

# ON REPRESENTING SEMANTICS IN FINITE MODELS

MARCIN MOSTOWSKI

*Warsaw University, Poland*

## Introduction

This paper is continuation of research presented in (Mostowski, 2001). It gives some new results related to finite order hierarchy in finite models. They are obtained by the method of truth-definitions in finite models. Additionally we give an application of FM-representability theorem for studying densities of spectra. We finish with philosophical discussion of some problems raised by the reported research.

We follow here works (Mostowski, 1993), (Mostowski, 2001)<sup>1</sup>, (Mostowski, 2003). In this paper we repeat the main ideas of (Mostowski, 1993) related to the method of truth-definitions in finite models. Applying this method, we give a new result related to finite order hierarchy in finite models. It gives an essential refinement of the theorem from (Mostowski, 2001) stating that there is  $(n + 2)$ -nd order truth definition for  $n$ -th order logic.

We show that there is  $(n + 1)$ -st order truth definition for  $n$ -th order logic with variables of restricted arity.

All the results considered are essentially simplified by the theorem about FM-representability (see Mostowski, 2001), which states that exactly those sets of natural numbers are representable in finite models, which are recursive with recursively enumerable oracle, that is of degree  $\leq 0'$ . We discuss shortly the concept of FM-representability, and we apply it for the description of densities of spectra of languages with decidable truth relation in finite models.

The last section contains some philosophical remarks motivated by the presented investigations mainly devoted to expressibility of main semantical concepts in finite worlds and a version of the Church thesis, which can be in a natural way justified by means of our FM-representability theorem.

## 1. THEORY *ST*

In (Mostowski, 1993) and (Mostowski, 2001) we have considered the theory *ST*, called so as an abbreviation from *Syntactic Theory* (or if you like *Semantic Theory*). Essentially it was intended as the theory of initial segments of the standard model for a suitable fragment of Peano Arithmetic. The task for which it has been designed is just giving a comfortable language for describing various combinatorial objects such as formulae, Turing machines, computations, and finite models. Unfortunately in both mentioned papers no finite axiomatization for *ST* was given. In a sense it was not necessary because we did not use any particular properties of it. We shortly repeat here the description of *ST* given in (Mostowski, 2003).

The language of *ST* consists of: one binary predicate  $<$ , three binary function symbols addition  $+$ , multiplication  $\times$ , concatenation over  $k$ -ary alphabet  $*$ , one unary function symbol successor  $S$ , and two individual constants  $0$  and  $MAX$ .  $MAX$  has to be  $<$ -greatest element. The  $k$ -ary concatenation  $*$  works as a concatenation of words over an alphabet with  $k$  characters. Other symbols have their intended meaning similar to those in Peano Arithmetic. In what follows we assume similar notational conventions as in elementary arithmetic, e.g. writing  $xy$  instead of  $(x \times y)$  or  $2$  instead of  $S(S(0))$ .

We have the following axioms:

- (ST1)  $\forall x \ S(x) \neq 0 \vee MAX = 0$ ,
- (ST2)  $\forall x \forall y ((S(x) = S(y) \wedge x \neq MAX \wedge y \neq MAX) \Rightarrow x = y)$ ,
- (ST3)  $\forall x \ x + 0 = x$ ,
- (ST4)  $\forall x \forall y \ x + S(y) = S(x + y)$ ,
- (ST5)  $\forall x \ x \cdot 0 = 0$ ,
- (ST6)  $\forall x \forall y \ x \cdot S(y) = (x \cdot y) + x$ ,
- (ST7)  $S(MAX) = MAX$ ,
- (ST8)  $\forall x \forall y (x < y \Rightarrow \neg y < x)$ ,
- (ST9)  $\forall x \forall y (x < S(y) \equiv (x \neq MAX \wedge (x = y \vee x < y \vee y = MAX)))$ ,
- (ST10)  $\forall x \forall y \forall z \ x * (y * z) = (x * y) * z$ ,
- (ST11)  $\forall x \ 0 * x = x * 0 = x$ ,
- (ST12<sub>*i*</sub>)  $\forall x \ x * S(i) = (x \cdot k) + S(i)$ , for  $i = 0, 1, \dots, k - 1$ .

We give here without proof the following straightforward lemma.<sup>2</sup>

**Lemma 1.** *For each natural number  $n > 0$  there is up to isomorphisms exactly one model for  $ST$  of cardinality  $n$ . This model can be built over numbers  $0, 1, \dots, n - 1 = MAX$ , the predicate  $<$  is interpreted as the standard ordering,  $S$  as  $<$ -successor with the exception given by (ST7).*

From now on we restrict our attention to finite models of  $ST$ .

## 2. FM-REPRESENTABILITY

In what follows we will essentially use the notion of *FM-representability*. This notion captures classes of combinatorial objects — such as natural numbers, words, finite models, and similar — which can be determined in finite models, provided they are sufficiently large for particular queries. It is known that all such combinatorial objects can be represented as natural numbers. So we can simplify our problem asking: for which  $R \subseteq \omega^n$  we can find a formula  $\varphi(x_1, \dots, x_n)$  correctly representing  $R$  in finite models.

By “correct” representation in finite models we mean such representation that for each  $a_1, \dots, a_n \in \omega$ , the question “ $R(a_1, \dots, a_n)$ ?” is equivalent to the question whether  $\varphi(a_1, \dots, a_n)$ <sup>3</sup> is true or false in all sufficiently large finite models.

This question motivates our notion of FM-representability.

**Definition 2.** *Let  $R \subseteq \omega^n$  and  $\varphi(x_1, \dots, x_n)$  be  $ST$ -formula, with no free variables besides  $x_1, \dots, x_n$ , we say that  $R$  is FM-represented by  $\varphi(x_1, \dots, x_n)$  if and only if*

$$\begin{aligned} \forall a_1, \dots, a_n \in \omega \exists m \forall M \models ST(card(M) > m \\ \Rightarrow (R(a_1, \dots, a_n) \equiv M \models \varphi(a_1, \dots, a_n))). \end{aligned}$$

*We say that  $R$  is FM-representable if and only if there is a formula such that  $R$  is represented by it.*

*The above  $m$  is called a witness for  $a_1, \dots, a_n$ , the relation, and the formula. Any function  $f : \omega^n \rightarrow \omega$  choosing such  $m$  for  $a_1, \dots, a_n$  will be called a witnessing function for  $R$  and  $\varphi(x_1, \dots, x_n)$ .*

We say that almost all finite models have a certain property if and only if there are finite models  $M_1, \dots, M_n$  such that for each finite model  $M$  if  $M$  does not possess this property then  $M$  is isomorphic to one of  $M_1, \dots, M_n$ . So the concept of FM-representability can be also characterized by the following:

**Proposition 3.** *Let  $R \subseteq \omega^n$ , then  $R$  is FM-represented by  $ST$ -formula  $\varphi(x_1, \dots, x_n)$  if and only if for each  $a_1, \dots, a_n \in \omega$ ,  $\varphi(a_1, \dots, a_n)$  is true*

Philosophical Dimensions of Logic and Science  
Selected Contributed Papers from the 11th  
International Congress of Logic, Methodology, and  
Philosophy of Science, Kraków, 1999  
Rojszczak, A.; Cachro, J.; Kurczewski, G. (Eds.)  
2003, XII, 396 p., Hardcover  
ISBN: 978-1-4020-1645-5