

Preface

This book deals with Model Theory. So the first question that a possible, recalcitrant reader might ask is just: What is Model Theory? Which are its intents and applications? Why should one try to learn it? Another, more particular question might be the following one. Let us assume, if you like, that Model Theory deserves some attention. Why should one use this book as a guide to it?

The answer to the former question may sound problematic, but it is quite simple, at least in our opinion. For, Model Theory has been developing, since its birth, a number of methods and concepts that do have their intrinsic relevance, but also provide fruitful and notable applications in various fields of Mathematics. We could mention here its role in Algebra and Algebraic Geometry, for instance the analysis of differentially closed fields (and the results on the differential closure of a differential field), or p -adic fields (and the asymptotic solution of Artin's Conjecture), as well as the recent Hrushovski's model theoretic approach to classical problems, like Mordell-Lang's Conjecture or Manin-Mumford's Conjecture.

So Model Theory is today a lively, sprightly and fertile research area, which surely deserves the attention of the mathematical world and, consequently, its own references. This recalls the latter question above. Actually there do exist some excellent textbooks explaining Model Theory, such as [56] and [57]. Also Poizat's book [131] should be mentioned; it was written more than ten years ago, but it is still up-to-date, and it has been recently translated in English. In addition more specialistic references treat adequately some particular fields in Model Theory, such as stability theory, simplicity theory, o-minimality, classification theory and so on.

Nevertheless, we believe that this book has its own role and its own originality in this setting. Indeed we wish to address this work not only to the experts of the area, but also, and mainly, to young people having a basic knowledge of model theory and wishing to proceed towards a deeper analysis, as well as to mathematicians which are not directly involved in Model

Theory but work in related and overlapping fields, such as Algebra and Geometry. Accordingly we will emphasize the frequent and fruitful connections between Model Theory and these branches of Mathematics (differentially closed fields, Artin's Conjecture, Mordell-Lang's Conjecture and so on). In each case, we aim at giving a detailed report or, at least, at sketching the main ideas and techniques of the model theoretic approach.

Our book wishes also to follow a historical perspective in introducing Model Theory. Of course, this does not mean to provide a full history of Model Theory (although such a project could be interesting and worthy of some attention), but just to insert any basic concept in the historical framework where it was born, and so to better clarify the reasons why it was introduced. Hence, after shortly recalling in Chapter 1 basic Model Theory (structures and theories, compactness and definability), we deal in Chapter 2 with quantifier elimination, in particular with the work of Alfred Tarski on algebraically closed fields and real closed fields. We will discuss the role of quantifier elimination in Model Theory, but we will treat briefly also its intriguing role in the $P = NP$ problem within the new models of computation (such as the Blum-Shub-Smale approach, and so on).

Chapter 3 will be concerned with Abraham Robinson's ideas: model completeness, model companions, existentially closed structures. We will consider again algebraically closed fields and real closed fields, but we will illustrate also other crucial classes, like differentially closed fields, separably closed fields, p -adically closed fields and, finally, existentially closed difference fields (a rather recent matter, with some remarkable applications to Algebraic Geometry).

Chapter 4 deals with imaginary elements. They are essentially classes of definable equivalence relations in a structure \mathcal{A} , so elements in some quotient structure. We describe Shelah's construction of \mathcal{A}^{eq} , englobing these classes as new elements in the whole structure, and we show that these imaginary elements can be sometimes eliminated, because the corresponding quotients can be simulated by some suitable definable subsets of \mathcal{A} .

Chapters 5 and 6 are devoted to Morley's Theorem on uncountable categoricity. Actually its proof will be given only in Chapter 7, but here we describe Morley's ideas -algebraic closure, totally transcendental theories, prime models, and so on- and we illustrate their richness and their applications.

We will be led in this way to one of the main topics in Model Theory, namely the Classification Problem. We will explain in Chapter 7 the more relevant ideas in the formidable work of Shelah on this matter (simplicity, stability, superstability, modularity), and we will discuss their significance in some

important algebraic classes, like differential fields, difference fields, and so on. We wish also to deal with the Zilber program of classifying structures up to biinterpretability, in particular with Zilber's Conjecture on strongly minimal sets, and its brilliant solution due to Hrushovski.

Also Chapter 8 largely owes to Hrushovski. In fact, after illustrating in more detail the natural connection between Model Theory and Algebraic Geometry, we will describe the Hrushovski proof of Mordell-Lang conjecture; we will refer very quickly also to the Hrushovski solution of the related Manin-Mumford conjecture. In particular we will realize how deeply Model Theory, actually both pure Model Theory and Model Theory applied to algebra are involved in these proofs.

The final Chapter is devoted to a (comparatively) recent and fertile area in Model Theory: ω -minimality. We will expound the basic results on ω -minimal theories, and we will discuss some intriguing developments, including Wilkie's solution of a classical problem of Tarski on the exponentiation in the real field.

We assume some familiarity with the basic notions of Algebra, Set Theory and Recursion Theory. [65], [66] or [78], and [121] respectively are good references for these areas. Incidentally, let us point out that we are working within the usual Zermelo - Fraenkel axiomatic system, including the Axiom of Choice. We also assume some acquaintance with basic Model Theory, such as it is usually proposed in any introductory course. However, Chapter 1 is devoted, as already said, to a short and somewhat informal sketch of these matters.

As its title states, this book aims at being only a guide. We do not claim to provide an exhaustive treatment of Model Theory; indeed our omissions are likely to be much more numerous and larger than the topics we deal with. But we have aimed at giving an almost complete report of at least two crucial subjects (ω -stability and ω -minimality), and at providing the basic hints towards some conspicuous generalizations (such as superstability, stability, and so on).

In a similar way, we have treated in detail some key algebraic examples (algebraically closed fields, real closed fields, differentially closed fields in characteristic 0), but we have provided at least some basic information on other relevant structures (like p -adic fields, existentially closed fields with an automorphism, differentially and separably closed fields in a prime characteristic). In conclusion, we do hope that the outcome of our work is a sufficiently clear and terse picture of what Model Theory is, and provides a report as homogeneous and general as possible. Incidentally, let us say

that this book is not a literal translation of the former italian version [108]; all the material was revised and rewritten; our treatment of some topics, like quantifier elimination and model completeness, are entirely new; and we have added some relevant matters, such as prime models and Morley's Theorem on uncountable categorical theories.

A Guide to Classical and Modern Model Theory

Marcja, A.; Toffalori, C.

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