Lecture 2

Kinematic Generalization

Suppose the medium is not homogeneous. For example, gravity waves impinging on a beach see of varying depth as the waves run up the beach, acoustic waves see fluid of varying pressure and temperature as they propagate vertically, etc. Then a pure plane wave in which all attributes of the wave are constant in space (and time) will not be a proper description of the wave field. Nevertheless, if the changes in the background occur on scales that are *long* and *slow* compared to the wavelength and period of the wave, a plane wave representation may be *locally* appropriate (Fig. 2.1). Even in a homogeneous medium, the wave might change its length if the wave is a superposition of plane waves (as we shall see later).

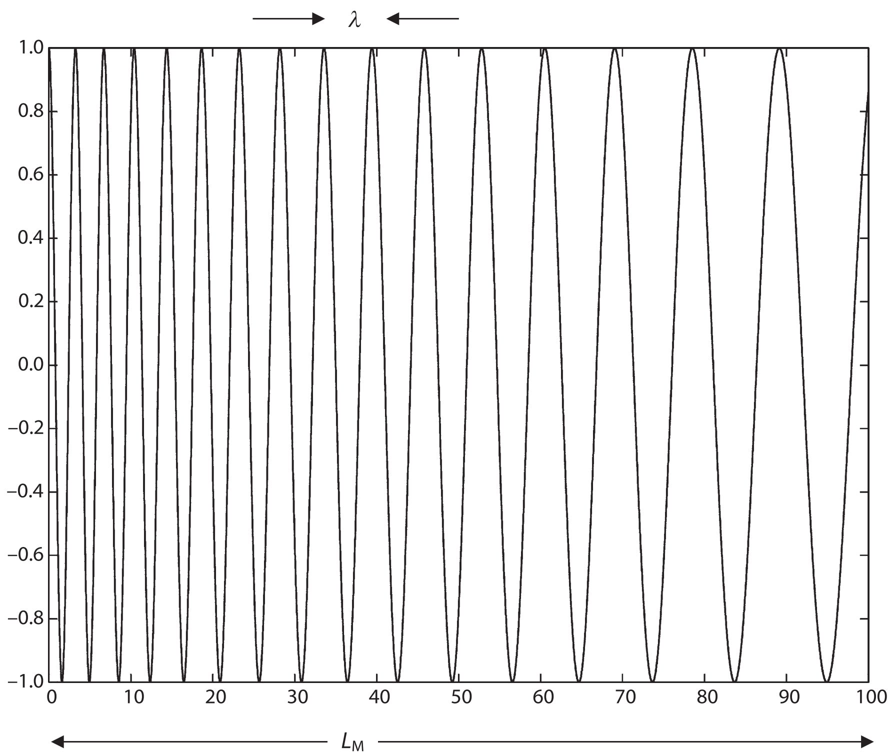

Fig. 2.1. Schematic of a slowly varying wave

Thus, locally the wave can still look like a plane wave if $\lambda / L_M \ll 1$. In that case, we might expect the wave to be described by the *form*:

$$\phi(\vec{x},t) = A(x,t)e^{i\theta(x,t)} \quad \text{(the real part of the expression is taken for granted),} \tag{2.1}$$

where A varies on the scale L_M while the phase varies on the scale λ. Thus,

$$\frac{1}{A}\frac{\partial A}{\partial x_i} = O\left(\frac{1}{L_M}\right) \tag{2.2a}$$

$$\frac{\partial \theta}{\partial x_i} = O\left(\frac{1}{\lambda}\right) \tag{2.2b}$$

so that

$$\nabla\phi = Ae^{i\theta}\nabla\theta + O\left(\frac{\lambda}{L_M}\right) \tag{2.3}$$

We *define* (guided by our experience with the plane wave):

$$\vec{K} = \nabla\theta \quad \text{local spatial increase of phase} \tag{2.4a}$$

$$-\omega = \frac{\partial\theta}{\partial t} \quad \text{local increase of phase with time} \tag{2.4b}$$

Since the wave vector is defined as the gradient of the scalar phase, it follows automatically that $\nabla \times \vec{K} = 0$.

