ERRATA and COMPLEMENTS for the corrected 2nd printing 2005 and the first edition 2003 of the Springer Monographs in Mathematics (ISSN 1439-7382):

CLASS FIELD THEORY: From Theory to Practice (by Georges Gras)

In this ‘ERRATA and COMPLEMENTS’ we indicate the improvements and corrections concerning the corrected 2nd printing 2005 (II). These precisions do not affect any statement, so that this errata will be usefull especially for those who work carefully on the text, except perhaps for some more important corrections that we give in the first part (**A**), before the less important ones given in the second part (**B**). When it is possible we indicate, in parentheses, the corresponding page and line for the 1st edition (I).
For the convenience of the reader we also precise, in the third part (**C**), the main changes which where written for the corrected 2nd printing 2005 (II) from the 1st edition 2003 (I). This part (**C**) is rather long since we reproduce all the paragraphs which were added to the 1st edition (especially some interpretations of the Spiegelungssatz).

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(**A**) MAIN CORRECTIONS TO THE CORRECTED 2nd PRINTING:

- II, p.5, ℓ.-11 (I, p.5, ℓ.-11), after: ‘because of the Schmidt–Chevalley theorem’ add the following: ‘(more precisely, the Theorem II.6.3.2 on the “going down” of p^e-powers in the cyclotomic extension by the p^eth roots of unity)’

- II, p.80, before subsection (c) (I, p.79–80), add the following exercise which is the local analog of Theorem 3.6 (global case):

‘**1.5.5 Exercise** (Galois action). Let M/K_v be a finite abelian extension and let g be an automorphism group of K_v with fixed subfield k. Let $N \subseteq K_v^\times$ be the norm group corresponding to M/K_v.
Prove the following facts:
(i) M/k is Galois if and only if g acts on N;
(ii) M/k is abelian if and only if g is commutative and there exists a subgroup n of finite index of $k^\times$ such that $N = \mathrm{N}_{K_v/k}^{-1}(n)$ in which case M is the compositum of K_v with the abelian extension of k corresponding to n. $\square$’

- II, p.228, ℓ.-8 (I, p.219, ℓ.1), the definition of g_v is incorrect for $v = \ell$; read: ‘We then have $g_\infty(\,\bullet\,) = (\,\bullet\,)^{\delta_\infty}$, $g_\ell(\,\bullet\,) = (\,\bullet\,)^{\frac{\ell-1}{2}\delta_\ell}$, $\delta_v = 0$ or 1 depending only on v. Let $a,\, b \in \mathbb{Z} \setminus \{0\}$. Taking $u_\infty = \left(\frac{a,\, b}{\infty}\right)$ and $u_\ell = \left(\frac{a,\, b}{\ell}\right)$ for all $\ell \neq 2$, which implies that $\left(\frac{a,\, b}{\ell}\right)^{\frac{\ell-1}{2}}$ is the quadratic Hilbert symbol $\left(\frac{a,\, b}{\ell}\right)_2 = \pm 1$, we get: …’

- II, p.294, ℓ.9/11 (I, p.283, ℓ.13/14), the explanation about 4.2.2 is not adequate: in fact we use the Spiegelungsrelation II.5.4.3 in $K' := K(\mu_p)$ for the class group $\mathcal{C}\!\ell_{K',S'_0}^{T'\cup\Delta'_\infty}$ and the corresponding radical $V'^{S'}_{T'}/K'^{\times p}_{T'}$, then the rank formula $\mathrm{rk}_p(V_T^S/K_T^{\times p}) = \mathrm{rk}_1(V'^{S'}_{T'}/K'^{\times p}_{T'}) = \mathrm{rk}_\omega(\mathcal{C}\!\ell^{S'}_{K',T'})$, and finally 4.2.

II, p.466–470 (I, p.454–457): This part concerns a minor modification in the Appendix, Section 3, Subsection (a), in view of Theorems 3.6 and 3.7 of the Subsection (b): The set $S_p := S \cap Pl_p$ is not taken into account in some computations, what is incorrect; this modifies the Notations 3.1 (pp. 466/467, 2nd ed.; p.454, 1st ed.) and the proofs of the Lemmas 1 and 2 (pp. 469-470, 2nd ed.; pp.456-457, 1st ed.) in an evident way given below; this does not affect any result since the cohomology on S_p is also trivial because of the total splitting of these places in the extensions under consideration:

- II, p.466, 3.1 and footnote (I, p.454), replace: ‘$T_\sharp$’ by: ‘$T_\sharp \setminus S_p$’ and: ‘$T'_\sharp$’ by: ‘$T'_\sharp \setminus S'_p$’; in the footnote supress: ‘is unimportant’; replace: ‘p’ by: ‘$Pl_p \setminus S_p$’

- II, p.467, ℓ.2 (I, p.454, ℓ.17), replace: ' p ' by: ' $Pl_p\backslash S_p$ '
- II, p.467, after ℓ.11 (I, p.454, after ℓ.-6), add the following notations:
  ' • $\mathcal{I}_\sharp := I_{T_\sharp}\otimes\mathbb{Z}_p,\ \ \mathcal{I}'_\sharp := I_{L,T'_\sharp}\otimes\mathbb{Z}_p$, '
- II, p.467, ℓ.14 (I, p.454 , ℓ.-3), replace: ' $\mathcal{I}$ ' and: ' $\mathcal{I}'$ ' respectively by: ' $\mathcal{I}_\sharp$ ' and: ' $\mathcal{I}'_\sharp$ '
- II, p.469, ℓ.12 (I, p.456, ℓ.-4), the exact sequence becomes:

$$`\ 1 \longrightarrow \mathcal{L}^\times_{\infty_\sharp} \longrightarrow \mathcal{L}^\times_\infty \xrightarrow{\ \overline{i}'_{\Delta'_p}\ } \bigoplus_{w\in\Delta'_p\backslash S'_p} U^1_w \bigoplus_{w\in S'_p} \big(\widehat{L^\times_w}\big)_p \longrightarrow 1.\ '$$

- II, p.469, ℓ.15–16 (I, p.456, ℓ.-1; p.457, ℓ.1), replace these two lines by:
  ' **Proof.** Since L/K is Galois, unramified in $\Delta_p\backslash S_p$ and split in S_p, the norm:

$$\mathrm{N} : \bigoplus_{w\in\Delta'_p\backslash S'_p} U^1_w \bigoplus_{w\in S'_p} \big(\widehat{L^\times_w}\big)_p \longrightarrow \bigoplus_{v\in\Delta_p\backslash S_p} U^1_v \bigoplus_{v\in S_p} \big(\widehat{K^\times_v}\big)_p\ '$$

- II, p.469, ℓ.-4 (I, p.457, ℓ.10), replace: ' Δ_p ' by: ' $\Delta_p\backslash S_p$ and split in S_p '
- II, p.469, ℓ.-1 (I, p.457, ℓ.13), the formula becomes:

$$`\ H^{-1}\Big(G, \bigoplus_{w\in\Delta'_p\backslash S'_p} U^1_w \bigoplus_{w\in S'_p} \big(\widehat{L^\times_w}\big)_p\Big) = \bigoplus_{v\in\Delta_p\backslash S_p} H^{-1}(D_{w_0}, U^1_{w_0}) = 1,$$

  since $D_{w_0} = 1$ in S_p, '
- II, p.470, ℓ.1 (I, p.457, ℓ.14), read:
  ' hence $\overline{i}'_{\Delta'_p}(y_\infty) \in I_G\Big(\bigoplus_{w\in\Delta'_p\backslash S'_p} U^1_w \bigoplus_{w\in S'_p} \big(\widehat{L^\times_w}\big)_p\Big)\ldots$ '
- II, p.471, ℓ.15 (I, p.458, ℓ.-1), replace: ' Pl'_p ' by: ' $(Pl'_p\backslash S'_p)$ '

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(**B**) OTHER CORRECTIONS TO THE CORRECTED 2nd PRINTING:

- II, p.8, ℓ.-12 and -22, replace: ' torsion groups ' by: ' torsion groups in p-ramification '
- II, p.26, ℓ.15 (I, p.26, ℓ.7, add: ' (see[Se4, §§14–16]) '); remark that this argument uses (i), the result in characteristic 0.
- II, p.26, ℓ.18 (I, p.26, ℓ.10),read: ' $S_0 := S\cap Pl_0$ '
- II, p.27, before 3.7 (I, p.27), add: ' Recall that E^S_K is the group of S-units, (r_1, r_2) the signature, of K, and μ_p the group of pth roots of unity. '
- II, p.27, after 3.7.2 (I, p.27), add the following comment: ' If K is real, $\mathbb{Q}\oplus(E^{\rm ord}_K\otimes_{\mathbb{Z}}\mathbb{Q}) \simeq \mathbb{Q}[G]$ and if K is complex, $\mathbb{Q}\oplus(E^{\rm ord}_K\otimes_{\mathbb{Z}}\mathbb{Q}) \simeq \mathbb{Q}[G](1+c_\infty)$ '
- II, p.30, ℓ.6 (I, p.29, ℓ.-2), read: ' Let $S =: S_0\cup S_\infty$ be a finite set of noncomplex places '
- II, p.32, 4.2.5 (I, p.32, ℓ.6): ' closure ' means: ' topological closure ' which will be denoted: ' adh '
- II, p.40, ℓ.3 (I, p.39, ℓ.-7), instead of: ' we ' read: ' will '
- II, p.41, ℓ.8–9, read: ' the above first exact sequence '
- II, p.44, ℓ.-6 (I, p.44, ℓ.17), instead of: ' II.5.4.3/ II.5.4.5 ' read: ' II.5.4.2 to II.5.4.6 '
- II, p.48, ℓ.14 (I, p.48, ℓ.3), add: ' (verify that $\big(\mathcal{C}\ell^S_{\mathfrak{m}} \longrightarrow \mathcal{C}\ell^{S\cup\delta_\infty}_{\mathfrak{n}}\big)\circ\gamma^S_{\mathfrak{m}} = \gamma^{S\cup\delta_\infty}_{\mathfrak{n}}$) '
- II, p.51, ℓ.-3 (I, p.51, ℓ.-13), at the end add: ' (see 4.2.5, 4.2.6) '

- II, p.59, ℓ.11 (I, p.58, ℓ.-4): there is a contradiction since we get $\mathrm{N}_{L/K}(y) \in \langle \zeta_{e-1} \rangle$.
- II, p.62, ℓ.1 (I, p.61, ℓ.-11), put the: ' (i) ' before: ' since... '
- II, p.67, ℓ.-11 (I, p.67, ℓ.-10/-11): note that $[G_w : G_w] \subseteq G_w^0$ because $L_w^{G_w^0} \subseteq L_w^{\mathrm{ab}}$.
- II, p.68, ℓ.13 (I, p.68, ℓ.13): remember that any Frobenius automorphism of a unramified local extension is the canonical lift of the Frobenius automorphism of the corresponding residue extension.
- II, p.73 (I, p.73), before 1.3.3, add the following remark: ' For $t \in G_w$, $\overline{y}^t := \overline{y^t}$ for all $\overline{y} \in F_w$ makes sense since $F_w = \mathcal{O}_w/(\pi_w)$ with $(\pi_w)^t = (\pi_w)$, and $F_w^{G_w} = F_v$. In fact, $\mathrm{Gal}(F_w/F_v)$ is canonically isomorphic to G_w/G_w^0 which is generated by the Frobenius automorphism (cf. 1.1.5). '
- II, p.82, ℓ.12 (I, p.82, ℓ.5): we have $\mathfrak{f}_v = \mathfrak{p}_v$ for the following reason: U_v^1 is a $\mathbb{Z}_p$-module and $\ell \neq p = [L_w : K_v]$; thus any element of U_v^1 is a pth power in U_v^1, then a norm.
- II, p.83, ℓ.3 (I, p.82, ℓ.-3), read: ' Since F_v is a finite field of characteristic p '
- II, p.85, ℓ.-1 (I, p.85, ℓ.-6), add the following remark: ' The maximal value of n is $|\mu(K_v)| = (q_v - 1) \times |\mu_\ell(K_v)|$ where ℓ is the residue characteristic of v '
- II, p.94, ℓ.10/11 (I, p.94, ℓ.10/11): it is usefull to see that the topology induced by that of $\bigoplus\limits_{w|v} L_w^\times$ on the diagonal embedding of L is the same as the topology of L identified with $K^{[L:K]}$ (by choosing a K-base for instance).
- II, p.96, 2.4 (I, p.96), read: ' $J_L^G = j_{L/K}(J_K) \simeq J_K$ ' and: ' $C_L^G = j_{L/K}(C_K) \simeq C_K$ '
- II, p.103, ℓ.-5 (I, p.103, ℓ.-4), read: ' but is not the norm of local units at 2 '
- II, p.107, ℓ.-3 (I, p.107, ℓ.-10), put the right: ') ' after: ' $\mathbb{Q}(\sqrt{2}/\mathbb{Q})$ '
- II, p.110, ℓ.1 (I, p.109, ℓ.-7): S is a set of noncomplex places.
- II, p.116, ℓ.-5 (I, p.116, ℓ.-14), replace: ' 1.8.3 ' by: ' 1.8.1 '
- II, p.120, ℓ.-2 (I, p.120, ℓ.17), read: ' n open subgroup in J_K '
- II, p.123, ℓ.-4 (I, p.123, ℓ.7), add: ' $= \mathrm{adh}(E^{\mathrm{ord}} U_\infty / E^{\mathrm{ord}})$ '
- II, p.129, ℓ.8 (I, p.128, ℓ.15): this normic equality comes from 2.6.7, (ii).
- II, p.133, 4.4 (I, p.132), add the usual hypothesis: ' For $\mathfrak{f}|\mathfrak{m}$ '
- II, p.139, ℓ.3 (I, p.138, ℓ.3), at the end put: ' (see 5.5.2; thus, the above symbol is trivial if and only if $y \equiv 1 \bmod (m)$) '
- II, p.141, ℓ.12 (I, p.140, ℓ.12), replace: ' $i \geq 1$ ' by: ' $i \geq 0$ '
- II, p.154, (iii) (I, p.153): we recall that $\Sigma_p := \Sigma \cap Pl_p$ for any set Σ of places.
- II, p.154, ℓ.-6 (I, p.153, ℓ.-3), instead of: ' I.4.5.1 ' read: ' I.4.5.4 '
- II, p.160, ℓ.8 (I, p.159, ℓ.14), read: ' (e.g., $\chi \neq 1$ and v nonsplit in K'/k for all $v \in S$, which implies $\left(\mathcal{C}\ell_{\mathfrak{m}}^S\right)_\chi = \left(\mathcal{C}\ell_{\mathfrak{m}}^{\mathrm{res}}\right)_\chi$) '
- II, p.162, ℓ.13, add the hypothesis: ' when $\mu_p \subset K$ '
- II, p.167, ℓ.-3, read: ' and the p-ranks of the class groups are equal '
- II, p.171, ℓ.12 (I, p.163, ℓ.7), read: ' [f, Neum1] '
- II, p.178, ℓ.-13 (I, p.170, ℓ.13), add the definition: ' $K^{(0)}(p) := K$ '. Moreover, all the indices: ' i ' are ≥ 0.
- II, p.179, Note: remark that the intermediate extensions $K^{(i+1)}(p)/K^{(i)}(p)$, in $\overline{H}_T^S(p)$, may be infinite.

