Vectors and Moments

Chapter 2

Vector functions are present in all mechanics. Forces, torques, velocities, angular velocities, accelerations, momenta, and angular momenta are some of the major examples of these kinds of quantities in the subject scope. Among the vector functions, some can be distinguished where the vector is directly associated with a certain point in space of those where this association is not significant. So, for instance, the velocity of a particle is a vector associated with the point in space occupied by it at each instant, while a torque applied to a rigid body is not necessarily associated with any particular point of the body.

This chapter addresses an especially important kind of vector in mechanics: the *moment* vectors. A torque applied to a body is a moment vector; the angular momentum of a body with respect to a given point is also a moment vector. Although torques and angular momenta are different concepts, the vector handling of both is exactly the same and the two functions are discussed together here.

The general purpose of this chapter is the study of *vector systems*. On the one hand, it seeks to give the reader the basic tools to correctly model the forces applied to a mechanical system. On the other, it offers a unified approach to the handling of vectors and their moments, which will make it easier to understand the dynamic properties of a mechanical system — especially the concepts of momentum and angular momentum — facilitating the formulation of equations that govern their motion.

For a systematic and unified approach, Section 2.1 discusses the free, sliding, and bound vector concepts while Section 2.2 defines the moment of a sliding or bound vector with respect to a point or axis, with examples. Section 2.3 introduces the fairly general concept of a vector system, including free and sliding (or bound) vectors and defines the resultant and resultant moment with respect to a point or axis, according to this general approach. The formulation is different from that usually found in the literature and has the advantage of suppressing ambiguities that, for instance, are found when discussing torques applied to a rigid body. Section 2.4 addresses the equivalence of vector systems and the reduction of systems at a given point. It is shown that any vector system can be substituted by a simpler system consisting of just one pair of vectors. Some special systems are also discussed, such as the couple and the null system. Section 2.5 shows the existence of the central axis of a vector system with a nonnull resultant and studies its properties and applications. Section 2.6 specifically discusses the force and torque systems. No attempt has been made to study statics but rather teach the reader how to model the forces and torques acting on a given mechanical system. The contact forces are discussed, paying special attention to the links and phenomenon of friction, field forces, and torques applied to a rigid body.

2.1 Free, Sliding, and Bound Vectors

In a three-dimensional Euclidean space, a given vector can be expressed in three components; if $\mathbf{v}$ is any vector and $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ is a basis of orthonormal vectors, the scalar components of $\mathbf{v}$ on this base, $v_j = \mathbf{v} \cdot \mathbf{n}_j,\ j = 1, 2, 3$, where the dot '$\cdot$' designates *scalar product* (see Appendix A), fully determine the vector $\mathbf{v}$. In its geometric representation, reference is usually made to its elements: magnitude and direction. If the vectors $\mathbf{u}$ and $\mathbf{v}$ are equal, they must have both elements equal and their respective components are necessarily equal on the same basis; in other words,

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_j = v_j, \quad j = 1, 2, 3. \tag{1.1}$$

Furthermore, all algebra for the vectors can be expressed in terms of their components on an arbitrary basis (see Appendix A). Vectors like those described above are called *free vectors*. Examples of free vectors are the angular velocity of a rigid body and a torque applied to a rigid body.

The effect of the action of a force on a rigid body depends on the former's line of action. As discussed in Chapter 1, Newton's third law states, among other things, that, given two particles P and Q, the force exerted, say, by Q over P is a vector associated to the straight line defined by P and Q. In other words, something besides the three components of the vector on a given basis must be specified to fully describe the applied force. So, from a dynamics viewpoint, two forces will be distinguished with the same components — therefore, with equal vectors — and different lines of action. Vectors associated to a certain straight line in the space are called *sliding vectors*. The characterization of a sliding vector requires its components on a given basis and the description of its line of action (the parameters of the equation of this straight line, coordinates of a point on the straight line, or any other form of determination). Two vectorially equal sliding vectors (equality in the usual sense, between free vectors) and associated to the same line of action are called *equivalents* or *equipollents*. Examples of sliding vectors are a force applied on a rigid body and the flow velocity of a fluid in a pipe with a uniform section.

The effect of a force on a deformable body depends, in addition to its line of action, on the point to which the force is applied. Vectors associated to an application point will be called *bound vectors*. To characterize a bound vector one must know its components on a given basis and the coordinates of its application point. Examples of bound vectors are the momentum of a particle and a force applied to the end of a spring.

