11
Schreier Varieties of Algebras

In this chapter, we consider the main properties of free algebras of Schreier varieties of algebras. A variety of algebras is said to be Schreier if any subalgebra of a free algebra of this variety is free in the same variety of algebras. In Section 11.1, we describe the main types of Schreier varieties and introduce universal multiplicative enveloping algebras of free algebras. Theorem 11.1.1 gives the main properties of the free algebras of these varieties. In Section 11.2, we expose the weak algorithm for free associative algebras and discuss Schreier's techniques for free algebras: ranks of left ideals of free associative algebras and Schreier-type formulas for ranks of subalgebras of free algebras.

Using free differential calculus, we introduce partial derivatives of elements of free algebras in Section 11.3. Theorem 11.3.1 shows that free algebras of the main types of Schreier varieties have the property of differential separability for subalgebras. Using the Poincaré–Birkhoff–Witt theorems, we prove in Theorem 11.3.4 that Lie superalgebras are differentially separable. Theorem 11.3.5 shows that free Lie superalgebras are residually finite. Theorems 11.3.6–11.3.8 give some necessary and sufficient conditions for a homogeneous variety of algebras to be Schreier. Theorem 11.3.11 gives an effective algorithm to find ranks of subalgebras of free algebras. In Section 11.4, we consider equivalence and stable equivalence of elements under the action of the automorphism group (Theorems 11.4.4 and 11.4.5).

Theorems 11.5.3 and 11.5.4 of Section 11.5 show that an injective endomorphism of maximal rank does not change the rank of a system of elements of maximal rank.

11.1 Main Types of Nielsen–Schreier Algebras

Let K be a field. A variety $\mathfrak{M}$ of algebras over K is a class of algebras closed under taking subalgebras, homomorphic images, and direct products. Let X be a set. An algebra $F_{\mathfrak{M}}(X)$ in $\mathfrak{M}$ is called the free algebra with the set X of free generators if $F_{\mathfrak{M}}(X)$ is generated by X, and if f is an arbitrary map from X to any algebra A of $\mathfrak{M}$, then f can be extended to a homomorphism $F_{\mathfrak{M}}(X) \to A$ (it is clear that such extension is unique). The algebra $F_{\mathfrak{M}}(X)$ is determined by X and $\mathfrak{M}$ uniquely up to an isomorphism. The cardinality of X is called the rank of $F_{\mathfrak{M}}(X)$. For any nontrivial variety $\mathfrak{M}$ and a set X there exists the free algebra $F_{\mathfrak{M}}(X)$. If $A \in \mathfrak{M}$, then there exists a set X such that A is a homomorphic image of $F_{\mathfrak{M}}(X)$. In particular, A is isomorphic to a factor algebra of $F_{\mathfrak{M}}(X)$.

Let X be a finite set, $X = \{x_1, \ldots, x_n\}$. A pair (u, v) of elements of $F_{\mathfrak{M}}(X)$ is an identity in an algebra A of $\mathfrak{M}$ if $u(a_1, \ldots, a_n) = v(a_1, \ldots, a_n)$ for all elements $a_1, \ldots, a_n$ of A. The following result is due to G. Birkhoff. Let $\mathfrak{M}$ be a variety of algebras. A subclass $\mathfrak{N}$ in $\mathfrak{M}$ is a (sub)variety if and only if there exists a set of identities (u_i, v_i), $i \in I$, such that $\mathfrak{N}$ consists of all algebras of $\mathfrak{M}$ satisfying these identities. Here u_i, v_i $(i \in I)$ are elements of $F_{\mathfrak{M}}$ with a countable set X of free generators. A variety of algebras is called homogeneous if it is defined by a set of homogeneous identities. For more details about varieties of algebras, we refer to [25, 79, 143, 178].

A variety of algebras is said to be Schreier if any subalgebra of a free algebra of this variety is free in the same variety of algebras; this notion came from group theory. In the 1920s, J. Nielsen [294] and O. Schreier [329] proved that any subgroup of a free group is free. A. G. Kurosh proved in [195] that subalgebras of free nonassociative algebras are free. A. I. Shirshov [345] showed that the variety of Lie algebras is Schreier (this result was also obtained by E. Witt in [403], and he also proved that the variety of all Lie p-algebras is Schreier). A. G. Kurosh in [197] extended this result to the variety of all Ω-algebras. An Ω-algebra is a vector space with a family Ω of basic multilinear multiplications of varieties at least 2. In this paper, he also classified subalgebras of free products of Ω-algebras. Earlier, for the case of all nonassociative algebras it was done in [196], and for the case of all commutative and all anticommutative algebras it was done in [139]. Yu. A. Bahturin [30, 31] and M. V. Zaicev [414] proved that Schreier varieties of Lie algebras are the variety of all Lie algebras and the variety of abelian Lie algebras, respectively. A. I. Shirshov showed in [346] that subalgebras of free nonassociative commutative and anticommutative algebras are free; that is, the varieties of commutative and anticommutative algebras are Schreier.

A. A. Mikhalev in [243] and A. S. Shtern in [369] showed that the variety of Lie superalgebras is Schreier. A. A. Mikhalev proved this result for the variety of color Lie p-superalgebras in [245]. Up to now there is no complete classification of Schreier varieties of algebras. U. U. Umirbaev [388, 392] obtained new examples of Schreier varieties of algebras and gave necessary and sufficient conditions for a variety of algebras to be Schreier. As for subalgebras of free algebras of varieties of linear Ω-algebras, see articles [26, 38].

Let K be a field, X a set, and let $F = F(X)$ be the free algebra of a homogeneous variety of algebras with the set X of free generators over the field K. For $u \in F(X)$, by $\ell(u) = \ell_X(u)$ we denote the degree of u. Consider also a generalized degree function $\mu\colon X \to \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. Let $\Gamma(X)$ be the free groupoid of nonassociative monomials in the alphabet X, $S(X)$ the free semigroup of associative words in X, and ${}^{\sim}\colon \Gamma(X) \to S(X)$ the bracket removing homomorphism. We set $\mu(x_1 \dots x_n) = \sum_{i=1}^n \mu(x_i)$ for $x_i \in X$. If $\mu(x) = 1$ for all $x \in X$, then μ is just the usual degree, $\mu = \ell$. If $a \in F$, $a = \sum \alpha_i a_i$, $0 \neq \alpha_i \in K$, a_i are basic monomials, $a_j \neq a_s$ with $j \neq s$, then we set $\mu(a) = \max_i\{\mu(\tilde{a}_i)\}$. By a° we denote the leading part of a: $a^\circ = \sum_{j,\mu(a_j)=\mu(a)} \alpha_j a_j$.

A subset M of F is called independent if M is a set of free generators of the subalgebra of F generated by M. A subset $M = \{a_i\}$ of nonzero elements of F is called reduced if for any i the leading part a_i° of the element a_i does not belong to the subalgebra of F generated by the set $\{a_j^\circ \mid j \neq i\}$.

Let $S = \{s_\alpha \mid \alpha \in I\}$ be a subset of F. A mapping $\omega\colon S \to S' \subseteq A$ is an elementary transformation of S if either ω is a nonsingular linear transformation of S, or $\omega(s_\alpha) = s_\alpha$ for all $\alpha \in I$, $\alpha \neq \beta$, and $\omega(s_\beta) = s_\beta + f(\{s_\alpha \mid \alpha \neq \beta\})$, where f is an element of a free algebra of the same variety of algebras. It is clear that elementary transformations of free generating sets induce automorphisms of the algebra F; such automorphisms are called elementary.

Let A be a free algebra of a homogeneous Schreier variety of algebras. One can transform any finite set of elements of the algebra A to a reduced set by using a finite number of elementary transformations and possibly cancelling zero elements. Every reduced subset of the algebra A is an independent subset (this is what is called the Nielsen property). Moreover, by using Kurosh's method, one can construct a reduced set of generators for any subalgebra of the free algebra A. Indeed, let B be a subalgebra of A, $M = \bigcup_{j=0}^\infty M_j$, where M_j consists of G-homogeneous elements of degree j, if nonempty. We set $M_0 = \varnothing$. Then, by induction, we suppose that $M_0, \dots, M_i$ have been defined. Let B_i be the subalgebra of B generated by $\bigcup_{j=0}^i M_j$ and $S_i = \{b \in B_i \mid \ell(b) \leq i+1\}$. Consider the G-homogeneous subsets of $\{b \in B \mid \ell(b) = i+1\}$ that are linearly independent modulo the subspace S_i, and let M_{i+1} be a maximal such subset. It is clear that M generates the subalgebra B. Moreover, M is a reduced set.

Hence any subalgebra of A is free in the same variety of algebras (this is what is called the Schreier property). In [212], J. Lewin proved that for a homogeneous variety of algebras, the Nielsen and Schreier properties are equivalent. By using this equivalence, it is easy to see that automorphism groups of the corresponding free algebras (of finite rank) are generated by elementary automorphisms (this result was obtained for free Lie algebras by P. M. Cohn in [77] and for free nonassociative algebras by J. Lewin in [212]).

Let $\mathfrak{M}$ be a homogeneous variety of linear algebras over a field K. Let C be an algebra of the variety $\mathfrak{M}$ and $\langle y\rangle$ the free algebra of rank 1 of the variety $\mathfrak{M}$. We consider the free product $B = \langle y\rangle * C \in \mathfrak{M}$. By B_1 we denote the subspace of elements of B with degree 1 with respect to y. Then B_1 is a free one-generated C-bimodule in $\mathfrak{M}$. Consider $C \oplus B_1$ with the multiplication given by

$$(a_1 + m_1)(a_2 + m_2) = a_1a_2 + a_1m_2 + m_1a_2,$$

where $a_1, a_2 \in C$, $m_1, m_2 \in B_1$. We have $C \oplus B_1 \in \mathfrak{M}$. For $a \in C$, by l_a and r_a we denote the universal operators of left and right multiplication, respectively,

$$b \cdot l_a = ab, \quad b \cdot r_a = ba, \quad b \in B_1.$$

Let $U(C) = U_{\mathfrak{M}}(C)$ be the subalgebra of the algebra $\mathrm{End}_F(B_1)$ of endomorphisms of the module B_1 generated by the set $\{1, l_a, r_a \mid a \in C\}$. If the algebra C has the unit element, then we set $r_1 = l_1 = 1$ in $U(C)$. The algebra $U(C)$ is the universal multiplicative enveloping algebra of C. The notion of a C-bimodule in the variety $\mathfrak{M}$ is equivalent to the notion of a right $U(C)$-module; see [157].

By $A = F_{\mathfrak{M}}\langle X\rangle$ we denote the free algebra of the variety $\mathfrak{M}$ with the set $X = \{x_1, \ldots, x_n\}$ of free generators. Let $\mathfrak{M}_0$ be the variety of all algebras and $\Gamma(X)$ the free groupoid of nonassociative monomials (free Magma) without unity element on the alphabet X; i.e., $X \subseteq \Gamma(X)$. If $u, v \in \Gamma(X)$, then $u \cdot v \in \Gamma(X)$, where $u \cdot v$ is the formal multiplication on nonassociative monomials. We consider the linear space $F(X)$ over K with the basis consisting of 1 and the elements of $\Gamma(X)$ with the multiplication given by

$$(\alpha a) \cdot (\beta b) = (\alpha\beta)(a \cdot b)$$

for all $\alpha, \beta \in K$, $a, b \in \Gamma(X)$. The algebra $F(X)$ is the free nonassociative algebra (the free algebra of the variety $\mathfrak{M}_0$). Let $W_0 = \Gamma(X)$, $A = F(X)$. Then the algebra $U(A)$ is the free associative algebra with the set $S_0 = \{r_w, l_w \mid w \in W_0\}$ of free generators (for the details, see [140, 388]).

Let $\mathfrak{M}_1$ be the variety of all commutative algebras, I the two-sided ideal of the free nonassociative algebra $F(X)$ generated by the set $\{ab - ba \mid a, b \in F(X)\}$. Then the factor algebra $A = F(X)/I$ is the free algebra of the variety $\mathfrak{M}_1$ with the set X of free generators (i.e., the free commutative nonassociative algebra).

