Preface

1. The Gauss Map. The Gauss map of an oriented smooth surface X^2 in Euclidean space $\mathbb{E}^3$ is the mapping of X^2 into the unit sphere $S^2 \subset \mathbb{E}^3$:

$$\gamma : X^2 \to S^2,$$

by means of the family of the unit normals $\boldsymbol{n}$ to X^2. This map carries a point $x \in X^2$ to a point $p \in S^2$, where p is the terminal point of the vector $\boldsymbol{n}$ emanating from some fixed point $O \in \mathbb{E}^3$, $\gamma(x) = p$ (see Figure 0.1).

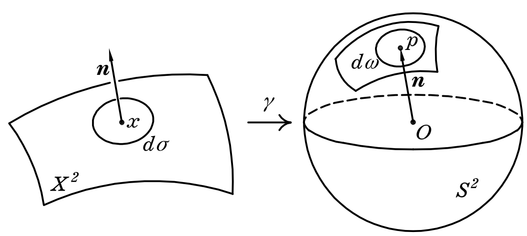

Figure 0.1

If $d\sigma$ is an area element of the surface X^2 and $d\omega$ is an area element of the spherical image of X^2, then

$$d\omega = K d\sigma,$$

where K is the Gaussian curvature of X^2 (see Gauss [Ga 27] or Stoker [Sto 61], p. 94).

The Gauss map γ is degenerate at a point $x \in X^2$ if $K = 0$ at this point, and the Gauss map γ is degenerate on the surface X^2 if the curvature K vanishes at all points of X^2. In this case the Gauss map γ maps the surface

X^2 into a curve $C \subset S^2$ (see Figure 0.2). The tangent planes to the surface X^2 depend on one parameter, and the surface X itself is an envelope of this family of tangent planes.

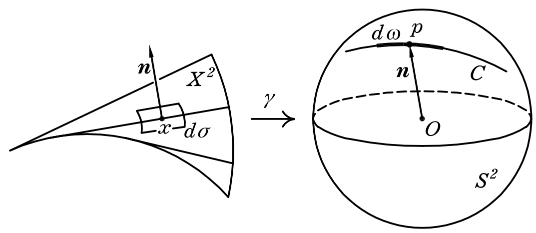

Figure 0.2

If the surface X^2 is defined in $\mathbb{E}^3$ by the equation $z = f(x, y)$, then the condition $K = 0$ is equivalent to the Monge–Ampère equation

$$rt - s^2 = 0,$$

where $r = z_{xx}, s = z_{xy}, t = z_{yy}$ (see Monge [Mon 50]). The surfaces with $K = 0$ are called *developable.* Such surfaces can be locally mapped isometrically into a plane. The latter property is the reason that surfaces with the vanishing Gaussian curvature are called developable: they can be "developed on the plane."

Developable surfaces are a well-known subject from the 19th century. Locally they are classified into three types: cones, cylinders, and torses (tangential developables). A torse is a one-parameter family of tangent lines to a fixed smooth space curve.

The definition of the Gauss map can be easily extended to a hypersurface $X = X^n$ of Euclidean space $\mathbb{E}^{n+1}$. The Gauss map of an oriented smooth hypersurface $X^n \subset \mathbb{E}^{n+1}$ is the mapping of V^n into the unit hypersphere $S^n \subset \mathbb{E}^{n+1}$:

$$\gamma : X \to S^n,$$

by means of the family of hypersurface normals $\boldsymbol{n}$. If $X \subset \mathbb{E}^{n+1}$ is given by the equation

$$z = f(x^1, \dots, x^n),$$

then the condition for its Gauss map to be degenerate has the form

$$\det(z_{ij}) = 0,$$

where $z_{ij} = \frac{\partial^2 z}{\partial x^i \partial x^j}$. If the submanifold X is of codimension $N - n > 1$, then the condition for its Gauss map γ to be degenerate has a more complicated form.

The fact that the Gauss map γ of $X \subset \mathbb{E}^{n+1}$ is degenerate is of projectively invariant nature. This is the reason that the degeneracy of the Gauss map can be defined in terms of projective differential geometry.

Let X be a smooth oriented submanifold of dimension n in the N-dimensional projective space $\mathbb{P}^N$, and let $\mathbb{G}(n, N)$ be the Grassmannian of n-dimensional subspaces of the space $\mathbb{P}^N$. Then the Gauss map γ of $X \subset \mathbb{P}^N$ is defined as the map

$$\gamma : X \to \mathbb{G}(n, N),$$

which carries a point $x \in X$ to the tangent subspace $T_x(X)$ to X at the point x, i.e.,

$$\gamma(x) = T_x(X).$$

The rank r of the map γ is called the *rank* of the submanifold X of dimension n. The rank r does not exceed n, and we assume that the rank r is constant on X.

