Preface

At the turn of the 20th century, mathematicians had a monopoly on well-defined complicated global problems, such as celestial mechanics, fixed points of high-dimensional nonlinear functions, the geometry of level sets of differentiable functions, algebraic varieties, distinguishing topological spaces, etc. Algebraic topology, of which homology is a fundamental part, was developed in response to such challenges and represents one of the great achievements of 20th-century mathematics. While its roots can be traced to the middle of the 18th-century with Euler's famous formula that for the surface of a convex polyhedron

$$\text{faces} - \text{edges} + \text{vertices} = 2,$$

it is fair to say that it began as a subject in its own right in the seminal works of Henri Poincaré on "Analysis Situs." Though Poincaré was motivated by analytic problems, as his techniques developed they took on a combinatorial form similar in spirit to Euler's formula.

The power of algebraic topology lies in its coarseness. To understand this statement, consider Euler's formula. Observe, for instance, that the size of the polyhedron is of no importance. In particular, therefore, small changes in the shape of the polyhedron do not alter the formula. On the other hand, if one begins with a polyhedron that is punctured by k holes as opposed to a convex polyhedron, then the formula becomes

$$\text{faces} - \text{edges} + \text{vertices} = 2 - 2k.$$

As a result, counting local objects—the faces, edges, and vertices—of a polyhedron allows us to determine a global property, how many holes it has. Furthermore, it shows that formulas of this form can be used to distinguish objects with important different geometric properties.

The potential of Poincaré's revolutionary ideas for dealing with global problems was quickly recognized, and this led to a broad development of the subject. However, as is to be expected, the form of development matched the problems of interest. As indicated above, a typical question might be concerned with the structure of the level set of a differentiable function. Solving such problems using purely combinatorial arguments suggested by Euler's formula, that is, cutting the set into a multitude of small pieces and then counting, is in general impractical. This led to the very formidable and powerful algebraic machinery that is now referred to as algebraic topology. In its simplest form this tool takes objects defined in terms of traditional mathematical formulas and produces algebraic invariants that provide fundamental information about geometric properties of the objects.

As we begin the 21st century, complexity has spread beyond the realm of mathematics. With the advent of computers and sophisticated sensing devices, scientists, engineers, doctors, social scientists, and business people all have access to, or through numerical simulation can create, huge data files. Furthermore, for some of these data sets the crucial information is geometric in nature, but it makes little sense to think of these geometric objects as being presented in terms of or derived from traditional mathematical formulas. As an example, think of medical imaging. Notice that even though the input is different, the problems remain the same: identifying and classifying geometric properties or abnormalities. Furthermore, inherent in numerical or experimental data is error. What is needed is a framework in which geometrical objects can be recognized even in the presence of small perturbations.

Hopefully these arguments suggest that the extraordinary success of algebraic topology in the traditional domains of mathematics can be carried over to this new set of challenges. However, to do so requires the ability to efficiently compute the algebraic topological quantities starting with experimental or numerical data—information that is purely combinatorial in nature.

The purpose of this book is to present a computational approach to homology with the hope that such a theory will prove beneficial to the analysis and understanding of today's complex geometric challenges. Naturally this means that our intended audience includes computer scientists, engineers, experimentalists, and theoreticians in nonlinear science. As such we have tried to keep the mathematical prerequisites to an absolute minimum. At the same time we, the authors, are mathematicians and proud of our trade. We believe that the most significant applications of the theory will be realized by those who understand the fundamental concepts of the theory. Therefore, we have insisted on a rigorous development of the subject. Thus this book can also be used as an introductory text in homology for mathematics students. It differs from the traditional introductory topology books in that a great deal of effort is spent discussing the computational aspects of the subject.

The broad range of background and interests of the intended readers leads to organizational challenges. With this in mind we have tried on a variety of levels to make the book as modular as possible. On the largest scale the book is divided into three parts: Part I, which contains the core material on computational homology; Part II, which describes applications and extensions; and Part III, which contains a variety of preliminary material. These parts are described in greater detail below. There is, however, another natural division—the homology of spaces and the homology of maps—with different potential applications. Cognizant of the fact that the applications are of primary interest to some readers, we have attempted to organize the chapters in Parts I and II along these lines. Thus, as is indicated in Figure 1, a variety of options concern the order in which the material can be read.