Suppose that the set $\Gamma(X)$ is totally ordered in such a way that, for $a, b \in \Gamma(X)$, if $\ell(a) > \ell(b)$, then $a > b$ ($\ell(u)$ denotes the degree of a monomial $u \in \Gamma(X)$). We construct the set W_1 of all commutative regular monomials inductively in the following way. First of all, $X \subset W_1$. Furthermore, $w \in W_1$

if $w = uv$, u and v are commutative regular monomials, and $u \leq v$. Then W_1 is a linear basis of $A = F(X)/I$ ([346]). The universal multiplicative enveloping algebra $U(A)$ is the free associative algebra with the set $S_1 = \{r_w \mid w \in W_1\}$ of free generators [140, 388].

Let $\mathfrak{M}_2$ be the variety of all anticommutative algebras and J the two-sided ideal of the free nonassociative algebra $F^+(X)$ without unit element generated by the set $\{aa \mid a \in F(X)\}$. Then the factor algebra $A = F^+(X)/J$ is the free algebra of the variety $\mathfrak{M}_2$ with the set X of free generators (i.e., the free anticommutative nonassociative algebra). We construct the set W_2 of all anticommutative regular monomials inductively: $X \subset W_2$; $w \in W_2$ if $w = uv$, u, and v are anticommutative regular monomials, and $u < v$. Then W_2 is a linear basis of $A = F(X)/J$ ([346]). The universal multiplicative enveloping algebra $U(A)$ is the free associative algebra with the set $S_2 = \{r_w \mid w \in W_2\}$ of free generators [140, 388].

Let G be an abelian semigroup, K a field of characteristic different from two, and $\varepsilon\colon G \times G \to K^*$ a skew symmetric bilinear form (a commutation factor, or a bicharacter); that is,

$$\varepsilon(g,h)\,\varepsilon(h,g) = 1,$$
$$\varepsilon(g_1 + g_2, h) = \varepsilon(g_1,h)\,\varepsilon(g_2,h), \quad \varepsilon(g, h_1 + h_2) = \varepsilon(g,h_1)\,\varepsilon(g,h_2)$$

for all $g, g_1, g_2, h, h_1, h_2 \in G$;

$$G_- = \{g \in G \mid \varepsilon(g,g) = -1\}, \quad G_+ = \{g \in G \mid \varepsilon(g,g) = +1\}.$$

For instance, if $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $K = \mathbb{C}$, $f = (f_1, f_2), g = (g_1, g_2) \in G$, then the following form is a commutation factor on G: $\varepsilon(f,g) = (-1)^{(f_1+f_2)(g_1+g_2)}$; $G_+ = \{(0,0),(1,1)\}$, $G_- = \{(0,1),(1,0)\}$. Suppose that $n = 2^k$, $k \geq 1$. Let ε_n be a primitive root of 1, $\varepsilon_n = e^{2\pi \mathbf{i}/n}$, $G = \langle f \rangle_n \oplus \langle g \rangle_n$ the direct sum of cyclic groups of order n generated by f and g, respectively, $\varepsilon(f,f) = 1$, $\varepsilon(f,g) = \varepsilon_n$, $\varepsilon(g,g) = -1$. Then ε is a commutation factor on G.

M. Scheunert [326] classified all commutation factors on groups $\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$, where p is a prime number. A commutation factor ε on an abelian group G is said to be nondegenerated if $\{g \in G \mid \varepsilon(g,h) = 1\ \forall h \in G\} = \{0\}$. A commutation factor ε' on a group G' is equivalent to a commutation factor ε'' on a group G'' if there is an isomorphism $\varphi\colon G' \to G''$ such that $\varepsilon'(g,h) = \varepsilon''(\varphi(g), \varphi(h))$ for all $g, h \in G'$. A. A. Zolotykh [419] obtained a classification (up to an equivalence) of all nondegenerate commutation factors on finite abelian groups.

A G-graded K-algebra $R = \bigoplus_{g \in G} R_g$ is a color Lie superalgebra if

$$[x,y] = -\varepsilon(d(x), d(y))[y,x], \quad [v,[v,v]] = 0,$$
$$[x,[y,z]] = [[x,y],z] + \varepsilon(d(x), d(y))[y,[x,z]]$$

with $d(v) \in G_-$ for G-homogeneous elements $x, y, z, v \in R$, where $d(a) = g$ if $a \in R_g$. The homogeneous elements of the components R_g, $g \in G_+$, are said to be even, and the homogeneous elements of the components R_g,

$g \in G_-$, are said to be odd. Let $R_+ = \bigoplus_{g\in G_+} R_g$, $R_- = \bigoplus_{g\in G_-} R_g$. We consider only G-homogeneous elements, homomorphisms preserving the G-graded structure, etc.

If $G = \mathbb{Z}_2$ and $\varepsilon(f,g) = (-1)^{fg}$, then a color Lie superalgebra is a Lie superalgebra. If $\varepsilon \equiv 1$, then we have a G-graded Lie algebra (if $G = \{e\}$, then we have a Lie algebra).

We denote $(\operatorname{ad} a)(b) = [a,b] = (a)\operatorname{Ad} b$ for all $a, b \in R$. Let $\operatorname{char} K = p > 2$. A color Lie superalgebra R over K is a color Lie p-superalgebra if on G-homogeneous components R_g, $g \in G_+$, we have a mapping $x \to x^{[p]}$, $d(x^{[p]}) = pd(x)$, such that, for all $\alpha \in K$ and all G-homogeneous elements $x, y, z \in R$ with $d(x) = d(y) \in G_+$, the following conditions are satisfied:

$$(\alpha x)^{[p]} = \alpha^p x^{[p]}, \quad (\operatorname{ad}(x^{[p]}))(z) = [x^{[p]}, z] = (\operatorname{ad} x)^p(z),$$
$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum s_j(x,y),$$

where $js_j(x,y)$ is the coefficient on t^{j-1} in the polynomial $(\operatorname{ad}(tx + y))^{p-1}(x)$.

If Q is a G-graded associative algebra over K, then $[Q]$ denotes the color Lie superalgebra with the operation $[\,,\,]$ where $[a,b] = ab - \varepsilon(d(a), d(b))ba$ for G-homogeneous elements $a, b \in Q$.

If $\operatorname{char} K = p > 2$, and $x^{[p]} = x^p$ for all G-homogeneous elements x of Q with $d(x) \in G_+$, then $[Q]$ with the operation $[p]$ is a color Lie p-superalgebra denoted by $[Q]^p$.

Let $X = \{x_1, \ldots, x_n\} = \bigcup_{g\in G} X_g$ be a G-graded set (that is, $X_g \cap X_f = \varnothing$ for $g \neq f$, $d(x) = g$ for $x \in X_g$), and let $K\langle X\rangle$ be the free G-graded associative K-algebra and $L(X)$ the subalgebra of $[K\langle X\rangle]$ generated by X. Then $L(X)$ is the free color Lie superalgebra with the set X of free generators. In the case where $\operatorname{char} K = p > 2$, let $L^p(X)$ be the subalgebra of $[K\langle X\rangle]^p$ generated by X. Then $L^p(X)$ is the free color Lie p-superalgebra on X. For more information on free Lie superalgebras, we refer to the monographs [33, 275].

We combine the main properties of free algebras of main types of Schreier varieties of algebras in one theorem.

Theorem 11.1.1 ([77, 195, 212, 243, 244, 245, 345, 346, 369, 403]). *Let X be a finite set, $X = \{x_1, \ldots, x_n\}$, K a field, $\operatorname{char} K \neq 2$, $F = F(X)$ the free algebra without the unity element on the set X of free generators of one of the following varieties of algebras over a field K: the variety of all algebras, the variety of Lie algebras, varieties of color Lie superalgebras, the variety of Lie p-algebras, varieties of color Lie p-superalgebras, or varieties of commutative and anticommutative algebras.*

1. *Any finite subset of F can be transformed into a reduced subset by a finite sequence of elementary transformations (with cancellation of possible zeros).*

2. *Any reduced subset of the algebra F is an independent subset.*
3. *The leading part of a polynomial on a reduced subset is a polynomial on leading parts of elements of this subset.*
4. *Any subalgebra of F is free.*
5. *A subset M of F is independent if and only if it is linearly independent modulo the square of the subalgebra of F generated by M (for free color Lie p-superalgebras, we add to the square the ideal all p-powers of elements of M).*
6. *If $|X| < \infty$, then the automorphism group of F is generated by elementary automorphisms.*

Problem 11.1.2 (P. M. Cohn [77]). Is it true that the automorphism group of a free associative algebra of finite rank is generated by elementary automorphisms?

Note that this problem has a positive solution for both polynomial algebra in two variables (H. W. E. Jung [164], W. van der Kulk [194]) and the free associative algebra of rank 2 (A. J. Czerniakiewicz [86], L. G. Makar-Limanov [233]).

Let $X = \{x_1, \ldots, x_n\}$. It follows from Theorem 11.1.1 that free algebras $F(X)$ are Hopfian; that is, if the algebra $F(X)$ is generated by elements $u_1, \ldots, u_n$, then these elements form a free generating set of $F(X)$.

11.2 Schreier Techniques

O. Schreier [329] introduced very useful techniques to obtain free generators for a subgroup of a free group from a special set of coset representatives. Later, this technique was extended by many authors for different types of free algebras.

Subalgebras of the free associative algebra are not necessarily free. For example, if we consider elements x^2 and x^3 in the free associative algebra $K\langle x\rangle$, then $x^2x^3 = x^3x^2$. Hence the subalgebra generated by x^2 and x^3 is not free. At the same time, we have P. M. Cohn's result that over a free associative algebra submodules of free left modules are free. P. M. Cohn obtained this result using a generalization of the Euclidean algorithm in the free associative algebra.

Let K be a field. The family $\{a_i \mid i \in I\}$ of elements of the free associative algebra $K\langle X\rangle$ is called left ℓ-dependent if there exist elements $b_i \in K\langle X\rangle$, almost all zero, such that

$$\ell\left(\sum_i b_i a_i\right) < \max\{\ell(a_i) + \ell(b_i)\}$$

or if some a_i is equal to zero. Otherwise the family $\{a_i\}$ is called left ℓ-independent.

An element $a \in K\langle X\rangle$ is said to be left ℓ-dependent on a family $\{a_i\}$ if either $a_i = 0$ or there exist $b_i \in K\langle X\rangle$, almost all zero, such that

$$\ell\bigg(a - \sum_i b_i a_i\bigg) < \ell(a), \quad \ell(a_i) + \ell(b_i) \leq \ell(a)$$

for all i. Otherwise a is said to be left ℓ-independent on a family $\{a_i\}$.

In the free associative algebra over a field K, the following analog of the Euclidean algorithm takes place:

Theorem 11.2.1 (weak algorithm [74, 75]). *Let $a_1, \dots, a_n \in K\langle X\rangle$ and $\ell(a_1) \leq \cdots \leq \ell(a_n)$. Then, if the family $\{a_1, \dots, a_n\}$ is left ℓ-dependent, there exists i, $1 \leq i \leq n$, such that the element a_i is left dependent on the family $\{a_1, \dots, a_{i-1}\}$.*

Note that the right version of this algorithm also takes place in $K\langle X\rangle$.

Since the weak algorithm takes place in $K\langle X\rangle$, the algebra $K\langle X\rangle$ is a free ideal ring (a fir) — i.e., $K\langle X\rangle$ is left fir and right fir; that is, left (right) ideals of $K\langle X\rangle$ are free left (right, respectively) $K\langle X\rangle$-modules of unique rank. Moreover, in a left fir every submodule of a free left module is free [76, 80].

Note that a very elegant proof of this fact is given by P. M. Cohn's theorem that the free product of firs is again fir. In [213] J. Lewin, with the use of Schreier's technique, gave a direct proof of the fact that $K\langle X\rangle$ is a fir.

Using the weak algorithm, P. M. Cohn [77] obtained the following sufficient condition for a subalgebra of a free algebra to be free. Let B be a subalgebra of $K\langle X\rangle$. We consider $K\langle X\rangle$ as a left B-module relative to usual multiplication. A system of elements $S = \{s_i \mid i \in I\}$ of $K\langle X\rangle$ is said to be B-independent if for almost all zero elements a_i

$$\ell\bigg(\sum_i a_i s_i\bigg) = \max\{\ell(v_i) + \ell(s_i)\}.$$

Theorem 11.2.2 ([77]). *If B is any subalgebra of $K\langle X\rangle$ such that $K\langle X\rangle$ is free as a left B-module, with a B-independent basis, then B is a free associative algebra over K. In particular, if B is a subalgebra of $K\langle X\rangle$ such that $K\langle X\rangle$ satisfies the weak algorithm as a B-module, then B is a free subalgebra of $K\langle X\rangle$.*

For more information on free ideal rings, we refer to P. M. Cohn's monograph [80].