All vector algebra is defined for free vectors; sums, scalar products, cross products, and other known operations have results exclusively dependent on the respective components of the vectors, thereby constituting free vectors. In other words, whatever the operation between vectors, following the rules of vector algebra, which results in a vector, this will necessarily be a free vector, since it will only depend on the components of the vectors involved in the operation. Algebra of sliding vectors or bound vectors is not, however, prohibited; the operation will only be done as if the vectors are free, and the result must necessarily be a free vector.

Example 1.1 Let us assume that forces $\mathbf{F}_1$ and $\mathbf{F}_2$ and torque $\mathbf{T}$ act on block B, illustrated in Fig. 1.1. $\mathbf{F}_1$ is applied along the axis x.

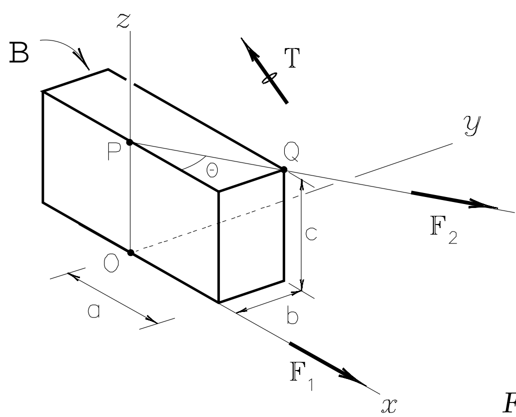

Figure 1.1

If the block can be considered as a rigid body, it makes no difference which is the point of the segment of x inside the block where the force is applied. $\mathbf{F}_1$ is, therefore, a sliding vector associated with axis x and its full characterization is given by its components — $(F_1, 0, 0)$, in the system of Cartesian axes in the figure — and by the axis with which it is associated, in the case x. Force $\mathbf{F}_2$ is also a sliding vector, associated with the straight line that contains points P and Q. It can be fully characterized, for example, by its magnitude, F_2, its direction (from P to Q), and the equation of its line of action: $bx = ay$; $z = c$. Torque $\mathbf{T}$ can be applied at any point of the block; therefore forming a free vector, characterized, for example, by its components (T_1, T_2, T_3). The vector sum $\mathbf{F}_1 + \mathbf{F}_2 = \big((F_1 + F_2 \cos\theta), F_2 \sin\theta, 0\big)$ is a free vector, not associated, therefore, with any straight line in space. The cross product $\mathbf{p}^{P/O} \times \mathbf{F}_2 = cF_2(-\sin\theta, \cos\theta, 0)$, where $\mathbf{p}^{P/O}$ is the position vector of point P with respect to O, is also a free vector.

2.2 Moments

Let us consider $\mathbf{v}$ as a sliding vector, associated with a straight line r, O any point in space, and P an arbitrary point on r (see Fig. 2.1). The cross product of vector $\mathbf{p}$, position of P with respect to O, with vector $\mathbf{v}$, is a free vector, called the *moment of* $\mathbf{v}$ *with respect to* O

$$\mathbf{M}^{\mathbf{v}/O} \rightleftharpoons \mathbf{p} \times \mathbf{v}. \tag{2.1}$$

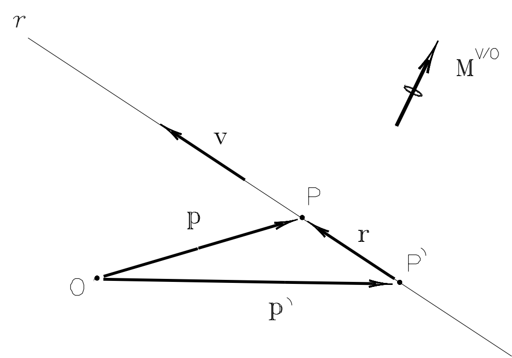

Figure 2.1

Of course, only a sliding vector (or bound vector, a particular case of sliding vector) admits a moment with respect to one point; the position vector $\mathbf{p}$ is not defined for a free vector. The moment of a vector $\mathbf{v}$ is always a free vector and orthogonal to $\mathbf{v}$. In fact, according to Eq. (2.1), the moment results in an algebraic operation and, as such, does not define a line of action for its result; moreover, as this operation is a cross product, the resulting vector must necessarily be orthogonal to $\mathbf{v}$ (see Appendix A).

The moment of a vector with respect to a point will be null if the vector is null or if the line of action of the vector contains the point. In fact, product $\mathbf{p} \times \mathbf{v}$ will be null if one of the vectors is null or if $\mathbf{v}$ is parallel to $\mathbf{p}$.