Consider the increase of phase on the curve C_1 from point A to point B in Fig. 2.2:

$$n_{C_1} = \frac{1}{2\pi}\int_A^B \vec{K}\cdot d\vec{x} = \frac{1}{2\pi}\int_{C_1} \vec{K}\cdot d\vec{x} \tag{2.5}$$

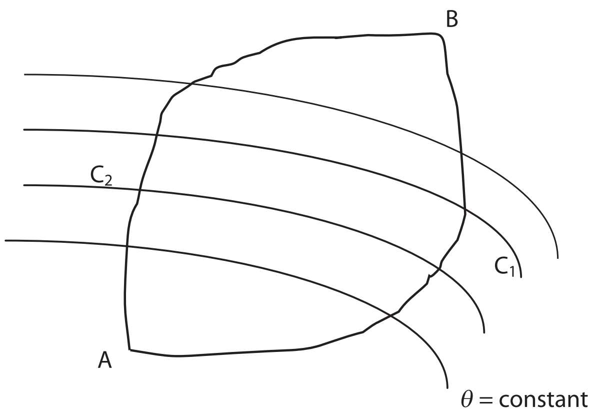

Fig. 2.2.
Counting crests on two paths AC_1B and AC_2B

Now consider the same increase calculated on curve C_2:

$$n_{C_2} = \frac{1}{2\pi}\int_A^B \vec{K}\cdot d\vec{x} = \frac{1}{2\pi}\int_{C_2} \vec{K}\cdot d\vec{x} \tag{2.6}$$

The difference between them is

$$\begin{aligned} n_{C_1} - n_{C_2} &= \frac{1}{2\pi}\int_{C_1} - \int_{C_2} \vec{K}\cdot d\vec{x} = \frac{1}{2\pi}\oint_{C_{total}} \vec{K}\cdot d\vec{x} \\ &= \iint_A \nabla\times\vec{K}\cdot\hat{n}\,dA \\ &= 0 \end{aligned} \tag{2.7}$$

Here we have used Stokes theorem relating the line integral of the tangent component of ***K*** with the area integral of its curl over the area bounded by the closed contour composed of the sum of the two curves C_1 and C_2. Since the curl is zero, the two calculations for the increase of phase must be *independent of the curve used to do the calculation*.

Note that since

$$\vec{K} = \nabla\theta \tag{2.8a}$$

$$\omega = -\frac{\partial\theta}{\partial t} \tag{2.8b}$$

it follows *by definition* that

$$\frac{\partial\vec{K}}{\partial t} + \nabla\omega = 0 \tag{2.9}$$

in those cases where the wave vector and the wave frequency are slowly varying functions of space and time (i.e., where it is sensible to define wavelength and frequency).

To better understand the consequences of the above equation, consider the fixed line element AB in Fig. 2.3.

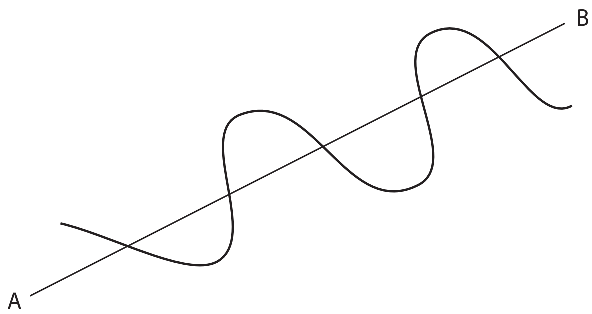

Fig. 2.3.
Conservation of crests along the line AB

Integrate the above conservation equation along the line element from A to B:

$$\frac{\partial}{\partial t}\int_{\mathrm{A}}^{\mathrm{B}} \vec{K}\cdot \mathrm{d}\vec{x} + \int_{\mathrm{A}}^{\mathrm{B}} \nabla\omega \cdot \mathrm{d}\vec{x} = 0 \tag{2.10}$$

Using our previous definitions, in particular that $Ks/2\pi$ is the number of crests in the interval s, it follows from the above that

$$\frac{\partial n_{\mathrm{AB}}}{\partial t} = \frac{\omega(\mathrm{A})}{2\pi} - \frac{\omega(\mathrm{B})}{2\pi} \tag{2.11}$$

That is to say, the rate of change of the number of crests in the interval (A,B) is equal to the rate of inflow of crests at point A minus the outflow of crests at point B, since the frequency (divided by 2π) is equal to the number of crests crossing a point at each moment. E.g.,

$\omega(\mathrm{A})$ = rate of decrease of phase at point A (see Fig. 2.4)