- II, p.181, ℓ.15 (I, p.172, ℓ.19), read: 'Hilbert class fields tower $\bigcup_{i\geq 0} K^{(i)}$ of K'
- II, p.189, 6.2.3 (I, p.180), add the following remark: 'Let L/K be a quadratic extension such that $(\mathcal{C}\!\ell_K^{\mathrm{res}})_2 = 1$; since $c \in (\mathcal{C}\!\ell_L^{\mathrm{res}})_2^G$ is equivalent to $c^2 = 1$ (use the algebraic norm $\nu_{L/K} = 1+s$), we have $\mathrm{rk}_2(\mathcal{C}\!\ell_L^{\mathrm{res}}) = t-1-\mathrm{rk}_2(E_K^{\mathrm{res}}/E_K^{\mathrm{res}} \cap \mathrm{N}_{L/K}(L^\times))$, where t is the number of finite places ramified in L/K. If $K = \mathbb{Q}$ this gives $\mathrm{rk}_2(\mathcal{C}\!\ell_L^{\mathrm{res}}) = t-1$. The ordinary sense is analogous. Of course, genus theory (Ch. IV, § 4) gives again these results.'
- II, p.194, ℓ.9 (I, p.185, ℓ.14), replace the two: 'G' by: 'G''
- II, p.196, ℓ.11 (I, p.187, ℓ.14), read: 'in $K' := K(\mu_{2^e})$'
- II, p.201, ℓ.11 (I, p.192, ℓ.15), remember that: 'f_v' is the residue degree of L_v/K_v.
- II, p.211, end of the footnote (I, p.202), add: '(hint: $K(\sqrt[4]{2}) = K(\sqrt[4]{-2})$)'
- II, p.214, ℓ.-7, after: '$n = 2$', add: '$v \in Pl_\infty^{\mathrm{r}}$'
- II, p.214, ℓ.-1 (I, p.205, ℓ.-11), read: '$\left(\dfrac{\tau(x)}{\tau(\mathfrak{b})}\right)_{\tau K,n} = \tau\left(\dfrac{x}{\mathfrak{b}}\right)_{K,n}$'
- II, p.215, (vii) (I, p.205): since the nth power residue symbol of x is an homomorphism of groups on I_{R_x}, it is equivalent to say that its kernel is N_x.
- II, p.225, ℓ.-5 (I, p.216, ℓ.3–4, Note), read/add: 'Moreover, the radical of $H^{\mathrm{ord}}_{Pl_p}[p]$ is $W_{Pl_p,\mathrm{pos}} := \{xK^{\times p},\ x \in K^\times_{\mathrm{pos}},\ (x) \in I^p\langle Pl_p\rangle\}$'
- II, p.226, 7.8.1 (I, p.216), read: 'Number fields K for which...'
- II, p.231, ℓ.-13 (I, p.221, ℓ.-13), Remember that $H_T^S = \bigcup_{\mathfrak{m}\in\langle T\rangle_{\mathbb{N}}} K(\mathfrak{m})^S$.
- II, p.242, ℓ.13, replace: 'In 1.1 for $t = T\backslash\{v\}$, $\delta_\infty = \emptyset$' by: 'From 1.1.2, (i)'
- II, p.245, ℓ.1 (I, p.234, ℓ.-7), the definition of the neighbourhoods is incorrect: the good neighbourhoods are the subgroups of finite index of A, but in the present case ($A = U_0$), the topology of the subspaces under consideration coincide with that induced by the $A^{n\widehat{\mathbb{Z}}}$, $n \geq 1$, so that the proof is unchanged.
- II, p.249, ℓ.-7 (I, p.239, ℓ.-14), refer to: 'Theorem 1.6 and Exercise 1.6.5'
- II, p.264, ℓ.16 (I, p.254, ℓ.7): we consider the product of the U_w only for $w \in Pl_{M,p}$, instead of the whole set of places, because the norms considered are surjective for the missing factors.
- II, p.264, ℓ.-2 (I, p.254, ℓ.-13), read: 'II.3.3, (iv) and (ii) in $Q_{n+1}L/L/\mathbb{Q}$'
- II, p.271, ℓ.-1/-2 (I, p.261, ℓ.1/2): this definition must be put in the Note above.
- II, p.283, ℓ.-6/-5 (I, p.272, ℓ.-3/-4), replace the exponent: 'n' by: '$n + \delta_{2,p}$'
- II, p.285, ℓ.7 (I, p.274, ℓ.9), add at the end: '(see 1.6.6, (i))'
- II, p.285, ℓ.11/12 (I, p.274, ℓ.13/14), after: 3.6.6, add the following exercise which has some interest:

  '**3.6.7 Exercise.** Suppose that $\mu_p \subset K$ and that there exists $v|p$ such that $\{\varepsilon \in E,\ \bar{i}_v(\varepsilon) \in K_v^{\times p}\} = (E^{\mathrm{ord}})^p$. Prove that K satisfies the Leopoldt conjecture at p. Hint: use 3.6.2, (vi) with a unit $\varepsilon \in \mathcal{E}^{\mathrm{ord}}$ such that $\bar{i}_p(\varepsilon) = 1$ and prove by induction that, for all $n \geq 1$, $\varepsilon = \varepsilon_n^{p^n}$, $\varepsilon_n \in \mathcal{E}^{\mathrm{ord}}$, with $\bar{i}_v(\varepsilon_n)^p = 1$ in $K_v^\times$ (to obtain $\varepsilon = \varepsilon_{n+1}^{p^{n+1}}$, modify ε_n by a suitable root of unity). □'
- II, p.290, 4.1.7 (I, p.279), the isomorphisms of the two exact sequences are not canonical and the factors: '$\mathbb{Z}_p^{r_2+1}$' correspond respectively to $\mathrm{Gal}(\widetilde{K}_p/K)$ and $\mathrm{Gal}(\widetilde{K}_p/\widetilde{K}_p \cap H^{\mathrm{ord}}(p))$.

- II, p.298, ℓ.-5 (I, p.287, ℓ.-9): as we have yet explained, the Schmidt–Chevalley Theorem uses more precisely the Theorem II.6.3.2 on the "going down" of p^e-powers in the cyclotomic extension by the p^eth roots of unity.
- II, p.299, ℓ.-1 (I, p.288, ℓ.-2), add the following remark: 'This is consistent with the criteria 1.4.3.'
- II, p.309, ℓ.-1 (I, p.298, ℓ.-4): the extension $\overline{K}^{\mathrm{ab}}/K$ is seen in $\mathbb{C}_\ell$, where ℓ is the residue characteristic of v finite, or in $\mathbb{C}_\infty = \mathbb{C}$ for v finite.
- II, p.310, ℓ.-15 (I, p.299, ℓ.15), read: 'the lemma follows since e is arbitrary'
- II, p.314, ℓ.8 (I, p.303, ℓ.6), read: 'we obtain from 4.6.3, (ii):'
- II, p.319, ℓ.3 (I, p.308, ℓ.1), read: 'We see from I.3.7.2 that'
- II, p.321, Proof of 4.10.6 (I, p.310): the use of Krasner lemma has some curious consequences: we can choose θ_{w_0} such that its global and local degrees coincide, but θ_{w_0} is not anymore necessarily in N ! This does not matter since the action of Γ on $(\theta_{w_0}, 0, \ldots, 0)$ only depends on the Galois action of D_{w_0} on θ_{w_0} (e.g., $N = \mathbb{Q}(\mu_5)$, $N^{D_{w_0}} = \mathbb{Q}(\sqrt{5})$ with $w_0|19$).
- II, p.322, ℓ.-11 (I, p.311, ℓ.-13), add: '$x' \in \mathbb{Q}'_p$'
- II, p.327, ℓ.-3 (I, p.316, ℓ.-1), replace: '(see § 4.9)' by: '(see 4.7.1)'
- II, p.335, ℓ.-10 (I, p.324, ℓ.-11), read: '(see II.6.3.4.2 and II.6.3.4.3)'
- II, p.337, ℓ.-13 (I, p.326, ℓ.-13), after: 'D_0' add: 'which is divisible (see 4.4.6)'
- II, p.341, ℓ.-7 (I, p.330, ℓ.-11), at the end of the line add: '(see II.7.1.5)'
- II, p.342, ℓ.5 (I, p.331, ℓ.1), read: 'is it possible, for a given number field K'
- II, p.349, ℓ.13 (I, p.338, ℓ.5), delete: 'in the (strong) sense of 1.4.3'
- II, p.361, ℓ.-10 (I, p.350, ℓ.12), delete: 'to p and'
- II, p.365, ℓ.-10, replace: '$\mathbb{Z}_2^\times \times \mathbb{Z}_2^\times$' by: '$\mathbb{Z}_2 \times \mathbb{Z}_2$'
- II, p.366, ℓ.18 (I, p.355, ℓ.7), read: 'Point (i) is evident from the formalism of the Log_p function (see 2.7, 2.8)'
- II, p.367, ℓ.-4/-5: invert the two sentences
- II, p.377, 2.4 (I, p.365): for the proof of the result stated in this remark, use 2.3 by going to the direct limit on T with $S_\infty = \emptyset$.
- II, p.379, ℓ.8 (I, p.367, ℓ.3), replace: 'U_w' and: 'U_v' respectively by: 'U_w^1' and: 'U_v^1'
- II, p.384, ℓ.5 (I, p.371, ℓ.-1), read: 'consider the $\mathbb{Z}_p$-lattices of the $\mathbb{Q}_p$-space $\mathcal{L}_p := \bigoplus_{v|p} K_v \big/ \mathbb{Q}_p \log_p(E)$'
- II, p.387, ℓ.13 (I, p.375, ℓ.8), in fact the prime number ℓ must be congruent to 1 modulo p (without this condition, ℓ does not ramify), so that the condition of the line 15 becomes: '$\ell \not\equiv 1 \bmod (p^2)$'
- II, p.394, ℓ.11 (I, p.382, ℓ.6), replace: 'modulo this norm group' by: 'modulo $(\mathcal{E}^S)^{[L:K]_p}$ contained in $\mathrm{N}_{L/K}(U_L^{S'})$'
- II, p.397, ℓ.1 (I, p.384, ℓ.-6), read: 'is unramified (and split at all the p_i)'
- II, p.409, ℓ.13 (I, p.396, ℓ.-3), read: 'The brutal lower bound (comming from the first exact sequence of 4.4.2):'
- II, p.412, ℓ.-8 (I, p.399, ℓ.-1), delete: 'since $\mathrm{N}(J_L) \cap U_K^{\mathrm{res}} = \mathrm{N}(U_L^{\mathrm{res}})$' which is useless
- II, p.414, ℓ.3 (I, p.401, ℓ.12), in: 'It follows that $\mathrm{N}(\boldsymbol{y}) \in K^\times . \mathrm{N}(U_L^{S'})$', replace: '$K^\times$' by: '$\mathrm{N}(L^\times)$'
- II, p.417, ℓ.12 (I, p.404, ℓ.16), read: 'extension F containing H'

