

## CHAPTER 3

# A Unified Approach To The Concept Of Fuzzy $L$ -Uniform Space <sup>1</sup>

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## Introduction and preliminaries

The theory of uniform structures is an important area of topology which in a certain sense can be viewed as a bridge linking metrics as well as topological groups with general topological structures. In particular, uniformities form, the widest natural context where such concepts as uniform continuity of functions, completeness and precompactness can be extended from the metric case. Therefore, it is not surprising that the attention of mathematicians interested in fuzzy topology constantly addressed the problem to give an appropriate definition of a uniformity in fuzzy context and to develop the corresponding theory. Already by the late 1970's and early 1980's, this problem was studied (independently at the first stage) by three authors: B. Hutton [21], U. Höhle [11, 12], and R. Lowen [30]. Each of these authors used in the fuzzy context a different aspect of the filter theory of traditional uniformities as a starting point, related in part to the different approaches to traditional uniformities as seen in [37, 2] *vis-a-vis* [36, 22]; and consequently, the applied techniques and the obtained results of these authors are essentially different. Therefore it seems natural and urgent to find a common context as broad as necessary for these theories and to develop a general approach containing the previously obtained results as special cases—it was probably S. E. Rodabaugh [31] who first stated this problem explicitly.

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To present such a general approach is the aim of this chapter. Our strategy consists in considering the uniformity as a characteristic morphism on the power set  $L^{X \times X}$  of all  $L$ -subsets of  $X \times X$ . As far as the lattice  $L$  is concerned, it will be enriched with an additional structure provided by binary operations  $*$  and  $\otimes$ . The first one of these operations is assumed to distribute over arbitrary joins and as a result it gives rise to a Galois connection and the corresponding implication.

Apart from this introduction and preliminaries, which serve for describing and specifying the lattice context in which our work is accomplished, this chapter consists of seven sections. The first three are devoted to the study of different generalizations of the concept of a filter in an  $L$ -valued context: prefilters,  $L$ -filters,  $L$ -filters of ordinary subsets and  $\top$ -filters and the relations between them. In sections four to seven we introduce the new concept of a uniformity in “fuzzy context”, called  $L$ -valued Hutton uniformity which is the central object of our work, study basic properties of the corresponding category **HL-UNIF** and establish the place of Hutton, Lowen, and Höhle approaches as special cases of **HL-UNIF**. Moreover, relations between the Hutton, Lowen and Höhle categories are obtained. This is done through the notion of  $L$ -uniformity which includes Lowen’s and Höhle’s uniformities. The notion of Hutton is then extended in such a way as to include  $L$ -uniformities. This extension of Hutton, for certain  $L$ , provides a fixed-basis unification of these diverse approaches, giving an affirmative answer to a fixed-basis analogue of the variable-basis question stated by S. E. Rodabaugh in [31]. More precisely, we can answer yes to this question: for appropriate  $L$ , can one find a common categorical framework which would include both the  $L$ -Lowen uniformities and the  $L$ -Hutton uniformities?

In what follows, we shall consider a quadruple  $(L, \leq, \otimes, *)$ , where  $(L, \leq)$  is a complete lattice with  $\top$  and  $\perp$  respectively being the universal upper and lower bound and  $*$  and  $\otimes$  two operations defined on  $L$  such that:

(i)  $(L, \leq, *)$  is a  $GL$ -monoid, that is:

- $(L, *)$  is a commutative semigroup;
- $*$  is distributive over *arbitrary* joins, i.e.

$$\left( \bigvee_{i \in J} \alpha_i \right) * \beta = \bigvee_{i \in J} (\alpha_i * \beta), \quad \beta * \left( \bigvee_{i \in J} \alpha_i \right) = \bigvee_{i \in J} (\beta * \alpha_i)$$

for all  $\beta \in L$  and all  $\{\alpha_i : i \in J\} \subseteq L$ ;

- $\top$  is the unit element in  $(L, *)$ ;
- $\beta \leq \alpha \implies \exists \delta \in L$  such that  $\beta = \alpha * \delta$ .

(ii)  $(L, \leq, \otimes)$  is a  $cl$ -quasi-monoid, that is:

- $\otimes$  is distributive over *non empty* joins, i.e.

$$\left( \bigvee_{i \in J} \alpha_i \right) \otimes \beta = \bigvee_{i \in J} (\alpha_i \otimes \beta), \quad \beta \otimes \left( \bigvee_{i \in J} \alpha_i \right) = \bigvee_{i \in J} (\beta \otimes \alpha_i)$$

for all  $\beta \in L$  and all  $\emptyset \neq \{\alpha_i : i \in J\} \subseteq L$ .

- $\alpha \leq \alpha \otimes \top, \quad \alpha \leq \top \otimes \alpha, \quad \forall \alpha \in L$

(iii)  $*$  is *dominated* by  $\otimes$ , that is:

- $(\alpha_1 \otimes \beta_1) * (\alpha_2 \otimes \beta_2) \leq (\alpha_1 * \alpha_2) \otimes (\beta_1 * \beta_2)$   
for all  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L$ .

The following conditions are also used and explicitly referenced in the text as necessary.

The quadruple  $(L, \leq, \otimes, *)$  is said to be *pseudo-bisymmetric* if and only if

$$\begin{aligned} (\alpha_1 * \beta_1) \otimes (\alpha_2 * \beta_2) &= \left( (\alpha_1 \otimes \alpha_2) * (\beta_1 \otimes \beta_2) \right) \vee \\ &\vee \left( (\alpha_1 \otimes \perp) * (\beta_1 \otimes \top) \right) \vee \left( (\perp \otimes \alpha_2) * (\top \otimes \beta_2) \right) \end{aligned}$$

for all  $\alpha_1, \alpha_2 \in L$  and  $\beta_1, \beta_2 \in L$ .

In any  $GL$ -monoid  $(L, \leq, *)$ ,  $*$  is distributive over *arbitrary* joins, and hence for each  $\alpha \in L$  the mapping  $- \mapsto \alpha * -$  has a right adjoint  $- \mapsto \alpha \rightarrow -$ . The *implication*  $\rightarrow$  is then determined by  $\alpha \rightarrow \beta = \bigvee \{\lambda \in L : \alpha * \lambda \leq \beta\}$ .

We recall that a  $GL$ -monoid is a complete  $MV$ -algebra if the following is satisfied:

$$(MV) \quad (\alpha \rightarrow \perp) \rightarrow \perp = \alpha, \quad \text{for each } \alpha \in L.$$

If  $L = [0, 1]$  is equipped with the Łukasiewicz  $t$ -norm  $T_m$  defined by  $T_m(\alpha, \beta) = \max\{\alpha + \beta - 1, 0\}$  for each  $\alpha, \beta \in [0, 1]$ , then  $([0, 1], \leq, T_m)$  is a complete  $MV$ -algebra.

Finally, given a lattice  $L$  used in this chapter and a mapping  $\varphi : X \mapsto Y$ , the Zadeh image and preimage operators

$$\varphi^\rightarrow : L^X \mapsto L^Y, \quad \varphi^\leftarrow : L^Y \mapsto L^X$$

are given by

$$\varphi^\rightarrow(a)(y) = \bigvee_{f(x)=y} a(x), \quad \varphi^\leftarrow(b) = b \circ \varphi$$

It is well-known that  $\varphi^\rightarrow \dashv \varphi^\leftarrow$ ; see [32].

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