Suppose that F is a free group of rank n, and let $H \subset F$ be a subgroup of a finite index. Then H is also the free group; moreover, the rank m of H is determined by Schreier's well-known formula [329]: $m-1 = (n-1)\cdot|F : H|$.

J. Lewin [213] obtained the following analog of Schreier's formula for free associative algebras. For a left ideal M of $K\langle X\rangle$, we denote by $\operatorname{rank}(M)$ the rank of the free left $K\langle X\rangle$-module M.

Theorem 11.2.3 ([213]). *Let $|X| = n$ and let I be a left ideal of $K\langle X\rangle$ such that $\dim_K(K\langle X\rangle/I) = k < \infty$. Then I is a free left $K\langle X\rangle$-module, $\operatorname{rank}(I) = k(n-1)+1$.*

For free Lie p-algebras, G. P. Kukin [192] obtained the following analog of Schreier's formula.

Theorem 11.2.4 ([192]). *Let K be a field, $p = \operatorname{char} K > 2$, $|X| = N < \infty$. Suppose that H is a subalgebra of the free Lie p-algebra $L^p(X)$, $\dim L^p(X)/H = t < \infty$. Then $\operatorname{rank}(H) = p^t(N-1)+1$.*

A. A. Mikhalev [243, 244, 245] obtained Schreier-type formulas for free Lie superalgebras and p-superalgebras.

Theorem 11.2.5 ([243]). *Let K be a field, $\operatorname{char} K \neq 2,3$, $L = L(X) = L_+ \oplus L_-$ the free color Lie superalgebra, $\operatorname{rank}(L) = |X| = N < \infty$, and let H be a G-homogeneous subalgebra of L, $H = H_+ \oplus H_-$, $H_+ = L_+$, $\dim(L_-/H_-) = s < \infty$. Then $\operatorname{rank}(H) = 2^s(N-1)+1$.*

Theorem 11.2.6 ([245]). *Let K be a field, $p = \operatorname{char} K > 3$, $|X| = N < \infty$. Suppose that $H = H_+ \oplus H_-$ is a G-homogeneous subalgebra of the free color Lie p-superalgebra $L^p(X)$, $\dim(L^p_+/H_+) = t < \infty$, and $\dim(L^p_-/H_-) = s < \infty$. Then $\operatorname{rank}(H) = 2^s p^t(N-1)+1$.*

A. A. Mikhalev showed that any subalgebra of finite rank of $L(X)$ is completely defined by its (nontrivial) even component.

Theorem 11.2.7 ([33, 275]). *Let K be a field, $\operatorname{char} K \neq 2,3$, $|X_-| \geq 1$, and let A, B be G-homogeneous subalgebras of finite rank in the free color Lie superalgebra $L(X)$, $A = A_+ \oplus A_-$, $B = B_+ \oplus B_-$. If $A_- = B_- \neq \{0\}$, then $A = B$.*

In the study of subalgebras of free algebras, the elimination of variables plays a very important role.

Let $x \in X$, $v \in X_+$, and $z \in X_-$, where $X_+ = \bigcup_{g\in G_+} X_g$, $X_- = \bigcup_{g\in G_-} X_g$. We denote

$$\begin{aligned} Z(x) &= \{y, y(\operatorname{Ad} x)^n \mid y \in X\setminus\{x\},\ n\in\mathbb{N}\} \subset L(X);\\ ZP(v) &= \{y, v^p, y(\operatorname{Ad} v)^n \mid y \in X\setminus\{x\},\ n = 1,\dots,p-1\} \subset L^p(X);\\ W(z) &= \{y, [y,z], [z,z] \mid y\in X\setminus\{x\}\} \subset L(X). \end{aligned}$$

For free Lie algebras, the elimination theorem was obtained by M. Lazard [202].

Theorem 11.2.8 ([202]). *Let $L(X)$ be the free Lie algebra, $x \in X$. Then $Z(x)$ is an independent subset of $L(X)$ and*

$$L(X) = Kx \oplus L(Z(x)).$$

W. Magnus [230] obtained elimination theorems for free associative rings. Elimination of variables in free color Lie superalgebras and p-superalgebras was considered by A. A. Mikhalev.

Theorem 11.2.9 ([243, 244, 245, 33, 275]).

1. *Let $L(X)$ be the free color Lie superalgebra, $x \in X$, $z \in X_-$. Then $Z = Z(x)$ and $W = W(z)$ are independent subsets of $L(X)$. The ideal of $L(X)$ generated by $\{y \in X \mid y \neq x\}$ is equal to $L(Z(x))$. Moreover, $L(X) = L(x) \oplus L(Z(x))$ is a semidirect product of the free algebras $L(x)$ and $L(Z(x))$ with the adjoint action of $L(x)$ on $L(Z(x))$. We also have $L(X) = Kz \oplus L(W(z))$ as vector spaces.*

2. *Let $L^p(X)$ be the free color Lie superalgebra, $x \in X$, $v \in X_+$, $z \in X_-$. Then $Z(x)$, $ZP(v)$, and $W(z)$ are independent subsets of $L^p(X)$. Moreover, $L^p(X) = L^p(x) \oplus L^p(Z(x))$ is a semidirect product with the adjoint action and $L^p(X) = Kv \oplus L^p(ZP(v))$, $L^p(X) = Kz \oplus L^p(W(z))$ as vector spaces.*

The elimination theorem for free Lie superalgebras was obtained also by J. Désarménien, G. Duchamp, D. Krob, and G. Melançon [91]. Recently N. Dobrynin [98, 99] obtained some generalizations of the elimination theorem for free Lie superalgebras. For free partially commutative Lie algebras elimination theorems were obtained by G. Duchamp and D. Krob [114], and for free partially commutative superalgebras — by N. Dobrynin [98, 99].

Note that a straightforward analog of Schreier's formula for the free Lie algebras does not exist. It is well-known that the rank of a subalgebra of finite codimension of a free Lie algebra is infinite; see, for example, [32]. But recently V. M. Petrogradsky found a form of Schreier's type formula for free Lie algebras and superalgebras in terms of formal power series [303, 305, 99]. He also established versions of the Schreier formula for these algebras using exponential generating functions (complexity functions) [302]. Recently he obtained Schreier formulas for free nonassociative algebras and free commutative and anticommutative nonassociative algebras [306]. We illustrate here his idea for free Lie algebras.

Let X be at most a countable set with a weight function $\operatorname{wt}\colon X \to \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) such that

$$X = \bigcup_{i=1}^{\infty} X_i, \quad X_i = \{x \in X \mid \operatorname{wt} x = i\}; \quad |X_i| < \infty, \quad i \in \mathbb{N}.$$

We call such a set finitely graded. For any monomial $y = x_{i_1} \dots x_{i_n}$, $x_{i_j} \in X$, we set $\operatorname{wt} y = \operatorname{wt} x_{i_1} + \dots + \operatorname{wt} x_{i_n}$. Suppose that Y is a set of monomials in X. Then we denote $Y_i = \{y \in Y \mid \operatorname{wt} y = i\}$, $i \in \mathbb{N}$. Also, we define the Hilbert–Poincaré series of Y as $\mathcal{H}_X(Y,t) = \sum_{i=1}^{\infty} |Y_i| t^i$.

Let $A = A(X)$ be an algebra generated by a finitely graded set X, and suppose that A inherits the gradation from X. Let $V \subset A$ be a homogeneous subspace; that is, $V = \bigoplus_{i=1}^{\infty} V_i$ with $V_i = \{v \in V \mid \operatorname{wt} v = i\}$. We define the Hilbert–Poincaré series of V as $\mathcal{H}_X(V,t) = \sum_{i=1}^{\infty} \dim(V_i) t^i$. We sometimes omit t, X and write $\mathcal{H}(Y)$, $\mathcal{H}(V)$.

Now let $Y \subset A$ be a not necessarily homogeneous subset. For $v \in A$, consider its decomposition into homogeneous components $v = v_1 + \cdots + v_n$, $\operatorname{wt}(v_i) = i$, $i = 1, \ldots, n$, $v_n \neq 0$. In this case, we write $\deg(v) = n$, $\operatorname{gr}(v) = v_n$. We define

$$\mathcal{H}_X(Y,t) = \sum_{n=1}^{\infty} |Y_n| t^n, \quad \text{where} \quad Y_n = \{y \in Y \mid \deg(y) = n\}.$$

Given a subspace $V \subset A$, one has a filtration $0 = V^0 \subset V^1 \subset \ldots$ with $V^i = \{v \in V \mid \deg(v) \leq i\}$, $i = 0, 1, 2, \ldots$. Let $\operatorname{gr}(V) = \bigoplus_{n=1}^{\infty} \operatorname{gr}_n(V)$, where

$$\operatorname{gr}_n(V) = \{\operatorname{gr}(v) \mid v \in V,\ \deg(v) = n\} \cup 0 \equiv V^n / V^{n-1}$$

denote the associated graded space. In the nonhomogeneous case, the Hilbert–Poincaré series is defined as follows:

$$\begin{aligned} \mathcal{H}_X(V,t) = \mathcal{H}(\operatorname{gr} V, t) \qquad &= \sum_{n=1}^{\infty} \dim_K(\operatorname{gr}_n(V)) t^n; \\ \mathcal{H}_X(A/V,t) = \mathcal{H}(A/\operatorname{gr} V, t) &= \sum_{n=1}^{\infty} \dim_K(A_n \operatorname{gr}_n(V)) t^n. \end{aligned}$$

We consider all series as elements of the ring of the formal power series

$$\mathbb{Z}[[t]] = \left\{ \sum_{n=0}^{\infty} b_n t^n \mid b_n \in \mathbb{Z} \right\}.$$

By $\mathbb{Z}_0[[t]]$ we denote a set of series with $b_0 = 0$. We consider the operator $\mathcal{E}\colon \mathbb{Z}_0[[t]] \to \mathbb{Z}[[t]]$

$$\mathcal{E}\colon \varphi(t) = \sum_{n=1}^{\infty} b_n t^n \to \mathcal{E}(\varphi(t)) = \sum_{n=0}^{\infty} a_n t^n = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^{b_n}}.$$

Let L be a Lie algebra generated by X, let $U(L)$ be its universal enveloping algebra, and suppose that

$$\mathcal{H}_X(L,t) = \sum_{n=1}^{\infty} b_n t^n, \quad \mathcal{H}_X(U(L),t) = \sum_{n=0}^{\infty} a_n t^n.$$

It is well-known that $\mathcal{H}_X(U(L)) = \mathcal{E}(\mathcal{H}_X(L))$ [384].

Theorem 11.2.10 ([303]). *Let L be a free Lie algebra freely generated by a finitely graded set X. Suppose that H is a subalgebra and Y is the set of free generators for H. Then*

$$\mathcal{H}(Y) - 1 = (\mathcal{H}(X) - 1) \cdot \mathcal{E}(\mathcal{H}(L/H)).$$

For more information on Schreier techniques, we refer to the following articles and monographs: [32, 33, 34, 57, 91, 99, 189, 190, 192, 195, 198, 202, 212, 213, 227, 230, 231, 243, 244, 245, 248, 252, 275, 302, 303, 304, 305, 306, 313, 322, 329, 345, 346, 369, 388, 392, 403, 414].

11.3 Free Differential Calculus

R. H. Fox [134] introduced free differential calculus in free group rings; see Section 1.4. Let $F = F(X)$ be the free group on a set X, $\mathbb{Z}F$ the integral group ring of F, and Δ_F the augmentation ideal of $\mathbb{Z}F$ (i.e., the kernel of the natural homomorphism

$$\sigma\colon \mathbb{Z}F \to \mathbb{Z}, \quad \sigma\biggl(\sum_i (n_i f_i)\biggr) = \sum_i n_i,$$

where $n_i \in \mathbb{Z}$, $f_i \in F$). The Fox partial derivation with respect to x_i is a mapping $d_i\colon \mathbb{Z}F \to \mathbb{Z}F$ that satisfies the following conditions:

$$\begin{aligned} d_i(x_j) &= \delta_{ij}, \\ d_i(uv) &= ud_i(v) + \sigma(v)\, d_i(u), \\ d_i(ku + lv) &= kd_i(u) + ld_i(v), \end{aligned}$$

$u, v \in \mathbb{Z}F$, $k, l \in \mathbb{Z}$.