- II, p.420, ℓ.-9, instead of: ‘F_v^*’ read: ‘$F_v^\times$’
- II, p.422, ℓ.7 (I, p.410, ℓ.2), the fields: ‘$K_{(7)}{}^{\mathrm{res}}{}_{(2)}$’ and: ‘$K_{(l)}{}^{\mathrm{res}}{}_{(2)}$’ must be permuted.
- II, p.433, ℓ.-6, replace: ‘$=$’ by: ‘$\leq$’
- II, p.439, ℓ.-7 (I, p.427, ℓ.-14): the above condition is only a necessary condition; more precisely, read: ‘It is easily checked that if this is the case, then $Pl_2^{\mathrm{ns}} \cap T_2 = \emptyset$’ (for $e > n$ it is a necessary and sufficient condition, but for $e = n$, there is a supplementary condition).
- II, p.445, footnote (I, p.433), read: ‘$E^S K^{\times 2^e}$’
- II, p.453, ℓ.13: Concerning the p-tower $\overline{H}_{T(p)}^S$, we refer to the first Note in II.5.9 (p.179).
- II, p.456, ℓ.-5 (I, p.444, ℓ.10): for these questions of duality, we also refer to II.2.4.1, (ii).
- II, p.461, ℓ.-12, after: ‘we have the exact sequences’ add: ‘(where δ' is the modification of δ depending on the identification of $(\mathbb{Z}/p^e\mathbb{Z})^*$ with $\mathbb{Z}/p^e\mathbb{Z}$)’
- II, p.475, ℓ.-12 (I, p.462, ℓ.-13), replace: ‘$\mathcal{P}_{L,T',\infty,\Delta'_\infty} . \mathcal{S}'_0$’ by: ‘$\mathcal{P}'_\infty . \mathcal{S}'_0$’

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(**C**) CORRECTIONS AND IMPROVEMENTS TO THE 1st EDITION:

- I, Introduction, p.3, ℓ.-2, add: ‘where I is the group of nonzero fractional ideals of K.’
- I, Introduction, p.4, ℓ.1, read: ‘for the subgroups of finite p-power index’
- I, Introduction, p.5, ℓ.-17, read: ‘the v-completion of K’
- I, p.9, ℓ.17, replace: ‘are homeomorphic’ by: ‘are equal’
- I, p.11, 1.4, replace the references: ‘1.1’ by: ‘1.2’
- I, p.13, ℓ.-14: Remark that we deduce from these isomorphisms the well-known formula $[K : \mathbb{Q}] = \sum\limits_{v|p} [K_v : \mathbb{Q}_p]$.
- I, p.14, end of 2.1.3, add the following remark: ‘For p prime, the ring of integers (resp. the maximal ideal) of $\mathbb{C}_p$ is then the subset of elements z such that $||z|| \leq 1$ (resp. $||z|| < 1$)’
- I, p.17, ℓ.-11, we give the proof of $i_v(K^{D_v}) = i_v(K) \cap \mathbb{Q}_p$: ‘Let $x \in K$ be such that $i_v(x) \in \mathbb{Q}_p$. Then for all $\varepsilon > 0$ there exists $a \in \mathbb{Q}$ such that $|x - a|_v < \varepsilon$; this yields $|x - a|_v = |sx - a|_{sv} = |sx - a|_v < \varepsilon$ for all $s \in D_v$, then $|sx - x|_v < 2\varepsilon$ proving that $sx = x$ for all $s \in D_v$ (i.e., $x \in K^{D_v}$); thus $[K : K^{D_v}] \leq [K_v : \mathbb{Q}_p]$. In the case p prime, the global theory of Dedekind rings yields $\sum_{v|p}[K : K^{D_v}] = [K : \mathbb{Q}] = \sum_{v|p}[K_v : \mathbb{Q}_p]$, which implies all the inclusions $i_v(K^{D_v}) \subset \mathbb{Q}_p$, $v|p$. The case $p = \infty$ is immediate.’
- I, p.21, ℓ.14, put on the left of the formula: ‘$U_v = \mu_{q_v - 1} \oplus U_v^1$,’
- I, p.24, ℓ.-1/-2, put: ‘(i)’ then: ‘(ii)’ on the left of the corresponding formula
- I, p.27, 3.7, note that the Teichmüller character is the character of the action of g on μ_p.
- I, p.27, 3.7.1, add the formula: ‘$\dim_{\mathbb{Q}}(F \otimes_{\mathbb{Z}} \mathbb{Q}) = r_1 + r_2 + |S_0| - 1$’
- I, p.27, ℓ.-13, read: ‘$E_K^{\mathrm{ord}} \otimes_{\mathbb{Z}} \mathbb{Q}$’
- I, p.32, ℓ.2, add: ‘Since J is locally compact, the Lemma 4.2.1 also gives $U_\infty \subseteq N$ in the proof of 4.2.3’
- I, p.34, ℓ.11, instead of: ‘U_v^{ord}’ read: ‘U_v’

- I, p.41, ℓ.11, instead of: '$v \in T \backslash T_p$' read: '$v \nmid p$' (since $T_p := T \cap Pl_p$ is not yet defined)
- I, p.51, footnote, read: 'choose a real quadratic field and, in the spirit of the proof of 4.2.8, (iv), use the powers of its totally positive fundamental unit ε in the idèle $(\ldots, 1, \ldots; \varepsilon^{-n}, \varepsilon^{n}) \in J$, so that $i(\varepsilon^n) . (\ldots, 1, \ldots; \varepsilon^{-n}, \varepsilon^{n}) = (\ldots, i_v(\varepsilon^n), \ldots; 1, 1)$.'
- I, p.54, ℓ.-9, add: 'The Subsection (a) is valid for any field whose characteristic is 0 or $p \nmid n$.'
- I, p.55, ℓ.3, read: 'for the same reason, since G is abelian,'
- I, p.61, 6.3.2, put: '(i)' at the begining of line -9
- I, p.71, after 1.2.3.1, add the following exercise:

  '**1.2.3.2 Exercise**. Let Σ be a finite set of places of L. Prove in the same way that $\langle D_w \rangle_{w \in \Sigma} = \varprojlim_{L'} \langle D_{w'}(L'/K) \rangle_{w' \in \Sigma'}$. □'
- I, p.76, ℓ.13, add: 'A direct proof may be found in [d, Lang1, Ch. 9, § 3]'
- I, p.77, ℓ.-10, after: 'where N corresponds to M,' add: 'since the kernel of $K_v^\times \longrightarrow \mathrm{Gal}(M/K_v)/\mathrm{Im}(U_v) = \mathrm{Gal}(M^{\mathrm{nr}}/K_v)$ is $U_v N$'
- I, p.85, ℓ.-6, add: 'In most books, the definition is the inverse of the more canonical present one.'
- I, p.97, ℓ.8, delete: 'and complexified' and remember that Σ' denotes the set of places of L above those of Σ.
- I, p.97, ℓ.11, add the following proof: 'We have $H^1(D_{w_0}, U_{w_0}) = 1$ in the unramified case, because π_v is a uniformizer of L_{w_0}, which yields ${}_{\nu}U_{w_0} \subseteq {}_{\nu}L_{w_0}^\times = (L_{w_0}^\times)^{1-\sigma} = U_{w_0}^{1-\sigma}$ for a generator σ of D_{w_0}'
- I, p.97, (iii), replace by: '(iii) For instance, the case $r = -2$, $A = \mathbb{Z}$ (with $\mathbb{Z}^* = \mathbb{Q}/\mathbb{Z}$), gives the canonical isomorphism $I_G/I_G^2 \simeq G^{\mathrm{ab}}$'
- I, p.104, after ℓ.7, add: 'we note that:

$$\begin{aligned} U_v \cap \bigcap_{w|v} \mathrm{N}_{L_w/K_v}(L_w^\times) &= \bigcap_{w|v} \big(U_v \cap \mathrm{N}_{L_w/K_v}(L_w^\times)\big) \\ &= \bigcap_{w|v} \mathrm{N}_{L_w/K_v}(U_w) \subseteq \langle \mathrm{N}_{L_w/K_v}(U_w) \rangle_{w|v}. \end{aligned}$$ '

- I, p.112, end of 3.3.5, add the following statement: 'The place v is totally split in L^{ab}/K if and only if $K_v^\times \subset N$.'
- I, p.116, (iii), this proof is not totally convincing; read: 'We have the product formula $\prod_v \Big(\dfrac{q\,,\,L/\mathbb{Q}}{v}\Big) = 1$, where $\Big(\dfrac{q\,,\,L/\mathbb{Q}}{v}\Big)$ is the canonical image of $(q\,,\,L_v/\mathbb{Q}_v)$ in $\mathrm{Gal}(L/\mathbb{Q})$, and we know that $(q\,,\,L_v/\mathbb{Q}_v) = 1$ except perhaps if $L_v/\mathbb{Q}_v$ is ramified (which occurs only for $v = \ell$) or if q is not a unit at v (hence only for $v = q$ since we chose $q > 0$); this yields $\Big(\dfrac{q\,,\,L/\mathbb{Q}}{\ell}\Big) = \Big(\dfrac{q\,,\,L/\mathbb{Q}}{q}\Big)^{-1} =: \sigma_q^{-1}$ by abuse of notation (we have also $\Big(\dfrac{q\,,\,L/\mathbb{Q}}{q}\Big)(\zeta) = \Big(\dfrac{L/\mathbb{Q}}{q}\Big)(\zeta) = \zeta^q$ in $L/\mathbb{Q}$ since $L/\mathbb{Q}$ is a cyclotomic field); interpreted in $\mathrm{Gal}(L_\ell/\mathbb{Q}_\ell)$, we get $(q\,,\,L_\ell/\mathbb{Q}_\ell) = \sigma_q^{-1}$.'
- I, p.118, 3.5.1, (ii), we give more complete comments for the first part of this point:

  '(ii) Similarly, if M corresponds to N, the decomposition subfield (resp. the inertia subfield) of a place v in M/K corresponds to $K_v^\times N$ (resp. to $U_v N$), i.e., is fixed under $\rho_{M/K}(K_v^\times)$ (resp. $\rho_{M/K}(U_v)$).

  For instance, the field corresponding to the closed subgroup of finite index $N := K^\times U^{\mathrm{res}}$ (resp. $K^\times U^{\mathrm{ord}}$) is the maximal abelian unramified (resp. unramified and Pl_∞-split) extension of K. This field H^{res} (resp. H^{ord}) is called the Hilbert class field of K in the restricted (resp. ordinary) sense. From I.5.1 or I.5.1.1 we deduce that $\mathrm{Gal}(H^{\mathrm{res}}/K) \simeq \mathcal{C}\ell^{\mathrm{res}}$ (resp. $\mathrm{Gal}(H^{\mathrm{ord}}/K) \simeq \mathcal{C}\ell^{\mathrm{ord}}$). We will find again these fields in the Paragraph 5 as particular cases of the ray class fields corresponding to the open subgroups $K^\times U_{\mathfrak{m}}^{\mathrm{res}}$.'

- I, p.120, ℓ.-10, read: 'the action of g on J/N'
- I, p.120, 3.6.1, we have the more detailed version of the first part: 'Let $K/\mathbb{Q}$ be Galois with Galois group $G =: g$, and let H (resp. $\mathcal{C}\ell$) be the restricted or the ordinary Hilbert class field (resp. class group) of K (see 3.5.1, (ii)). If $|G|$ and $|\mathcal{C}\ell|$ are coprime, $\mathrm{Gal}(H/\mathbb{Q}) \simeq \mathcal{C}\ell \rtimes G$ is characterized by the relations:

$$s' \circ \rho_{H/K}(c\ell(\boldsymbol{x})) \circ s'^{-1} = \rho_{H/K}(c\ell(s\boldsymbol{x})),$$

for any s' extending $s \in G$ and any idèle $\boldsymbol{x}$ (with $c\ell(\,\bullet\,) \in J/K^\times U \simeq \mathcal{C}\ell$), which become, in terms of Artin symbols that we will introduce in Subsection (b):

$$s' \circ \left(\frac{H/K}{c\ell(\mathfrak{a})}\right) \circ s'^{-1} = \left(\frac{H/K}{c\ell(s\,\mathfrak{a})}\right),$$

for any s' extending $s \in G$ and any ideal $\mathfrak{a}$ (with $c\ell(\,\bullet\,) \in \mathcal{C}\ell$).'

- I, p.124, after 3.8.2, add the following example:

'For instance, if K is equal to $\mathbb{Q}$ or to a *principal* imaginary quadratic field, this yields $\overline{G}^{\mathrm{ab}} \simeq U_0^{\mathrm{ord}}/i_0(\mu(K))$.'

- I, p.132, after 4.3.2, add the following remark:

'**4.3.3 Remark.** As for the composite map $\alpha_{L/K}^S\ :\ I_T \longrightarrow G^{\mathrm{ab}\,S}$, its kernel is equal to:

$$A_{L/K,T}^S := P_{T,\mathfrak{m},\mathrm{pos}}\langle\, S\,\rangle \mathrm{N}_{L/K}(I_{L,T}) := P_{T,\mathfrak{m},\Delta_\infty}\,.\,\langle\, S_0\,\rangle \mathrm{N}_{L/K}(I_{L,T}),$$

for any $\mathfrak{m}$ multiple of $\mathfrak{f}$, where $\Delta_\infty := Pl_\infty^{\mathrm{r}} \backslash S_\infty$ (see I.4.4).