In a projective space $\mathbb{P}^N$, a variety X of dimension n is said to be a *variety with a degenerate Gauss map* or a *tangentially degenerate variety* if the rank of its Gauss map $\gamma : X \to \mathbb{G}(n, N)$ is less than n. We use the term "variety" here instead of "submanifold" because X has a degenerate Gauss map, and hence it is differentiable almost everywhere (see Section 2.1) while a submanifold is differentiable everywhere.

In this book we study the geometry of varieties with degenerate Gauss maps, construct a classification of such varieties based on the structure of their focal images, and consider applications of the theory of such varieties to different problems of differential geometry and its applications.

Note that in higher dimensions the property through which developable surfaces can be mapped isometrically into a plane is not valid any longer. This is why we prefer to call a variety $X \subset \mathbb{P}^N$ for which $\operatorname{rank} \gamma < n$ a variety with a degenerate Gauss map or a tangentially degenerate variety. Note that some authors (Fisher, Ishikawa, Piontkowski, Mezzetti, Tommasi, Rogora, Wu, Zheng) call such varieties developable.

2. Developments in the Theory of Varieties with Degenerate Gauss Maps. As we mentioned earlier, the developable surfaces in the three-dimensional Euclidean space are a well-known subject from the 19th century. The torses (tangential developables) form a special class of ruled surfaces, namely developable ruled surfaces, and of necessity have singularities, at least along the original curve. There are numerous publications on developable surfaces. The main properties of developable surfaces can be found in most textbooks on differential geometry.

Mathematically developable surfaces are the subject of several branches of mathematics, especially of differential geometry and algebraic geometry. Recently developable surfaces have attracted attention through their relation with computer science (see, for example, the book by Pottmann and Wallner [PW 01]). They are widely used in industry, and are fundamental objects in computer-aided design (see for example, the paper by Hoschek and Pottmann [HoP 95]). Though singularities can be avoided in practical situations, the appearance of singularities in developable surfaces is essential to their nature. Thus the complete description of the structure of developable surfaces involves the singularity theory which was developed in the 20th century (see, for example, the books Bruce and Giblin [BG 92] and Porteous [Por 94]).

The multidimensional varieties X with degenerate Gauss maps of rank $r < n$ were considered by É. Cartan in [C 16] in connection with his study of metric deformation of hypersurfaces, and in [C 19] in connection with his study of manifolds of constant curvature. Yanenko [Ya 53] encountered these varieties in his study of metric deformation of submanifolds of arbitrary classes. Akivis [A 57, 62], Savelyev [Sa 57, 60], and Ryzhkov [Ry 60] systematically studied this kind of variety in a projective space $\mathbb{P}^N$. Brauner [Br 38], Wu [Wu 95], and Fischer and Wu [FW 95] studied such varieties in a Euclidean N-space $\mathbb{E}^N$. Akivis and Goldberg in their book [AG 93] investigated the multidimensional varieties with degenerate Gauss maps in Chapter 4.

Note that a relationship of the rank of varieties X and their deformation in a Euclidean N-space was indicated by Bianchi [Bi 05] who proved that a necessary condition for X to be deformable is the condition rank $X \leq 2$. Allendörfer [Al 39] introduced the notion of type t, $t = 0, 1, , \ldots, m = \dim X$, of X and proved that varieties X^{N-p}, $p > 1$, of type $t > 2$ in $\mathbb{E}^N$ are rigid. Note that both notions, the type and the rank, are projectively and metrically invariant, and that for a hypersurface, the type coincides with the rank.

Griffiths and Harris in their classical paper [GH 79] considered the varieties X with degenerate Gauss maps from the point of view of algebraic geometry. The paper [GH 79] was followed by Landsberg's paper [L 96] and book [L 99] and by the recently published book [FP 01] by Fischer and Piontkowski. The books [L 99] and [FP 01] have special sections devoted to varieties with degenerate Gauss maps. They are in some sense an update to the paper [GH 79]. In both books, following [GH 79], the authors employed a second fundamental form for studying developable varieties, gave detailed and more elementary proofs of some results in [GH 79], and reported on some recent progress in this area. In particular, in [FP 01] the authors gave a classification of developable varieties of rank two in codimension one.