Lastly, it is worth noting that the moment of a vector with respect to the point O is independent of point P chosen on the line of action of $\mathbf{v}$. To check this, one only needs to choose any other point P′ over r and see that $\mathbf{p}' \times \mathbf{v} = \mathbf{p} \times \mathbf{v} - \mathbf{r} \times \mathbf{v} = \mathbf{p} \times \mathbf{v}$, since $\mathbf{r} \times \mathbf{v} = 0$ (see Fig. 2.1).

The physical dimension of the moment vector will always be equal to the physical dimension of the sliding vector that originated it, multiplied by dimension [L], a characteristic of the position vector $\mathbf{p}$, that is,

$$\text{Dim}\,[\mathbf{M}^{\mathbf{v}/O}] = \text{Dim}\,[\mathbf{v}] \times [\text{L}]. \tag{2.2}$$

If $\mathbf{F}$ is a *force*, a sliding or bound vector with dimension $[\text{MLT}^{-2}]$, that is, Newtons (N), in SI units, its moment with respect to a point will be a *torque*, with dimension $[\text{ML}^2\text{T}^{-2}]$, that is, Newtons-meter (Nm), in the same units. If $\mathbf{G}$ is a *momentum* vector of a particle, a bound vector with dimension $[\text{MLT}^{-1}]$, its moment with respect to a point will be the *angular momentum* vector of the particle with respect to the point, with dimension $[\text{ML}^2\text{T}^{-1}]$.

Given a point O, an axis (straight line) E passing through O and parallel to a certain adimensional unit vector $\mathbf{n}$ and a sliding vector $\mathbf{v}$ associated with the line of action r (see Fig. 2.2), the *moment of the vector* $\mathbf{v}$ *with respect to the axis* E, $\mathbf{M}^{\mathbf{v}/E}$, is defined as the component of the moment of vector $\mathbf{v}$ with respect to the point O, in the direction of the axis, that is (see Appendix A),

$$\mathbf{M}^{\mathbf{v}/E} \rightleftharpoons \mathbf{M}^{\mathbf{v}/O} \cdot \mathbf{n}\,\mathbf{n}. \tag{2.3}$$

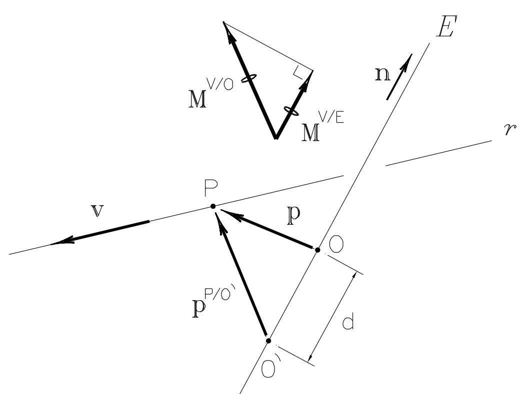

Figure 2.2

The moment of a vector $\mathbf{v}$ with respect to an axis E is a free vector (result of an algebraic operation) parallel to the axis (its direction is given by the unit vector $\mathbf{n}$). The physical dimension of $\mathbf{M}^{\mathbf{v}/E}$ is the same as $\mathbf{M}^{\mathbf{v}/O}$, since $\mathbf{n}$ is adimensional. Lastly, the moment of a vector with respect to an axis does not depend on the point on the axis chosen for its calculation, which justifies no reference to point O in the notation made for a moment with respect to an axis. In fact, if O′ is another point on the axis E (see Fig. 2.2), $\mathbf{M}^{\mathbf{v}/O'} = \mathbf{p}^{P/O'} \times \mathbf{v}$, where $\mathbf{p}^{P/O'}$ is the position vector of point P with respect to point O′, as shown. But if d is the distance between points O and O′, $\mathbf{p}^{P/O'} = \mathbf{p} + d\mathbf{n}$, so $\mathbf{M}^{\mathbf{v}/O'} \cdot \mathbf{n}\,\mathbf{n} = \mathbf{p} \times \mathbf{v} \cdot \mathbf{n}\,\mathbf{n} + d\mathbf{n} \times \mathbf{v} \cdot \mathbf{n}\,\mathbf{n}$ and, as the mixed product $\mathbf{n}\times\mathbf{v}\cdot\mathbf{n}$ is null, then $\mathbf{M}^{\mathbf{v}/O'} \cdot \mathbf{n}\,\mathbf{n} = \mathbf{M}^{\mathbf{v}/O} \cdot \mathbf{n}\,\mathbf{n} = \mathbf{M}^{\mathbf{v}/E}$ (see Appendix A).