By definition, since $A_{L/K,T}^S$ corresponds to $L^{\mathrm{ab}\,S}/K$, we have:

$$P_{T,\mathfrak{m},\mathrm{pos}}\langle\, S\,\rangle \mathrm{N}_{L/K}(I_{L,T}) = P_{T,\mathfrak{m},\mathrm{pos}} \mathrm{N}_{L^{\mathrm{ab}\,S}/K}(I_{L^{\mathrm{ab}\,S},T}),$$

for any $\mathfrak{m}$ multiple of the conductor of $L^{\mathrm{ab}\,S}$. □'

- I, p.136, 4.4.3.1, (ii), we give more justification of $\rho_{L/K}(i(x')) = 1$: 'moreover $\mathfrak{f} \,|\, \mathfrak{m}' := \mathfrak{m}\,\mathfrak{p}_v^{-m_v}$ and $(x') \in P_{T,\mathfrak{m}'}$ with $i_{v'}(x') > 0$ on Pl_∞^{rc}; since by definition $L^{\mathrm{ab}\,S} = L^{\mathrm{ab}}$ for $S = Pl_\infty^{\mathrm{r}} \backslash Pl_\infty^{\mathrm{rc}}$, the formula given in 4.3.3 yields $P_{T,\mathfrak{m}',Pl_\infty^{\mathrm{rc}}} \subseteq P_{T,\mathfrak{m}',\mathrm{pos}} \mathrm{N}_{L^{\mathrm{ab}}/K}(I_{L^{\mathrm{ab}},T})$, giving the result'
- I, p.156, ℓ.-1, read: '$W = V_T^S/K_T^{\times p}$'
- I, p.157, ℓ.-10, read: 'we can use the Kummer interpretation $\mathrm{rk}_{\chi^*}(W) = \mathrm{rk}_\chi(A)$ of the Remark 5.4.3, with $W = V_T^S/K_T^{\times p}$ and $A = \mathcal{C}\ell_{\mathfrak{m}^*}^{T\cup\Delta_\infty}$, giving the reflection theorem'
- I, p.159, ℓ.-8, note that the maximal abelian p-ramified pro-p-extension of $\mathbb{Q}$ is, for $p \neq 2$, the cyclotomic $\mathbb{Z}_p$-extension; then read: '$\rho_1(\emptyset, Pl_p) = -1$'
- I, p.161, before Subsection (c), we give a large complement of 6 pages, concerning the representation theory and the Spiegelungssatz, that we reproduce here:

'**(5.4.9)** ADDITIONAL MATERIAL. In 5.4.9.1, we will go into more details about the computation of $\mathrm{rk}_{\chi^*}\Big(\bigoplus_{v\in T} U_v \bigoplus_{v\in\Delta_\infty} \{\pm 1\}\Big) - \mathrm{rk}_{\chi^*}(E^{S_0\,\mathrm{ord}})$ when $\mu_p \subset K$, and in 5.4.9.2, we will give some comments on the interpretation of the classical reflection theorem. The notations are given in 5.4.2 and 5.4.5.

(5.4.9.1) p-RANKS COMPUTATIONS. We use the following properties of representation theory of g over $\mathbb{F}_p$, in the semi-simple case (i.e., $p \nmid |g|$):

(i) In the exact sequence of $\mathbb{Z}_p[g]$-modules $1 \to N \to M \to M/N \to 1$ we suppose that $N \cap M^p = N^p$; then this yields the exact sequence:

$$1 \longrightarrow N/N^p \longrightarrow M/M^p \longrightarrow M/M^pN \longrightarrow 1,$$

and by semi-simplicity: $M \otimes_{\mathbb{Z}_p} \mathbb{F}_p \simeq N \otimes_{\mathbb{Z}_p} \mathbb{F}_p \,\oplus\, (M/N) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ (isomorphism of representations).

(ii) If the $\mathbb{Z}_p[g]$-module M is a free $\mathbb{Z}_p$-module of finite type, the $\mathbb{F}_p$-representation $M \otimes_{\mathbb{Z}_p} \mathbb{F}_p \simeq M/M^p$ and the $\mathbb{Q}_p$-representation $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ have the same character (see [Se4, §§ 14-16]).

(iii) If M is a *finite* $\mathbb{Z}_p[g]$-module then the $\mathbb{F}_p$-representations M/M^p and ${}_pM := \{x \in M,\ x^p = 1\}$ are isomorphic: from the above reference, we know that, in the semi-simple case, the representation theories over $\mathbb{Z}_p$ and $\mathbb{F}_p$ are "the same" by reduction modulo p, so that for any $\chi \in \mathfrak{X}_p(g)$ we can write the exact sequence of $\mathbb{Z}_p[g]$-modules:

$$1 \longrightarrow ({}_pM)^{e_\chi} \longrightarrow M^{e_\chi} \xrightarrow{\ p\ } (M^{e_\chi})^p = (M^p)^{e_\chi} \longrightarrow 1$$

(the idempotents e_χ being those of $\mathbb{Z}_p[g]$); since M is finite, we get:

$$|({}_pM)_\chi| := |({}_pM)^{e_\chi}| = |M^{e_\chi}|\,|(M^p)^{e_\chi}|^{-1} = |(M/M^p)^{e_\chi}| =: |(M/M^p)_\chi|,$$

which proves that $(M/M^p)_\chi$ and $({}_pM)_\chi$ are isomorphic.

(iv) Let d be a subgroup of g and V_d the permutation representation of g modulo d ($V_d \simeq \mathbb{F}_p[g] \sum\limits_{t \in d} t$ for instance); then the character of V_d is $\mathrm{Ind}_d^g(1_d) =: \sum\limits_{\chi \in \mathfrak{X}_p(g)} \rho_\chi\, \chi$, where we recall that $\rho_\chi := \frac{1}{|d|} \sum\limits_{t \in d} \psi(t),\ \psi | \chi$. If d is normal in g, V_d is the regular representation of g/d, then $\rho_\chi = \psi(1)$ (resp. 0) if $d \subseteq \mathrm{Ker}(\chi)$ (resp. $d \not\subseteq \mathrm{Ker}(\chi)$).

(v) Let d be a subgroup of g and W a representation of d whose character is equal to the restriction of the Teichmüller character ω (since $\mu_p \subset K$, then μ_p is such a representation). Then, the character of the representation V of g induced by W is $(\mathrm{Ind}_d^g(1_d))^*$, and $\mathrm{rk}_{\chi^*}(V) = \rho_\chi$: indeed, the s_i denoting a complete system of representatives of g/d, by definition (see [Se4, § 3.3, Th. 12]) we have for all $s \in g$:

$$\begin{aligned}\mathrm{Ind}_d^g(\omega)(s) &= \sum_{\substack{s_i \in g/d \\ s_i^{-1} s s_i \in d}} \omega(s_i^{-1} s s_i) = \sum_{\substack{s_i \in g/d \\ s_i^{-1} s s_i \in d}} \omega(s) \\ &= \omega(s)\, \mathrm{Ind}_d^g(1_d)(s) = \omega(s)\, \mathrm{Ind}_d^g(1_d)(s^{-1}),\end{aligned}$$

giving the first part of the claim; then, $\mathrm{rk}_{\chi^*}(V)$ is given by the scalar product $\langle\, (\mathrm{Ind}_d^g(1_d))^*, \psi^* \,\rangle$ with $\psi^* := \omega\psi^{-1}$ (note that ψ^* is also absolutely irreducible), and an elementary computation yields $\langle\, (\mathrm{Ind}_d^g(1_d))^*, \psi^* \,\rangle = \langle\, \mathrm{Ind}_d^g(1_d), \psi \,\rangle$. Therefore $\mathrm{rk}_{\chi^*}(V) = \rho_\chi$.

Let $u \in Pl_{p,k}$, and consider the induced representation $\bigoplus\limits_{v|u} U_v \otimes \mathbb{F}_p$ of g; from (i) with $N = \bigoplus\limits_{v|u} \mathrm{tor}(U_v)$ we have:

$$\bigoplus_{v|u} U_v \otimes \mathbb{F}_p \simeq \bigoplus_{v|u} \mathrm{tor}(U_v) \otimes \mathbb{F}_p \,\oplus\, \bigoplus_{v|u} (U_v/\mathrm{tor}(U_v)) \otimes \mathbb{F}_p.$$

Using the log map defined in III.2.2.1, which is a g-module homomorphism, injective on $\bigoplus\limits_{v|u}(U_v/\mathrm{tor}(U_v))$, we see from (ii) that the character of $\bigoplus\limits_{v|u}(U_v/\mathrm{tor}(U_v)) \otimes \mathbb{F}_p$ is the character of the $\mathbb{Q}_p$-representation $\bigoplus\limits_{v|u} K_v$ which is $[k_u : \mathbb{Q}_p]$ times the regular representation. The corresponding χ^*-rank is thus $[k_u : \mathbb{Q}_p]\psi^*(1) = [k_u : \mathbb{Q}_p]\psi(1)$.

Since $\mu_p \subset K$, $\bigoplus\limits_{v|u} \mathrm{tor}(U_v) \otimes \mathbb{F}_p$ is induced by $\mathrm{tor}(U_v) \otimes \mathbb{F}_p$ whose character is ω; from (v), the character of the above representation of g is $(\mathrm{Ind}_{d_u}^g(1_{d_u}))^*$, where we recall that d_u is the decomposition group in K/k of a fixed place $v|u$, and the χ^*-rank is $\rho_{u,\chi}$.

Let $u \in Pl_{\mathrm{ta}}$. In this case $U_v^1 \otimes \mathbb{F}_p = 1$ and the character of $\bigoplus_{v|u} U_v \otimes \mathbb{F}_p$ is the character of the torsion part giving a χ^*-rank equal to $\rho_{u,\chi}$.

Let $u \in Pl_{k,\infty}^{\mathrm{r}}$. In this case, $\bigoplus_{v|u}\{\pm 1\} \otimes \mathbb{F}_p$ is nontrivial only for $p = 2$ and gives the regular representation since $d_u = 1$ (by assumption $|g|$ is odd); this yields a χ^*-rank equal to $\delta_{2,p}\psi(1)$.

We now compute the χ^*-rank of $E^{S_0\,\mathrm{ord}}$ which is given by the Dirichlet–Herbrand Theorem I.3.7. We have:

$$\mathrm{rk}_{\chi^*}(E^{S_0\,\mathrm{ord}}) = \sum_{u \in Pl_{k,\infty}} \rho_{u,\chi^*} + \sum_{u \in S_{0,k}} \rho_{u,\chi^*} + \delta_{\omega,\chi^*} - \delta_{1,\chi^*}.$$

We remark that if u is a complex infinite place of k or a real infinite place of k, totally split in K/k, then $\rho_{u,\chi^*} = \psi(1)$ since $d_u = 1$; if u is a real infinite place of k, complexified in K/k, then $\rho_{u,\chi^*} = \frac{1}{2}(\psi(1) + \psi(c_u))$ where c_u generates d_u. Note that $\delta_{\omega,\chi^*} = \delta_{1,\chi}$ and $\delta_{1,\chi^*} = \delta_{\omega,\chi}$.

We have obtained:

$$\mathrm{rk}_{\chi^*}\Big(\bigoplus_{v\in T} U_v \bigoplus_{v\in\Delta_\infty}\{\pm1\}\Big) = \sum_{u\in T_k}\rho_{u,\chi} + \sum_{u\in T_{p,k}}[k_u : \mathbb{Q}_p]\psi(1) + \delta_{2,p}\psi(1)|\Delta_{\infty,k}|,$$

and:

$$\mathrm{rk}_{\chi^*}(E^{S_0\,\mathrm{ord}}) = r_2(k)\psi(1) + \sum_{u\in Pl_{k,\infty}^{\mathrm{r}} \cup S_{0,k}} \rho_{u,\chi^*} + \delta_{1,\chi} - \delta_{\omega,\chi}.$$

This yields the second expression of $\rho_\chi(T,S)$ given at the end of 5.4.2.

Note. We have $\sum_{u\in Pl_{k,\infty}} (\rho_{u,\chi} + \rho_{u,\chi^*}) = \psi(1)[k : \mathbb{Q}] + \delta_{2,p}\psi(1)r_1(k)$: we check that $\rho_{u,\chi} + \rho_{u,\chi^*} = \frac{1}{2}(\psi(1) + \psi(c_u) + \psi(1) + \psi(c_u)\omega(c_u))$; if $c_u = 1$, this sum is equal to $2\psi(1)$; otherwise, if $c_u \neq 1$ (which supposes $p \neq 2$), $\omega(c_u) = -1$, and this sum is equal to $\psi(1)$; let $r_1^{\mathrm{c}}(k) := |Pl_{k,\infty}^{\mathrm{rc}}|$; $\sum_{v\in Pl_{k,\infty}} (\rho_{u,\chi} + \rho_{u,\chi^*}) - \psi(1)[k : \mathbb{Q}] = \psi(1)\big(2(r_2(k)+r_1(k)-r_1^{\mathrm{c}}(k))+r_1^{\mathrm{c}}(k)\big)-\psi(1)\big(r_1(k)+2r_2(k)\big) = \psi(1)\big(r_1(k)-r_1^{\mathrm{c}}(k)\big) = \delta_{2,p}\psi(1)r_1(k)$. Finally we use the relation $[k : \mathbb{Q}] = r_1(k) + 2r_2(k) = \sum_{u|p}[k_u : \mathbb{Q}_p]$ to obtain the first expression of $\rho_\chi(T,S)$ (note that $r_1^{\mathrm{c}}(k) = 0$ if $p = 2$ since $|g|$ is odd, and $r_1^{\mathrm{c}}(k) = r_1(k)$ if $p \neq 2$ since $K \supset \mu_p$ is totally complex).