In recent years many papers devoted to varieties with degenerate Gauss maps have appeared. Zak [Za 87] studied the Gauss maps of submanifolds of the projective space from the point of view of algebraic geometry. Ishikawa and Morimoto [IM 01] investigated the connection between such varieties and solutions of Monge–Ampère equations. Ishikawa [I 98, 99b] found real algebraic cubic nonsingular hypersurfaces with degenerate Gauss maps in $\mathbb{RP}^N$ for $N = 4, 7, 13, 25$, and in [I 99a] he studied singularities of C^∞-hypersurfaces with degenerate Gauss maps. Rogora [Rog 97] and Mezzetti and Tommasi [MT 02a, 02c] also considered varieties with degenerate Gauss maps from the point of view of algebraic geometry. Piontkowski [Pio 01, 02a, 02b] considered in $\mathbb{P}^N$ complete varieties with degenerate Gauss maps with rank equal to two, three, and four and with all singularities located on a hyperplane at infinity. The reader can find more details on all these results in the Notes to Chapter 2.

The contents of this book are connected with the theory of singularities of differentiable mappings. There are numerous publications on this topic. In particular, in the book [AVGL 89] by Arnol'd, Vasil'ev, Goryunov, and Lyashko, which is devoted to investigations of singularities of differentiable mappings, their classification, and their applications, the authors consider the singularities of the Grassmann mappings of submanifolds of the Euclidean space and the projective space. Many papers (for example, [Sh 82] by Shcherbak and [I 00b] by Ishikawa) are devoted to a classification of isolated singular points of curves in the Euclidean space and the projective space.

As a rule, the singular points we consider on a variety with a degenerate Gauss map are not isolated (see Section 2.4).

We outline here what distinguishes our book on varieties with degenerate Gauss maps from other literature on this subject:

i) In the current book the authors systematically study the differential geometry of varieties with degenerate Gauss maps. They apply the main methods of differential geometry: the tensor analysis, the method of exterior forms, and the moving frame method.

ii) Western authors were not familiar with the results obtained by Russian geometers in the 1960s (Akivis, Ryzhkov, Savelyev). Some of the results presented by western geometers had been known for years. We present all these results in their historical perspective.

iii) In the study of varieties with degenerate Gauss maps, the authors *systematically* use the focal images (the focal hypersurfaces and the focal hypercones) associated with such varieties. These images were first introduced by Akivis in [A 57]. They allow the authors to describe the geometry of the varieties with degenerate Gauss maps and give their classification. Note that in algebraic geometry, the focal hypersurfaces are called the discriminant varieties.

iv) In the complex projective space, every plane generator L of a variety with a degenerate Gauss map carries singular points. The question is whether these singular points should be included in L. Our point of view is that it is very useful to include them in L; this simplifies the exposition. Many algebraic geometers who study this subject do not consider singular points as a part of L, and this makes their exposition of the results more complicated.

    Note also that in most of the books and papers where the singularities of differentiable mappings are considered, the authors investigate only isolated singularities. But the singularities of Gauss maps comprise algebraic curves or hypersurfaces in the plane generators of varieties with degenerate Gauss maps.

v) It appeared that the Griffiths–Harris conjecture on the structure of varieties with degenerate Gauss maps is not complete. As we show in this book (see also our paper [AG 01a] and the paper [AGL 01] by Akivis, Goldberg, and Landsberg), the basic types of varieties with degenerate Gauss maps include not only cones and torses but also hypersurfaces with degenerate Gauss maps. Note that such hypersurfaces form a very wide class of varieties with degenerate Gauss maps.

vi) When the authors were writing this book, they found some new results on the varieties with degenerate Gauss maps. Some of them were already published and some are in papers submitted for publication. Among these results are a new classification of such varieties (see Akivis and Goldberg [AG 01a]), a detailed investigation of Sacksteder–Bourgain hypersurfaces (see Akivis and Goldberg [AG 01b]), finding an affine analogue of the Hartman–Nirenberg cylinder theorem (see [AG 02a]), establishing the relation between the smooth lines on projective planes over two-dimensional algebras and the varieties with degenerate Gauss maps (see Akivis and Goldberg [AG 02b]), application of the duality principle for construction of varieties with degenerate Gauss maps (see Akivis and Goldberg [AG 02b]), and a description of a new class varieties with degenerate Gauss maps (twisted cones) (see Akivis and Goldberg [AG 03b]).

vii) In this book we consider a very large number of examples. Some of these examples (such as the twisted cones and some algebraic hypersurfaces in $\mathbb{P}^4$) are considered here for the first time, and other examples (such as the cubic symmetroid in $\mathbb{P}^5$ and its projection onto $\mathbb{P}^4$) were known earlier but are considered here from a new point of view.

viii) The authors give a new definition for the dual defect of a variety with a degenerate Gauss map and for dually degenerate varieties with degenerate Gauss maps (see p. 72). This new definition is better than the usual definition of the dual defect given on p. 71: while by old definition all varieties with degenerate Gauss maps are dually degenerate, by the new definition, they can be both dually degenerate and dually nondegenerate. Moreover, while by the old definition, the dual defect δ_* of a dually nondegenerate variety with degenerate Gauss map equals its Gauss defect, $\delta_* = \delta_\gamma > 0$, by the new definition, the dual defect δ_* of such a variety equals 0, $\delta_* = 0$, and this is more appropriate for a *dually nondegenerate* variety.

