

# INTRODUCTION

As stated in the introduction of [45], “the mathematics of fuzzy sets is the mathematics of lattice-valued maps”; and consequently, the mathematics of fuzzy sets lends itself to a variety of structures, both topological and algebraic. Closely related to the standardization of the mathematics of fuzzy sets begun in [45], this volume continues the work of [45] in topology as well as gives several important developments in algebraic structures not available when [45] was published. At the same time, the chapters of the present work are motivated in significant measure by the presentations, informal discussions, and roundtables of the *Twentieth International Seminar on Fuzzy Set Theory*, or *Twentieth Linz Seminar*, held in Linz, Austria, February 1999.

The chapters in this volume are arranged into three main parts: Part I, topological structures; Part II, algebraic structures; and Part III, shorter chapters on topics related to Parts I and II. However, the chapters of this volume make so many substantive contributions across such a broad range of interrelated mathematical disciplines, that we have chosen to arrange our introductory remarks by the major themes of this volume in the order of their first appearance by chapter. We therefore discuss the chapters of this book as grouped by, and in the order of, the following major themes:

1. Uniformities and convergence structures
2. Fundamental examples in fuzzy topology
3. Modifications and extensions of sobriety
4. Categorical aspects
5. Logic and foundations of mathematics
6. Triangular norms and associated structures

Many chapters touch on more than one theme; however, most chapters have a primary emphasis which determines their place in the following remarks.

## 1 Uniformities and convergence structures

The first theme of this volume is uniformities and convergence structures, the former treated within both point-free and point-set contexts. To place the uniformity and later topology chapters into an overall context, we recall notions of fixed-basis *vis-a-vis* variable-basis topology. Frameworks for fixed-basis topology [47], e.g. **L-Top** or **L-Ftop**, are based on the same underlying lattice  $L$  of membership values, yet allow change of the underlying set for spaces. Frameworks for variable-basis topology [97], e.g. **C-Top** or **C-Ftop**, allow both underlying lattices and underlying sets to change from space to space (where **C** is some category of lattice-theoretic structures), thus providing a common framework for both point-set and point-free contexts, the latter including locales [60],

Hutton spaces [53], and topological molecular lattices [117]. It is known [89, 97] each point-free context is isomorphic to a subcategory of **C-Top** of *singleton* spaces (for prescribed **C** and singleton): e.g., **C = Loc** is isomorphic to the full subcategory of all singleton spaces of **Loc-Top** using a prescribed singleton. We note:

1. Point-free contexts, i.e. singleton spaces, have weaker requirements in foundations (see Chapters 1,11); cf. the localic Tihonov theorem [58].
2. Neither **C** nor **L-Top** (for  $L \in |\mathbf{C}|$ , which includes **Top** as **2-Top**) generalize the other and both are generalized by **C-Top**.
3. Each morphism in **C-Top** is a morphism embedded from **C** followed by a morphism embedded from some **L-Top**, and conversely [20, 21, 22].

Representing the point-free approach to uniformities is Chapter 1 of Prof. Banaschewski, a systematic tutorial which builds upon [12] and concerns uniform completions in the point-free context. This chapter deals with nearness and metric frames [10, 4, 11], a generalized sense of (regular) Cauchy filters, completeness of the natural uniformity on the localic reals [3], compactification [5, 6, 9], and other topics in which completeness plays a role. Needed specific foundations are pinpointed, including the role of Countable Dependent Choice. Noting completeness in spaces is equivalent to dense-embeddings always being isomorphisms, completeness is defined for nearness frames using strict surjections and it is proved that each nearness and metric frame has a completion.

Notions of filter and neighborhood play key roles in the chapters on uniformities, as well as in the theory of convergence developed in Chapter 2 of Prof. Gähler using partially-ordered monads of [24] and the *L*-filters of [43]. This theory includes probabilistic convergence structures, limit towers, and graded fuzzy convergence structures, and uses characterizations of neighborhood and closure operators to construct a scheme of separation axioms for convergence spaces [25, 26, 27].

Representing the point-set approach to uniformities are two chapters, Chapter 3 of Profs. Gutiérrez García, de Prada Vicente, and Šostak and Chapter 7 of Prof. Rodabaugh.

Chapter 3 categorically unifies in the fixed-basis case the entourage approach of [76], the *T*-uniformities approach of [38], and the uniform operator approach of [52] by extending the original approach of [52] to include the other approaches. The super category unifying these approaches is a topological construct. This is a significant advance in the unification of uniform spaces in fuzzy sets—the first such advance since [39]—and relies on a generalized notion of *L*-filter, cf. [23, 35, 36, 42, 47], which incorporates the notions of filter implicit in previous approaches to uniformities.