There is another interpretation of d_i as follows. The ideal Δ_F is a free left $\mathbb{Z}F$-module with a free basis $\{(x_i - 1) \mid 1 \le i \le n\}$, and the mappings d_i are projections to the corresponding free cyclic direct summands. Every element $u \in \Delta_F$ can be uniquely written in the form

$$u = \sum_{i=1}^{n} d_i(u)(x_i - 1).$$

11.3.1 Free differential calculus and Schreier varieties

As before, by $A = F_{\mathfrak{M}}\langle X\rangle$ we denote the free algebra of the variety $\mathfrak{M}$ with the set $X = \{x_1, \ldots, x_n\}$ of free generators.

Let I_A be a free right $U(A)$-module with a basis $y_1, \ldots, y_n$,

$$I_A = y_1 U(A) \oplus \cdots \oplus y_n U(A).$$

The linear mapping $\mathcal{D}\colon A \to I_A$ given by $\mathcal{D}(x_i) = y_i$, $1 \le i \le n$,

$$\mathcal{D}(ab) = \mathcal{D}(a) \cdot r_b + \mathcal{D}(b) \cdot l_a,$$

$a, b \in A$, is called the universal derivation of the algebra A in the variety $\mathfrak{M}$. The partial derivatives $\frac{\partial f}{\partial x_i}$ of an element f of A are uniquely determined by

$$\mathcal{D}(f) = \sum_{i=1}^{n} y_i \frac{\partial f}{\partial x_i}.$$

We set

$$\partial(f) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)^{\mathrm{T}},$$

where T is the operator of transposition. From the definition, we get $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$, the Kronecker symbol,

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} r_v + \frac{\partial v}{\partial x_i} l_u, \quad u, v \in A. \tag{11.1}$$

For a subalgebra H of the algebra A by J_H, we denote the $U(A)$-submodule of the module I_A generated by the elements $D(h)$, $h \in H$. A variety $\mathfrak{M}$ has the property of differential separability for subalgebras if for any subalgebra H of A and $a \in A$, $D(a) \in J_H$ if and only if $a \in H$.

Theorem 11.3.1. *The following varieties have the property of differential separability for subalgebras:*

1. *the variety of all algebras;*
2. *the variety of all commutative algebras* (char $K \neq 2$)*;*
3. *the variety of all anticommutative algebras;*
4. *the varieties of Lie* (*super*)*algebras and p-*(*super*)*algebras;*
5. *the variety of algebras with zero multiplication.*

The first time the property of differential separability for subalgebras for Lie algebras was mentioned was in [385] (for Lie superalgebras, see [249]). In general form, this theorem was proved by U. U. Umirbaev in [388].

For Lie algebras and superalgebras, the property of differential separability for subalgebras follows from the Poincaré–Birkhoff–Witt theorems.

Let L be a color Lie superalgebra, U a G-graded associative K-algebra with 1, and $\delta\colon L \to [U]$ a homomorphism of color Lie superalgebras. We say that the algebra U (with δ) is the universal enveloping algebra of L (we denote it $U(L)$) if for any homomorphism σ of the color Lie superalgebra L into a color Lie superalgebra $[R]$ (with the same ε and G) for some G-graded associative algebra R with 1 there exists a unique homomorphism $\theta\colon U \to R$ of G-graded associative algebras with 1 such that $\sigma = \theta\delta$. Let K be a field, $p = \operatorname{char} K > 2$, L a color Lie p-superalgebra, and U a G-graded associative K-algebra with 1, $\delta\colon L \to [U]^p$ a homomorphism of color Lie p-superalgebras. We say that U (with δ) is the restricted universal enveloping algebra of L (notation $u(L)$) if for any homomorphism $\sigma\colon L \to [R]^p$ of color Lie p-superalgebras, where R is a G-graded associative K-algebra with 1 (with the same ε and G), there exists a unique homomorphism $\theta\colon U \to R$ of G-graded associative K-algebras with 1 such that $\sigma = \theta\delta$. Suppose now that L is a color Lie superalgebra with a G-homogeneous basis $X = X_+ \cup X_-$, $K\langle X\rangle$ is the free G-graded associative algebra, and I is the two-sided ideal in $K\langle X\rangle$ generated by the elements of the form $ab - \varepsilon(d(a), d(b))\, ba - [a, b]$ for all homogeneous $a, b \in L$. Consider the canonical mapping $\delta\colon L \to K\langle X\rangle/I = U(L)$. It is clear that $U(L)$ is the universal enveloping algebra of L. Let $\leq$ be a total ordering of X.

Theorem 11.3.2 (PBW type). *The universal enveloping algebra $U(L)$ constructed above has a G-homogeneous basis consisting of 1 and all monomials $\delta(x_1)\dots\delta(x_n)$ where $n \in \mathbb{N}$, $x_i \in X$, $x_i \leq x_{i+1}$, $x_i \neq x_{i+1}$ with $x_i \in X_-$ for all $i = 1, \dots, n-1$ (in particular, δ is an embedding).*

Let K be a field, $\operatorname{char} K = p > 2$, L a color Lie p-superalgebra with a G-homogeneous linear basis X, J the two-sided ideal of $K\langle X\rangle$ generated by elements $ab-\varepsilon(d(a)a, d(b))ba-[a, b]$ (where a, b are homogeneous, $a, b \in L$), and $a^p - a^{[p]}$ ($a \in L_g$, $g \in G_+$). Let $\delta\colon L \to K\langle X\rangle/J$ be the canonical mapping. It is clear that $K\langle X\rangle/J = u(L)$ is the restricted universal enveloping algebra for L.

Theorem 11.3.3 (PBW type). *The monomials $\delta(x_1)^{\lambda_1}\dots\delta(x_n)^{\lambda^n}$, where $x_i \in X$, $x_1 < \dots < x_n$, $0 \leq \lambda_i \leq p-1$ for $x_i \in X_+$, $\lambda_i = 0, 1$ for $x_i \in X_-$ give us a linear basis of $u(L)$.*

It is clear that the universal enveloping algebra of the free color Lie superalgebra $L(X)$ is the free associative algebra $K\langle X\rangle$. This algebra is also the universal multiplicative enveloping algebra of $L(X)$. For the restricted universal enveloping algebra of the free color Lie p-superalgebra $L^p(X)$, we have $u(L^p(X)) = K\langle X\rangle$. This algebra is also the universal multiplicative enveloping algebra of $L^p(X)$. For more results concerning PBW theorems for Lie superalgebras, we refer to the monographs [33, 275, 327].

Theorem 11.3.4. *Let K be a field, L a color Lie superalgebra, and B a subalgebra of L. Then B is differentially separable; i.e., if J is the left ideal of $U(L)$ generated by B, then $L \cap J = B$.*

In the case where $\operatorname{char} K = p > 2$, let M be a color Lie p-superalgebra and D a subalgebra of M. Then D is differentially separable (relative to $u(M)$).

Proof. Let $E = \{e_1, e_2, \dots\}$ be a G-homogeneous ordered K-linear basis of the subalgebra B. We extend the subset E up to a G-homogeneous ordered K-linear basis $E \cup H$ of L such that $H = \{h_1, h_2, \dots\}$ and $h_i < e_j$ for all i, j. By Theorem 11.3.2, the set of all monomials

$$h_{i_1}\dots h_{i_m} \cdot e_{i_1}\dots e_{i_n}$$

where

$$h_{i_1} \leq \cdots \leq h_{i_m}, \quad e_{i_1} \leq \cdots \leq e_{i_n},$$

$i_r \neq i_{r+1}$ with $d(h_{i_r}) \in G_-$, $j_s \neq j_{s+1}$ with $d(e_{j_s}) \in G_-$, gives us a K-linear basis of $K\langle X\rangle$. Since B is a subalgebra of L, these monomials with $n \geq 1$ form a K-linear basis of J. Therefore $L \cap J = B$.

The same arguments work for the color Lie p-superalgebra M (we need only use Theorem 11.3.3). ■

Using the property of differential separability of subalgebras, it is possible to prove that free Lie algebras and superalgebras are residually finite.

Theorem 11.3.5 ([249, 385]). *Let $L(X)$ be a free color Lie superalgebra over a field K (in the case* $\operatorname{char} K = p > 2$*, we consider also a free color Lie p-superalgebra $L^p(X)$). Then the algebra $L = L(X)$ ($L = L^p(X)$, respectively) is residually finite; i.e., if a is a G-homogeneous element of L, B is a G-homogeneous finitely generated subalgebra of L, $a \notin B$, then there exists a finite-dimensional color Lie (p-)superalgebra H and an epimorphism $\varphi\colon L \to H$ such that $\varphi(a) \notin \varphi(B)$.*

Proof. It is sufficient to prove the statement in the case where $|X| < \infty$. At the beginning, we consider the free color Lie superalgebra $L = L(X)$. The free G-graded associative algebra $K\langle X\rangle$ is the universal enveloping algebra of L. Consider the left ideal J of $K\langle X\rangle$ generated by the subalgebra B. Since B is a finitely generated subalgebra, J is a finitely generated left ideal of $K\langle X\rangle$. Since $a \in L$ and $a \notin B$, by Theorem 11.3.4 $a \notin J$. Since the ideal J is finitely generated, there exists a left G-graded ideal J' of finite codimension in $A(X)$ such that $a \notin J'$.

Consider the finite-dimensional left $U(L)$-module $M = U(L)/J'$, where $U(L) = K\langle X\rangle$. Let $\operatorname{Ann}(M)$ be the annihilator of the module M and $R = U(L)/\operatorname{Ann}(M)$. Then R is a finite-dimensional G-graded associative algebra, M is a faithful R-module, and $\operatorname{Ann}(M) \subseteq J'$ since $1 \in U(L)$. Let $\xi\colon U(L) \to R$ be the canonical epimorphism. It is obvious that $H = [\xi(L)]$ is a finite-dimensional color Lie superalgebra. Assuming now that $\xi(a) \in \xi(B)$, one obtains for $\{u_1, \ldots, u_r\}$ a set of generators of the subalgebra B such that $\xi(a)$ belongs to the left ideal of R generated by $\xi(u_1), \ldots, \xi(u_r)$. Therefore there exist elements $v_1, \ldots, v_r \in U(L)$ such that

$$\xi(a) = \sum_{i=1}^{r} \xi(v_i)\,\xi(u_i),$$

in other words

$$a - \sum_{i=1}^{r} \xi(v_i)\,\xi(u_i) \in \operatorname{Ann}(M) \subseteq J',$$

and thus $a \in J'$. This contradiction completes the proof of our theorem in the case $L = L(X)$.

In the case $\operatorname{char} K = p > 2$ and $L = L^p(X)$, the proof is the same using the fact that the free G-graded associative algebra $K\langle X\rangle$ is the restricted universal enveloping algebra of L. ■

In group theory, a well-known theorem by M. Hall shows that free groups are residually finite. G. V. Kryazhovskih [188] showed that free nonassociative algebras and free commutative and anticommutative nonassociative algebras are residually finite. This result for free Lie algebras over fields of positive characteristic and for free restricted Lie algebras was obtained by G. P. Kukin [193]. A similar statement does not hold for free rings. In the review article [126] is the following example, which is due to K. Mandelberg. If F is a free ring on one generator x, then in any homomorphism of F onto a finite ring, the subring B generated by $2x$ and $2x^2 + x$ maps onto the image of F. At the same time $x \notin B$.

The following result by U. U. Umirbaev gives a necessary condition for a variety of algebras to be Schreier.

Theorem 11.3.6 ([388]). *If $\mathfrak{M}$ is a homogeneous Schreier variety of algebras, then*

1. *for any free algebra A of $\mathfrak{M}$ the universal multiplicative enveloping algebra $U(A)$ is a free associative algebra;*
2. *for any homogeneous subalgebra H of a free algebra A of the variety $\mathfrak{M}$ the algebra $U(A)$ is a free right $U(H)$-module.*

If $\operatorname{char} K = 0$, *then a variety satisfying conditions* 1 *and* 2 *is Schreier.*

Point 1 of Theorem 11.3.6 for free nonassociative algebras and for free commutative and anticommutative nonassociative algebras was proved by A. T. Gainov [140].

