2 \supset \ldots \supset M_r = \{0\}$ is called a *composition series* if the set of quotients $(M_i/M_{i+1})_{i=1}^{r-1}$ consists of irreducible modules. Two composition series for M yield isomorphic sets of irreducible quotients, up to a permutation of the indices.

A (monic) differential operator L can be written as a product $L_1 \cdots L_r$ of irreducible monic differential operators. For any other factorization $L = R_1 \cdots R_s$ with irreducible operators R_i, one has $r = s$ and there exists a permutation π of $\{1, \ldots, r\}$ such that L_i is equivalent to $R_{\pi(i)}$. Indeed, the factorization $L = L_1 \cdots L_r$ induces for the module $k[\partial]/k[\partial]L$ the composition series $k[\partial]/k[\partial]L \supset k[\partial]L_r/k[\partial]L \supset \cdots$.

A monic differential operator has, in general, many factorizations into irreducible monic operators. Consider $k = C(z)$ and $L = \partial^2$. Then all factorizations are $\partial^2 = (\partial + \frac{f'}{f})(\partial - \frac{f'}{f})$ with f a monic polynomial in z of degree ≤ 1.

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