Developing the axiomatic foundations of a generalized Hutton approach to quasi-uniformities, Chapter 7 weakens Hutton's intersection operator to allow tensor products [47, 97] other than binary meets—e.g. cross products of  $t$ -norms, removes any distributivity of the underlying quasi-monoidal lattice, and constructs new classes of fundamental examples captured by such an axiomatization, including  $\mathbb{R}$  and  $\mathbb{I}$  for certain co-quantales  $L$  and  $L$ -2 soberifications of  $\mathbb{R}$  and  $\mathbb{I}$  for certain complete quasi-monoidal lattices and quasi-uniform monoidal lattices  $L$  [98, 99].

It is an open question how much of Chapter 7 can be recovered within (a modification of) the unification of Chapter 3.

## 2 Fundamental examples in fuzzy topology

The importance and role of fundamental examples in fuzzy topology occur throughout many chapters; but three chapters are specifically dedicated to this topic, namely Chapter 4 of Prof. Höhle, Chapter 5 of Prof. Kubiak, and Chapter 17 of Profs. Pultr and Rodabaugh. Chapter 17 is a trailer for Chapter 6 by the same authors and will be discussed later.

Chapter 4 proposes a new intersection axiom for  $\mathbb{I}$ -valued spaces using the arithmetic mean (monoidal mean operator derived from Łukasiewicz conjunction [41]) and constructs  $\mathbb{I}$ -valued rigid topological spaces, a setting in which Boolean negation  $\neg : \{0, 1\} \rightarrow \mathbb{I}$  has Łukasiewicz negation  $' : \mathbb{I} \rightarrow \mathbb{I}$  as its *unique* continuous extension, uniqueness not being possible in traditional or standard  $\mathbb{I}$ -topology [47]. This remarkable result comes from the construction of new classes of examples by considering  $\tau$ -smooth Borel probability measures on ordinary spaces and Radon measures on ordinary compact Hausdorff spaces, examples which include  $\mathbb{I}$  interpreted as the  $\mathbb{I}$ -rigid topological space comprising all Radon measures on  $\{0, 1\}$ , a space in which  $\{0, 1\}$  is densely embedded—thus providing  $' : \mathbb{I} \rightarrow \mathbb{I}$  in this setting as the unique continuous extension of  $\neg : \{0, 1\} \rightarrow \mathbb{I}$ .

The technique of constructing lattice-valued spaces by considering all measures of some kind on an ordinary topological space has also been considered in [77, 78] where saturated  $\mathbb{I}$ -topological spaces of probability measures on the Borel subsets of ordinary separable metric spaces are constructed. Such constructions are part of the broader phenomenon of fuzzification and fuzzy duals [91]. Perhaps the most studied fuzzifications are the  $L$ -fuzzy real line  $\mathbb{R}(L)$  and the  $L$ -fuzzy unit interval  $\mathbb{I}(L)$ , the well-known fuzzy duals of  $\mathbb{R}$  and  $\mathbb{I}$  constructed using probability distributions on  $\mathbb{R}$  and  $\mathbb{I}$ .

Chapter 5 is a tutorial dedicated to the study of  $\mathbb{R}(L)$  and  $\mathbb{I}(L)$  as codomains of particular families of  $L$ -continuous mappings, especially to the role played by these families in Urysohn Lemmas, insertion theorems, Tietze and Urysohn extension theorems, separation axioms, and the  $L$ -Tychonoff cube  $\prod_{\gamma} \mathbb{I}(L)_{\gamma}$  [66, 67, 69, 70, 71, 72, 74]. Striking consequences of this approach include the

following: the Urysohn lemma and the insertion theorem for normality characterize normality (for  $L$  a complete deMorgan algebra), but the extension theorem does not—normality guarantees extension (the “hard” direction in traditional topology) but the converse fails; and the  $L$ -Tychonoff cube  $\prod_{\gamma} \mathbb{I}(L)_{\gamma}$  is shown to have a Brouwer fixed point theorem if  $L$  is completely distributive.

### 3 Modifications and extensions of sobriety

Sobriety has been an active area in the topology of fuzzy sets since [40] and [90, 92, 93]. Chapters 6 and 17 of Profs. Pultr and Rodabaugh, the Appendix to Chapter 6 of Profs. Höhle and Rodabaugh, and Chapter 16 of Prof. Kotzé develop modifications or extensions of sobriety as originally defined for  $L$ -topological spaces.

Chapter 6 introduces lattice-valued frames or  $L$ -frames, related to traditional frames analogously to how  $L$ -topological spaces are related to traditional topological spaces, motivated by the insight that level sets and level mappings associated with  $L$ -topologies are special systems of collection-wise monomorphic and collection-wise extremally epimorphic frame morphisms. The notion of an  $L$ -frame allows new descriptions of previously known classes sober spaces as well as generating a new class of sober spaces—which surprisingly turn out to be the ultra-sober or  $\iota$ -sober spaces (in the notation of [68]), justifying examples for which are extensively documented in Chapter 17 primarily using results of [68] applied to various well-known spaces or their modifications.

Chapter 6 requires a complete chain  $L$  for much of its development. The Appendix to Chapter 6 outlines how this requirement may be relaxed to a spatial frame  $L$  for many constructions of Chapter 6 by focusing on the meet-irreducibles of  $L$ , which in the case of a complete chain are dually isomorphic to  $L_{\top} \equiv \{\alpha \in L : \alpha < \top\}$ .

