SPECTRA OF FORMULAE WITH HENKIN QUANTIFIERS

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Introduction

Scholz defined the spectrum of a formula φ as the set of cardinalities of all finite structures in which φ is true and the spectrum of a logic as the set of spectra of all formulae of this logic. The spectrum problem is usually considered as one of the following:

1 Scholz problem: to give a characterization of the spectrum of a given logic.

2 Asser problem: is the spectrum of a given logic closed under complement?

We can generalize the Asser problem in the following way: under which operations are the spectra of a given logic closed? Of course, a good description of the spectra of a logic would be very useful in any attempt of solving this kind of problems.

The most important and most interesting theorem related to the spectrum problem is Fagin's theorem. Fagin considered in (Fagin, 1974) generalized spectra of existential second-order formulae. A generalized spectrum of φ is the class of all finite models $\boldsymbol{M}$ such that φ holds in $\boldsymbol{M}$. Fagin proved that a given class of finite models is a generalized spectrum of an existential second-order formula if and only if it is in NP. Of course, we should restrict our attention to classes of models closed under isomorphisms. Blass and Gurevitch in (Blass and Gurevitch, 1986) showed that the same holds for positive formulae with Henkin quantifiers.

The subject of our investigation is the spectrum problem formulated for the logic with branched quantifiers in the empty vocabulary. In this case generalized spectra and spectra in the sense of Scholz are equivalent. This is so because for the empty vocabulary models are uniquely determined, up to isomorphism, by their cardinalities. We consider languages with Henkin quantifiers of various kinds. For background, basic notions and results see (Krynicki and Mostowski, 1995).

A. Rojszczak[†], *J. Cachro and G. Kurczewski (eds.), Philosophical Dimensions of Logic and Science*, 29-45.

1. BASIC NOTATIONS AND OLD RESULTS

We repeat here shortly the basic definitions for Henkin quantifiers.

Definition 1 (Henkin prefixes as dependency relations). A Henkin prefix (a branched prefix) *is a triple* $Q = (A_Q, E_Q, D_Q)$, *where* A_Q *and* E_Q *are disjoint finite sets of variables called respectively universal and existential variables of* Q, *and* D_Q *is a relation between universal and existential variables of* Q $(D_Q \subseteq (A_Q \times E_Q))$, *called* the dependency relation of Q. *If* $(x, y) \in D_Q$ *then we say that the existential variable* y *depends on the universal variable* x *in* Q.

Example
Let $Q = (\{x_1, x_2\}, \{y_1, y_2\}, \{(x_1, y_1), (x_2, y_2)\})$. The prefix Q can be written down in a more intuitive way as follows:

$$\begin{pmatrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{pmatrix}.$$

Definition 2 (Skolemization of branched prefixes). *Let* Q *be a Henkin prefix binding universal variables* $x_1, ..., x_n$ *and existential variables* $y_1, ..., y_k$. *Let* $\boldsymbol{x}_i$ *be the sequence of the universal variables of* Q *on which* y_i *depends in* Q, *for* $i = 1, .., k$ *and let* $x_1, \ldots, x_n, y_1, \ldots, y_k$ *be free in* φ. *Then we define* a skolemization of φ relative to Q, $\boldsymbol{sk}(Q, \varphi)$, *as the result of substituting* $f_i(\boldsymbol{x}_i)$ *in* φ *for* y_i, *for* $i = 1, ..., k$. *The function symbols* $f_1, ..., f_k$ *introduced in this way have to be new and distinct one from another. They are called the* Skolem functions *introduced by skolemization of* φ *relative to* Q.

Definition 3 (Logic with branched quantifiers). *We define a* logic L^* *as an assignment to every vocabulary of* σ *a pair* $L^*_\sigma = (F_\sigma(H), \models_{L^*_\sigma})$, *where* $F_\sigma(H)$ *is the set of all formulae with Henkin quantifiers and* $\models_{L^*_\sigma}$ *is an extension of the satisfaction relation for elementary logic by the following condition:*

> $\boldsymbol{M} \models_{L^*_\sigma} Q\varphi\,[\mathbf{p}]$ *if and only if there are operations* $F_1, ..., F_k$ *defined on the universe of* $\boldsymbol{M}$ *such that* $(\boldsymbol{M}, F_1, ..., F_k) \models_{L^*_{\sigma'}}$ $\forall \boldsymbol{x}\, \boldsymbol{sk}(Q, \varphi)\,[\mathbf{p}]$, *where* $F_1, ..., F_k$ *interpret the respective Skolem functions introduced by skolemization of* φ *relative to* Q, σ' *is the extension of the vocabulary* σ *by these Skolem functions, and* $\boldsymbol{x}$ *is the sequence of all universal variables of* Q. *If some variables of* Q *are bounded in* φ, *then of course before skolemization we should rename these variables in* φ.

We consider the following classes of branched quantifiers. The class of quantifiers H_n of the following form:

$$\begin{pmatrix} \forall x_1 \exists y_1 \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \forall x_n \exists y_n \end{pmatrix}.$$

and the class of quantifiers A_n^k of the following form:

$$\begin{pmatrix} \forall x_1 \exists y_1 \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \forall x_n \exists y_n \\ \forall z_{11} \forall z_{12} \exists w_1 \\ \cdots\cdots\cdots \\ \forall z_{k1} \forall z_{k2} \exists w_k \end{pmatrix}.$$

By $L(\mathsf{H}_n)_{n\in\omega}$, $L(\mathsf{A}_n^1)_{n\in\omega}$, $L(\mathsf{A}_n^k)_{n,k\in\omega}$ we denote the extensions of the elementary logic by all formulae with quantifiers H_n, A_n^1, A_n^k respectively, for $n, k = 2, 3, \ldots$

Definition 4. *By a* simple positive formula *of a given logic with a branched quantifier we mean a formula of the form $Q\varphi$, where φ is a quantifier free formula and Q is a quantifier prefix.*

We consider only spectra of simple positive formulae. By the theorem of Blass and Gurevitch (1986) each class of models in NP is definable by such a formula. So this restriction does not restrict the problem considered from the complexity point of view.

Definition 5 (The spectrum of a formula). *Let φ be any sentence of a logic L in the empty vocabulary.* The spectrum of a formula φ, $\mathrm{Spec}(\varphi)$, *is the set:*

$\mathrm{Spec}(\varphi) =_{df} \{n \in \omega :$ *there is a model M such that* $\mathrm{card}(M) = n$ *and* $M \models \varphi\}$.

Definition 6 (The spectrum of a logic). *Let F_L be the set of sentences of a logic L in the empty vocabulary.* The spectrum of the logic L, $\mathrm{Spec}(L)$, *is the set:*

$$\mathrm{Spec}(L) =_{df} \{\mathrm{Spec}(\varphi) : \varphi \in F_L\}.$$

If there is a formula φ of the logic L such that the spectrum of φ is the set $A \subseteq \omega$, we will say that the set A is an L-spectrum.

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