(5.4.9.2) Interpretation of the Reflection Theorem for Usual Class Groups. We consider the case where $\mu_p =: \langle\zeta\rangle \subset K$, $T = \emptyset$, $S = S_\infty \subseteq Pl_\infty^{\mathrm{r}}$; then the reflection theorem becomes for any $\chi \in \mathfrak{X}_p(g)$:

$$\mathrm{rk}_{\chi^*}(\mathcal{C}\ell^{S_\infty}) - \mathrm{rk}_\chi(\mathcal{C}\ell_{\mathfrak{m}^*}^{\Delta_\infty}) = \rho_\chi(\emptyset, S_\infty), \tag{1}$$

where $\mathfrak{m}^* := \prod_{v|p}\mathfrak{p}_v^{pe_v} = p(1-\zeta)$ is the modulus of p-primarity of K which characterizes the non-ramification at p of Kummer extensions of degree p (review I.6.3), and where $\Delta_\infty := Pl_\infty^{\mathrm{r}} \backslash S_\infty$.

We can take the χ-parts of the exact sequences given in the proof of I.4.5, (ii) (beware that the notations are permuted because of the reflection situation: $\mathfrak{m} \mapsto \mathfrak{m}^*$, $T \mapsto Pl_p$, $S_\infty \mapsto \Delta_\infty$, $\Delta_\infty \mapsto S_\infty$); then since $(U_v/U_v^1)_p = 1$ for $v|p$, taking $\mathfrak{n} = 1$, $\Sigma_\infty \subseteq S_\infty$, we obtain:

$$\begin{aligned}&\mathrm{rk}_\chi(\mathcal{C}\ell_{\mathfrak{m}^*}^{\Delta_\infty}) - \mathrm{rk}_\chi(\mathcal{C}\ell^{\Delta_\infty\cup\Sigma_\infty}) = \\ &\mathrm{rk}_\chi\Big(\bigoplus_{v|p} U_v^1/(U_v^1)^p U_v^{pe_v} \bigoplus_{v\in\Sigma_\infty}\{\pm1\}\Big) - \mathrm{rk}_\chi\big(Y_{Pl_p}^{\Delta_\infty\cup\Sigma_\infty}/Y_{Pl_p,\mathfrak{m}^*}^{\Delta_\infty}\big),\end{aligned} \tag{2}$$

where we recall that:

$$\begin{aligned} Y_{Pl_p}^{\mathrm{ord}} &:= \{\alpha \in K_{Pl_p}^\times,\ \ (\alpha) = \mathfrak{a}^p\},\\ Y_{Pl_p}^{\Delta_\infty\cup\Sigma_\infty} &:= \{\alpha \in Y_{Pl_p}^{\mathrm{ord}},\ \ i_v(\alpha) > 0\ \forall v \in S_\infty\backslash\Sigma_\infty\},\\ Y_{Pl_p,\mathfrak{m}^*}^{\Delta_\infty} &:= \{\alpha \in Y_{Pl_p}^{\mathrm{ord}},\ \ i_v(\alpha) \in (U_v^1)^p U_v^{pe_v}\ \forall v|p,\ \ i_v(\alpha) > 0\ \forall v \in S_\infty\}\end{aligned}$$

(the subgroup of p-primary "Δ_∞-pseudo-units"); this group will be denoted $Y^{\Delta_\infty}_{\rm prim}$ (and for simplicity, the indices Pl_p will be omitted). Recall that $Y^{\rm ord} = K^{\times p} Y^{\rm ord}_{Pl_p}$ is given by the exact sequence:

$$1 \longrightarrow E^{\rm ord}/(E^{\rm ord})^p \longrightarrow Y^{\rm ord}/K^{\times p} \longrightarrow {}_p\mathcal{C}\!\ell^{\rm ord} \longrightarrow 1, \tag{3}$$

where $({}_p\mathcal{C}\!\ell^{\rm ord})_\chi \simeq \mathcal{C}\!\ell^{\rm ord}_\chi$ by (iii). Thus we easily obtain from (1) and (2):

$$\begin{aligned}&{\rm rk}_{\chi^*}(\mathcal{C}\!\ell^{S_\infty}) - {\rm rk}_\chi(\mathcal{C}\!\ell^{\Delta_\infty \cup \Sigma_\infty}) = \rho_\chi(\emptyset, S_\infty)\\ &+{\rm rk}_\chi\Big(\bigoplus_{v \vert p} U_v^1/(U_v^1)^p U_v^{pe_v} \bigoplus_{v \in \Sigma_\infty} \{\pm 1\}\Big) - {\rm rk}_\chi\big(Y^{\Delta_\infty \cup \Sigma_\infty}/Y^{\Delta_\infty}_{\rm prim}\big).\end{aligned}$$

We now compute the χ-rank of $\bigoplus_{v \vert p} U_v^1/(U_v^1)^p U_v^{pe_v}$. Consider the exact sequences:

$$1 \longrightarrow U_v^{pe_v}/U_v^{pe_v} \cap (U_v^1)^p \longrightarrow U_v^1/(U_v^1)^p \longrightarrow U_v^1/(U_v^1)^p U_v^{pe_v} \longrightarrow 1,$$

and:

$$1 \longrightarrow U_v^{pe_v} \cap (U_v^1)^p \longrightarrow U_v^{pe_v} \xrightarrow{\ \tau\ } \mu_p \longrightarrow 1,$$

where the map τ associates with $\alpha = 1 + p(1-\zeta)\eta$ the root of unity ζ^t with $t := {\rm tr}_{F_v/\mathbb{F}_p}(\overline{\eta})$ (see I.6.3.5).

If $s \in d_u$, $s(\alpha) = 1 + p(1-\zeta^{\omega(s)})s(\eta) = 1 + p(1-\zeta)\frac{1-\zeta^{\omega(s)}}{1-\zeta} s(\eta) \equiv 1 + p(1-\zeta)\omega(s)s(\eta) \bmod p(1-\zeta)\mathfrak{p}_v$, thus ${\rm tr}_{F_v/\mathbb{F}_p}(\overline{\omega(s)}\ \overline{s(\eta)}) = \overline{\omega(s)}{\rm tr}_{F_v/\mathbb{F}_p}(\overline{\eta})$ since $\overline{\omega(s)} \in \mathbb{F}_p$; therefore τ is an homomorphism of d_u-modules and the character of the representation $\bigoplus_{v \vert u} U_v^{pe_v}/U_v^{pe_v} \cap (U_v^1)^p$ is induced by ω; then, using 5.4.9.1, (v), we get:

$$\begin{aligned}{\rm rk}_\chi\Big(\bigoplus_{v \vert u} U_v^1/(U_v^1)^p U_v^{pe_v}\Big) &= {\rm rk}_\chi\Big(\bigoplus_{v \vert u} U_v^1/(U_v^1)^p\Big) - {\rm rk}_\chi\Big(\bigoplus_{v \vert u} U_v^{pe_v}/U_v^{pe_v} \cap (U_v^1)^p\Big)\\ &= \rho_{u,\chi^*} + [k_u : \mathbb{Q}_p]\psi(1) - \rho_{u,\chi^*} = [k_u : \mathbb{Q}_p]\psi(1),\end{aligned}$$

and we get:

$$\begin{aligned}&{\rm rk}_{\chi^*}(\mathcal{C}\!\ell^{S_\infty}) - {\rm rk}_\chi(\mathcal{C}\!\ell^{\Delta_\infty \cup \Sigma_\infty})\\ &\quad = \rho_\chi(\emptyset, S_\infty) + \sum_{u \vert p}[k_u : \mathbb{Q}_p]\psi(1) + \delta_{2,p}\psi(1)\vert\Sigma_{\infty,k}\vert - {\rm rk}_\chi\big(Y^{\Delta_\infty \cup \Sigma_\infty}/Y^{\Delta_\infty}_{\rm prim}\big)\\ &\quad = \rho_\chi(\emptyset, S_\infty) + \psi(1)\big([k : \mathbb{Q}] + \delta_{2,p}\vert\Sigma_{\infty,k}\vert\big) - {\rm rk}_\chi\big(Y^{\Delta_\infty \cup \Sigma_\infty}/Y^{\Delta_\infty}_{\rm prim}\big).\end{aligned}$$

Using the expression of $\rho_\chi(\emptyset, S_\infty)$ and the relation $[k : \mathbb{Q}] = r_1(k) + 2r_2(k)$, we finally obtain:

$$\begin{aligned}&{\rm rk}_{\chi^*}(\mathcal{C}\!\ell^{S_\infty}) - {\rm rk}_\chi(\mathcal{C}\!\ell^{\Delta_\infty \cup \Sigma_\infty}) = \delta_{\omega,\chi} - \delta_{1,\chi} - \sum_{u \in Pl^{\rm r}_{k,\infty}} \rho_{u,\chi^*}\\ &+\psi(1)\big(r_1(k) + r_2(k) + \delta_{2,p}\vert\Delta_{\infty,k} \cup \Sigma_{\infty,k}\vert\big) - {\rm rk}_\chi\big(Y^{\Delta_\infty \cup \Sigma_\infty}/Y^{\Delta_\infty}_{\rm prim}\big)\ ;\end{aligned} \tag{4}$$

since ${\rm rk}_\chi\big(Y^{\Delta_\infty \cup \Sigma_\infty}/Y^{\Delta_\infty}_{\rm prim}\big)$ is the χ-rank of the diagonal image of $Y^{\Delta_\infty \cup \Sigma_\infty}$ in $\bigoplus_{v \vert p} U_v^1/(U_v^1)^p U_v^{pe_v} \bigoplus_{v \in \Sigma_\infty} \{\pm 1\}$, we have the inequality:

$${\rm rk}_\chi\big(Y^{\Delta_\infty \cup \Sigma_\infty}/Y^{\Delta_\infty}_{\rm prim}\big) \leq \psi(1)\big([k : \mathbb{Q}] + \delta_{2,p}\vert\Sigma_{\infty,k}\vert\big). \tag{4$'$}$$

We will explain the interest of such formulas (4), (4$'$), by giving two classical examples which are not always well understood since, in general, in (4) the term $\psi(1)\big([k : \mathbb{Q}] + \delta_{2,p}\vert\Sigma_{\infty,k}\vert\big) - {\rm rk}_\chi\big(Y^{\Delta_\infty \cup \Sigma_\infty}/Y^{\Delta_\infty}_{\rm prim}\big)$ is replaced by 0 (for a lower bound) or by $\psi(1)\big([k : \mathbb{Q}] + \delta_{2,p}\vert\Sigma_{\infty,k}\vert\big)$ (for an upper bound), giving again the Leopoldt's Spiegelungssatz 5.4.6, (i) with inequalities.

For $p = 2$, to obtain the inequalities 5.4.6, (ii), we substract the equality (4) with $S_\infty = Pl^{\rm r}_\infty$ from the equality (4) with $S_\infty = \emptyset$ ($\Sigma_\infty = \emptyset$ in each case); then we check that $\mathrm{rk}_\chi(Y^{\mathrm{ord}}/Y^{\mathrm{ord}}_{\mathrm{prim}}) - \mathrm{rk}_\chi(Y^{\mathrm{res}}/Y^{\mathrm{res}}_{\mathrm{prim}})$ is the χ-rank of the quotient of the images of Y^{ord} and Y^{res} in $\bigoplus_{v|p} U_v^1/(U_v^1)^p U_v^{pe_v}$.

Analysis of the Theorem of Scholz. We refer to 5.4.6.1 and 5.4.6.2 to review that $\mathrm{rk}_\chi(\mathcal{C}\!\ell) = \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{d}\,)})$, $\mathrm{rk}_{\chi^*}(\mathcal{C}\!\ell) = \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{-3d}\,)})$.

Let $E =: \langle\, \varepsilon, \zeta \,\rangle$, where ε is the fundamental unit of $\mathbb{Q}(\sqrt{d}\,)$; to simplify, we merge the notation of an element with that of its class modulo $K^{\times 3}$. Thus:

$$(E/E^3)_\chi = \langle\, \varepsilon \,\rangle, \quad (E/E^3)_{\chi^*} = 1,$$

$$(E/E^3)_\omega = \langle\, \zeta \,\rangle, \quad (E/E^3)_1 = 1\ ;$$

Formulas (4), (4′) yield (since $\rho_{\infty,\chi^*} = 0$):

$$\mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{-3d}\,)}) - \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{d}\,)}) = 1 - \mathrm{rk}_3(Y_{\mathbb{Q}(\sqrt{d}\,)}/Y_{\mathbb{Q}(\sqrt{d}\,),\mathrm{prim}}),$$

with $\mathrm{rk}_3(Y_{\mathbb{Q}(\sqrt{d}\,)}/Y_{\mathbb{Q}(\sqrt{d}\,),\mathrm{prim}}) \le 1$. Let:

$$Y_{\mathbb{Q}(\sqrt{d}\,)}/\mathbb{Q}(\sqrt{d}\,)^{\times 3} =: \langle\, y_1, \ldots, y_r, \varepsilon \,\rangle,$$

where r is the 3-rank of the class group of $\mathbb{Q}(\sqrt{d}\,)$ (see (3)); we have:

$$\mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{-3d}\,)}) - \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{d}\,)}) = 1$$

if and only if the y_i as well as ε are 3-primary; otherwise $Y_{\mathbb{Q}(\sqrt{d}\,),\mathrm{prim}}$ is of index 3 in $Y_{\mathbb{Q}(\sqrt{d}\,)}$ and we have:

$$\mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{-3d}\,)}) = \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{d}\,)}).$$

If we only know that ε is 3-primary, then we have:

$$0 \le \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{-3d}\,)}) - \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{d}\,)}) \le 1\ ;$$

oterwise, if we know that ε *is not* 3-primary, then:

$$\mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{-3d}\,)}) = \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{d}\,)}).$$

Symmetrically, we can start from the character χ^* and write (with $\rho_{\infty,\chi} = 1$):

$$\mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{d}\,)}) - \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{-3d}\,)}) = -\mathrm{rk}_3(Y_{\mathbb{Q}(\sqrt{-3d}\,)}/Y_{\mathbb{Q}(\sqrt{-3d}\,),\mathrm{prim}}),$$

with $\mathrm{rk}_3(Y_{\mathbb{Q}(\sqrt{-3d}\,)}/Y_{\mathbb{Q}(\sqrt{-3d}\,),\mathrm{prim}}) \le 1$, then we can put, in an analogous manner, $Y_{\mathbb{Q}(\sqrt{-3d}\,)}/\mathbb{Q}(\sqrt{-3d}\,)^{\times 3} = \langle\, y'_1, \ldots, y'_{r'} \,\rangle$, where r' is the 3-rank of the class group of $\mathbb{Q}(\sqrt{-3d}\,)$, which gives the following reasoning. We have:

$$\mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{d}\,)}) = \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{-3d}\,)})$$

if and only if all the y'_i are 3-primary, otherwise $Y_{\mathbb{Q}(\sqrt{-3d}\,),\mathrm{prim}}$ is of index 3 in $Y_{\mathbb{Q}(\sqrt{-3d}\,)}$ and this yields:

$$\mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{d}\,)}) - \mathrm{rk}_3(\mathcal{C}\!\ell_{\mathbb{Q}(\sqrt{-3d}\,)}) = -1.$$

This methodology was initiated in: Bull. Soc. Math. France 100 (1972), 177–193; in this paper we gave many numerical examples.