In addition to varieties with degenerate Gauss maps, algebraic geometry studies other kinds of degenerate varieties (such as secantly degenerate and dually degenerate varieties; see, for example, the paper [GH 78] by Griffiths and Harris; the books [L 99] by Landsberg, pp. 4, 16, and 52; [T 01] by Tevelev, Chapters 6, 9; and [Ha 92] by Harris, pp. 197–199). Not as many secantly degenerate, dually degenerate, and degenerate varieties of other kinds are known. For example, there is only one secantly degenerate variety in the projective space $\mathbb{P}^5$, namely, the Veronese variety (see Sasaki [Sas 91] and Akivis [A 92]). In this connection, note also that all smooth dually degenerate varieties of dimension $n \leq 10$ are listed (see for example, the book [T 01] by Tevelev, Chapter 9, or Notes to Section 2.5 of this book where the appropriate references are given).

Unlike the classes of these degenerate varieties, the varieties with degenerate Gauss maps compose a much wider class. In particular, the arbitrariness of the class of torsal varieties is equal to some number of functions of two variables, and the arbitrariness of the class of hypersurfaces with degenerate Gauss maps of rank r in the space $\mathbb{P}^N$ (as well as their dual image, smooth tangentially nondegenerate subvarieties, for which $r = n$) is equal to $N - r$ functions of r variables. Hence, the study of the varieties with degenerate Gauss maps in the space $\mathbb{P}^N$ is of considerable interest.

Note that in the book only dually nondegenerate varieties with degenerate Gauss maps are under investigation. For such varieties, the system of second fundamental forms always contains at least one nondegenerate form of rank r, and for them not only the focus hypersurfaces but also the focus hypercones whose vertices are the tangent subspaces of the variety X are correctly defined.

3. The Contents of the Book. The book consists of five chapters. In Chapter 1, we give the basic notions and results of vector spaces and projective space, consider the main topics associated with differentiable manifolds, and study some algebraic varieties, namely, Grassmannians and determinant submanifolds.

In Chapter 2, we introduce the basic notions associated with a variety in a projective space $\mathbb{P}^N$, define the rank of a variety and varieties with degenerate Gauss maps, present the main examples of varieties with degenerate Gauss maps (cones, torses, hypersurfaces, joins, etc.), study the duality principle and its applications, consider another example of submanifolds with degenerate Gauss maps (the cubic symmetroid) and correlative transformations, and investigate a hypersurface with a degenerate Gauss map associated with a Veronese variety and find its singular points. The reader can find more details on Chapters 1 and 2 in the Contents.

In Chapter 3, we define the Monge–Ampère foliation associated with a variety with a degenerate Gauss map of dimension n, derive the basic equations of varieties with degenerate Gauss maps, prove a characteristic property of such varieties (the leaves of the Monge–Ampère foliation are flat), and consider focal images of such varieties (the focus hypersurfaces and the focus hypercones). In this chapter we also study varieties with degenerate Gauss maps not only in the complex projective space but also in the real projective space, the affine space, the Euclidean space, and the non-Euclidean spaces. We prove that in these spaces there are varieties with degenerate Gauss maps without singularities, and we introduce and investigate an important class of varieties with degenerate Gauss maps without singularities, the so-called the Sacksteder–Bourgain hypersurface. Note that Sacksteder and Bourgain constructed examples of hypersurfaces with degenerate Gauss maps in the affine space $\mathbb{A}^4$. In Section 3.4 (see also the paper by Akivis and Goldberg [AG 01b]), we prove that the hypersurfaces constructed by them are locally equivalent, and we construct a series of hypersurfaces with degenerate Gauss maps in the affine space $\mathbb{A}^N$ generalizing the Sacksteder–Bourgain hypersurface.