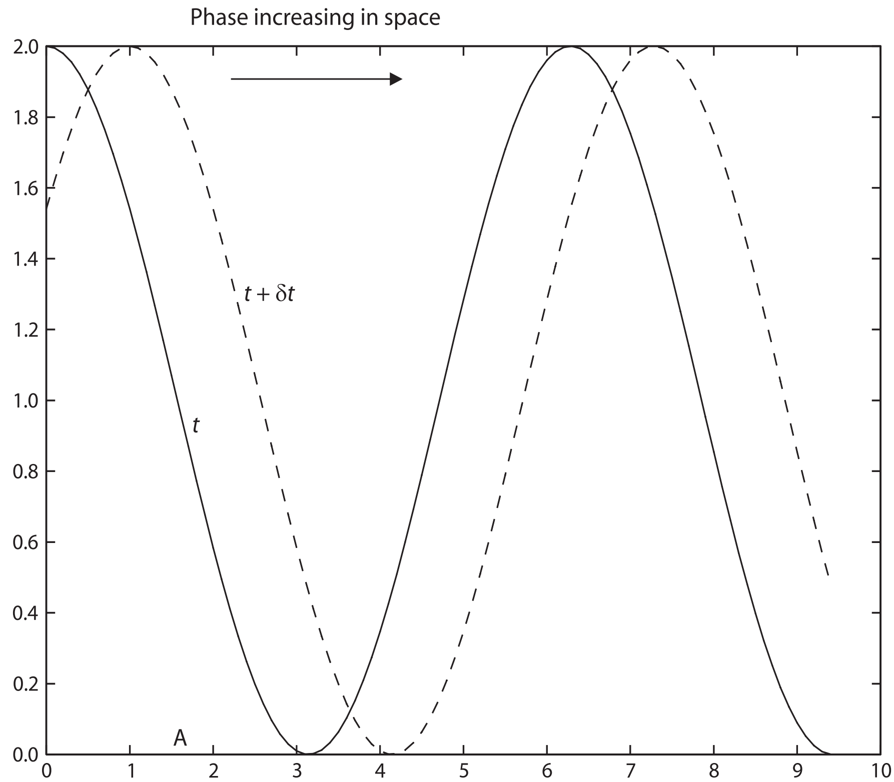

Fig. 2.4. The movement of the phase through the interval AB

We may think of this as a statement of the *conservation of wave crests*. Namely, the number of wave crests in a smoothly varying function ϕ as given above does not change. The number in any local region increases or decreases solely due to the arrival of preexisting crests, not to the creation or destruction of existing crests.

Now, let's suppose that we still have a local dispersion relation between frequency and wave number but that the relationship slowly changes on scales that are long compared with a wavelength or period due to changes, perhaps, in the nature of the medium in which the wave is embedded.

In that case, the natural generalization of the dispersion relation is

$$\omega = \Omega(k_j, x_i, t) \tag{2.12}$$

where the wave vector components and the frequency may themselves be functions of space and time (slowly), and the dispersion relation is *explicitly* dependent on space and time.

Thus,

$$\frac{\partial \omega}{\partial t} = \left.\frac{\partial \Omega}{\partial t}\right)_{\vec{K},\vec{x}} + \frac{\partial \Omega}{\partial k_j}\frac{\partial k_j}{\partial t} \tag{2.13}$$

where the first term on the right-hand side is due to the *explicit* dependence of the dispersion relation on time, as might happen if the temperature of a region through which an acoustic wave were traveling were increasing with time.

We *define* the *group velocity* by the formula for each of its Cartesian components:

$$c_{g_j} = \frac{\partial \Omega}{\partial k_j} \tag{2.14}$$

for the component of the group velocity in the j^{th} direction, or

$$\vec{c}_g = \nabla_{\vec{K}} \Omega \tag{2.15}$$

It follows from a fundamental theorem in vector analysis that since the phase is a scalar and the gradient operator is a vector, the group velocity is a true vector (distinct from the phase speed). That is, it follows the law of vector decomposition.

Since, by our earlier definitions

$$\frac{\partial k_j}{\partial t} = -\frac{\partial \omega}{\partial x_j} \tag{2.16}$$

we thus obtain

$$\frac{\partial \omega}{\partial t} = \frac{\partial \Omega}{\partial t} - \frac{\partial \Omega}{\partial k_j}\frac{\partial \omega}{\partial x_j} \tag{2.17}$$

It therefore follows that

$$\frac{\partial \omega}{\partial t} + \vec{c}_g \cdot \nabla \omega = \frac{\partial \Omega}{\partial t} \Leftarrow \text{explicit derivative with time} \tag{2.18}$$

