

Chapter 3

Model Completeness

3.1 An introduction

We already defined model completeness in Chapter 1: a theory T is called *model complete* if every embedding between models of T is elementary. We dealt with this notion also in Chapter 2, where we considered its connection with quantifier elimination and completeness. But now we wish to examine model completeness in a closer and more direct way, to discuss its genesis and motivations, as well as its importance and applications.

Model completeness deals with embeddings between structures. This perspective might look slightly oblique with respect to the fundamental purpose in model theory, namely to connect sentences and structures via truth; under this point of view, the most genuine relation among structures is elementary equivalence (that is, to satisfy the same sentences). Nevertheless some basic theorems in model theory, such as the Löwenheim-Skolem theorems, involve pretty naturally extensions, substructures, embeddings, and so draw attention to this subject. Furthermore, as we will see in Section 3.2, there are other possible ways of introducing model completeness. The first one still deals with embeddings and says that a theory T is model complete when each embedding between models of T preserves existential formulas. But another characterization is quite syntactical and resembles the way we defined quantifier elimination; it says that a theory T is model complete exactly when any formula $\varphi(\vec{v})$ in the language of T is equivalent in T to an appropriate existential formula $\varphi'(\vec{v})$.

The main motivations leading to model completeness come from algebra. For instance, consider field theory. Given a field \mathcal{K} , one looks at the irreducible polynomials $f(x) \in K[x]$. Algebra builds richer and richer extensions of

\mathcal{K} , equipping these polynomials with a single root, or all the possible roots. Eventually, one reaches the *algebraic closure* \overline{K} of K : a minimal extension where every nonconstant polynomial $f(x)$ in $K[x]$, and even in $\overline{K}[x]$ itself, splits into linear factors, and so gets its own roots. Notice that, from a logical point of view, adding a root of a polynomial $f(x)$ means to satisfy the sentence $\exists w(f(w) = 0)$ with parameters from K (the coefficients of $f(x)$). \mathcal{K} is algebraically closed when it equals \overline{K} and hence when it is able to satisfy all these sentences when $f(x)$ ranges over the nonconstant polynomials over K itself. Pursuing this logical approach, one can generalize and look at arbitrary L -structures \mathcal{A} instead of pure fields, towards two possible objects:

- ★ to enlarge \mathcal{A} to a richer $\overline{\mathcal{A}}$ satisfying every existential sentence $\exists \vec{w}\alpha(\vec{w})$ (with a quantifier free $\alpha(\vec{w})$), or even every sentence in $L(\mathcal{A})$, or (why not?) in $L(\overline{\mathcal{A}})$, too;
- ★ to examine closely the structures $\overline{\mathcal{A}}$.

This program recalls A. Weil's notion of *universal domains* in [176]. Weil's idea (for the class of fields) was to determine large and rich structures, embedding every field under consideration. Of course, in the case of fields, universal domains are just algebraically closed fields of infinite transcendence degree. This strategy has now fallen into disuse within Algebraic Geometry, but it is still alive in Model Theory (and certainly it was in the sixties). Model completeness arises quite naturally in this framework: for, in a model complete theory T , for every model \mathcal{A} and for every $L(\mathcal{A})$ -sentence φ , whenever φ is true in some model extending \mathcal{A} , then \mathcal{A} itself satisfies φ ; so, it is worth devoting some specific pages to this matter. This is what we will do in this chapter. First we will give an abstract analysis of model completeness. Then we will emphasize its strong connection with Algebra. In fact, Algebra inspires the notion of model completeness, and several related concepts; but, conversely, we will see that some developments in Model Theory concerning model completeness do produce a significant progress in Algebra; indeed some alternative elegant proofs of the celebrated Hilbert Nullstellensatz, or of the Hilbert Seventeenth Problem, and, more notably, the solution of Artin's Conjecture on p -adic fields witness these fruitful contributions. Actually, this was the dream of Abraham Robinson (the father of model completeness): to quote his own words in his address to the 1950 ICM,

"Symbolic Logic can produce useful tools for the developments of actual mathematics, more particularly of Algebra and, it would

appear, of Algebraic Geometry. This is the realization of an ambition... expressed by Leibniz in a letter to Huyghens as long ago as 1679".

This point of view is developed one year later in [140]. The algebraic theorems recalled before do corroborate this program. Other deep confirmations (also concerning Geometry) will be provided in the next chapters.

Let us conclude this section by recalling some connections between model completeness, elimination of quantifiers and completeness.

First of all, remember that elimination of quantifiers implies model completeness. The converse is not true. For instance, we saw in the last chapter that the theory of real closed fields RCF loses the quantifier elimination property if one removes the relation symbol for \leq from its language L : actually the order \leq is definable in the restricted language $L_0 = \{+, -, \cdot, 0, 1\}$, as $v \geq 0$ is equivalent in RCF to

$$\exists w(w^2 = v),$$

but any possible definition needs quantifiers. However to forget \leq does not affect model completeness: in fact every embedding $f : \mathcal{A} \rightarrow \mathcal{B}$ of real closed fields in the restricted language L_0 enlarges naturally and involves \leq (because the nonnegative elements must equal the squares), so is elementary in both L and L_0 .

On the other side, model completeness can yield completeness under some suitable additional hypotheses. We saw that this happens, for instance, for real closed fields (or also for discrete linear orders). The reason was that RCF has a "minimal" model, embeddable in every real closed field: the ordered field of real algebraic numbers (hence the real closure of the rationals). To extend this example towards a general setting, we need the following

Definition 3.1.1 *Let T be a theory. A model of T is prime if it is embeddable in every model of T .*

- Examples 3.1.2**
1. The (complex) algebraic numbers are a prime model of ACF_0 .
 2. The real algebraic numbers are a prime model among real closed ordered fields.
 3. $(\mathbb{N}, \leq, 0, s)$ is a prime model of dLO^+ .

Proposition 3.1.3 *Let T a model complete theory. If T has a prime model \mathcal{A} , then T is complete.*

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