In Chapter 4, in the projective space $\mathbb{P}^N$, we consider the basic types of varieties with degenerate Gauss maps: cones, torsal varieties, hypersurfaces with degenerate Gauss maps. For each of these types, we consider the structure of their focal images and find sufficient conditions for a variety to belong to one of these types (for torsal varieties our condition is also necessary). The classification of varieties X with degenerate Gauss maps presented in this chapter is based on the structure of the focal images of X. In a series of theorems, we establish this connection. We prove that varieties with degenerate Gauss maps that do not belong to one of the basic types are foliated into varieties of basic types. Finally, we prove an embedding theorem for varieties with degenerate Gauss maps and find sufficient conditions for such a variety to be a cone. In this chapter, we also consider varieties with degenerate Gauss maps in the affine space $\mathbb{A}^N$ and find a new affine analogue of the Hartman–Nirenberg cylinder theorem. We consider here parabolic hypersurfaces in the space $\mathbb{P}^4$ (i.e., the hypersurfaces X with degenerate Gauss maps of rank $r = 2$ that have a double focus F on each rectilinear generator L). We also prove existence theorems for some varieties with degenerate Gauss maps, for example, for twisted cones in $\mathbb{P}^4$ and $\mathbb{A}^4$, and we establish a structure of twisted cones in $\mathbb{P}^4$. This structure allows us to find a procedure for construction of twisted cylinders in $\mathbb{A}^4$.

Chapter 5 is devoted to further examples and applications of the theory of varieties with degenerate Gauss maps. As the first application, we prove that lightlike hypersurfaces in the de Sitter space $\mathbb{S}_1^{n+1}$ have degenerate Gauss maps, that their rank $r \leq n-1$, and that there are singular points and singular submanifolds on them. We classify singular points and describe the structure of lightlike hypersurfaces carrying singular points of different types. Moreover, we establish the connection of this classification with that of canal hypersurfaces of the conformal space. As the second application, we establish a relation of the theory of varieties with degenerate Gauss maps in projective spaces with the theory of congruences and pseudocongruences of subspaces and show how these two theories can be applied to the construction of induced connections on submanifolds of projective spaces and other spaces endowed with a projective structure. As the third application, we consider smooth lines on projective planes over the complete matrix algebra $\mathbb{M}$ of order two, the algebra $\mathbb{C}$ of complex numbers, the algebra $\mathbb{C}^1$ of double numbers, and the algebra $\mathbb{C}^0$ of dual numbers. For the algebras, $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$, in the space $\mathbb{RP}^5$, to these smooth lines there correspond families of straight lines describing three-dimensional point varieties X^3 with degenerate Gauss maps of rank $r \leq 2$. We prove that they represent examples of different types of varieties X^3 with degenerate Gauss maps.

Sections, formulas, and figures in the book are numbered within each chapter. Each chapter is accompanied by notes containing remarks of historical and bibliographical nature and some supplementary results pertinent to the main content of the book. A fairly complete bibliography on multidimensional varieties with degenerate Gauss maps, a list of notations, an author index, and a subject index are at the end of the book.

Bibliographic references give the author's last name followed by the first letter(s) of the author's last name and the last two digits of the year in square brackets, for example, Blaschke [Bl 21]. Note that in the bibliography, in addition to the original article being cited, reviews of the article in major mathematical review journals (*Jahrbuch für Fortschritte der Mathematik*, *Zentralblatt für Mathematik*, *Mathematical Reviews*) are referenced.

4. General Remarks for the Reader. This book is intended for graduate students whose field is differential geometry, as well as for mathematicians and teachers conducting research in this subject. It can be used in special graduate courses in mathematics.

In our presentation we use the tensorial methods in combination with the methods of exterior differential forms and moving frames of Élie Cartan. The reader is assumed to be familiar with these methods, as well as with the basics of modern differential geometry. However, in Chapter 1 we recall basic facts of tensor calculus and the method of moving frame in the form in which they will be used in the book. Many other concepts of differential geometry are explained briefly in the text; some are given without explanation. As references, the books [KN 63] by Kobayashi and Nomizu, [BCGGG 91] by Bryant et al., and [C 45] by É. Cartan are recommended. In the book [Sto 69] by Stoker, the reader can find the main notions and theorems of elementary differential geometry that are necessary for reading this book. We also recommend our book Akivis and Goldberg [AG 93], in which the projective differential geometry of general submanifolds and some of their most important special classes were developed systematically. We will often refer to this book.

All functions, vector and tensor fields, and differential forms are assumed to be differentiable almost everywhere. As a rule, we use the index notations in our presentation. We believe this allows us to obtain a deeper understanding of the essence of problems in local differential geometry.

Note also that if we impose a restriction on a variety, then, as a rule, we assume that this condition holds at all points of this variety. More precisely, we consider only the domain of the variety where this restriction holds.

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Jerusalem, Israel *Maks A. Akivis*

Newark, New Jersey, USA *Vladislav V. Goldberg*

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