

Chapter 2

Quantifier Elimination

2.1 Elimination sets

Let L be a language. It may happen that two different L -formulas $\varphi(\vec{v})$ and $\varphi'(\vec{v})$ admit the same meaning in a structure \mathcal{A} of L , or in a class of L -structures, for instance among the models of a given L -theory T . For example, in the ordered field of reals (and even in every real closed field), the formula $\varphi(v) : v \geq 0$ (being nonnegative) is the same thing as $\varphi'(v) : \exists w(v = w^2)$ (being a square). Similarly, in the ordered domain of integers, $\varphi(v) : v \geq 0$ (being positive) has the same interpretation as $\varphi'(v) : \exists w_1 \exists w_2 \exists w_3 \exists w_4 (v = \sum_{1 \leq i \leq 4} w_i^2)$ (being the sum of four squares): this is a celebrated theorem of Lagrange, already mentioned in the last chapter.

So, fix a consistent, possibly incomplete theory T in a countable L . We shall say that two L -formulas $\varphi(\vec{v})$ and $\varphi'(\vec{v})$ are equivalent with respect to T , and we shall write $\varphi(\vec{v}) \sim_T \varphi'(\vec{v})$, when

$$\forall \vec{v} (\varphi(\vec{v}) \leftrightarrow \varphi'(\vec{v})) \in T,$$

equivalently when

$$\varphi(\mathcal{A}^n) = \varphi'(\mathcal{A}^n)$$

for all models \mathcal{A} of T .

The notion of elimination set arises quite naturally at this point. An elimination set for T is a set F of L -formulas such that every L -formula $\varphi(\vec{v})$ is T -equivalent to a suitable Boolean combination of formulas of F .

Clearly the set of all the L -formulas is an elimination set for T . But, of course, this is not an interesting case, and we reasonably expect simpler sets

F . In particular, when the set of atomic formulas in L is an elimination set for T , we say that T has the *quantifier elimination* in L . In detail

Definition 2.1.1 *Let T be a theory in a language L . T has the **elimination of quantifiers** (q.e.) in L if and only if every formula $\varphi(\vec{v})$ of L is equivalent in T to a quantifier free L -formula $\varphi'(\vec{v})$ (so to a finite Boolean combination of atomic formulas).*

One easily realizes that every T gets the elimination of quantifiers in a suitable language extending L . In fact, put $L = L_0$, $T = T_0$, and enlarge L_0 to a language L_1 containing an n -ary relation symbol R_φ for every formula $\varphi(\vec{v})$ of L_0 (n is the length of \vec{v} , of course); then add the following sentences to T_0

$$\forall \vec{v}(\varphi(\vec{v}) \leftrightarrow R_\varphi(\vec{v}))$$

for every $\varphi(\vec{v})$, and get a new theory T_1 ; it is clear that the atomic formulas of L_1 form an elimination set in T_1 for the formulas in L_0 . By repeating this procedure countably many times, one eventually defines a language $L' \supseteq L$ and a theory T' of L' "naturally" extending T and having the elimination of quantifiers in L' .

Unfortunately this procedure has a quite artificial and abstract flavour. Indeed, what we would like to obtain, given a theory T in a language L , is showing that T has the quantifier elimination directly in L or, otherwise, determining a smallest extension $L' \supseteq L$, possibly suggested by the algebraic analysis of the models of T , where T (or, more exactly, its natural extension to L') has the elimination of quantifiers, or also a reasonably simple elimination set of formulas, in L' . In fact, there are good reasons to believe that such a language L' is, in some sense, "the" proper language of T .

Which are the main advantages of an elimination set, in particular of quantifier elimination? They concern several applications.

1. The main one (at least from a historical point of view) is *decidability*. Actually the first and most celebrated quantifier elimination results are related to the decision theme. Let us explain why. Recall that a theory T is decidable if there is an algorithm checking in finitely many steps, for every sentence α in the language L of T , whether α is in T or not. Now suppose that F is an elimination set for T and that the following are available:

- an *effective* procedure translating any L -sentence into a T -equivalent Boolean combination of sentences in F (or even an effective

reduction of any L -formula into a T -equivalent Boolean combination of formulas in F);

- an algorithm to decide, for every Boolean combination α of sentences of F , whether α is or not in T .

Then, clearly, T is decidable, and actually we have got a decision algorithm (by successively applying the previous two procedures).

2. Another noteworthy application of quantifier elimination concerns *definability*. In fact, if F is an elimination set for T , then the definable sets of a model \mathcal{A} of T reduce to

$$\varphi(\mathcal{A}^n, \vec{x})$$

where $\varphi(\vec{v}, \vec{w})$ is a finite Boolean combination of formulas of F and $\vec{x} \in A$; in particular, if T has the quantifier elimination in L , then the definable sets of \mathcal{A} are just the ones of the form

$$\varphi(\mathcal{A}^n, \vec{x})$$

where $\varphi(\vec{v}, \vec{w})$ is a quantifier free formula and \vec{x} in A .

3. A third application regards the classification of *completions* of T . Recall that T is possibly incomplete; but we know that T has some (non-unique!) complete extension in L . So we are led to consider the problem of finding all the complete extensions of T in L , in other words classifying the isomorphism classes of models of T up to elementary equivalence. Now, if \mathcal{A} and \mathcal{B} are two models and \mathcal{A} is not elementarily equivalent to \mathcal{B} , then there is some sentence φ in L such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \models \neg\varphi$. As F is an elimination set for T in L , we can assume that φ is a Boolean combination of sentences in F . Indeed, one easily realizes that one can choose φ directly in F .

For instance, we will see in this chapter that the theory ACF of algebraically closed fields has the quantifier elimination in $L = \{+, \cdot, -, 0, 1\}$. Consequently the classification of algebraically closed fields up to elementary equivalence depends on the quantifier free sentences in L , which are of the form $m = n$, where m and n are integers (m abbreviates in the previous formula the addition of m summands equal to 1 if $m > 1$, and $-(-m)$ if $m < -1$; similarly for n). This implies that the complete extensions of ACF are fully determined by the characteristic of their models, and hence coincide with the theories ACF_p where p is 0, or a prime.

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Marcja, A.; Toffalori, C.

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