FEW NON MINIMAL TYPES ON NON STRUCTURE*

SH603

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Abstract We deal with abstract elementary classes $\mathfrak{K}$ which has amalgamation in λ. Our main result is that if ($2^\lambda < 2^{\lambda^+}$ and) the minimal types on members of K_λ are not dense (among non algebraic (complete) types over models in K_λ, extending our given model), <u>then</u> the number of models in K of cardinality λ^+ or λ^{++} is maximal. For this we deal with some claims in pcf. This improves a result in Shelah (2001), but the amount of relying is small, mostly of a "black box" character.

Keywords: model theory, abstract elementary classes, classification theory, categoricity, nonstructure theory, pcf theory

Annotated Content

0. Introduction [We explain our aim and define our framework.]

1. Non minimal types and nonstructure [We define unique amalgamation, UQ, and try to use it for building many models in λ^+ when $2^\lambda < 2^{\lambda^+}$ (so the weak diamond holds). If this approach fails we still get the many models in λ^{++} by the "easy" criterion of Shelah (2001, §3) but it works only if the weak diamond ideal on λ^+ is not λ^{++}-saturated.]

2. Remarks on pcf [We prove some pcf observations needed here.]

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3. Finishing the many models [We prove the result of Section 1 without the extra assumption on the saturation of the weak diamond ideal.]

4. A minor debt [There was one point in Shelah (2001) where we use $\lambda > \aleph_0$, though our aim there was to generalize theorem known for $\lambda = \aleph_0$. We eliminate this use.]

0. Introductions

In Shelah (2001) there was an important point where we used as assumption $I(\lambda^{+3}, K) = 0$. This was fine for the purpose there, but is unsuitable in other frameworks, like Shelah (200x): we want to analyze what occurs in higher cardinals, so our main aim here is to eliminate its use and add to our knowledge on non-structure.

The point was "the minimal triples in K^3_λ are dense" (Shelah 2001, 3.17t). For this we assume we have a counterexample, and try to build many nonisomorphic models. Hence we get cases of amalgamation which are necessarily unique. Those "unique amalgamations" are normally too strong (even for first order superstable theories), but here they help us to prove positive theorems, controlling omitting types. So we try to build many models in λ^+ by omitting "types" over models of size λ, in a specific way where unique amalgamation holds. If this argument fails, we prove $\boldsymbol{C}^1_{\mathfrak{K},\lambda}$ has weak λ^+-coding (see Shelah 2001, §3) and by it get $2^{\lambda^{++}}$ non-isomorphic models except when the weak diamond ideal on λ^+ is λ^{++}-saturated; this is done in Section 1. In Section 3 we work harder and by partition to cases relying on pcf theory we succeed to get the full result. We work also to get large IE (many models no one $\leqslant_{\mathfrak{K}}$-embedding to another). The pcf lemmas (which are pure infinite combinatorics) are dealt with in Section 2.

There was also another point left in Shelah (2001, 4.2t), for the case $\lambda = \aleph_0$ only, this is filled in Section 4.

* * *

Definition 0.1 We say $\mathfrak{K} = (K, \leqslant_{\mathfrak{K}})$ is an abstract elementary class, aec or a.e.c. in short, if ($\tau = \tau_K$ is a fixed vocabulary, K a class of τ-models (and *Ax* 0 holds and)) *Ax* $I - VI$ hold where:

Ax 0: The holding of $M \in K, N \leqslant_{\mathfrak{K}} M$ depends on N, M only up to isomorphism i.e. $[M \in K, M \cong N \Rightarrow N \in K]$, and [if $N \leqslant_{\mathfrak{K}} M$ and f is an isomorphism from M onto the τ-model M' mapping N onto N' <u>then</u> $N' \leqslant_{\mathfrak{K}} M'$].

Ax I: If $M \leqslant_{\mathfrak{K}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

Ax II: $M_0 \leqslant_{\mathfrak{K}} M_1 \leqslant_{\mathfrak{K}} M_2$ implies $M_0 \leqslant_{\mathfrak{K}} M_2$ and $M \leqslant_{\mathfrak{K}} M$ for $M \in K$.

Ax III: If λ is a regular cardinal, $M_i\,(i<\lambda)$ is $\leqslant_{\mathfrak{K}}$-increasing (i.e. $i<j<\lambda$ implies $M_i \leqslant_{\mathfrak{K}} M_j$) and continuous (i.e. for limit ordinal $\delta<\lambda$ we have $M_\delta = \bigcup_{i<\delta} M_i$) then $M_0 \leqslant_{\mathfrak{K}} \bigcup_{i<\lambda} M_i$.