Analysis of a Result of Hecke. We are now concerned with the case $K = \mathbb{Q}(\mu_p)$, $p \ne 2$, with $g = \mathrm{Gal}(K/\mathbb{Q})$. For an even character $\chi \ne 1$ (i.e., $\chi = \omega^k$, k even, $1 < k < p-1$), we have $\chi^* = \omega\chi^{-1} = \omega^{1-k} \ne \omega$. Since $\rho_{\infty,\chi^*} = 0$, formulas (4), (4′) yield:

$$\mathrm{rk}_{\chi^*}(\mathcal{C}\!\ell) - \mathrm{rk}_\chi(\mathcal{C}\!\ell) = 1 - \mathrm{rk}_\chi(Y/Y_{\mathrm{prim}}),$$

with $\mathrm{rk}_\chi\big(Y/Y_{\mathrm{prim}}\big) \le 1$. Let $(E/E^p)_\chi =: \langle\, \varepsilon_\chi E^p \,\rangle$ denoted $\langle\, \varepsilon_\chi \,\rangle$, and let:

$$\big(Y/K^{\times p}\big)_\chi =: \langle\, y_1, \ldots, y_{r_\chi}, \varepsilon_\chi \,\rangle,$$

where r_χ is the χ-rank of the class group, and where all the numbers are prime to p. Then:

$$\mathrm{rk}_{\chi^*}(\mathcal{C}\ell) - \mathrm{rk}_\chi(\mathcal{C}\ell) = 1$$

if and only if all the elements $y_1, \ldots, y_{r_\chi}, \varepsilon_\chi$ are p-primary, otherwise, the group $\big(Y_{\mathrm{prim}}/K^{\times p}\big)_\chi$ is of index p in $\big(Y/K^{\times p}\big)_\chi$ and the p-ranks of the class groups are equal.

If ε_χ is p-primary, we only have $0 \le \mathrm{rk}_{\chi^*}(\mathcal{C}\ell) - \mathrm{rk}_\chi(\mathcal{C}\ell) \le 1$, otherwise $\mathrm{rk}_{\chi^*}(\mathcal{C}\ell) = \mathrm{rk}_\chi(\mathcal{C}\ell)$.

The result of Hecke (1910) was (with classical notations) the inequality $\mathrm{rk}_p(\mathcal{C}\ell^+) \le \mathrm{rk}_p(\mathcal{C}\ell^-)$ that we easily obtain from (4) by summation over the even $\chi \ne 1$; the use of (4′) yields $0 \le \mathrm{rk}_p(\mathcal{C}\ell^-) - \mathrm{rk}_p(\mathcal{C}\ell^+) \le \frac{p-3}{2}$. Some insights into representation aspects were given by Pollaczek (1924) after Kummer.

If we start from the odd character $\chi^* \ne \omega$, we obtain, since $\rho_{\infty,\chi} = 1$:

$$\mathrm{rk}_\chi(\mathcal{C}\ell) - \mathrm{rk}_{\chi^*}(\mathcal{C}\ell) = -\mathrm{rk}_{\chi^*}\big(Y/Y_{\mathrm{prim}}\big)\ ;$$

then if $\big(Y/K^{\times p}\big)_{\chi^*} =: \langle\, y'_1, \ldots, y'_{r_{\chi^*}} \,\rangle$, the reasoning is the same, but with pseudo-units (which are not units) comming from an odd component.

We do not know examples with $r_\chi \ge 1$ (see [(c), Wa, Ch. 8, § 3]). To find an $r_\chi \ge 1$, we must check that $r_{\chi^*} \ge 1$ and (when it is the case) that y'_{χ^*} (generator of $(Y/K^{\times p})_{\chi^*}$ when $r_{\chi^*} = 1$) is p-primary (the case $r_{\chi^*} \ge 2$ automatically yields $r_\chi \ge 1$); the first condition is equivalent to the triviality modulo (p) of the generalized Bernoulli number b_{χ^*} or to the p-primarity of the cyclotomic unit η_χ, giving a probability equal to $\dfrac{1}{p}$; [1] the second condition (when the first one is realized with $r_{\chi^*} = 1$) has also a probability equal to $\dfrac{1}{p}$. If we assume that these two conditions are independent, this gives the probability $\dfrac{1}{p^2}$ (for χ fixed). We have neglected the case $r_{\chi^*} \ge 2$ whose probability is less than $\dfrac{1}{p^2}$ (the principal theorem of Ribet–Mazur–Wiles–Kolyvagin implies that $|\mathcal{C}\ell_K^{e_{\chi^*}}| = |b_{\chi^*}|_p^{-1}$, but if $b_{\chi^*} \equiv 0 \bmod (p^2)$, $\mathcal{C}\ell_K^{e_{\chi^*}}$ may be cyclic), so that we can consider the probability $\dfrac{2}{p^2}$ as a wide upper bound.

This heuristic reasoning, involving *congruences*, is more convincing than the direct interpretation $r_\chi \ge 1$ if and only if $p \,|\, (\langle\, \varepsilon_\chi \,\rangle : \langle\, \eta_\chi \,\rangle)$ (principal theorem of Ribet–Mazur–Wiles–Kolyvagin–Greither), since we do not know efficient heuristics for global p-powers; the above gives for $p \,|\, (\langle\, \varepsilon_\chi \,\rangle : \langle\, \eta_\chi \,\rangle)$ a probability less than $\dfrac{2}{p^2}$ (see [Scho2] for the non-p-parts of the class goup).

But there are $n := \dfrac{p-3}{2}$ even characters $\ne 1$ for $p \ge 3$; perhaps they do not have the same "weight" because of the subfields of "small" degree whose p-class number can be limited. To be more precise, we may estimate the index of irregularity $i(p)$ (i.e., the number of odd characters χ^* giving $r_{\chi^*} \ge 1$): it seems clear that the density of prime numbers p, for which $i(p) \ge 1$, exists (its value is discussed in [(c), Wa, Ch. 5, § 3] after the Theorem 5.17; see [Ri4] and

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[1] The equivalence of these two conditions is classical and comes from the congruence properties of p-adic L-functions. Let $\mu_p =: \langle\, \zeta \,\rangle$, and for any $a \in \mathbb{Z}$ prime to p, let $\sigma_a \in g$ be such that $\sigma_a(\zeta) = \zeta^a$. By definition, we have $b_{\chi^*} := \frac{1}{p}\sum_{a=1}^{p-1} \chi^*(\sigma_a^{-1})a \in \mathbb{Z}_p$, and $\eta_\chi := (1-\zeta)^{e_\chi}$ seen in $(E/E^p)_\chi$, with the idempotent $e_\chi := \frac{1}{p-1}\sum_{a=1}^{p-1} \chi(\sigma_a^{-1})\sigma_a \in \mathbb{Z}_p[g]$; for $u = p$, $e_v = 1$, $\varphi \notin \{1, \omega\}$, we have $(U_v^1/(U_v^1)^p U_v^p)_\varphi \simeq \mathbb{F}_p$; since η_χ is p-primary if and only if its image in $(U_v^1/(U_v^1)^p U_v^p)_\chi$ is trivial, this gives one possibility out of p. In an analogous way, the p-primarity of y'_{χ^*} only depends on its image in $(U_v^1/(U_v^1)^p U_v^p)_{\chi^*}$.

[BCEMS] for some numerical computations). In this context, the probability that $i(p) = i \geq 0$ is $\mathsf{C}_n^i \left(1 - \frac{1}{p}\right)^{n-i} \left(\frac{1}{p}\right)^i$, for $p \geq 3$.

Since these values are in accordance with all numerical data, this "proves" that the above phenomena can be neglected [2], which yields a number of favourable cases $p < B$ around $\sum\limits_{p<B} \sum\limits_{i=0}^{n} \mathsf{C}_n^i \left(1 - \frac{1}{p}\right)^{n-i} \left(\frac{1}{p}\right)^i \left(1 - \left(1 - \frac{2}{p}\right)^i\right) = \sum\limits_{p<B} \left(1 - \left(1 - \frac{2}{p^2}\right)^n\right) < \sum\limits_{p<B} \frac{2n}{p^2} < \sum\limits_{p<B} \frac{1}{p} < \log(\log(B))$ (of course, it is then equivalent to use directly the probability that $p \,|\, (\langle \varepsilon_\chi \rangle : \langle \eta_\chi \rangle)$ for *at least* one character χ). See in [Th2] a criterion for the p-triviality of this index.

In conclusion, the classical Kummer–Vandiver conjecture is probably false for probabilistic reasons. [3] (see [Iw5] for another approach with Gauss sums). '

- I, p.170, ℓ.-6, we give the following Note to precise some forthcoming notations:

'**Note.** We will introduce the notation $\overline{H}_{T(p)}^{S}$ for the maximal T-ramified S-split pro-p-extension of K; as in the case $T = \emptyset$, $S = Pl_\infty^r$ (the p-Hilbert class fields tower $K^{(\infty)}{}_{(p)}$ in the ordinary sense), $\overline{H}_{T(p)}^{S}$ is also the p-tower of the successive maximal T-ramified S-split abelian pro-p-extensions defined in 5.3, and the maximal pro-p-subextension of the corresponding tower $\overline{H}_T^S$ (same proof). The groups $\mathcal{G}_T^S := \mathrm{Gal}(\overline{H}_{T(p)}^{S}/K)$ will be studied in the Appendix. Warning: $\overline{H}_T^S$ may be strictly contained in the maximal T-ramified S-split Galois extension of K (e.g., take $k/\mathbb{Q}$ with Galois group A_5 and let T be the set of ramified primes in $k/\mathbb{Q}$; then $k \cap \overline{H}_T = \mathbb{Q}$ with $\overline{H}_T \neq \mathbb{Q}$).'

- I, p.172, 5.9.2, we have a new redaction of the point (i) as follows:

'(i) The Hilbert class field is a particular solution to the principalization problem of the ideal group of a field K; we will not expand on this, but it is clear that the classes of K can principalize in many other abelian extensions of K, and we now have quite a precise understanding of the ideal extension map for the extension $\overline{K}^{\mathrm{ab}}/K$ (see [Gr9], [Kur], [Bos]). For instance, from the above papers we can state the following results:

(α) Let $K/\mathbb{Q}$ be a real abelian extension of degree prime to $p \neq 2$. For any $\mathbb{Q}_p$-irreducible character χ of $g := \mathrm{Gal}(K/\mathbb{Q})$, let $\mathcal{C}\ell_\chi := (\mathcal{C}\ell_K)_p^{e_\chi}$ where $e_\chi \in \mathbb{Z}_p[g]$ is the corresponding idempotent. Let $\psi | \chi$; ψ is of degree 1, of order m_χ prime to p, and $\mathbb{Z}_p[g]e_\chi \simeq \mathbb{Z}_p[\mu_{m_\chi}] =: R_\chi$; put $\mathcal{C}\ell_\chi \simeq \bigoplus\limits_{i=1}^{r_\chi} R_\chi / p^{n_{\chi,i}} R_\chi$.

Then there exist infinitely many abelian extensions $M/\mathbb{Q}$ such that $\mathrm{Gal}(KM/K) \simeq \bigoplus\limits_{\chi} \bigoplus\limits_{i=1}^{r_\chi} \mathbb{Z}/p^{n_{\chi,i}}\mathbb{Z}$ and $j_{KM/K}((\mathcal{C}\ell_K)_p) = 1$.

(β) Let k be a non-totally real number field; then for all finite extension K of k there exists an abelian extension M of k such that $j_{KM/K}(\mathcal{C}\ell_K^{\mathrm{ord}}) = 1$.

(γ) For any totally real number field K there exists a real abelian extension M of $\mathbb{Q}$ such that $j_{KM/K}(\mathcal{C}\ell_K^{\mathrm{ord}}) = 1$.

