A CLASSIFICATION OF LOGICS OVER $\mathbf{FL_{ew}}$ AND ALMOST MAXIMAL LOGICS

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Introduction

Let $\mathbf{FL_{ew}}$ be the logic obtained from the intuitionistic propositional logic by deleting contraction rule if we formulate it in a sequent system. Sometimes, this logic is called *intuitionistic affine logic.* The class of logics over $\mathbf{FL_{ew}}$, i.e. logics stronger than or equal to $\mathbf{FL_{ew}}$, includes many interesting logics, e.g., intermediate logics, Łukasiewicz's many-valued logics, Grišin's logic and product logic, etc. (See, e.g., Cignoli et al., 2000; Grišin, 1976; Hájek, 1998.) The study of logics over $\mathbf{FL_{ew}}$ will enable us to discuss these different kinds of logics within a uniform framework (see Ono and Komori, 1985; Ono, 1999 for the detail).

In (Ono, 1999), the first author introduced a classification $\{W_k\}_{k\leq\omega}$ of logics over $\mathbf{FL_{ew}}$. For each $k < \omega$, let E_k be the formula $p^k \supset p^{k+1}$, which is called *the axiom for k-potency.* Here, p^i is an abbreviation of $p * \cdots * p$ with i times p, where $*$ denotes the *fusion.* (For convenience's sake, we assume also that E_ω denotes the formula $p \supset p$.) It is easy to see that E_1 is equivalent to the formula $(p \supset (p \supset q)) \supset (p \supset q)$ in $\mathbf{FL_{ew}}$, which plays the same role as contraction rule. By using the sequence of formulas $\{\mathrm{E}_k\}_k$, we define the classification $\{W_k\}_{k\leq\omega}$ as follows: *A logic L belongs to the class W_k if and only if k is the smallest number among these j such that* E_j *is in L.* Note that when any of E_j is not in L for $j < \omega$, L belongs to W_ω.

It is obvious that W_1 is exactly the class of all intermediate logics and that for each $k \leq \omega$, W_k has the minimum logic $\mathbf{FL_{ew}}[\mathrm{E}_k]$, the logic obtained from $\mathbf{FL_{ew}}$ by adding E_k as the axiom. Note here that $\mathbf{FL_{ew}}[\mathrm{E}_1]$ and $\mathbf{FL_{ew}}[\mathrm{E}_\omega]$ are equal to the intuitionistic logic and $\mathbf{FL_{ew}}$, respectively. On the other hand, any W_k doesn't have the maximum logic when $k > 1$.

A. Rojszczak[†], *J. Cachro and G. Kurczewski (eds.), Philosophical Dimensions of Logic and Science,* 3-13.

Main purpose of the present paper is to study further the classification $\{W_k\}_{k\leq\omega}$, in particular in relation to *almost maximal logics*. Here, we say that a logic L over $\mathbf{FL_{ew}}$ is almost maximal if it is properly weaker than the classical logic **Cl**, whose proper, consistent extension is only **Cl**. Our results say that a great many almost maximal logics exist in W_k for $k > 1$, even among logics with some particular additional axioms. Recently T. Kowalski succeeded to extend our results and showed that in many cases there exist *uncountably many* almost maximal logics. These results will be announced as a joint paper of Kowalski and the second author.

1. LOGICS OVER $\mathbf{FL_{ew}}$ AND RESIDUATED LATTICES

In this section, we will make a quick survey of logics over $\mathbf{FL_{ew}}$ and of the class of *residuated lattices* which defines algebraic semantics for these logics. (For more information, see, e.g., Ono, 1999).

As mentioned in the previous section, $\mathbf{FL_{ew}}$ is the system obtained from Gentzen's sequent calculus **LJ** for the intuitionistic logic by deleting the contraction rule. The language of $\mathbf{FL_{ew}}$ consists of a logical constant $\perp$, logical connectives $\supset, \wedge, \vee$ and $*$ (called *fusion* or *multiplicative conjunction*). The negation $\neg A$ of a formula A is defined as $A \supset \perp$. The system $\mathbf{FL_{ew}}$ consists of the following initial sequents:

1 $A \to A$,

2 $\perp, \Gamma \to C$;

with structural rules except *contraction rule* and rules for logical connectives. For fusion, we will take the following rules:

