Preface

On several occasions I and colleagues have found ourselves teaching a one-semester course for students at the second year of graduate study in mathematics who want to gain a general perspective on Jordan algebras, their structure, and their role in mathematics, or want to gain direct experience with nonassociative algebra. These students typically have a solid grounding in first–year graduate algebra and the Artin–Wedderburn theory of associative algebras, and a few have been introduced to Lie algebras (perhaps even Cayley algebras, in an offhand way), but otherwise they have not seen any nonassociative algebras. Most of them will not go on to do research in nonassociative algebra, so the course is not primarily meant to be a training or breeding ground for research, though the instructor often hopes that one or two will be motivated to pursue the subject further.

This text is meant to serve as an accompaniment to such a course. It is designed first and foremost to be *read* by students on their own without assistance by a teacher. It is a direct mathematical conversation between the author and a reader whose mind (as far as nonassociative algebra goes) is a tabula rasa. In keeping with the tone of a private conversation, I give more heuristic and explanatory comment than is usual in graduate texts at this level (pep talks, philosophical pronouncements on the proper way to think about certain concepts, historical anecdotes, mention of some mathematicians who have contributed to our understanding of Jordan algebras, etc.), and employ a few English words which do not standardly appear in mathematical works. I have tried to capture to colloquial tone and rhythm of an oral presentation, and have not felt bound (to my copy editor's chagrin) to always adhere to the formal "rules" of English grammar.

This book tells the story of one aspect of Jordan structure theory: the origin of the theory in an attempt by quantum physicists to find algebraic systems more general than hermitian matrices, and ending with the surprising proof by Efim Zel'manov that there is really only one such system, the 27-dimensional Albert algebra, much too small to accommodate quantum mechanics. I try to give students a feeling for the historical development of the subject, so they will realize that mathematics is an *ongoing process*, but I have not tried to write a history of Jordan theory; I mention people only as they contribute directly to the particular story I am telling, so I have had to leave out many important colleagues. I also try to give students a sense of how the subject of Jordan structures has become intertwined with other aspects of mathematics, so students will realize that mathematics *does not develop in isolation*; I describe some applications outside algebra which have been enriched by Jordan theory, and which have influenced in turn purely algebraic developments, but I have not tried to give a compendium of applications, and I have had to leave out many important ones.

It is important for the reader to develop a visceral intuitive feeling for the living subject. The reader should see isomorphisms as cloning maps, isotopes as subtle rearrangements of an algebra's DNA, radicals as pathogens to be isolated and removed by radical surgery, annihilators as biological agents for killing off elements, Peircers as mathematical enzymes ("Jordan-ase") which break an algebra down into its Peirce spaces. Like Charlie Brown's kite-eating trees, Jordan theory has Zel'manov's tetrad-eating ideals (though we shall stay clear of these carnivores in our book). The reader must think of both mathematicians and abstract ideas as active participants in the theory. Just as the mathematicians have proper names that need to be recognized, so too the results need to be appreciated and remembered. To this end, I have christened all statements (theorems, examples, definitions, etc.) and basic equations with a proper name (using capital letters as with ordinary proper names). Instead of saying "by Lemma 21.2.1(1), which of course you will remember," I say "by Nuclear Slipping 21.2.1(1)," hoping to trigger long-repressed memories of a formula for how nuclear elements of alternative algebras slip in and out of associators. The reader should get on a first-name basis with these characters in our story, and be able to comfortably use locutions like "Nuclear Slipping says that such-and-such holds".

While I wind up doing most of the talking, there is some room in Parts II and III for the reader to participate (and stay mathematically fit) by doing exercises. The Exercises give slight extensions, or alternate proofs, of results in the text, and are placed immediately after the results; they give practice in proving variations on the previous mathematical theme. At the end of each chapter I gather a few problems and questions. The Problems usually take the form "Prove that something-or-other"; they involve deeper investigations or lengthier digressions than exercises, and develop more extensive proof skills on a new theme. The Questions are more open-ended, taking the form "What can you say about something-or-other" without giving a hint as to which way the answer goes; they develop proof skills in uncharted territories, in composing a mathematical theme from scratch (most valuable for budding researchers). Hints are given at the back of the book for the starred exercises, problems, and questions (though these should be consulted only after a good-faith effort to prove them).

The Introduction *A Colloquial Survey of Jordan Theory* is in the nature of an extended colloquium talk, a brief survey of the life and times of Jordan algebras, to provide appreciation of the role Jordan algebras play on the broader stage of mathematics. It is divided into eight sections: the origin of the species, the genus of related Jordan structures, and links to six other areas of mathematics (Lie algebras, differential geometry, Riemannian symmetric spaces, bounded symmetric domains, functional analysis, and projective geometry). Since the students at this level cannot be assumed to be familiar with all these areas, the description has to be a bit loose; readers can glean from this part just enough respect and appreciation to sanction and legitimate their investment in reading further. There are no direct references to this material in the rest of the book.

Part I *A Historical Survey of Jordan Structure Theory* is designed to provide an overview of Jordan structure theory in its historical context. It gives a general historical survey, divided chronologically into eight chapters, from the origins in quantum mechanics in 1934 to Efim Zel'manov's breathtaking description of arbitrary simple algebras in 1983 (which later played a role in his Fields Medal work on the Burnside Problem). I give precise definitions and examples, but no proofs, except in the last chapter where I give brief sketches of Zel'manov's revolutionary proof techniques. In keeping with its nature as a survey, I have not included any exercises.