A method to test that a variety of algebras is Schreier is given by the following theorem.

Theorem 11.3.7 ([388]). *Let $\mathfrak{M}$ be a homogeneous variety of algebras such that:*

1. *For any free algebra A of $\mathfrak{M}$, the universal multiplicative enveloping algebra $U(A)$ is a free associative algebra.*
2. *The variety $\mathfrak{M}$ has the property of differential separability for subalgebras.*

Then $\mathfrak{M}$ is a Schreier variety.

It follows that:

Theorem 11.3.8 ([388]). *The variety of algebras defined by the identity $x \cdot x^2 = 0$ is Schreier.*

Problem 11.3.9. To give the complete classification of Schreier varieties of algebras.

An algebra A over a field K with an anticommutative bilinear operation $[x, y]$ and a trilinear operation $\mathcal{A}(x, y, z)$ is called an Akivis algebra if

$$\begin{aligned}[[x, y], z] + [[y, z], x] + [[z, x], y] &= \mathcal{A}(x, y, z) + \mathcal{A}(y, z, x) + \mathcal{A}(z, x, y) \\ &\quad - \mathcal{A}(y, x, z) - \mathcal{A}(x, z, y) - \mathcal{A}(z, y, x).\end{aligned}$$

These algebras were introduced by M. A. Akivis as tangent algebras of local analytic groups. I. P. Shestakov and U. U. Umirbaev [338] proved that the variety of Akivis algebras is Schreier, automorphisms of finitely generated free Akivis algebras are tame, the occurrence problem for free Akivis algebras is decidable, finitely generated subalgebras of free Akivis algebras are residually finite, and the word problem is decidable for the variety of Akivis algebras.

For any element a of the free associative algebra $K\langle X\rangle$, we have the unique presentation in the form $a = \alpha \cdot 1 + x_1 a_1 + x_2 a_2 + \dots$, where only a finite number of elements $a_i \in K\langle X\rangle$ are nonzero, $\alpha \in K$. We call the element a_i the right Fox partial derivative of the element a by x_i, and we use the notation $a_i = \frac{\partial a}{\partial x_i} = \frac{\partial}{\partial x_i} a$. Thus we have the operators $\frac{\partial}{\partial x_i}$. These operators are exactly linear mappings $\frac{\partial}{\partial x_i} \colon K\langle X\rangle \to K\langle X\rangle$ such that $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$ and

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial}{\partial x_i}(u)v + \sigma(u)\frac{\partial}{\partial x_i}(v),$$

where $\sigma \colon K\langle X\rangle \to K\langle X\rangle$ is the homomorphism defined by $\sigma(x_i) = 0$ for all $x_i \in X$. For one of the applications of these derivatives, see Section 9.2.1.

For all $a \in L(X)$ and $b \in K\langle X\rangle$, we set $l_a \cdot b = -\varepsilon(a, b)ba$ and $a \cdot r_b = a \cdot b$. Then $K\langle X\rangle$ is an $L(X)$-bimodule. It is clear that the universal multiplicative enveloping algebra of $L(X)$ is exactly the free G-graded associative algebra $K\langle X\rangle$ with the actions l_a and r_a defined above (at the same time, $K\langle X\rangle$ is the universal enveloping algebra of the color Lie superalgebra $L(X)$). The derivation

$$D \colon L(X) \to I_A = \bigoplus_{i=1}^{n} dx_i \cdot K\langle X\rangle$$

given by

$$D(a) = \sum_{i=1}^{n} dx_i \cdot \frac{\partial a}{\partial x_i}$$

for all $a \in L(X)$ (only a finite number of summands are nonzero) is the universal derivation of $L(X)$.

For more information on Schreier varieties of algebras, we refer to the articles [388, 392] (for the varieties of linear Ω-algebras, see [26, 38]).

Free differential calculus can be applied to a study of free abelian extensions in a variety $\mathfrak{V}$. Let A be an algebra in $\mathfrak{V}$ with a fixed set of generators X. We say that an algebra B in $\mathfrak{V}$ is a free abelian extension of A in $\mathfrak{V}$ if

1. B is generated by the same set X.
2. There exists a surjective homomorphism $\gamma\colon B \to A$ that is identical on X.
3. If C is an algebra in $\mathfrak{V}$ generated by the set X having a surjective homomorphism $\eta\colon C \to A$ that is identical on X, then there exists a homomorphism $\xi\colon B \to C$ such that $\eta\xi = \gamma$.

Take a free $\mathfrak{V}$-algebra A with the set X of free generators. Since A is generated by the set X, there exists a surjective homomorphism of algebras $\rho\colon F \to A$ that is identical on X. Denote by I the kernel $\ker\rho$. It is not difficult to show that the free abelian extension B is precisely F/I^2, where I^2 is the ideal in F generated by all products ab, $a, b \in I$. Thus, free abelian extensions are analogs of wreath products in group theory. An extension of a study of free abelian extensions to any congruence-modular variety in universal algebra and a development of generalized free differential calculus is exposed in [23].

11.3.2 Ranks of subalgebras

Using free differential calculus (universal derivations), A. A. Mikhalev, V. Shpilrain, and A. A. Zolotykh [257] obtained matrix algorithms to find ranks of subalgebras of free Lie algebras. A. A. Mikhalev and A. A. Zolotykh [279] constructed such algorithms for subalgebras of free Lie superalgebras and free Lie p-superalgebras. Recently, K. Champagnier [67] obtained similar results for ranks of subalgebras of free nonassociative algebras and free (anti)commutative nonassociative algebras.

Let K be a field, $\operatorname{char} K \neq 2$, and $L(X)$ the free color Lie superalgebra (if $\operatorname{char} K = p > 2$, also let $L^p(X)$ be the free color Lie p-superalgebra). For the algebra $A = L(X)$ $(A = L^p(X))$, the universal multiplicative enveloping algebra $U(A)$ of A is the free associative algebra $K\langle X\rangle$. Let $H = \{h_1, \dots, h_m\}$ be a subset of G-homogeneous elements of A. Consider the matrix $J(H) = (\frac{\partial h_j}{\partial x_i})$, $1 \leq i \leq n$, $1 \leq j \leq m$. Here partial derivatives $\frac{\partial h_j}{\partial x_i}$ are components of the (right) universal derivation. Consider the columns $\partial(h_j)$, $1 \leq j \leq m$, of the matrix $J(H) = (\partial(h_1), \dots, \partial(h_m))$.

Theorem 11.3.10 ([279]). *A subset $H = \{h_1, \dots, h_m\}$ of A is independent if and only if the columns of the matrix $J(H)$ are right independent over $U(A)$.*

Theorem 11.3.11 ([279]). *Let $H = \{h_1, \dots, h_m\} \subset A$. Then the rank of the subalgebra of A generated by H is equal to the rank of the free right $U(A)$-submodule of $U(A)^n$ generated by the columns of the matrix $J(H)$.*

Theorem 11.3.10 gives an effective algorithm to decide whether a given subset of A is independent or not. Note that U. U. Umirbaev [386] showed that this problem for free associative algebras is algorithmically undecidable. Theorem 11.3.11 gives an effective algorithm to find the rank of a finitely generated subalgebra of A. These algorithms are based on the right weak algorithm. Another algorithm could be derived from Theorem 11.1.1. Indeed, using combinatorics on words, by elementary transformation we could transform the set H to a reduced set. The cardinality of this set is the rank of the subalgebra of A generated by H.

11.4 Stable Equivalence

An element u of a free algebra F is said to be primitive if it is an element of some set of free generators of the algebra F. A set of nonzero pairwise distinct elements of F is said to be a primitive system of elements if it is a subset of some set of free generators of F. The rank of $u \in F$ is the smallest number of generators from X on which an element $\varphi(u)$ depends, where φ runs through the automorphism group of F (in other words, $\operatorname{rank}(u)$ is the smallest rank of a free factor of F containing u). In the same way, the rank of a system of elements is defined.

In this section, we are interested in the following problems.

Problem 11.4.1. Let A be a free subalgebra of a free algebra F. Given two elements $u, v \in A$ such that there exists an automorphism φ of F with $\varphi(u) = v$, does there exist an automorphism ψ of A such that $\psi(u) = v$?

Problem 11.4.2. Let A be a free subalgebra of a free algebra F, u a primitive element of F, and $u \in A$. Does it imply that u is actually a primitive element of the free algebra A?

For free groups F of finite rank, the affirmative answer to Problem 11.4.1 with A a free factor of F and to Problem 11.4.2 follows from [227, Propositions 1.4.17 and 1.5.4]. V. Shpilrain and J.-T. Yu [365] solved Problem 11.4.2 for polynomial algebras in two and three variables with A being a free factor of F; see Proposition 6.0.4.

Problem 11.4.2 has a positive solution for the main types of Schreier varieties of algebras.

Let K be a field, $\operatorname{char} K \neq 2$, $X = \{x_1, \ldots, x_n\}$, and $F = F(X)$ the free algebra without the unity element on the set X of free generators of one of the following varieties of algebras over a field K: the variety of all algebras, the variety of Lie algebras, varieties of color Lie superalgebras, the variety of Lie p-algebras, varieties of color Lie p-superalgebras, and varieties of commutative and anticommutative algebras.

Proposition 11.4.3 ([265]). *Let $A = \{a_1, \dots, a_l\}$ be a primitive system of elements of F and B a finitely generated subalgebra of F such that $A \subset B$. Then A is a primitive system in B.*

Proof. Let H be a finite generating set of B, and $H' = H \cup A$. Using a finite number of elementary transformations of H', we will come to a free generating set of B. We may suppose that $A \subseteq X$. We may use only the following elementary transformations: if $h_1, \dots, h_s \in H'$ and the leading part h_1° of the element h_1 belongs to the subalgebra generated by the leading parts $h_2^\circ, \dots, h_s^\circ$, $h_1^\circ = \theta(h_2^\circ, \dots h_s^\circ)$, then we set $h_1' = h_1 - \theta(h_2, \dots h_s)$. It is clear that using these elementary transformations (with possible cancellation of zeros) we may come to a free generating set Y of B such that $A \subseteq Y$. Thus A is a primitive system of elements of B. ■

Now we consider Problem 11.4.1. At first we show that in general form this problem has a negative solution. We illustrate this for free Lie algebras. Let $B = L(X)$ be the free Lie algebra, where $X = \{x, y, z\}$, A the subalgebra of B generated by the elements $a = x + [y, z]$, $b = y$, $c = [x, y]$. The set $\{a, b, c\}$ is reduced (relative to the ordinary degree function). By Theorem 11.1.1, this set is an independent subset of A. Let φ be the automorphism of B given by $\varphi(x) = a$, $\varphi(y) = y$, $\varphi(z) = z$. Let $u = c$ and $v = [a, b]$. Then $\varphi(u) = v$. At the same time $u, v \in A$, u is a primitive element of A. For any automorphism ψ of A, the element $\psi(u)$ is a primitive element of A. Thus $\psi(u) \neq v = \varphi(u) = [a, b] \in [A, A]$. We solve Problem 11.4.1 in the affirmative supposing that A is a free factor of the free algebra $F(X)$.

Theorem 11.4.4 ([267]). *Let X and Y be nonempty sets, $X \cap Y = \varnothing$, $u_i, v_i \in F(X)$, $u_i \neq 0$, $v_i \neq 0$, $i = 1, \dots, m$. Suppose that there is an automorphism φ of the algebra $F(X \cup Y)$ such that $\varphi(u_i) = v_i$, $i = 1, \dots, m$. Then there is an automorphism ψ of $F(X)$ with $\psi(u_i) = v_i$, $i = 1, \dots, m$. In other words, if two finite systems of elements of the free algebra $F(X)$ are stably equivalent, then they are equivalent.*

Proof. Let $X = \{x_1, \dots, x_n\}$, $U = \{u_1, \dots, u_m\}$, and $\operatorname{rank}_{F(X)}(U) = l$. We may suppose that elements of U depend only on $x_1, \dots, x_l \in X$. On the contrary, suppose that the system of elements $\Phi = \{\varphi(x_1), \dots, \varphi(x_l)\}$ depends on a free generator $y \in Y$. Let $d = \max_{i=1,\dots,m} \deg(v_i)$. Consider the generalized degree function μ on $X \cup Y$ given by $\mu(x) = 1$ with $x \in X$, $\mu(z) = 1$ with $z \in Y$, $z \neq y$, and $\mu(y) = d + 1$.