Chapter 16 extends the semi-sobriety for  $L$ -topological spaces given in [64]—defined using irreducible  $L$ -closed subsets and the concept of  $\beta$ -closure of singletons—to  $(L, M)$ -topological spaces in the sense of [65, 102, 103, 73, 105]. Viewing  $(L, M)$ -topologies as towers of  $L$ -topologies ordered by the way below relation, where  $M$  is a continuous lattice, this chapter is the first extension of sobriety-like concepts to  $(L, M)$ -topologies.

It should be noted that traditional sobriety is equivalent to  $T_0$  + semi-sobriety (restricted to the crisp case). On one hand, [64] shows that in  $L$ -topology—with  $L$  a complete De Morgan algebra—the standard  $L$ -sobriety [40, 62, 90, 92, 93, 94, 98] implies  $T_0$  + semi-sobriety; but on the other hand, Chapter 17 shows that the converse fails for a rather large class of simple spaces. Thus for  $L$  a complete De Morgan algebra, [64] and Chapter 16 are implicitly proposing a new sobriety axiom for  $L$ -topological spaces— $T_0$  + semi-sobriety—which is more general than standard  $L$ -sobriety.

To summarize, Chapters 6, 16, 17 propose several new axioms of sobriety. Many questions concerning their interrelationships are resolved by the examples of Chapter 17. But open questions remain: e.g., it is an open question how the new sobriety of Chapter 6— $\iota$ -sobriety—is related to the new sobriety of Chapter 16— $T_0$  + semi-sobriety.

## 4 Categorical aspects

Although categorical aspects of fuzzy sets, topology, and related areas play a critical role in many chapters of this volume—see chapters 1,2,3,4,6,11, developments in this area are a primary emphasis of Chapter 8 of Prof. Stout, Chapter 12 of Prof. Šostak, Chapter 13 of Prof. Alderton, and Chapter 15 of Prof. Guido.

Chapter 8 develops a fully fuzzy topological point of view by considering topological space objects [108, 109] in Goguen’s category  $\mathbf{SET}(L)$  [33], where  $L$  is  $[0, 1]$  equipped with a left-continuous  $t$ -norm. Since  $L$  has a closed structure,  $\mathbf{SET}(L)$  has a closed structure [83, 84, 85], and remarkably, using its unbalanced subobjects, has sufficient structure to internize the operations needed for topology [48, 110, 111] despite the fact  $\mathbf{SET}(L)$  is not a topos (since  $L$  is not  $\mathbf{2}$ ). The resulting category  $\mathbf{FFTOP}(L)$  of fully fuzzy topological spaces is topological over  $\mathbf{SET}(L)$ . If the  $t$ -norm is  $\wedge$ , then fully topological spaces on a crisp set are saturated  $\mathbb{I}$ -topological spaces (cf. [40]). Interior operators, neighborhoods, and convergence are developed in this setting. Extensive computational examples are provided with the Likert sublattice of  $L$  used in opinion research.

Motivated by the crisp classification—induced by the standard forgetful functor  $V : \mathbf{L-Top} \rightarrow \mathbf{Set}$ —of functions between  $L$ -topological spaces as being either  $L$ -continuous or not  $L$ -continuous, Chapter 12 applies the notion of fuzzy category [104, 106, 107] to construct the degree to which such a function is  $L$ -continuous, thereby constructing the  $L$ -fuzzy category  $\mathcal{FL}\text{-}\mathbf{Top}$ , where  $L$  is a  $GL$ -monoid [46]. The issue of degree of objecthood is also addressed, including the degree to which an  $L$ -preinterior space is an  $L$ -topological space and the degree to which  $L$ -kernel space is an  $L$ -preinterior space, the concept of fuzzy functor is given, several  $L$ -fuzzy categories are constructed related to  $L$ -interior,  $L$ -preinterior, and  $L$ -kernel spaces, and initial and final structures in the fuzzy categories of  $L$ -kernel and  $L$ -preinterior spaces are obtained.

Categorical definitions of compactness rooted in certain categorical closure operators [16, 2] are applied in Chapter 13 to the fixed-basis categories  $\mathbf{SI-Top}$  and  $\mathbf{I-Top}$  [47] to generate corresponding compactness notions for  $\mathbb{I}$ -topology, namely the well-known  $\alpha$ -compactness and  $\alpha^*$ -compactness axioms of [28], as well as a new axiom termed semi- $\alpha$ -compactness arising via the closure spaces of Čech. The  $\alpha$ -compactness axiom is also known to be categorically generated via factorization structures [1, 2] in the sense of [37].

Topological and Algebraic Structures in Fuzzy Sets

A Handbook of Recent Developments in the

Mathematics of Fuzzy Sets

Rodabaugh, S.E.; Klement, E.P. (Eds.)

2003, XI, 470 p., Hardcover

ISBN: 978-1-4020-1515-1