CHAPTER 1

Uniform Completion In Pointfree Topology

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Introduction

Pointfree topology deals with certain complete lattices, called frames, which may be viewed as abstractly defined lattices of open sets, sufficiently resembling the concrete lattices of this kind that arise from topological spaces to make the treatment of a variety of topological questions possible. It turns out that a remarkable number of topological facts derive from results in this pointfree setting while the proofs of the latter are often more suggestive and transparent than those of their classical counterparts. But there is a deeper aspect of frames which endows them with a very specific significance: various topological spaces classically associated with other entities (such as several types of rings, or Banach spaces, or lattices) are actually the spectra of appropriate frames which themselves require weaker logical foundations for the proofs of their basic properties than those needed for the actual spaces but which can still serve much the same purposes as the spaces in question. In this way, pointfree topology acquires an autonomous rôle and appears as more fundamental than classical topology.

The subject as such originated in the Séminaire Ehresman in 1957 with early results by Bénabou [19] and S. Papert and D. Papert [36] and was subsequently pursued by Dowker and D. Papert-Strauss in a sequence of joint papers. A further, particularly important step in its development was provided by Isbell [29] who put the precise relationship between frames and spaces into categorical perspective, introduced a wealth of new concepts, and established a range of remarkable results. Of particular note was his introduction of uniformities into this setting, leading among other things to the unique existence of completions.

S.E. Rodabaugh and E.P. Klement (eds.), Topological and Algebraic Structures in Fuzzy Sets, 19–56.

The present chapter provides a systematic exposition of the basic aspects of completion as they have evolved over the years, in particular taking into consideration the generalization from uniformity to nearness as well as the notion of metric frames. The material is organized as follows. After reviewing the general concepts and results required (Section 1), we introduce the specific notions to be treated here: nearness, uniformity, and metric diameter (Section 2), and then establish the central results, the unique existence of the completion for the corresponding types of frames (Section 3). Next, we give an account of the rôle of (regular) Cauchy filters, in an appropriately generalized sense, in connection with the completion, using them to establish that completion is a coreflection under suitable hypotheses—though not in general, as examples show—and to provide a completeness criterion which is an abstract variant of the classical Cauchy condition (Section 4). Further, we give a characterization of the nearness frames with compact completion and derive several consequences regarding compactifications of frames from this (Section 5), to be followed by a number of completeness results including the completeness of the frame $\mathcal{L}(\mathbb{R})$ of reals in its natural uniformity (Section 6). The chapter concludes with a brief survey of a few further topics in which completeness plays an important rôle (Section 7).

We conclude with a comment on foundations. Although it might be desirable to provide a constructive treatment, in the sense of topos theory, of the subject considered here, this is not quite available at this stage. It therefore seemed natural to settle for the next best approach, using as basis classical set theory in its least stringent form, that is, Zermelo-Fraenkel set theory without the Axiom of Choice, treated in ordinary logic. There are, however, a few instances where even this seems to be too weak a foundation. In these cases, the principle which has to be invoked is that of Countable Dependent Choice

(CDC) *Given any relation R on a set E such that, for each $x \in E$, $(x,y) \in R$ for some $y \in E$, there exists a sequence $x_0, x_1, \ldots$ in E for which $(x_n, x_{n+1}) \in R$ for all $n = 0, 1, \ldots$.*

Whenever this is required, this will be explicitly stated as well as indicated by an asterisk.

1 Background

1.1. A *frame* is a complete lattice L in which

$$a \wedge \bigvee S = \bigvee \{a \wedge t \mid t \in S\}$$

for all $a \in L$ and $S \subseteq L$, and a frame *homomorphism* is a map $h : L \to M$ between frames which preserves finitary meets including the unit (= top) e and arbitrary joins including the zero (= bottom) 0.

As standard examples of frames we mention: the finite distributive lattices, the complete Boolean algebras, the complete totally ordered sets, and for any topological space X the lattice $\mathfrak{O}X$ of its open sets.

Frames isomorphic to some $\mathfrak{O}X$ are called *spatial*. They are otherwise characterized as those frames L for which the homomorphisms from L into the two-element chain $\mathbf{2}$ distinguish the elements of L.

As a general reference to frames we suggest Johnstone [31] or Vickers [42].

1.2. We list some properties of frames which will be relevant here. A frame L is

compact if $e = \bigvee S$ implies $e = \bigvee T$ for some finite $T \subseteq S$,

regular if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, where $x \prec a$ ("rather below" or "well inside") means that $x \wedge y = 0$ and $a \vee y = e$ for some $y \in L$, alternatively expressed as $a \vee x^* = e$, using the pseudocomplement

$$x^* = \bigvee\{z \in L \mid x \wedge z = 0\}$$

completely regular if $a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}$ for each $a \in L$, where $x \prec\!\!\prec a$ ("completely below" or "really inside") means there exists a sequence $(x_{ik})_{i=0,1,\ldots;k=0,\ldots,2^i}$ such that

$$x_{00} = x,\ x_{01} = a,\ x_{ik} = x_{i+1\,2k},\ x_{ik} \prec x_{ik+1}$$

and

normal if $a \vee b = e$ implies there exist $u, v \in L$ for which $a \vee u = e = b \vee v$ and $u \wedge v = 0$.

Concerning these properties we have

1.2.1. *Any compact regular frame is normal.*

***1.2.2.** *Any normal regular frame is completely regular.*

Regarding the first, if $a \vee b = e$, then, by regularity, compactness, and the fact that $\prec$ is stable under $\vee$, there exist $x \prec a$ and $y \prec b$ such that $x \vee y = e$, and $u = x^*$, $v = y^*$ are then the desired elements.

As to the second, given $x \prec y$ there exist v such that $x \prec v \prec y : y \vee x^* = e$ implies that $y \vee u = e = x^* \vee v$ for some u and v such that $u \wedge v = 0$, and then indeed $x \prec v \prec y$. Now, it follows by CDC that $x \prec\!\!\prec y$ whenever $x \prec y$, and this proves the result.

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