In a forthcoming paper, Bosca proves (β) and (γ) in a unified way: "if K/k is totally split at a (real or complex) infinite place, then the ordinary class group of K principalizes in an abelian compositum of K/k". In [Bos], we also find analogous results for the logarithmic class group defined in (Ch. III, § 7)... '

- I, p.186, ℓ.5, add: 'Let τ be a generator of $\mathrm{Gal}(K_1/K)$'

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[2] However, see [Sou] giving some insights into this aspect.

[3] A more precise computation involving the Cohen–Lenstra–Martinet heuristics on class groups would certainly give less than $c \log(\log(B))$ with $c < 1$. Moreover, discarding the small primes, we would obtain $c \log(\log(B)) - c'$, $c' > 1$, which explains that only very large p can disprove the conjecture.

- I, p.189, ℓ.-17, replace: '$\Sigma_2 \neq \emptyset$' by: '$\Sigma_2 \supseteq Pl_2^{\mathrm{ns}}$'
- I, p.202, footnote, add: '$a_2 = -2$'
- I, p.205, ℓ.11, read: 'power residue symbol for $n|m := |\mu(K)|$'
- I, p.205, ℓ.-2, replace: '$(x, \pi_v)_v$' by: '$\big(i_v(x)^{\frac{m_v}{n}}, \pi_v\big)_v$'
- I, p.206, ℓ.10, after: 'separately' add: '(the case $\mathrm{v}(x) \neq 0$ implies $\mathrm{v}(y) = 0$ and, since $v \nmid n$, this yields $v \notin R_y$; the case $\mathrm{v}(y) \neq 0$ is symmetrical)'
- I, p.213, 7.6.1: this result comes from the snake lemma applied to the fundamental diagram.
- I, p.216, ℓ.4, replace: 'this rank' by: 'the p-rank of $R_2^{\mathrm{ord}}(K)$' ; read: '(see III.2.1.1 for $S = Pl_\infty^{\mathrm{r}}$, $T = Pl_p$, and III.4.2.2).'
- I, p.223, 1.1.3, this remark is written as follows:

'**1.1.3 Remarks.** (i) Note that in 1.1 we have, by II.5.2.2, (i):

$$\mathrm{Gal}(K(\mathfrak{m})^S / K(\mathfrak{n})^{S \cup \delta_\infty}) = \langle\, I_v(K(\mathfrak{m})^S/K)\,\rangle_{v \in T \backslash t} \,.\, \langle\, D_v(K(\mathfrak{m})^S/K)\,\rangle_{v \in \delta_\infty},$$

the group generated by the inertia groups of the places $v \in T\backslash t$ and the decomposition groups of the places $v \in \delta_\infty$, in $K(\mathfrak{m})^S/K$.

This interpretation is valid only if $\mathfrak{n}$ is a pure divisor of $\mathfrak{m}$, in other words if $\mathfrak{n}$ and $\frac{\mathfrak{m}}{\mathfrak{n}}$ are coprime (see more general situations in 1.1.6 and 1.1.8).

(ii) If $m_v = 1$ for all $v \in T\backslash t$, then $U_v/U_v^{m_v} \simeq F_v^\times$ for these places, the embedding $i_{T\backslash t, \delta_\infty}$ may be identified with the family of residual maps, and the exact sequence of the Theorem 1.1 yields the isomorphism:

$$\mathrm{Gal}(K(\mathfrak{m})^S / K(\mathfrak{n})^{S \cup \delta_\infty}) \simeq \bigoplus_{v \in (T\backslash t) \cup \delta_\infty} F_v^\times \Big/ i_{T\backslash t, \delta_\infty}(E_{\mathfrak{n}}^{S \cup \delta_\infty})$$

(with $\frac{\mathfrak{m}}{\mathfrak{n}} = \prod_{v \in T\backslash t} \mathfrak{p}_v$ prime to $\mathfrak{n}$, $\delta_\infty \subseteq \Delta_\infty$, $F_v^\times = \{\pm 1\}$ if $v \in \delta_\infty$). $\square$'

- I, p.224, ℓ.-10/-9, put: '(α)' before: 'Using II.3.3, (iii), or II.3.3.5…'
- I, p.236, 1.6, this theorem is written as follows:

'**1.6 Theorem** (structure of $\mathrm{Gal}(H_T^S/H^S)$). *We have the homeomorphism:*

$$\mathrm{Gal}(H_T^S/H^S) \simeq \bigoplus_{v \in T} U_v \Big/ \mathrm{adh}_T(E^S),$$

where $\mathrm{adh}_T(E^S)$ *is the topological closure of* $i_T(E^S)$ *in* $\bigoplus_{v \in T} U_v$. *This Galois group is a* $\widehat{\mathbb{Z}}$*-module of finite type. We have* $\mathrm{Gal}(H_T^S/H^S) = \langle\, I_v(H_T^S/K)\,\rangle_{v \in T}$, $I_v(H_T^S/K) \simeq U_v/U_v \cap \mathrm{adh}_T(E^S)$.

For any prime number p *we have:*

$$\mathrm{Gal}(H_T^S(p)/H^S(p)) \simeq \Big(\bigoplus_{v \in T_p} U_v^1 \bigoplus_{v \in T_{\mathrm{ta}}} (F_v^\times)_p\Big) \Big/ \mathrm{adh}_T(E'^S),$$

where $\mathrm{adh}_T(E'^S)$ *is the topological closure of* $i_T(E'^S)$ *in* $\bigoplus_{v \in T_p} U_v^1 \bigoplus_{v \in T_{\mathrm{ta}}} (F_v^\times)_p$. *This Galois group is a* $\mathbb{Z}_p$*-module of finite type.* $\square$'

- I, p.239, ℓ.16, replace: '$\bar{i}_0(E^{\mathrm{ord}} \otimes \mathbb{Z}_p)$' by: '$\bar{i}_0(E^{\mathrm{ord}} \otimes \widehat{\mathbb{Z}})$'
- I, p.255, ℓ.-7, read: 'The fact that we consider (still in the totally real case):'
- I, p.257, 2.8, we give a more detailed version of this page (after the end of 2.7):

'This is consistent since, for all n such that $\mathfrak{m}(n)$ is a multiple of the conductor of M (see 2.3.5), we have $A = P_{T,\mathfrak{m}(n),\mathrm{pos}} \mathrm{N}_{M/K}(I_{M,T})$ which yields $A \otimes \mathbb{Z}_p = \mathcal{P}_{T,\mathfrak{m}(n),\mathrm{pos}} \mathrm{N}_{M/K}(\mathcal{I}_{M,T}) = \mathcal{P}_{T,\infty,\mathrm{pos}} \mathrm{N}_{M/K}(\mathcal{I}_{M,T})$.[4]

[4] Since $\mathcal{I}_T/\mathcal{P}_{T,\infty,\mathrm{pos}}$ is of finite type, $\mathcal{P}_{T,\infty,\mathrm{pos}} \,.\, \mathrm{N}_{M/K}(\mathcal{I}_{M,T}) = \mathcal{P}_{T,\infty,\mathrm{pos}} \,.\, \mathcal{C}$, where $\mathcal{C} \subset \mathcal{I}_T$ is of finite type (i.e. compact); thus we apply I.5.4 to the relations $A \otimes \mathbb{Z}_p = \mathcal{P}_{T,\mathfrak{m}(n),\mathrm{pos}} \,.\, \mathcal{P}_{T,\infty,\mathrm{pos}} \,.\, \mathrm{N}_{M/K}(\mathcal{I}_{M,T}) = \mathcal{P}_{T,\mathfrak{m}(n),\mathrm{pos}} \,.\, \mathcal{C}$ for all $n \gg 1$.

In the general case of abelian pro-p-extensions with restricted ramification, and mainly when $T_p \neq \emptyset$, we can state the correspondence of class field theory in the following way. [5]

2.8 Theorem. *There is a bijective Galois correspondence between the subextensions M/K of $H_T^{\rm res}(p)/K$ and the closed subgroups $\mathcal{N}$ of $\mathcal{I}_T$ containing $\mathcal{P}_{T,\infty,\rm pos}$. The field $H_T^{\rm res}(p)$ corresponds to $\mathcal{P}_{T,\infty,\rm pos}$. For a given M, we have:*

$$\mathcal{N} = \bigcap_{\substack{M'\subseteq M \\ M'/K \text{ finite}}} \mathcal{P}_{T,\infty,\rm pos}{\rm N}_{M'/K}(\mathcal{I}_{M',T}) = \bigcap_{\substack{M'\subseteq M \\ M'/K \text{ finite}}} {\rm adh}\,({\rm N}_{M'/K}(\mathcal{I}_{M',T}))$$

since the $\mathcal{P}_{T,\mathfrak{m}(n),\rm pos}$ form a fundamental system of neighbourhoods of 1 in $\mathcal{I}_T$ containing $\mathcal{P}_{T,\infty,\rm pos}$.

We have the following properties where $K \subseteq M' \subseteq M \subseteq H_T^{\rm res}(p)$:

(i) We have the exact sequence:

$$1 \longrightarrow \mathcal{N}/\mathcal{P}_{T,\infty,\rm pos} \longrightarrow \mathcal{I}_T/\mathcal{P}_{T,\infty,\rm pos} \xrightarrow{\alpha_{M/K}} {\rm Gal}(M/K) \longrightarrow 1\ ;$$

if M/K is finite, $\mathcal{N} = \mathcal{P}_{T,\infty,\rm pos}{\rm N}_{M/K}(\mathcal{I}_{M,T})$;

(ii) the composition of $\alpha_{M/K}$ and of the projection ${\rm Gal}(M/K) \longrightarrow {\rm Gal}(M'/K)$ is equal to $\alpha_{M'/K}$;

(iii) the decomposition group (resp. the inertia group) of $v \in T$ in M/K is the image under $\alpha_{M/K}$ of $(\mathcal{P}_{T\backslash\{v\},\infty_{T\backslash\{v\}},\rm pos}\,.\,\langle\, \mathfrak{p}_v\,\rangle \otimes \mathbb{Z}_p) \cap \mathcal{I}_T$ (resp. of $\mathcal{P}_{T,\infty_{T\backslash\{v\}},\rm pos}$); if $v \notin T$ is finite, the decomposition group of v is the image of $\langle\, \mathfrak{p}_v\,\rangle \otimes \mathbb{Z}_p$; if $v \in Pl_\infty^{\rm r}$, the decomposition group of v is the image of $\mathcal{P}_{T,\infty,Pl_\infty^{\rm r}\backslash\{v\}}$;

(iv) if M'/K is finite, for all $\mathfrak{a}' \in \mathcal{I}_{M',T}$, the image of $\alpha_{M/M'}(\mathfrak{a}')$ in ${\rm Gal}(M/K)$ is $\alpha_{M/K}({\rm N}_{M'/K}(\mathfrak{a}'))$; in particular, we have:

$${\rm Gal}(M/M') = \alpha_{M/K}({\rm N}_{M'/K}(\mathcal{I}_{M',T}))\ ;$$

(v) if M'/K is finite, for all $\mathfrak{a} \in \mathcal{I}_T$, the image of $\alpha_{M/K}(\mathfrak{a})$ under the transfer map (from ${\rm Gal}(M/K)$ to ${\rm Gal}(M/M')$) is $\alpha_{M/M'}(\mathfrak{a}')$, where $\mathfrak{a}'$ is obtained by extending $\mathfrak{a}$ to M';

(vi) for any $\mathbb{Q}$-isomorphism τ of M in $\overline{\mathbb{Q}}$, we have for all $\mathfrak{a} \in \mathcal{I}_T$:

$$\alpha_{\tau M/\tau K}(\tau\mathfrak{a}) = \tau \circ \alpha_{M/K}(\mathfrak{a}) \circ \tau^{-1} \text{ on } \tau M. \qquad \square$$

The finite extensions correspond to closed subgroups of finite index of the form $A \otimes \mathbb{Z}_p$ as in 2.7 above. For instance $A = P_{T,\mathfrak{m},\rm pos}$, $\mathfrak{m} \in \langle\, T\,\rangle_{\mathbb{N}}$, for ray class fields, giving the open subgroups $\mathcal{P}_{T,\mathfrak{m},\rm pos}$.

Then, the function ${\rm Log}_{T_p}$ allows us to characterize the $\mathbb{Z}_p$-torsion and the $\mathbb{Z}_p$-free part of any relative extension, and hence of any subextension (see 2.7).

2.8.1 Example. Take $T = Pl_p$. Then it is immediate to check that the subgroup $\mathcal{N}$ of $\mathcal{I}_p$, corresponding to the cyclotomic p-extension $K\mathbb{Q}(\mu_{p^\infty})(p)$ of K, is the subgroup $\{\mathfrak{a} \in \mathcal{I}_p,\ \ {\rm N}_{K/\mathbb{Q}}(\mathfrak{a}) \in \mathcal{P}_{\mathbb{Q},p,\infty,\rm pos}\}$, in other words $\mathcal{N} = \{\mathfrak{a} \in \mathcal{I}_p,\ \ \bar{i}_p({\rm N}\mathfrak{a}) = 1 \text{ in } \mathbb{Z}_p^\times\}$, where N now denotes the absolute norm taking values in $\mathcal{Q}_{\rm pos}^\times := \mathbb{Q}_{\rm pos}^\times \otimes \mathbb{Z}_p$.