Again, by similarly using

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0$$

it follows that

$$\frac{\partial k_i}{\partial t} + \frac{\partial \Omega}{\partial x_i} + \frac{\partial \Omega}{\partial k_j}\frac{\partial k_i}{\partial x_j} = 0 \quad \text{or} \tag{2.19a}$$

$$\frac{\partial k_i}{\partial t} + \frac{\partial \Omega}{\partial k_j}\frac{\partial k_i}{\partial x_j} = -\frac{\partial \Omega}{\partial x_i} \tag{2.19b}$$

Since the wave vector has no curl, it follows that

$$\frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i}$$

so the above equation can be rewritten:

$$\frac{\partial \vec{K}}{\partial t} + (\vec{c}_g \cdot \nabla)\vec{K} = -\nabla \Omega \Leftarrow \text{explicit dependence on space} \tag{2.20}$$

Note that the sum of derivatives on the left in the equations for the rate of change of wave vector and frequency are the rate of change for *an observer moving with the group velocity*.

So,

1. If the medium is independent of time, $\longrightarrow \omega$ propagates with the group velocity;
2. If the medium is independent of space, $\longrightarrow \boldsymbol{K}$ propagates with the group velocity.

If both (*1*) and (*2*) are true, both frequency and wave number propagate with the group velocity:

$$c_{g_i} = \frac{\partial \Omega}{\partial k_i}$$

This is a vector, and we see here that real wave attributes propagate with this velocity. If the dispersion relation is a function of space and/or time, the above equations tell us *how* the frequency and wave number change as we move with the group velocity following a wave. Further discussion can be found in Bretherton (1971) and Pedlosky (1987).

Example

We will soon see that free surface gravity waves (short enough so that rotation is unimportant but long enough so that the wavelength is large) compared to the depth have a dispersion relation:

$$\omega = k\sqrt{gH}$$

where H is the depth of the fluid and k is the wave number for this one-dimensional example (Fig. 2.5).

The phase speed and group velocity are equal in this case:

$$c_g = c = (gH)^{1/2}$$

If the depth is a function of x, then following a signal, since the dispersion relation is *independent of time,* the frequency will be constant for an observer moving with the velocity $c_g = c = (gH)^{1/2}$. For such an observer, with frequency constant, $k = \text{const.} / H^{1/2}$, which implies that the wave will grow shorter (larger k) as the wave enters shallow water. (It may become so short that it might break). Note that the observer, following a particular frequency moving with the group speed will proceed at a rate:

$$\frac{dx}{dt} = (gH(x))^{1/2} \tag{2.22}$$

For example, if $H(x)$ is of the form $H = H_0(1 - x / x_0)$ where x is measured positive shoreward from some offshore position a distance x_0 from the waterline (see Fig. 2.5), the signal corresponding to a given frequency will proceed onshore such that at a point x after an elapsed time t, the relationship between the elapsed time and its onshore progress is

$$t = 2x_0\left(1 - \sqrt{1 - x / x_0}\right)/(gH_0)^{1/2} \tag{2.23}$$

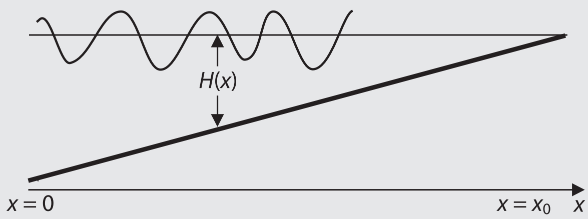

Fig. 2.5. Water wave running up a sloped beach

The above kinematic discussion doesn't tell us how the amplitude of the wave propagates or, equivalently, how the energy in the wave moves. In some simple cases that are general enough to be of interest, we can actually describe how the amplitude and hence energy moves.

Consider the case of a *homogeneous* medium in which the governing equation for the wave function ϕ is of the form

$$\Pi(\partial/\partial t, \partial/\partial x_i)\phi(x_i, t) = 0 \tag{2.24}$$

where Π is a polynomial in the partial derivatives with respect to space and time. A simple example would be the Rossby wave equation:

$$\left(\frac{\partial^3}{\partial x^2 \partial t} + \frac{\partial^3}{\partial y^2 \partial t} + \beta \frac{\partial}{\partial x}\right)\phi = 0 \tag{2.25}$$

so that in this case,

$$\Pi = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)\right] + \beta \frac{\partial}{\partial x}$$

i.e., the polynomial in the partial derivatives are in respect of x, y and t.