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A * B}\ (\to *), \qquad \frac{A, B, \Gamma \to C}{A * B, \Gamma \to C}\ (* \to).$$

The provability of a given sequent is defined in the usual way. In particular, we say that a formula A is provable in $\mathbf{FL_{ew}}$ when the sequent $\to A$ is provable in it. By a *logic over* $\mathbf{FL_{ew}}$ (or simply a *logic*) we mean any set of formulas which is closed under substitution and *modus ponens* and which includes all formulas provable in $\mathbf{FL_{ew}}$. In the following, we always assume the *consistency* of a given logic. Thus, it never contains $\perp$. We sometimes identify a formal system with the set of all formulas which are provable in it. In the following, **Cl** and **Int** denote the classical logic and the intuitionistic logic, respectively. Interesting subclasses of the class of all logics over $\mathbf{FL_{ew}}$ are the class of intermediate logics (or, superintuitionistic logics), which are logics over the intuitionistic logic, and the class of Łukasiewicz's many-valued logics.

The class W of logics over $\mathbf{FL_{ew}}$ is ordered by the set inclusion $\subseteq$. Of course, $\mathbf{FL_{ew}}$ is the smallest and the classical logic **Cl** is the greatest among them. It is easy to see that W forms a complete lattice with respect to the above order. As defined already in the previous section, a logic L over $\mathbf{FL_{ew}}$ is said to be almost maximal if it is properly weaker than the classical logic **Cl** and any proper, consistent extension of L is **Cl**. In other words, L is almost maximal if and only if L is a *coatom* of the lattice W. It is well-known that there exists only one almost maximal logic over **Int**, which is determined by the 3-valued Heyting algebra $\mathbf{H_3}$.

To study almost maximal logics over $\mathbf{FL_{ew}}$, we need to use algebraic tools. Here, we introduce algebraic structures, called *residuated lattices*, which are shown to be algebraic counterparts of logics over $\mathbf{FL_{ew}}$.

Definition 1. *An algebra* $\mathbf{M} = \langle M, \cap, \cup, \cdot, \rightarrow, 0, 1\rangle$ *is a* residuated lattice *if*

1 $\langle M, \cap, \cup, 0, 1\rangle$ *is a bounded lattice with the greatest element 1 and the least 0,*

2 $\langle M, \cdot, 1\rangle$ *is a commutative monoid,*

3 for $x, y \in M$, $\quad x \cdot y \leq z$ *if and only if* $x \leq y \rightarrow z$.

The third condition in the above is called *the law of residuation.* In the following, we consider only residuated lattices in which $0 \neq 1$ holds. We define $\sim x$ by $\sim x = x \rightarrow 0$. It is easy to see that a residuated lattice $\mathbf{M}$ satisfies that $x \cdot x = x$ for any $x \in M$, or equivalently $\mathbf{M}$ satisfies that $x \cdot y = x \cap y$ for all $x, y \in M$, if and only if $\mathbf{M}$ is a Heyting algebra.

A *valuation* on a residuated lattice $\mathbf{M}$ is a mapping from the set of all propositional variables to the set M. By interpreting logical symbols $\wedge, \vee, *, \rightarrow$, and $\perp$ as $\cap, \cup, \cdot, \rightarrow$, and 0, respectively, we can extend any valuation v uniquely to a mapping from the set of all formulas to M. A formula A is *valid* in a residuated lattice $\mathbf{M}$ if $v(A) = 1$ holds for any valuation v on $\mathbf{M}$. The set of valid formulas in a given residuated lattice $\mathbf{M}$ is denoted by $L(\mathbf{M})$, and is called the logic *determined by* $\mathbf{M}$. It is in fact a logic over $\mathbf{FL_{ew}}$ as shown in the following.

Proposition 2. *For any residuated lattice* $\mathbf{M}$, $L(\mathbf{M})$ *is a logic over* $\mathbf{FL_{ew}}$. *Conversely, any logic over* $\mathbf{FL_{ew}}$ *can be represented as* $L(\mathbf{M})$ *for some residuated lattice* $\mathbf{M}$.

Let $\mathbf{M}$ and $\mathbf{N}_i$ $(i \in I)$ be residuated lattices. By a *subdirect representation* of $\mathbf{M}$ with factors $\mathbf{N}_i$ we mean an embedding f from $\mathbf{M}$

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