We have already argued that algebraic topology holds tremendous potential for applications, but homology is also a beautiful subject in that topology is transformed into algebra and from the algebra one can recover aspects of the topology. However, this process involves some deep ideas and it is easy to lose track of the big picture in the midst of the mathematical technicalities. With this in mind we begin Part I with a preview both to the applications and to the homology of spaces. The ideas are sketched through very simple examples and without too much concern for rigor. Since the homology of maps depends deeply on the homology of spaces, we postpone a preview of this subject to Chapter 5.

As mentioned above, Part I contains the core material. It provides a rigorous introduction to the homology of spaces and continuous functions and is meant to be read sequentially. In Chapter 2 we define homology and investigate its most elementary properties. In particular, we explain how to each topological space we can assign a sequence of abelian groups called the homology groups of the space. There is a caveat: that the topological spaces we consider must be built out of d-dimensional unit cubes with vertices on the integer lattice. This is in contrast to the standard combinatorial approach, which is based on simplices. There are two reasons for this. The first comes from applications. Consider digital images. The basic building blocks are pixels, which are easily identified with squares. Similarly, an experimental or numerically generated data point comes with errors. Thus a d-dimensional data point can be thought of as lying inside a d-dimensional cube whose width is determined by the error bounds. The second reason—which will be made clear in the text—has to do with the simplicity of algorithms.

In Chapter 3 we show that homology is computable by presenting in detail an algorithm based on linear algebra over the integers. This is essential because it demonstrates that the homology group of a topological space made up of cubes is computable. However, for spaces made up of many cubes, the algorithm is of little immediate practical value. Therefore, in Chapter 4 we introduce combinatorial techniques for reducing the number of elements involved in the computation.

The contents of Chapter 4 also naturally foreshadow questions concerning maps between topological spaces and maps between the associated homology groups. The construction of homology maps is done in Chapter 6. We approach this problem from the viewpoint of multivalued maps. This has an extremely important consequence. One can efficiently compute homology maps—an absolutely essential feature given the purposes of this endeavor. We know of no practical algorithms for producing simplicial maps that approximate continuous maps. Algorithms for the computation of homology maps are discussed in Chapter 7.

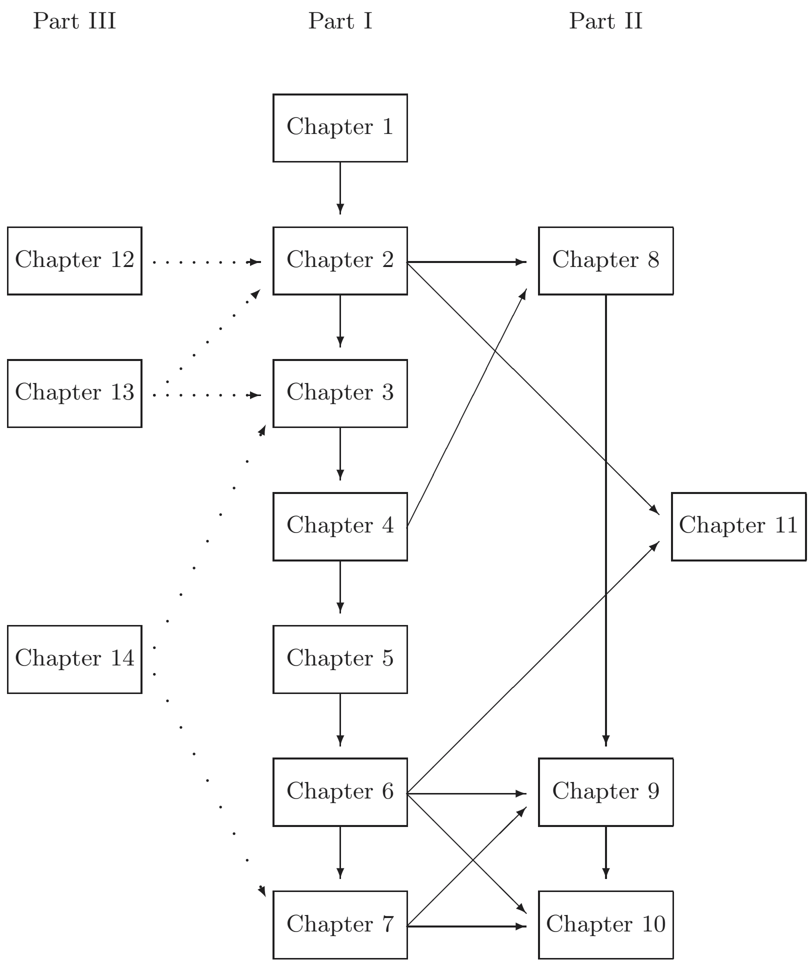

Fig. 1. Chapter dependence chart.