Suppose we look for an *approximate* solution of the form

$$\phi = A e^{i\theta}$$

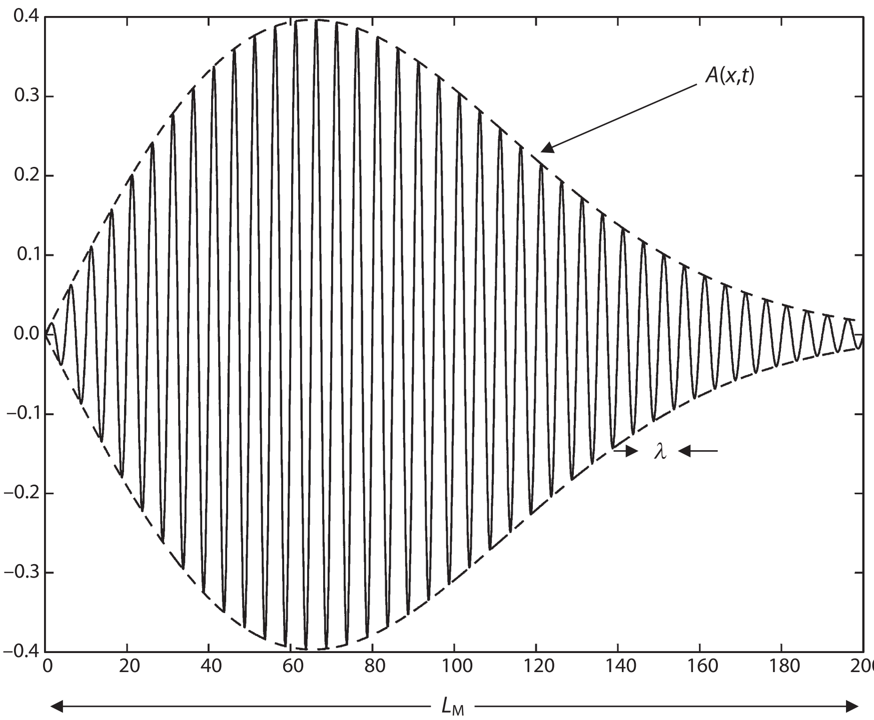

Fig. 2.6. A wave packet. The wave has wavelength λ while its *envelope* has a scale L_M

where A, k and ω are slowly varying functions of time, i.e., where the solution has the form of a one-dimensional *wave packet* (see Fig. 2.6), then

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= \left(i\frac{\partial \theta}{\partial x}A + \frac{\partial A}{\partial x}\right)e^{i\theta} \\ &= \left(ikA + \frac{\partial A}{\partial x}\right)e^{i\theta} \quad \text{, etc.}\end{aligned} \tag{2.26}$$

or

$$\Pi\phi = 0 \Rightarrow \Pi\left(-i\omega + \frac{\partial}{\partial t}, ik + \frac{\partial}{\partial x}\right)A = 0 \tag{2.27}$$

Expanding the polynomial using the fact that the time and space derivatives of A are small compared to ω and k,

$$\Pi(-i\omega, ik) + \frac{\partial \Pi}{\partial(-i\omega)}\frac{\partial A}{\partial t} + \frac{\partial \Pi}{\partial(ik)}\frac{\partial A}{\partial x} = 0 \tag{2.28}$$

The dispersion relation for plane waves comes from the disappearance of the first term (which is the dominant one), namely

$$\Pi(-i\omega, ik) = 0 \longrightarrow \textit{Linear dispersion relation} \tag{2.29}$$

In the case above, this yields $\omega = -\beta / k$.

When this dispersion relation is satisfied, the remaining term yields the condition:

$$\frac{\partial A}{\partial t} - \frac{\partial \Pi / \partial k}{\partial \Pi / \partial \omega}\frac{\partial A}{\partial x} = 0 \tag{2.30}$$

where the derivatives of Π in the equation occur when Π is evaluated as a function of frequency and wave number as in Eq. 2.29.

Since

$$\frac{\partial \Pi / \partial k}{\partial \Pi / \partial \omega} = -\left.\frac{\partial \omega}{\partial k}\right)_{\Omega} \tag{2.31}$$

it follows that

$$\frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial x} = 0 \tag{2.32}$$

Thus, the amplitude (and we can suppose) energy will propagate with the group velocity and not the phase speed. Where the *envelope (that is A)* of the wave goes, that is where the energy is. There is clearly no energy outside the wave envelope.

The reader should calculate the group velocity for this simple case of one-dimensional Rossby waves to see that the group and phase velocities are not the same. Similarly, the argument presented here can be extended to any number of dimensions (try it).

It is also clear that one might be able to use similar ideas for inhomogeneous media.

Once again we see here the physical primacy of the group velocity over the phase speed for the propagation of physical attributes of the wave.

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