

CHAPTER 2

Monadic Convergence Structures

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Introduction

A basic notion for monadic convergence structures is that of partially ordered monad $\Phi = (\varphi, \leq, \eta, \mu)$, where (φ, η, μ) is a monad over **SET** and (φ, \leq) is a functor from **SET** to the category **acSLAT** of almost complete semilattices. We introduce the notion of sup-inverse of an **acSLAT**-morphism and present some of its properties which are important in our theory. After defining the notion of partially ordered monad, there are given some examples, e.g. that of partially ordered fuzzy filter monad and of partially ordered fuzzy stack monad. Because of the generality considered here, microobjects may appear. For each partially ordered monad the notion of stratification is introduced.

One of the aims of this chapter is to demonstrate that a lot of different types of convergence structures can be included in our general theory, for instance probabilistic convergence structures, limit towers and graded fuzzy convergence structures. For showing this, the notion of partially ordered product monad is introduced.

Fuzzy pretopologies and extended fuzzy pretopologies are characterized by their interior operators. The related open fuzzy sets fulfill different conditions, that is (O_f) and (O_s) , whereas in every case of a monadic convergence structure the open φ -objects fulfill one and the same condition, that is (O) .

There is given a characterization of the neighborhood operator by sup-inverses which leads to a characterization of monadic topologies by a diagonal axiom. In some sense dually by means of the closure operator regularity is characterized. The last part of the chapter is devoted to separation axioms, applying previous results.

1 Sup-inverses and Galois connections

In our theory we are interested in posets (X, \leq) in which all non-empty suprema exist. They are called *almost complete semilattices*. It is important that infima need not exist.

Let **acSLAT** denote the category of almost complete semilattices, where morphisms are the mappings between almost complete semilattices which preserve non-empty suprema.

In the following fix an **acSLAT**-morphism $f : (X, \leq) \rightarrow (Y, \leq)$. We assign to each element y of $D = \{y \in Y \mid \exists x \in X f(x) \leq y\}$ the greatest element x of X for which $f(x) \leq y$ holds. Let $g : D \rightarrow X$ denote this mapping. D equipped with the induced partial ordering of (Y, \leq) is an almost complete subsemilattice of (Y, \leq) and $g : (D, \leq) \rightarrow (X, \leq)$ is an **acSLAT**-morphism, called the *sup-inverse* of f .

Since $f[X] \subseteq D$ holds, f has the range restriction $f' : (X, \leq) \rightarrow (D, \leq)$.

For each $x \in X$ and each $y \in Y$ we have that $f(x) \leq y$ is equivalent to $y \in D$ and $x \leq g(y)$. Hence, f' together with g define a *Galois connection*. If f is surjective, then $D = Y$ and $g \circ f = 1_X$.

In the theory of monadic convergence structures a lot of results on sup-inverses appear. Most of these results are specializations of well-known results on Galois connections.

2 Partially ordered monads

There is a series of examples of partially ordered monads which are important in general topology.

By a *partially ordered monad* (over **SET**) (cf. [6]) we mean a quadrupel $\Phi = (\varphi, \leq, \eta, \mu)$ with the following properties.

Φ consists at first of a covariant functor $(\varphi, \leq) : \mathbf{SET} \rightarrow \mathbf{acSLAT}$, $X \mapsto (\varphi X, \leq)$ with $\varphi : \mathbf{SET} \rightarrow \mathbf{SET}$ the underlying set functor.

Moreover, Φ consists of two natural transformations $\eta = (\eta_X)_{X \in \mathbf{ObSET}}$ and $\mu = (\mu_X)_{X \in \mathbf{ObSET}}$ of mappings $\eta_X : X \rightarrow \varphi X$ and $\mu_X : \varphi \varphi X \rightarrow \varphi X$, respectively.

We assume that the triple (φ, η, μ) is a monad over **SET** and that the following conditions are fulfilled:

- (M0) φX is empty in case X is empty.
- (M1) For any set X and each pair of different elements x and y of X , the infimum of $\eta_X(x)$ and $\eta_X(y)$ does not exist.
- (M2) For all mappings $f, g : Y \rightarrow \varphi X$, $f \leq g$ implies $\mu_X \circ \varphi f \leq \mu_X \circ \varphi g$, where \leq is defined argumentwise with respect to the partial ordering of φX .
- (M3) For each set X , $\mu_X : (\varphi \varphi X, \leq) \rightarrow (\varphi X, \leq)$ preserves non-empty suprema.

Condition (M1) appears as a useful separation condition.

The partial orderings \leq of the sets φX are considered as *finer relations*. For each set X , the elements of φX are called φ -objects on X , and the minimal elements of φX also *ultra objects*.

A φ -object \mathcal{M} on a set X for which $\mathcal{M} < \eta_X(x)$ holds for some $x \in X$, will be called a *microobject* at x . Because of condition (M1), x is uniquely associated to \mathcal{M} . If there is a microobject at an element x of X , then there is a microobject at any element y of X , which follows by means of a bijection $f : X \rightarrow X$ for which $f(x) = y$.

Microobjects are in some sense properly finer than points. They may exist or do not. The conditions (M1) and that Φ has not any microobjects can be formulated together as follows:

(M1') For each set X , $\eta_X : X \rightarrow \varphi X$ is an injection and all values $\eta_X(x)$ are ultra objects on X .

Let $\Phi = (\varphi, \leq, \eta, \mu)$ be a partially ordered monad. By a *partially ordered submonad* of Φ we mean a partially ordered monad $\Psi = (\varphi', \leq, \eta', \mu')$ such that

- (1) $(\varphi', \leq) : \mathbf{SET} \rightarrow \mathbf{acSLAT}$ is a subfunctor of (φ, \leq) (in particular, for each set X and each non-empty set $A \subseteq \varphi' X$ the suprema of A with respect to $(\varphi' X, \leq)$ and to $(\varphi X, \leq)$ coincide) and
- (2) (φ', η', μ') is a submonad of (φ, η, μ) .

Instead of Ψ being a partially ordered submonad of Φ we also say that Φ is an *extension* of Ψ .

In the next sections we will give some examples of partially ordered monads and of their partially ordered submonads.

3 Filter case

Many classical topological structures can be completely described by means of the *partially ordered filter monad* (F, \leq, η, μ) . F is the *filter functor*, which assigns to each set X the set FX of all filters on X .

\leq indicates that the sets FX are equipped with the finer relations of filters, that is, the inversion of the inclusion.

η and μ are natural transformations consisting of all mappings $\eta_X : X \rightarrow FX$ and $\mu_X : FF X \rightarrow FX$ respectively, where for each $x \in X$, $\eta_X(x) = \dot{x}$, and for each filter \mathcal{L} on FX , $\mu_X(\mathcal{L}) = \bigcup_{A \in \mathcal{L}} \bigcap_{M \in A} \mathcal{M}$. In this example microobjects do not exist.

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