In the above, for $p = 2$, using $K\mathbb{Q}^{\rm cycl}(2)/K$ instead of $K\mathbb{Q}(\mu_{2^\infty})/K$, where $\mathbb{Q}^{\rm cycl}(2)$ is also equal to $H_2^{\rm ord}(2)$, we obtain:

$${\rm Gal}(\widetilde{K}_2/K\mathbb{Q}^{\rm cycl}(2)) \simeq \{{\rm Log}_2(\mathfrak{a}) \in {\rm Log}_2(\mathcal{I}_2),\ \ {\rm Log}_{\mathbb{Q},2}({\rm N}_{K/\mathbb{Q}}(\mathfrak{a})) = 0\}.$$

[5] We leave the reader to establish the details for the functorial properties of this correspondence (restriction, decomposition and inertia groups, norm lifting, transfer, etc ...), in the spirit of II.4.5, where ray groups mod $\mathfrak{m}$ or $\frac{\mathfrak{m}}{\mathfrak{m}_v}$ become infinitesimal ray groups for $\infty := \infty_T$ or for $\infty_{T\backslash\{v\}}$ in an evident meaning. For instance, remark that $\mathcal{K}_{T,\infty}^\times = \mathcal{K}_{T,\infty_{T_p}}^\times \cap \mathcal{K}_{T,\mathfrak{m}_{\rm ta}}^\times$.

We always have $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a}) = (\mathrm{N}\mathfrak{a})$ but possibly with $\bar{\imath}_2(-\mathrm{N}\mathfrak{a}) = 1$, which means that the Artin symbol of $\mathfrak{a}$ in $H_2^{\mathrm{res}}(2)/K$ fixes $K\mathbb{Q}^{\mathrm{cycl}}(2)$ but not $K\mathbb{Q}(\mu_{2^\infty})$. □'

- I, p.269, ℓ.-15, instead of: 'p-rank' read: '$\mathbb{Z}_p$-rank'
- I, p.273: In 3.6.3 and 3.6.4, we still assume that $E^{S_0\mathrm{ord}}$ is monogeneous (= monogenic in II).
- I, p.278, 4.1.5, the statement is given in a more general form:

'**4.1.5 Theorem.** *Let K be a number field satisfying the Leopoldt conjecture for p. Let T be a finite set of finite places of K containing Pl_p, and let $S_\infty \subseteq Pl_\infty^{\mathrm{r}}$. Put $T_{\mathrm{ta}} := T\backslash Pl_p$. We then have the exact sequence:*

$$1 \longrightarrow \bigoplus_{v\in T_{\mathrm{ta}}\cup\,\delta_\infty} (F_v^\times)_p \longrightarrow \mathcal{T}_{T\cup t}^{S_\infty} \longrightarrow \mathcal{T}_{Pl_p\cup t}^{S_\infty\cup\delta_\infty} \longrightarrow 1,$$

for any finite set $t \subset Pl_0\backslash T$, and $\delta_\infty \subseteq Pl_\infty^{\mathrm{r}}\backslash S_\infty$, giving the following particular cases:

$$1 \longrightarrow \bigoplus_{v\in T_{\mathrm{ta}}} (F_v^\times)_p \longrightarrow \mathcal{T}_T^{\mathrm{ord}} \longrightarrow \mathcal{T}_p^{\mathrm{ord}} \longrightarrow 1,$$
$$1 \longrightarrow \bigoplus_{v\in T_{\mathrm{ta}}\cup Pl_\infty^{\mathrm{r}}} (F_v^\times)_p \longrightarrow \mathcal{T}_T^{\mathrm{res}} \longrightarrow \mathcal{T}_p^{\mathrm{ord}} \longrightarrow 1,$$
$$1 \longrightarrow \{\pm1\}^{r_1} \longrightarrow \mathcal{T}_T^{\mathrm{res}} \longrightarrow \mathcal{T}_T^{\mathrm{ord}} \longrightarrow 1 \ \ (p=2).$$ □'

- I, p.299, ℓ.-1, read: 'by 1.1.6, (ii), (α) for $S = \emptyset$ and $m_v = 0$ (i.e., $\mathfrak{m}_v = 1$), we have:'
- I, p.305, ℓ.16, read: 'if and only if $|Pl_p| > 1$ or $|Pl_p| = 1$ and $\mu_p(K) = 1$'
- I, p.308–310, we say that the coefficients of $e \in \mathbb{Q}_p[\Gamma]$ are in $\mathbb{Q}(\theta_{w_0})$; this is not correct: these coefficients are in general in the Galois closure of $\mathbb{Q}(\theta_{w_0})$.
- I, p.327, Note, after: 'can be reptesented by $c \in \mathrm{tor}(C_K)$' put: 'since D_K is divisible'
- I, p.328, the proof of 4.15.4 is given in a more detailed way:

'**Proof.** Denote by $\chi \in \overline{G}^{\mathrm{ab}\,*}$ a character of order p^r of $\overline{G}^{\mathrm{ab}}$ such that the orthogonal complement $\langle\,\chi\,\rangle^\perp$ (i.e., the kernel of χ) fixes L. The existence of M is equivalent to that of $\psi \in \overline{G}^{\mathrm{ab}\,*}$ such that $\chi = \psi^{p^e}$. Since a profinite abelian group is reflexive, we can say that $\chi \in \overline{G}^{\mathrm{ab}\,*\,p^e}$ if and only if:

$$\begin{aligned}\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/L) &\supseteq (\overline{G}^{\mathrm{ab}\,*\,p^e})^\perp \\ &= \{\sigma \in \overline{G}^{\mathrm{ab}},\ \varphi^{p^e}(\sigma) = \varphi(\sigma^{p^e}) = 1\ \ \forall \varphi \in \overline{G}^{\mathrm{ab}\,*}\} = {}_{p^e}\overline{G}^{\mathrm{ab}}.\end{aligned}$$

Thus this is equivalent to ${}_{p^e}\overline{G}^{\mathrm{ab}} \subseteq \mathrm{Ker}(\chi)$, hence (apart from the special case) to $\rho_{\overline{K}^{\mathrm{ab}}/K}({}_{p^e}J) \subseteq \mathrm{Ker}(\chi)$ which means, by restriction to L/K, $\rho_{L/K}({}_{p^e}J) = 1$, in other words to ${}_{p^e}\mu(K_v) \subset \mathrm{Ker}(\rho_{L/K})$ for all v since ${}_{p^e}J = \prod_v {}_{p^e}\mu(K_v)$, which is again equivalent to: ${}_{p^e}\mu(K_v) \subset \mathrm{Ker}(\rho_{L_v/K_v}) = \mathrm{N}_{L_v/K_v}(L_v^\times)$ for all v, by definition of $\rho_{L/K}$ (or because "$N \cap K_v^\times = N_v$" seen in II.3.3.1).

In the special case we add the condition $\rho_{L/K}(s) = 1$. □'

- I, p.340, 4.18.1, read: 'Let K be a number field; consider the following objects which are $\widehat{\mathbb{Z}}$-modules or homomorphisms of $\widehat{\mathbb{Z}}$-modules:'
- I, p.353, ℓ.4, add the following hint: 'For the computation of $\left(\frac{7}{\mathfrak{l}_7}\right)$, check that the Frobenius of $\mathfrak{l}_7$ in $K(\sqrt{7})/K$ is that of 7 in $\mathbb{Q}(\sqrt{-23})/\mathbb{Q}$; for the computation of $\left(\frac{\pi}{\mathfrak{l}_3}\right)$, use the relation $\pi\pi' = 2\,.\,3^4$.'
- I, p.354, 6.7.2, we add the following hint: 'show that $\left(\dfrac{x+\sqrt{-\ell}}{2}\right) = \mathfrak{p}\mathfrak{a}^2$, $\mathfrak{p}|2$, and $(\mathcal{C}\ell)_2 = 1$ (genera theory), deduce that $W_2 = \langle -1, 2, a\,\rangle K^{\times 2}/K^{\times 2}$, then use the equality $\mathbb{Z}_2\mathrm{Log}_2(I_2) = \mathbb{Z}_2\mathrm{Log}_2(P_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ to compute $\widetilde{W}_2$.'

- I, p.356, 7.3, after: '(i)' add: 'Let $\mathbb{Q}_p^{\mathrm{cycl}}$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$.'. Then in the sequel replace: '$\mathbb{Q}_p^{\mathrm{cycl}(p)}$' by: '$\mathbb{Q}_p^{\mathrm{cycl}}$'
- I, p.356, ℓ.-14, add: 'For this, the embeddings i_v for $v|p$ must be extended to $\overline{i}_v : \mathcal{K}^\times \longrightarrow \big(\widehat{K_v^\times}\big)_p = \pi_v^{\mathbb{Z}_p} \oplus (U_v)_p$.'
- I, p.360, ℓ.3, add the following justification of the existence of v_0: 'This is valid because the extension is cyclic since $\mu_{2p} \subset K$.'
- I, p.360, ℓ.-10, add the following remark: 'The map $1 \otimes (\tilde{\mathrm{v}})_v$ is also the map $1 \otimes \widetilde{\mathrm{div}}$ (see 7.5, (iii)).'
- I, p.361, ℓ.-5, we give the well-known proof: 'if $\mathfrak{a} \in I_K$ is such that $(\mathfrak{a}) = (\alpha) \in P_L$, then $(\alpha^{\sigma-1})_{\sigma\in G} =: (\varepsilon_\sigma)_{\sigma\in G}$ defines a 1-cocycle in E_L^{ord}; this yields a map which factors through P_K; if $(\varepsilon_\sigma)_{\sigma\in G} = (\eta^{\sigma-1})_{\sigma\in G}$ with $\eta \in E_L^{\mathrm{ord}}$, then we get $(\mathfrak{a}) = (\alpha\eta^{-1})$ with $\alpha\eta^{-1} \in K^\times$ giving the injectivity of the map'
- I, p.398, ℓ.17, we give the well-known proof of this algebraic property in the following Note:

  '**Note.** Let H be an abelian normal subgroup of a group Γ and let $G := \Gamma/H$. Since H is abelian, G acts by conjugation on H (for $s \in G$, $\sigma \in H$, $\sigma^s := s'\sigma s'^{-1}$ for any $s' \in \Gamma$ extending s). Let h be a subgroup of H, normal in Γ; then H/h is in the center of Γ/h if and only if $s'\sigma s'^{-1}\sigma^{-1} \in h$ for all $s' \in \Gamma$ and all $\sigma \in H$, which is equivalent to $\sigma^{s-1} \in h$ for all $s \in G$ and all $\sigma \in H$; thus the minimal solution h is H^{I_G}, where I_G is the augmentation ideal of G. Here, $H = \mathrm{Gal}(H_L^{S'}/L)$.'
- I, p.403, ℓ.2, read: 'then $[C^S_{L/K^{(p)}} : L] = |(\mathcal{C}\ell_L^{S'})_p^{*G}|$;'
- I, p.416, ℓ.9, instead of: 'Kummer theory for the case $e = 1$' read: 'the Hensel lemma in K_v'
- I, p.419, ℓ.-10, read: 'In the case $p = 2$, if K is a real quadratic field with a trivial restricted 2-class group,'
- I, p.420, in 2.4.6, when we put $v|p$ we have of course $v \notin S_0$.
- I, p.421, ℓ.-13, add the following remark: 'More generally, we must have $\langle\, \delta^S_{1,v}\,\rangle_{v\in t} \simeq \bigoplus_{v\in t} U_v/(U_v)^p$ for all $t \subset T$ (which is equivalent to be satisfied for all maximal subsets $t \subset T$); thus, minimality takes place if and only if $\mathrm{rk}_p(\langle\, \delta^S_{1,v}\,\rangle_{v\in t}) = \sum_{v\in t} \mathrm{rk}_p(U_v)$ for all maximal subsets $t \subset T$, and if $\mathrm{rk}_p(\langle\, \delta^S_{1,v}\,\rangle_{v\in T}) \leq \sum_{v\in T} \mathrm{rk}_p(U_v) - 1$.'
- I, p.435, ℓ.-1, add: '(since $\mathrm{Gal}(Q_{1(\infty)}/K)$ is the decomposition group of $v_0|2$, we are reduced to the tame case)'
- I, p.443, 1.2, we give the well-known proof of this lemma: 'Let c be a continuous 2-cocycle of $\mathcal{G}$; the inverse images of the elements a of the discrete *finite* set $\mathbb{Z}/p^e\mathbb{Z}$ are thus open and closed sets in $\mathcal{G}^2$; since the profinite group $\mathcal{G}^2$ is compact (Ch. I, § 5, (c)), these sets are compact of the form $\bigcup_{i\in I \text{ finite}} x_{a,i}\,\mathcal{U}_{a,i}$, $x_{a,i} \in \mathcal{G}^2$, for open normal subgroups $\mathcal{U}_{a,i}$ of $\mathcal{G}^2$, so that c is locally constant; since all the above classes $x_{a,i}\,\mathcal{U}_{a,i}$ are finite in number, by taking suitable *finite* intersections, we can find an open normal subgroup $\mathcal{H}$ of $\mathcal{G}$ such that c factors through $(\mathcal{G}/\mathcal{H})^2$, yielding the nontrivial inclusion ($\subseteq$) (for more general situations, see [g, Se3, Ch. I, § 2.1] or [g, NSW, Ch. I, § 5]).'
- I, p.443, ℓ.-12, the Note is modified as follows:

  '**Note.** If A is a finite G-module, we denote by A^* the dual G-module of A; this notation, defined in I.5.7, will be compatible with those used elsewhere, subject to some identifications since we consider dualities with values in various cyclic groups of order p^e.'
- I, p.462, 3.8: For this theorem we still suppose that the p-tower L satisfies the Leopoldt conjecture at p.

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