Using a sequence of elementary transformations, we transform the set Φ to a μ-reduced set $\Phi' = \{\varphi'(x_1), \dots, \varphi'(x_l)\}$. Since φ is an automorphism, Φ' is a primitive system of elements of $F(X \cup Y)$. It is clear that Φ' depends on y. We may assume that the element $\varphi'(x_1)$ depends on y. The sequence of elementary transformations $\Phi \to \Phi'$ induces the sequence of elementary automorphisms θ of $F(x_1, \dots, x_l)$, namely $\varphi'(x_i) = \varphi(\theta(x_i))$, $i = 1, \dots, l$. Since $\operatorname{rank}_{F(X)}(U) = l$ and $v_i = \varphi(u_i)$, $i = 1, \dots, m$, at least one element of $V = \{v_1, \ldots, v_m\}$ depends on $\varphi'(x_1)$. One may assume that v_1 depends on $\varphi'(x_1)$. Since $v_1 \in F(X)$, $\mu(v_1) = \deg(v_1)$. At the same time, Φ' is a μ-reduced set and $\mu(\varphi'(x_1)) \geq \mu(y) = d + 1$. By Theorem 11.1.1,

$$\deg(v_1) = \mu(v_1) \geq \mu(\varphi'(x_1)) \geq d + 1 > d = \max_{i=1,\ldots m}\{\deg(v_i)\},$$

and we have come to a contradiction. This contradiction shows that actually $\Phi \subset F(X)$.

Since φ is an automorphism, Φ is a primitive system of $F(X \cup Y)$. Since $\Phi \subset F(X)$, by Proposition 11.4.3 we get that Φ is a primitive system of $F(X)$. Let $\{\bar{x}_{l+1}, \ldots, \bar{x}_n\}$ be a complement of Φ with respect to a free generating set of $F(X)$. Now we consider the automorphism ψ of $F(X)$ given by $\psi(x_i) = \varphi(x_i)$, $i = 1, \ldots, l$, and $\psi(x_j) = \bar{x}_j$, $j = l + 1, \ldots, n$. It is clear that ψ is an automorphism of $F(X)$ and $\psi(u_i) = v_i$, $i = 1, \ldots, m$. ■

Theorem 11.4.5 ([267]). *Let X and Y be nonempty sets, $X \cap Y = \varnothing$, and U a finite set of nonzero elements of $F(X)$. Then*

$$\operatorname{rank}_{F(X)}(U) = \operatorname{rank}_{F(X\cup Y)}(U).$$

In other words, the rank of the system U is equal to its stable rank.

Proof. Using free differential calculus, matrix criteria for systems of elements of $F(X)$ to have a given rank are obtained in Section 12.4 of Chapter 12, Theorems 12.4.7–12.4.10 (free Lie algebras and superalgebras were considered over fields of characteristic zero). The statement of the theorem follows from these criteria. Here we give the proof without use of free differential calculus (and without restriction of the characteristic of the ground field for free Lie algebras and superalgebras).

Let $l = \operatorname{rank}_{F(X)}(U)$ and $k = \operatorname{rank}_{F(X\cup Y)}(U)$. It is clear that $k \leq l$. Suppose that $k < l$. Then there is an automorphism φ of $F(X \cup Y)$ such that the set $V = \varphi(U)$ depends only on free generators $x_1, \ldots, x_k \in X$. In particular, $V \subset F(X)$. By Theorem 11.4.4, there is an automorphism ψ of the algebra $F(X)$ such that $V = \psi(U)$. Hence $\operatorname{rank}_{F(X)}(V) = \operatorname{rank}_{F(X)}(U) = l > k$, and we have come to a contradiction (the set V depends only on $x_1, \ldots, x_k$). It completes the proof. ■

11.5 The Rank of an Endomorphism

In this section, we consider the following problem.

Problem 11.5.1. Let φ be an injective endomorphism of a free algebra F and $0 \neq u \in F$. Is it true that $\operatorname{rank}(\varphi(u)) \geq \operatorname{rank}(u)$?

In order to solve this problem, we define the rank of an endomorphism of a free algebra of Schreier variety of algebras and prove in Theorems

11.5.3 and 11.5.4 that an injective endomorphism of maximal rank does not change the rank of systems of elements with maximal rank.

Lemma 11.5.2. *Let U be a system of elements of a free algebra $F(X)$. Suppose that a system V is obtained from U by a finite sequence of elementary transformations. Then* $\operatorname{rank}(V) = \operatorname{rank}(U)$.

Proof. Suppose that $\operatorname{rank}(V) < \operatorname{rank}(U)$. Then there is an automorphism φ of $F(X)$ such that the system of elements $\varphi(V)$ depends only on $\operatorname{rank}(V)$ free generators from X. Since elementary transformations are invertible, the set U belongs to the subalgebra of $F(X)$ generated by V. Therefore the set $\varphi(U)$ belongs to the subalgebra of $F(X)$ generated by the set $\varphi(V)$, and

$$\operatorname{rank}(U) = \operatorname{rank}(\varphi(U)) \leq \operatorname{rank}(\varphi(V)) = \operatorname{rank}(V) < \operatorname{rank}(U).$$

This contradiction shows that $\operatorname{rank}(V) \geq \operatorname{rank}(U)$. Since there is a sequence of elementary transformations $V \to U$, we get $\operatorname{rank}(U) \geq \operatorname{rank}(V)$. Thus $\operatorname{rank}(V) = \operatorname{rank}(U)$. ■

Let φ be an endomorphism of $F(X)$, where $X = \{x_1, \ldots, x_n\}$. For the rank of φ ($\operatorname{rank}(\varphi)$), we take the rank of the system of elements $\varphi(X) = \{\varphi(x_1), \ldots, \varphi(x_n)\}$. This notion does not depend on the choice of free generators of the free algebra $F(X)$. Indeed, let $Y = \{y_1, \ldots, y_n\}$ be another set of free generators of $F(X)$. By Theorem 11.1.1, there is a sequence of elementary automorphisms $\psi\colon X \to Y$. This sequence induces the sequence of elementary transformations ψ' of the set $\varphi(X)$, $\psi'\colon Y = \varphi(X) \to \varphi(Y) = \{\varphi(y_1), \ldots, \varphi(y_n)\}$. By Lemma 11.5.2, $\operatorname{rank}(\varphi(Y)) = \operatorname{rank}(\varphi(X))$.

Let φ be an endomorphism of $F(X)$, $0 \neq u \in F(X)$. If u or φ is not of maximal rank, or φ is not injective, then in general there is no relation between $\operatorname{rank}(\varphi(u))$ and $\operatorname{rank}(u)$. We illustrate this for free Lie algebras.

Let $L(X)$ be the free Lie algebra, where $X = \{x, y\}$, φ the endomorphism of $L(X)$ given by $\varphi(x) = x$, $\varphi(y) = 0$, and $u = x + [x, y]$. Then $\varphi(u) = x$ and $\operatorname{rank}(\varphi(u)) = 1 < 2 = \operatorname{rank}(u) = |X|$. If ψ is the endomorphism of $L(X)$ given by $\psi(x) = u$, $\psi(y) = 0$, then $\psi(u) = u$, and $\operatorname{rank}(\psi(u)) = 2 = \operatorname{rank}(u)$. If $v = x$, then $\psi(v) = u$, and $\operatorname{rank}(\psi(v)) = 2 > \operatorname{rank}(v) = 1$.

If φ is an injective endomorphism of maximal rank, but $\operatorname{rank}(u) < n = |X|$, then we consider the following examples. Let $X = \{x, y, z, w\}$, and let φ be the endomorphism of the free Lie algebra $L(X)$ given by $\varphi(x) = x$, $\varphi(y) = [x, y]$, $\varphi(z) = [[x, y], y]$, $\varphi(w) = [z, w]$. Since the set $\{\varphi(x), \varphi(y), \varphi(z), \varphi(w)\}$ is reduced relative to the ordinary degree, φ is injective. Let $u_1 = [[x, y], z]$, $u_2 = [x, y]$, and $u_3 = [x, w]$. Then $\varphi(u_1) = [[x, [x, y]], [[x, y], y]]$, $\varphi(u_2) = [x, [x, y]]$, $\varphi(u_3) = [x, [z, w]]$, and we get

$$\begin{aligned}\operatorname{rank}(\varphi(u_1)) &= 2 < 3 = \operatorname{rank}(u_1),\\ \operatorname{rank}(\varphi(u_2)) &= 2 = 2 = \operatorname{rank}(u_2),\\ \operatorname{rank}(\varphi(u_3)) &= 3 > 2 = \operatorname{rank}(u_3).\end{aligned}$$

In the case where φ is an injective endomorphism of $F(X)$, $0 \neq u \in F(X)$, both u and φ have maximal rank and we are able to prove that $\operatorname{rank}(\varphi(u)) = \operatorname{rank}(u)$.

Theorem 11.5.3 ([267]). *Let* $X = \{x_1, \ldots, x_n\}$, $0 \neq u \in F(X)$, *and* $\operatorname{rank}(u) = n$. *Let also* φ *be an injective endomorphism of* $F(X)$, $\operatorname{rank}(\varphi) = n$. *Then* $\operatorname{rank}(\varphi(u)) = \operatorname{rank}(u) = n$.

Proof. Suppose that $\operatorname{rank}(\varphi(u)) < n$. Then there is an automorphism ψ of $F(X)$ such that the element $\psi(\varphi(u))$ does not depend on x_n. It is clear that the endomorphism $\varphi' = \psi \circ \varphi$ is injective and $\operatorname{rank}(\varphi') = \operatorname{rank}(\varphi) = n$. Consider the generalized degree function μ given by

$$\mu(x_1) = \cdots = \mu(x_{n-1}) = 1, \quad \mu(x_n) = \deg(\varphi'(u)) + 1.$$

We transform the set $\varphi'(X) = \{\varphi'(x_1), \ldots, \varphi'(x_n)\}$ to a μ-reduced set $\varphi''(X)$ by a finite sequence of elementary transformations. Since φ is injective, the set $\varphi''(X)$ consists of n nonzero elements. It is clear that the set $\varphi''(X)$ depends on x_n, and the element $\varphi'(u)$ is a polynomial in free generators $\varphi''(X)$ $(F(\varphi''(X)) = F(\varphi'(X)))$. Since $\operatorname{rank}(u) = n$, the element $\varphi'(u)$ depends on every free generator from $\varphi''(X)$. Since the set $\varphi''(X)$ depends on x_n, by Theorem 11.1.1 we get

$$\deg(\varphi'(u)) = \mu(\varphi'(u)) \geq \mu(x_n) = \deg(\varphi'(u)) + 1.$$

This contradiction completes the proof. ∎

The idea of the proof of Theorem 11.5.3 works also for systems of elements. Thus we have:

Theorem 11.5.4. *Let* $X = \{x_1, \ldots, x_n\}$. *Let also* $U = \{u_1, \ldots, u_m\}$ *be a system of nonzero elements of the free algebra* $F(X)$, $\operatorname{rank}(U) = n$, φ *an injective endomorphism of* $F(X)$, $\operatorname{rank}(U) = \operatorname{rank}(\varphi) = n$. *Then*

$$\operatorname{rank}(\{\varphi(u_1), \ldots, \varphi(u_m)\}) = \operatorname{rank}(U) = n.$$

In connection with Theorem 11.5.4, see Theorem 12.7.2 in Section 12.7 of Chapter 12 about inverse images of primitive systems of elements.

12

Rank Theorems and Primitive Elements

In this chapter, we consider automorphic orbits and combinatorial properties of primitive elements in free algebras of the main types of Schreier varieties of algebras. A system of elements of a free algebra F is called primitive if it is a subset of some set of free generators of F. The rank of a system of elements of a free algebra F is the minimal number of generators from X on which automorphic images of these elements can depend.

Let A be a finitely generated free algebra of the variety $\mathfrak{M}_0$ (the variety of all algebras), $\mathfrak{M}_1$ (the variety of all commutative algebras, $\operatorname{char} K \neq 2$), or $\mathfrak{M}_2$ (the variety of all anticommutative algebras).

In Sections 12.1 and 12.2, we consider basic properties of partial derivatives of elements of A. Some of these properties are based on differential separability of subalgebras of A and on the fact that the universal multiplicative enveloping algebra $U(A)$ is a free associative algebra that is a free ideal ring.

