

Chapter 1

Structures

1.1 Structures

The aim of this chapter is to sketch out basic model theory. We wish to summarize some key facts for people already acquainted with them, but also, at the same time, to introduce them to people unfamiliar to logic, and perhaps disliking too many logical details. Accordingly we will use a rather colloquial tone. The fundamental question to be answered is: what is Model Theory? As we will see in more detail in Section 1.2, Model Theory is -or, more precisely, was at its beginning- the study of the relationship between mathematical formulas and structures satisfying or rejecting them. But, in order to fully appreciate this matter, it is advisable for us preliminarily to recall what a structure is, and which kind of formulas we are dealing with. This section is devoted to the former concept.

Structures are an algebraic notion. Actually, since Galois, Algebra is not only the solving of equations, or literal calculus, but becomes the science of structures (groups, rings, fields, and so on). This new direction gets clearer at the beginning of the last century, with Steinitz's work on fields and, later, the publication of the Van der Waerden book. What is a structure? Basically, it is a non-empty set A , with a collection of distinguished elements, operations, and relations. For instance, the set \mathbf{Z} of integers with the usual operations of addition $+$ and multiplication \cdot is a structure, as well as the same set \mathbf{Z} with the order relation \leq . Note that, in these examples, the underlying set is the same (the integers), but, of course, the structure changes: in the former case we have the ring of integers, in the latter the integers as an ordered set. To make this kind of difference among structures clearer, we have to choose a *language*, in other words to specify how many distinguished

elements, how many n -ary operations and relations (for every natural $n \neq 0$) we want to involve in building our structure. So, when we discuss the integral domain of integers, our language needs two binary operations (for addition and multiplication), while, in the latter case, a binary relation (for the order) is enough. Notice that the language of the ring case works as well for all the structures admitting two binary operations, and hence possibly for structures which are not rings; for instance, the reals with the functions

$$f(x, y) = \sin(x - y), \quad g(x, y) = e^{x \cdot y}$$

for all x and y in \mathbf{R} provide a new structure for our language, but, of course, the algebraic features of this structure are very far from the basic properties of integral domains. Accordingly it is advisable, from a general point of view, to distinguish the constant, operation and relation symbols of a language L and the elements, operations and relations embodying these symbols in a given structure for L . Symbols are something like the characters in a tragedy (like Hamlet), while their interpretations in a structure are the actors playing on the stage (Laurence Olivier, or Kenneth Branagh, or your favourite "Hamlet").

In this framework, we can at last provide a sharp definition of *structure*. We fix a language L . For simplicity, we assume that L is countable, hence either finite or denumerable (but most of what we shall say can be extended without problems to uncountable languages).

Definition 1.1.1 *A structure \mathcal{A} for L is a pair consisting of a non empty set A , called the **universe** of \mathcal{A} , and a function mapping*

- (i) *every constant c of L into an element $c^{\mathcal{A}}$ of A ,
and, for any positive integer n ,*
- (ii) *every n -ary operation symbol f of L into an n -ary operation $f^{\mathcal{A}}$ of A (hence a function from A^n into A),*
- (iii) *every n -ary relation symbol R of L into an n -ary relation $R^{\mathcal{A}}$ of A (hence a subset of A^n).*

The structure \mathcal{A} is usually denoted as follows

$$\mathcal{A} = (A, (c^{\mathcal{A}})_{c \in L}, (f^{\mathcal{A}})_{f \in L}, (R^{\mathcal{A}})_{R \in L}).$$

Let us propose some examples, which will be useful later in this book.

Examples 1.1.2 1. A graph is a non empty set A with a binary relation P both irreflexive and symmetric. Hence a graph can be viewed as a structure \mathcal{A} in the language L consisting of a unique binary relation symbol R , with $R^{\mathcal{A}} = P$. Also a non empty set A partially ordered by some relation \leq can be regarded as a structure in the same language L ; this time, $R^{\mathcal{A}} = \leq$.

2. A (multiplicative) group \mathcal{G} is a structure of the language $L = \{1, \cdot, ^{-1}\}$, where 1 is a constant, \cdot and $^{-1}$ are operation symbols of arity 2 and 1 respectively. $1^{\mathcal{G}}$ represents the identity element in \mathcal{G} , while $\cdot^{\mathcal{G}}$ and $^{-1\mathcal{G}}$ denote the product and the inverse operation in \mathcal{G} . Actually one might enrich L with some additional symbols; for instance, one might introduce a new binary operation symbol $[\ , \]$ corresponding to the commutator operation in \mathcal{G} . But, for a and b in G , $[a, b]$ is just $a \cdot b \cdot a^{-1} \cdot b^{-1}$, so $[\ , \]$ is not really new, and is implicitly defined by L . Actually we will prefer L later; but it is noteworthy that L can capture and express some further operations (and relations and constants) of \mathcal{G} besides those literally interpreting its symbols.

3. A field \mathcal{K} is a structure of the language $L = \{0, 1, +, -, \cdot\}$ where 0 and 1 are constant, and $+$, $-$ and \cdot are operation symbols (each having an obvious interpretation in \mathcal{K}). Alternatively, \mathcal{K} can be viewed as a structure in the language $L' = L \cup \{^{-1}\}$ with a new operation symbol $^{-1}$; obviously, $^{-1}$ has to be interpreted within \mathcal{K} in the inverse operation for nonzero elements of K . However, according to the general definition of structure given before, $^{-1\mathcal{K}}$ should denote a 1-ary operation with domain K . So we run into the problem of defining 0^{-1} ; this can be overcome by agreeing, for instance, $0^{-1} = 0$, but this solution may sound slightly artificial. So we will prefer to adopt below the language L when dealing with fields. Indeed, when a and b are two elements in a field \mathcal{K} , then $a = b^{-1}$ can be equivalently expressed by saying $a \cdot b = 1$.

4. An ordered field is a structure in the language $L = \{+, -, \cdot, 0, 1, \leq\}$ obtained by adding a new binary relation symbol \leq . Its interpretation in a given ordered field is clear: the order relation in the field.

5. Let \mathbb{N} denote the set of natural numbers. 0 is an element of \mathbb{N} ; the successor s (mapping each natural n into $n + 1$) is a 1-ary function from \mathbb{N} to \mathbb{N} . Giuseppe Peano pointed out that the Induction Principle (together with the auxiliary conditions that s is 1 – 1 but 0 is not in

A Guide to Classical and Modern Model Theory

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2003, XI, 371 p., Hardcover

ISBN: 978-1-4020-1330-0