As mentioned earlier, we have attempted to keep the necessary prerequisites for this book to a minimum. Ideally the reader would have had an introductory course in point-set topology, an abstract algebra course, and familiarity with computer algorithms. On the other hand, such a reader would know far more topology, algebra, and computer science than is really necessary. Furthermore, the union of these subjects in the typical curriculum of our desired readers is probably fairly rare. For this reason we have included in Part III brief introductions to these subjects. Perhaps a more traditional organization would have placed Chapters 12, 13, and 14 at the beginning of the book. However, it is our opinion that nothing kills the excitement of studying a new subject as rapidly as the idea of multiple chapters of preliminary material. We suggest that the reader begins with Chapter 1 and consults the last three chapters on a need-to-know basis.

We argue at the beginning of this preface that algebraic topology has an important role to play in the analysis of numerical and experimental data. In Part II we elaborate on these ideas. It should be mentioned that applications of homology to these types of problems is a fairly new idea and still in a fairly primitive state. Thus the focus of this part is on conveying the potential rather than elaborate applications. We begin in Chapter 8 with a discussion that relates cubical complexes to image data and numerically generated data. Even the simple examples presented here suggest the need for more sophisticated ideas from homology. Therefore, Chapter 9 introduces more sophisticated algebraic concepts and computational techniques that are at the heart of homological algebra. In Chapter 10 we indicate how homology can be used to study nonlinear dynamics. In some sense this is one of the most traditional applications of the subject—the focus of much of Poincaré's work was related to differential equations and, in fact, he set the foundations for what is known today as dynamical systems. The modern twist is that these ideas can be tied to numerical methods to obtain computer-assisted proofs in this subject. Finally, though the cubical theory presented in this book has concrete advantages, it is also in many ways too rigid. In Chapter 11 we extend the cubical theory to polyhedra, thereby tying the results and techniques of this book to the enormous body of work known as algebraic topology.

Preliminary versions of this book have been used by the authors to teach one-semester or full-year courses at both the undergraduate and graduate levels and for a mixed audience of mathematics, physics, computer science, and engineering students. As mentioned several times, Part I contains the essential material. It can be covered in a single quarter, but at the expense of any applications or extensions presented in Part II. On the other hand, as is indicated in Figure 1, the material in Chapter 8 is conceptually accessible immediately following Chapter 2. However, the description of the computational techniques used to analyze the images in Chapter 8 is presented in Chapters 3 and 4. The first two sections of Chapter 11 concerning simplicial complexes can be read right after (or even parallel with) Chapter 2, but the last section on the homology functor can only be attained after reading Chapter 6. Similarly, Sections 10.4 and 10.5 of Chapter 10 can be read following Chapter 6. However, the material of Sections 10.6 and 10.7 depends heavily on topics in Chapter 9.

This book provides the conceptual background for computational homology. However, interesting applications also require efficient code, which as one might expect, is evolving rapidly. Software and examples are available at the *Computational Homology Program* (CHomP) web site, accessible via *www.springeronline.com.* This site also contains errata and additional links.

Preparing this book has been an exciting and challenging project. The material presented here is a unique combination of current research and classical mathematics that fundamentally depends on an interplay among mathematical rigor, computation, and application. We could not have completed this project without the support and assistance of numerous individuals.

Results presented here for the first time have been shaped by our collaboration with students and colleagues: Madjid Allili, Sarah Day, Marcio Gameiro, Bill Kalies, Howard Karloff, Paweł Pilarczyk, Andrzej Szymczak, and Thomas Wanner. We would like also to thank Stanisław Sędziwy for his continuous encouragement to write this book.

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Sherbrooke, Quebec, Canada — *Tomasz Kaczynski*
Atlanta, Georgia, USA — *Konstantin Mischaikow*
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