Section 12.3 is devoted to elimination of variables. Proposition 12.3.2 shows that if we have a dependence of derivatives of a homogeneous element, then there is an automorphism that eliminates a variable in the image of this element.

In Section 12.4, we obtain the rank theorems. Theorems 12.4.7 and 12.4.8 show that the rank of a system of elements is equal to the rank of the free module generated by the elements whose components are partial derivatives of elements of this system. Theorems 12.4.9 and 12.4.10 are the rank theorems for free Lie algebras and superalgebras.

In Section 12.5, we study primitive systems of elements and actions of endomorphisms on automorphic orbits of elements. Theorem 12.5.1 gives a matrix characterization of primitive systems of elements: a system of elements of A is primitive if and only if the matrix of partial derivatives of elements of the system is left invertible over $U(A)$. In particular, an element of A is primitive if and only if its gradient is unimodular. Based on Theorems 12.4.7–12.5.1, we obtain in Theorem 12.5.2 algorithms to find the rank of a system of elements and to decide whether this system is primitive or not. Theorems 12.5.3–12.5.8 give matrix characterizations of primitive systems of elements in free Lie algebras and superalgebras. Using the Freiheitssatz, we obtain another characterization of primitive elements in Theorem 12.5.15: an element a of A is primitive if and only if the factor algebra of A by the ideal generated by a is a free algebra in the same variety of algebras.

In Section 12.6, we consider actions of endomorphisms on primitive elements. Theorem 12.6.2 shows that endomorphisms of A preserving primitivity of elements are automorphisms (Theorem 12.6.3 shows that this statement is true for free Lie algebras and superalgebras). Using this result, we obtain a more general statement in Section 13.3 of Chapter 13. Theorem 13.3.3 says that if an endomorphism of A preserves the automorphic orbit of a nonzero element, then it is an automorphism of A. In Theorems 12.6.6–12.6.8 and Problems 12.6.9 and 12.6.10, we consider images of nonzero linear combinations of free generators.

Section 12.7 deals with inverse images of primitive elements of free algebras of the main types of Schreier varieties of algebras under the action of endomorphisms. Theorems 12.7.1 and 12.7.2 show that inverse images of primitive systems of elements under the action of injective endomorphisms are primitive. In the general case, inverse images of primitive elements generate a retract (Theorem 12.7.3).

12.1 Basic Properties of Partial Derivatives

Let A be a finitely generated free algebra of the variety $\mathfrak{M}_0$ (the variety of all algebras), $\mathfrak{M}_1$ (the variety of all commutative algebras, $\operatorname{char} K \neq 2$), or $\mathfrak{M}_2$ (the variety of all anticommutative algebras).

Let H be a subalgebra of A. By J_H we denote the submodule of the module I_A generated by the elements $\{\mathcal{D}(h) \mid h \in H\}$. A variety $\mathfrak{M}$ has a property of differential separability for subalgebras if for any $a \in A$ we have $a \in H$ if and only if $\mathcal{D}(a) \in J_H$. The varieties $\mathfrak{M}_0$ and $\mathfrak{M}_2$ have the property of differential separability for subalgebras. If $\operatorname{char} K \neq 2$, then the variety $\mathfrak{M}_1$ also has this property. For the details, we refer to [388].

In all of what follows, by $\mathfrak{M}$ we will denote one of the varieties $\mathfrak{M}_0$, $\mathfrak{M}_1$ ($\operatorname{char} K \neq 2$), and $\mathfrak{M}_2$. Also, we use the notations W_i for the linear basis of the algebra A and S_i for the set of free generators of the algebra $U(A)$.

The structure of a $U(A)$-module on A is given by $a \cdot r_w = a \cdot w$, $a \cdot l_w = w \cdot a$ $(a, w \in A)$.

Let H be the subalgebra of A generated by $x_1, \dots, x_{n-1}$, $U(H)$ the subalgebra with the unit element of $U(A)$ generated by $\{r_u, l_u \mid u \in H\}$, and $S_H = S \cap U(H)$. In the general case, if H is a subalgebra of A, then $U(H)$ is not necessarily a subalgebra of $U(A)$. But for Schreier varieties of algebras, $U(H)$ is a subalgebra of $U(A)$. Let V_H be the left ideal of $U(A)$ generated by the elements r_u and l_u, where u runs through all monomials in $x_1, \dots, x_{n-1}$ of the degree not less than 1.

Lemma 12.1.1. *For any $a \in A$ and $u \in U(H)$*

$$\frac{\partial}{\partial x_n}(a \cdot u) = \frac{\partial}{\partial x_n}(a) \cdot u.$$

Proof. One may assume that u is a monomial in S_H. Let $u = u_1 \cdot s$, where either $s = r_h$ or $s = l_h$, $h \in H$. Since $h \in H$, $\frac{\partial h}{\partial x_n} = 0$. Hence

$$\frac{\partial}{\partial x_n}((a \cdot u_1) \cdot r_h) = \frac{\partial a u_1}{\partial x_n} \cdot r_h + \frac{\partial h}{\partial x_n} \cdot l_{a u_1} = \frac{\partial a u_1}{\partial x_n} \cdot r_h,$$
$$\frac{\partial}{\partial x_n}((a \cdot u_1) \cdot l_h) = \frac{\partial h}{\partial x_n} \cdot r_{a u_1} + \frac{\partial a u_1}{\partial x_n} \cdot l_h = \frac{\partial a u_1}{\partial x_n} \cdot l_h;$$

that is, $\frac{\partial a \cdot u}{\partial x_n} = \frac{\partial a u_1}{\partial x_n} s$. Applying the induction on the S-degree, we complete the proof. ■

Lemma 12.1.2. *If $a \in A$ and $\frac{\partial a}{\partial x_n} = 0$, then $a \in H$.*

Proof. If $\frac{\partial a}{\partial x_n} = 0$, then

$$\mathcal{D}(a) = y_1 \frac{\partial a}{\partial x_1} + \cdots + y_{n-1} \frac{\partial a}{\partial x_{n-1}} \in J_H.$$

Since the variety $\mathfrak{M}$ has the property of differential separability for subalgebras, it follows that $a \in H$. ■

Lemma 12.1.3. *If $a \in A$ and $\frac{\partial a}{\partial x_n} \in V_H$, then the element a has a presentation $a = \sum \alpha_i u_i v_i + a'$, where $\alpha_i \in K$, $u_i, v_i \in W$, $a' \in H$, and for every i either $u_i \in H$ or $v_i \in H$.*

Proof. Since algebras A and $U(A)$ and the ideal V_H are homogeneous with respect to any $x_i \in X$, it is enough to prove the statement for a multihomogeneous element a. If a does not depend on x_n, then $\frac{\partial a}{\partial x_n} = 0$. If $a = \alpha x_n$ with $\alpha \neq 0$, then $\frac{\partial a}{\partial x_n} = \alpha \notin V_H$. Therefore we suppose that a is a multihomogeneous element of degree not less than 2, and a depends on x_n. Then a has the unique presentation in the form $a = \sum \alpha_i u_i v_i$, where $u_i, v_i \in W$, $\alpha_i \in K$.

If $\mathfrak{M} = \mathfrak{M}_0$, then

$$a = f_1 w_1 + f_2 w_2 + \dots f_k w_k, \tag{12.1}$$

where f_i are linear combinations of words of W_0, $w_i \in W_0$, $w_1 > \cdots > w_k$. Then

$$\frac{\partial a}{\partial x_n} = \sum_{i=1}^{k} \frac{\partial f_i}{\partial x_n} \cdot r_{w_i} + \sum_{i=1}^{k} \frac{\partial w_i}{\partial x_n} l_{f_i}.$$

Since elements of $S_H = S \cap V_H$ are left independent over $U(A)$, it follows from $\frac{\partial a}{\partial x_n} \in V_H$ that $\frac{\partial f_i}{\partial x_n} r_{w_i} \in V_H$. Hence $w_i \in H$ or $\frac{\partial f_i}{\partial x_n} = 0$. By Lemma 12.1.2, we get either $f_i \in H$ or $w_i \in H$.

If $\mathfrak{M} = \mathfrak{M}_1$, then the element a has a presentation in the form (12.1), where f_i are linear combinations of monomials of W_1 less than or equal to w_i. Let $f_i = \alpha_i w_i + g_i$, $1 \leq i \leq k$, where g_i are linear combinations of monomials less than w_i. Then

$$\frac{\partial a}{\partial x_n} = \sum_{i=1}^{k} \left(2\alpha_i \frac{\partial w_i}{\partial x_n} + \frac{\partial g_i}{\partial x_n} \right) r_{w_i} + \sum_{i=1}^{k} \frac{\partial w_i}{\partial x_n} r_{g_i}.$$

Since $w_1 > w_2 > \cdots > w_n$, r_{g_i} has no terms of the form αr_{w_1}, $\alpha \in K$. Hence, if $\frac{\partial a}{\partial x_n} \in V_H$, then

$$\left(2\alpha_1 \frac{\partial w_1}{\partial x_n} + \frac{\partial g_1}{\partial x_n} \right) r_{w_1} \in V_H.$$

It follows that either $w_1 \in H$ or $\frac{\partial(2\alpha_1 w_1 + g_1)}{\partial x_n} = 0$; that is, either $w_1 \in H$ or $2\alpha_1 w_1 + g_1 \in H$. Denote $b = a - (2\alpha_1 w_1 + g_1) w_1$. We have $b = -\alpha_1 w_1 w_1 + \sum_{i=2}^{k} f_i w_i$, $\frac{\partial b}{\partial x_n} \in V_H$, and

$$\frac{\partial b}{\partial x_n} = -2\alpha_1 \frac{\partial w_1}{\partial x_n} r_{w_1} + \sum_{i=2}^{k} \left(\frac{\partial f_i}{\partial x_n} r_{w_i} + \frac{\partial w_i}{\partial x_n} r_{f_i} \right) \in V_H.$$

Therefore $2\alpha_1 \frac{\partial w_1}{\partial x_n} r_{w_1} \in V_H$. We consider the variety $\mathfrak{M}_1$ over a field K of characteristic different from 2. Thus $\alpha_1 = 0$, $\frac{\partial w_1}{\partial x_n} = 0$, or $w_1 \in H$. The condition $\frac{\partial w_1}{\partial x_n} = 0$ is equivalent to $w_1 \in H$, so we get $\alpha_1 = 0$ or $w_1 \in H$, and it is enough to prove the statement of the lemma for the element $b + \alpha_1 w_1 w_1 = \sum_{i=2}^{k} f_i w_i$. The induction on k for the presentation (12.1) completes the proof.

If $\mathfrak{M} = \mathfrak{M}_2$, then the element a has the presentation (12.1), where f_i are linear combinations of monomials less than w_i. As in the proof for $\mathfrak{M}_1$, it follows from $\frac{\partial a}{\partial x_n} \in V_H$ that $\frac{\partial f}{\partial x_n} r_{w_1} \in V_H$; that is, $w_1 \in H$ or $f_1 \in H$. Considering the element $b = a - f_1 w_1$ and using the induction on the length of the presentation (12.1), we complete the proof of the lemma. ∎

12.2 Homogeneous Admissible Elements

Let $\mathbb{Z}_+ = \{k \in \mathbb{Z} \mid k \geq 0\}$, $\mathbb{Q}_+ = \{\alpha \in \mathbb{Q} \mid \alpha \geq 0\}$. We say that a monomial $w \in W$ has the multidegree $m(w) = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ if for each $i = 1, \ldots, n$ the generator x_i occurs in w exactly k_i times. A linear mapping $\mu\colon \mathbb{Z}_+^n \to \mathbb{Q}_+$ is a functional if $\mu(s) \neq 0$ for all nonzero $s \in \mathbb{Z}_+^n$. For $w \in W$, we introduce the μ-degree by $\mu(w) = \mu(m(w))$. An element $a = \sum_{i=1}^k \alpha_i w_i$, $w_i \in W$ is said to be μ-homogeneous (m-homogeneous) if $\mu(w_1) = \cdots = \mu(w_k)$ ($m(w_1) = \cdots = m(w_k)$, respectively). For $w \in W$ we set $m(r_w) = m(l_w) = m(w)$ and $\mu(r_w) = \mu(l_w) = \mu(w)$. This gives us a possibility to consider μ-homogeneous and m-homogeneous elements of the universal multiplicative enveloping algebra $U(A)$. The algebra $U(A)$ is the free associative algebra with the set S of free generators. Therefore elements of $U(A)$ have the S-degree.