In contrast to the Introduction, the definitions and results in the Survey will be recur in Parts II and III when material from the Survey is being restated. These restatements not only make Part II and Part III fully independent units, capable of serving as course texts, but the repetition itself helps solidify the material in students' minds. All statements (theorems, examples, definitions, etc.) and named equations throughout the book have been christened with a proper name, and readers should try to remember statements by their verbal mnemonic tags. When material from the Survey is being repeated, this will be quietly noted in a footnote. I have been careful to try to keep the same name as in the Survey. Hopefully the name itself will trigger memory of the result, but a numerical tag is included in the reference to help locate the result when the mnemonic tag has not been memorable enough. For the purpose of navigating back to the tagged location, each chapter in the Survey is divided into numbered sections.

Part II *The Classical Theory* and Part III *Zel'manov's Exceptional Theorem* are designed to provide direct experience with nonassociativity, and either one (in conjunction with Part I) could serve as a basis for a one-semester course. Throughout, I stick to linear Jordan algebras over rings of scalars containing $\frac{1}{2}$, but give major emphasis to the quadratic point of view.

The Classical Part gives a development of Jacobson's classical structure theory for nondegenerate Jordan algebras with capacity, in complete detail and with full proofs. It is suitable for a one-semester course aiming to introduce students to the methods and techniques of nonassociative algebra. The (sometimes arcane) details of Peirce decompositions, Peirce relations, and coordinatization theorems are the key tools leading to Jacobson's Classical Structure Theory for Jordan algebras with capacity. The assumption of nondegeneracy allows me to avoid a lengthy discussion of radicals and the passage from a general algebra to a semisimple one.

Zel'manov's Part gives a full treatment of his Exceptional Theorem, that the only simple i-exceptional Jordan algebras are the Albert algebras, closing the historical search for an exceptional setting for quantum mechanics. This part is much more concerned with understanding and translating to the Jordan setting some classical ideas of associative theory, including primitivity; it is suitable for a one-semester course aiming to introduce students to the modern methods of Jordan algebras. The ultrafilter argument, that if primitive systems come in only a finite number of flavors then a prime system must come in one of those pure flavors, is covered in full detail; ultrafilters provide a useful tool that many students at this level are unacquainted with. Surprisingly, though the focus is entirely on prime and simple algebras, along the way we need to introduce and characterize several different radicals. Due to their associative heritage, the techniques in this Part seem more intuitive and less remote than the minutiae of Peirce decompositions.

The book contains five appendices. The first three establish important results whose technical proofs would have disrupted the narrative flow of the main body of the text. We have made free use of these results in Parts II and III, but their proofs are long, combinatorial or computational, and do not contribute ideas and methods of proof which are important for the mainstream of our story. These are digressions from the main path, and should be consulted only after gaining a global picture. A hypertext version of this book would have links to the appendices which could only be opened after the main body of text had been perused at least once. Appendix A *Cohn's Special Theorems* establishes the useful Shirshov–Cohn Theorem which allows us to prove results involving only two elements entirely within an associative context. Appendix B *Macdonald's Theorem* establishes Macdonald's Theorem that likewise reduces verification of operator identities in two variables to an associative setting. Appendix C *Jordan Algebras of Degree* 3 gives detailed proofs that the constructions of cubic factors in Section II.4 do indeed produce Jordan algebras. I have made the treatment of strict simplicity in Part II and prime dichotomy in Part III independent of the Density Theorem, but their proofs could have been streamlined using this powerful result; in Appendix D *The Jacobson–Bourbaki Density Theorem* I give a proof of this theorem. A fifth Appendix E *Hints* gives hints to selected exercises and problems (indicated by an asterisk).

In addition to the appendices, I include several indexes. The first index is a very brief *Index of Collateral Reading*, listing several standard reference books in Jordan theory and a few articles mentioned specifically in the text. In keeping with the book's purpose as a textbook, I do not attempt a detailed bibliography of monographs and research articles; students wishing to pursue a topic of research in more detail will be guided by an advisor to the relevant literature.

A *Pronouncing Index of Names* lists the mathematicians mentioned in the book, and gives references to places where their work is mentioned (but not to every occurrence of a theorem named after them). In addition, it gives a phonetic description of how to pronounce their name correctly — a goal more to strive for than to achieve. (In preparing the guide I have learned to my chagrin that I have been mispronouncing my colleagues' names for years: Let not the sins of the father pass on to the children!)

An *Index of Notations* contains symbols other than words which are defined in the text, with a helpful but brief description of their meaning, and a reference to the location of their formal introduction. An *Index of Named Statements* provides an alphabetical list of the given names of all statements or equations, with a reference to the location of their statement, but does restate them or list all references to them in the text. All other boldface terms are collected in a final *Index of Definitions*, where again reference is given only to their page of definition.

I have dedicated the book to the memory of Nathan and Florie Jacobson, both of whom passed away during this book's long gestation period. They had an enormous influence on my mathematical development. I am greatly indebted to my colleague Kurt Meyberg, who carefully read through Part II and made many suggestions which vastly improved the exposition. I am also deeply indebted to my colleague Wilhelm Kaup, who patiently corrected many of my misconceptions about the role of Jordan theory in differential geometry, improving the exposition in Part I and removing flagrant errors. My colleague John Faulkner helped improve my discussion of applications to projective geometries. I would also like to thank generations of graduate students at Virginia who read and commented upon the text, especially my students Jim Bowling, Bernard Fulgham, Dan King, and Matt Neal. Several colleagues helped correct my pronunciation of the names of foreign mathematicians. Finally, I wish to thank David Kramer for his careful and illuminating copyediting of the manuscript, and Michael Koy of Springer-Verlag for his patience as editor.

Charlottesville, Virginia, USA *Kevin McCrimmon*

http://www.springer.com/978-0-387-95447-9

A Taste of Jordan Algebras
McCrimmon, K.
2004, XXV, 563 p., Hardcover
ISBN: 978-0-387-95447-9