Ax IV: If λ is a regular cardinal, $M_i\,(i<\lambda)$ is $\leqslant_{\mathfrak{K}}$-increasing continuous, $M_i \leqslant_{\mathfrak{K}} N$ then $\bigcup_{i<\lambda} M_i \leqslant_{\mathfrak{K}} N$.

Ax V: If $M_0 \subseteq M_1$ and $M_\ell \leqslant_{\mathfrak{K}} N$ for $\ell = 0,1$, then $M_0 \leqslant_{\mathfrak{K}} M_1$.

Ax VI: $LS(\mathfrak{K})$ exists[1], where $LS(\mathfrak{K})$ is the minimal cardinal λ such that: if $A \subseteq N$ and $|A| \leqslant \lambda$ then for some $M \leqslant_{\mathfrak{K}} N$ we have $A \subseteq |M| \leqslant \lambda$ and we demand for simplicity $|\tau| \leqslant \lambda$.

Notation 0.2 1) $K_\lambda = \{M \in K : \|M\| = \lambda\}$ and $K_{<\lambda} = \bigcup_{\mu<\lambda} K_\mu$.
See more in Shelah (2001, §0).

Definition 0.3

1) For $\mu \geqslant LS(\mathfrak{K})$ and $M \in K_\mu$ we define $\mathscr{S}(M)$ as $\{\mathrm{tp}(a,M,N) : M \leqslant_{\mathfrak{K}} N \in K_\mu \text{ and } a \in N\}$ where $\mathrm{tp}(a,M,N) = (M,N,a)/E_M$ where E_M is the transitive closure of E^{at}_M, and the two-place relation E^{at}_M is defined by:

$$(M,N_1,a_1)E^{at}_M(M,N_2,a_2) \text{ iff there is } N \in K_\mu \text{ and } \leqslant_{\mathfrak{K}}\text{-embeddings}$$
$$f_\ell : N_\ell \to N \text{ for } \ell = 1,2 \text{ such that:}$$
$$f_1 \restriction M = \mathrm{id}_M = f_2 \restriction M \text{ and } f_1(a_1) = f_2(a_2).$$
$$(\text{of course } M \leqslant_{\mathfrak{K}} N_1, M \leqslant_{\mathfrak{K}} N_2 \text{ and } a_1 \in N_1, a_2 \in N_2)$$

2) We say "a realizes p in N" if $a \in N, p \in \mathscr{S}(M)$ and for some $N' \in K_\mu$ we have $M \leqslant_{\mathfrak{K}} N' \leqslant_{\mathfrak{K}} N$ and $a \in N'$ and $p = \mathrm{tp}(a,M,N')$; so $M,N' \in K_\mu$ but possibly $N \notin K_\mu$.

3) We say "a_2 strongly realizes $(M,N^1,a^1)/E^{at}_M$ in N" if for some N^2, a^2 we have $M \leqslant_{\mathfrak{K}} N^2 \leqslant_{\mathfrak{K}} N$ and $a_2 \in N^2$ and $(M,N^1,a^1)E^{at}_M(M,N^2,a^2)$. (Note: if M_0 is an amalgamation base, see below, then the difference between realize and strongly realize disappears).

4) We say $M_0 \in \mathfrak{K}_\lambda$ is an amalgamation base if: for every $M_1, M_2 \in \mathfrak{K}_\lambda$ and $\leqslant_{\mathfrak{K}}$-embeddings $f_\ell : M_0 \to M_\ell$ (for $\ell = 1,2$) there is $M_3 \in \mathfrak{K}_\lambda$ and $\leqslant_{\mathfrak{K}}$-embeddings $g_\ell : M_\ell \to M_3$ (for $\ell = 1,2$) such that $g_1 \circ f_1 = g_2 \circ f_2$.

5) We say $\mathfrak{K}$ is stable in λ if $LS(\mathfrak{K}) \leqslant \lambda$ and $M \in K_\lambda \Rightarrow |\mathscr{S}(M)| \leqslant \lambda$.

6) We say N is λ-universal over M if for every $M', M \leqslant_{\mathfrak{K}} M' \in K_\lambda$, there is a $\leqslant_{\mathfrak{K}}$-embedding of M' into N over M. If we omit λ we mean $\|N\|$.

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