We say that a set $\{u_1, \ldots, u_k\}$ of μ-homogeneous elements of $U(H)$ is admissible if it follows from the equation

$$\frac{\partial a}{\partial x_n} = \sum_{i=1}^{k} m_i u_i, \tag{12.2}$$

where $a \in A$, $m_1, \ldots, m_k \in U(A)$, that there exist $a_1, \ldots, a_k \in A$ such that

$$\frac{\partial a}{\partial x_n} = \sum_{i=1}^{k} \frac{\partial a_i}{\partial x_n} u_i.$$

Lemma 12.2.1.

(i) *A set with zero elements is an admissible set.*

If $\{u_1, \ldots, u_k\}$ is an admissible set, then:

(ii) *The set $\{u_1, \ldots, u_k, 0\}$ is admissible.*

(iii) *The set $\{u_{\sigma(1)}, \ldots, u_{\sigma(k)}\}$ is admissible for any element σ of the symmetric group S_k.*

(iv) *If $0 \neq \alpha \in K$, then the set $\{u_1, \ldots, u_{k-1}, \alpha u_k\}$ is admissible.*

(v) *If $u \in U(H)$ and the element $u_k + uu_{k-1}$ is μ-homogeneous, then $\{u_1, \ldots, u_{k-1}, u_k + uu_{k-1}\}$ is an admissible set.*

Proof. The statements of points (i)–(iv) are obvious. To prove (v), suppose that

$$\frac{\partial a}{\partial x_n} = \sum_{i=1}^{k-1} m_i u_i + m_k(u_k + uu_{k-1}) = \sum_{i=1}^{k-2} m_i u_i + (m_{k-1} + m_k u)u_{k-1} + m_k u_k.$$

By the conditions of the lemma, there are $a_1, a_2, \ldots, a_k \in A$ such that

$$\frac{\partial a}{\partial x_n} = \sum_{i=1}^{k} \frac{\partial a_i}{\partial x_n} u_i.$$

Therefore

$$\begin{aligned}\frac{\partial a}{\partial x_n} &= \sum_{i=1}^{k}\frac{\partial a_i}{\partial x_n}u_i + \frac{\partial a_k}{\partial x_n}uu_{k-1} - \frac{\partial a_k}{\partial x_n}uu_{k-1}\\ &= \sum_{i=1}^{k-2}\frac{\partial a_i}{\partial x_n}u_i + \left(\frac{\partial a_{k-1}}{\partial x_n} - \frac{\partial a_k}{\partial x_n}u\right)u_{k-1} + \frac{\partial a_k}{\partial x_n}(u_k + uu_{k-1})\\ &= \sum_{i=1}^{k-2}\frac{\partial a_i}{\partial x_n}u_i + \frac{\partial}{\partial x_n}(a_{k-1} - a_k u)u_{k-1} + \frac{\partial a_k}{\partial x_n}(u_k + uu_{k-1}).\end{aligned}$$

Thus the set $\{u_1, \ldots, u_{k-1}, u_k + uu_{k-1}\}$ is admissible. ■

Lemma 12.2.2. *Suppose that elements $a \in A$, $m_1, \ldots, m_k \in U(A)$, and μ-homogeneous elements $u_1, \ldots, u_k \in U(H)$ satisfy the equation (12.2), and $\mu(u_1) \leq \mu(u_i)$ with $i \geq 2$. If there is an S-monomial u having a nonzero coefficient in the presentation of u_1 such that elements $u_2, \ldots, u_k$ have no monomials ending on u, then $m_1 \in \frac{\partial}{\partial x_n}(A)$.*

Proof. We use the induction on the S-degree of the monomial u. If $u = 1$, then $u_1 = \alpha u$, $0 \neq \alpha \in K$, and $u_2 = \cdots = u_k = 0$, and

$$m_1 = \frac{1}{\alpha}\sum_{i=1}^{k} m_i a_i = \frac{1}{\alpha}\frac{\partial a}{\partial x_n} \in \frac{\partial}{\partial x_n}(A).$$

It is clear that $u \in U(H)$. Let $u = u_0 s$, where $s \in S$. Consider $w \in H$ such that either $s = r_w$ or $s = l_w$. Since $0 < \mu(u_1) \leq \mu(u_i)$, $i \geq 2$, it follows from (12.2) that $\frac{\partial a}{\partial x_n} \in V_H$. By Lemma 12.1.3, $a = \sum_i \alpha_i w_i v_i$, where for any i either $w_i \in H$ or $v_i \in H$. We write down the element a in the form

$$a = \alpha w \cdot w + w \cdot f + g \cdot w + \sum \beta_j w_j v_j,$$

where f and g are linear combinations of basic monomials different from w and $w_j, v_j \neq w$, and for any i either $w_j \in H$ or $v_j \in H$. We get

$$\frac{\partial a}{\partial x_n} = \frac{\partial f}{\partial x_n}l_w + \frac{\partial g}{\partial x_n}r_w + \sum_j \beta_j\left(\frac{\partial w_j}{\partial x_n}r_{v_j} + \frac{\partial v_j}{\partial x_n}l_{w_j}\right).$$

The elements u_i have the unique presentation in the form $u_i = u_i' s + u_i''$, $1 \leq i \leq k$, where elements u_i'' have no monomials ending on s.

Let $s = r_w$. Since $U(A)$ is the free algebra on S, comparing in (12.2) monomials ending on s, we obtain $\frac{\partial g}{\partial x_n} = \sum_{i=1}^{k} m_i u_i'$. It is easy to see that u_0 has a nonzero coefficient in u_1' and elements $u_2', \ldots, u_k'$ have no S-monomials ending on u_0. Hence we have proved the statement for u_0. Since the S-degree of u_0 is less than the S-degree of u, by induction we get $m_1 \in \frac{\partial}{\partial x_n}(A)$. ■

Lemma 12.2.3. *Any subset $\{u_1, u_2, \ldots, u_k\}$ of μ-homogeneous elements of $U(H)$ is admissible.*

Proof. We apply the induction on k. Let $k = 1$. If $u_1 = 0$, then our statement follows from Lemma 12.2.1. Suppose that $u_1 \neq 0$, and for some $a \in A$ and $m_1 \in U(A)$ we have $\frac{\partial a}{\partial x_n} = m_1 u_1$. Consider any S-monomial u that has a nonzero coefficient in u_1. By Lemma 12.2.2, $m_1 \in \frac{\partial}{\partial x_n}(A)$.

Let $k > 1$. By Lemma 12.2.1, we may assume that $\mu(u_1) \leq \mu(u_i)$ with $i \geq 2$. Consider an S-monomial u entering in u_1 with a nonzero coefficient α. One may suppose that $\alpha = 1$. For the elements u_i, $i \geq 2$, we have the presentation in the form $u_i = u_i' u + u_i''$, where u_i'' have no monomials ending on u. By Lemma 12.2.1, the set $\{u_1, \ldots, u_k\}$ is admissible if and only if the set $\{u_1, u_2 - u_2' u_1, \ldots, u_k - u_k' u_1\}$ is admissible. Since the element u_1 is μ-homogeneous, the element $u_1 - u$ has no monomials ending on u. Therefore, for $i \geq 2$, the element

$$u_i - u_i' u_1 = u_i' u + u_i'' - u_i' u_1 = u_i'(u - u_1) + u_i''$$

has no monomials ending on u. Hence, we may suppose that elements $u_2, \ldots, u_k$ have no monomials ending on u. By Lemma 12.2.2, $m_1 = \frac{\partial a_1}{\partial x_n}$ for some $a_1 \in A$. Thus

$$\frac{\partial a}{\partial x_n} = \frac{\partial a_1}{\partial x_n} u_1 + \sum_{i=2}^{k} m_i u_i.$$

Applying Lemma 12.1.1, we get

$$\frac{\partial a}{\partial x_n} - \frac{\partial a_1}{\partial x_n} u_1 = \frac{\partial}{\partial x_n}(a - a_1 u_1) = \sum_{i=2}^{k} m_i u_i.$$

By the induction hypothesis, there are $a_2, \ldots, a_k \in A$ such that

$$\frac{\partial a}{\partial x_n} - \frac{\partial a_1}{\partial x_n} u_1 = \sum_{i=2}^{k} \frac{\partial a_i}{\partial x_n} u_i;$$

that is, the set $\{u_1, \ldots, u_k\}$ is admissible. ■

Lemma 12.2.4. *Suppose that elements $u_1, u_2, \ldots, u_k$ of $U(H)$ are left independent over $U(A)$. Then, if for some $a \in A$, $m_1, \ldots, m_k \in U(A)$ the equation* (12.2) *is fulfilled, then $m_i \in \frac{\partial}{\partial x_n}(A)$, $1 \leq i \leq k$.*

Proof. By Lemma 12.2.3, there are $a_1, \ldots, a_k$ of A such that $\frac{\partial u}{\partial x_n} = \sum_{i=1}^{k} \frac{\partial a_i}{\partial x_n} u_i$. Then $\sum_{i=1}^{k}(m_i - \frac{\partial a_i}{\partial x_n}) u_i = 0$. Since elements $u_1, \ldots, u_k$ are left independent over $U(A)$, $m_i = \frac{\partial a_i}{\partial x_n} \in \frac{\partial}{\partial x_n}(A)$, $1 \leq i \leq k$. ■

12.3 Elimination of Variables

An endomorphism φ of the algebra A is said to be μ-homogeneous if $\varphi(x_i)$, $i = 1, \ldots, n$ are μ-homogeneous elements, and $\mu(\varphi(x_i)) = \mu(x_i)$, $x_i \in X$.

The matrix

$$J(\varphi) = (\partial(\varphi(x_1)), \partial(\varphi(x_2)), \dots, \partial(\varphi(x_n))) \in M_n(U(A))$$

is the Jacobian matrix of the endomorphism φ. An endomorphism φ is an automorphism of A if and only if the matrix $J(\varphi)$ is invertible over $U(A)$ ([388, 406]). For any $k \leq n$, we set

$$J_k(\varphi) = \begin{pmatrix} \frac{\partial\varphi(x_1)}{\partial x_1} & \dots & \frac{\partial\varphi(x_k)}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial\varphi(x_1)}{\partial x_k} & \dots & \frac{\partial\varphi(x_k)}{\partial x_k} \end{pmatrix}.$$

It is clear that $J_n(\varphi) = J(\varphi)$.

Lemma 12.3.1. *Let k be an integer, $1 \leq k \leq n$, and let φ be a μ-homogeneous automorphism of A such that $\varphi(x_i) = x_i$ for all $i > k$. Then the matrix $J_k(\varphi)$ is invertible. If $B = (b_{ij}) = (J_k(\varphi))^{-1}$, then all elements b_{ij} are μ-homogeneous and $\mu(b_{ij}) = \mu(x_j) - \mu(x_i)$ (here, if $\mu(x_j) - \mu(x_i) < 0$, then $b_{ij} = 0$).*

The proof of this lemma is straightforward.

The main result of this section is the following proposition:

Proposition 12.3.2 ([259]). *Let a be a μ-homogeneous element of A and $k < n$. Suppose that there are μ-homogeneous elements $m_1, m_2, \dots, m_k$ of $U(A)$ such that $\mu(m_i) = \mu(x_i) - \mu(x_n)$ for all $i \leq k$ and*

$$\frac{\partial a}{\partial x_n} = \sum_{i=1}^{k} m_i \frac{\partial a}{\partial x_i}. \tag{12.3}$$

Then there is a μ-homogeneous automorphism φ of A such that $\varphi(x_i) = x_i$ for all $i > k$ and $\varphi(a) \in H$, where H is the subalgebra of A generated by $x_1, \dots, x_{n-1}$.

In order to prove Proposition 12.3.2, at first we prove the following lemmas.

Lemma 12.3.3. *Suppose that a positive integer k and an element a of A satisfy the conditions of Proposition 12.3.2. If φ is a μ-homogeneous automorphism of A such that $\varphi(x_i) = x_i$ for all $i > k$, then k and $\varphi(a)$ also satisfy the conditions of Proposition 12.3.2.*

Proof. Since

$$\frac{\partial(\varphi(a))}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial(\varphi(x_i))}{\partial x_j} \varphi\left(\frac{\partial a}{\partial x_i}\right)$$

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