Example 2.1 Consider a particle P, of mass m, moving in the reference system shown in Fig. 2.3. At the instant represented, the position of P is given by the Cartesian coordinates $(s\cos\theta, y_0, s\sin\theta)$; its magnitude velocity v is parallel to straight line r; and a force of magnitude F, directed to point O, acts on the particle. The momentum vector of the particle, defined as $\mathbf{G} = mv\mathbf{n}$, is bound to P.

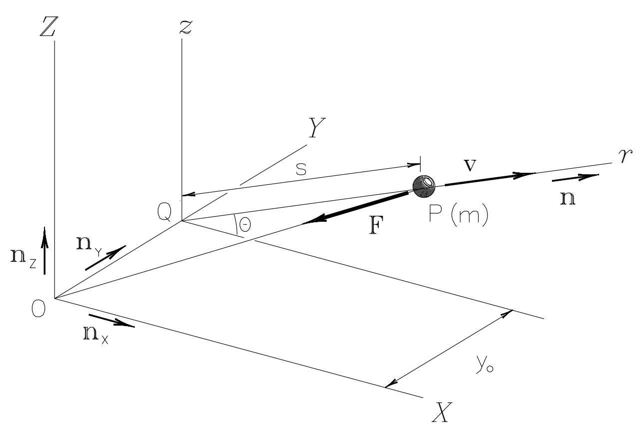

Figure 2.3

The moment of this vector with respect to point O is

$$\begin{aligned}\mathbf{M}^{\mathbf{G}/O} &= \mathbf{p}^{P/O} \times \mathbf{G} = \mathbf{p}^{Q/O} \times \mathbf{G} \\ &= y_0\mathbf{n}_y \times mv(\cos\theta\mathbf{n}_x + \sin\theta\mathbf{n}_z) \\ &= mvy_0(\sin\theta\mathbf{n}_x - \cos\theta\mathbf{n}_z).\end{aligned}$$

The moment of vector $\mathbf{G}$ with respect to axis X is

$$\mathbf{M}^{\mathbf{G}/X} = \mathbf{M}^{\mathbf{G}/O} \cdot \mathbf{n}_x\,\mathbf{n}_x = mvy_0\sin\theta\mathbf{n}_x,$$

and the moment with respect to axis Z is

$$\mathbf{M}^{\mathbf{G}/Z} = \mathbf{M}^{\mathbf{G}/O} \cdot \mathbf{n}_z\, \mathbf{n}_z = -mvy_0 \cos\theta \mathbf{n}_z.$$

Force F, applied on P, is a bound vector at P. The moment of this force with respect to point O is null, because the support of the force passes through O. The moment of this force with respect to point Q is

$$\begin{aligned}
\mathbf{M}^{\mathbf{F}/Q} &= \mathbf{p}^{P/Q} \times \mathbf{F} \\
&= s(\cos\theta \mathbf{n}_x + \sin\theta \mathbf{n}_z) \\
&\quad \times \frac{-F}{(s^2 + y_0^2)^{1/2}} (s\cos\theta \mathbf{n}_x + y_0 \mathbf{n}_y + s \sin\theta \mathbf{n}_z) \\
&= \frac{Fsy_0}{(s^2 + y_0^2)^{1/2}} (\sin\theta \mathbf{n}_x - \cos\theta \mathbf{n}_z).
\end{aligned}$$

Note that the moment of **F** with respect to Q can also be obtained (and more easily) by

$$\begin{aligned}
\mathbf{M}^{\mathbf{F}/Q} &= \mathbf{p}^{O/Q} \times \mathbf{F} \\
&= -y_0 \mathbf{n}_y \times \frac{-F}{(s^2 + y_0^2)^{1/2}} (s\cos\theta \mathbf{n}_x + y_0 \mathbf{n}_y + s \sin\theta \mathbf{n}_z) \\
&= \frac{Fsy_0}{(s^2 + y_0^2)^{1/2}} (\sin\theta \mathbf{n}_x - \cos\theta \mathbf{n}_z).
\end{aligned}$$

The moment of vector **F** with respect to the vertical axis z passing through Q is

$$\mathbf{M}^{\mathbf{F}/z} = \mathbf{M}^{\mathbf{F}/Q} \cdot \mathbf{n}_z\, \mathbf{n}_z = -\frac{Fsy_0}{(s^2 + y_0^2)^{1/2}} \cos\theta \mathbf{n}_z,$$

and the moment of **F** with respect to axis Y is

$$\mathbf{M}^{\mathbf{F}/Y} = \mathbf{M}^{\mathbf{F}/Q} \cdot \mathbf{n}_y\, \mathbf{n}_y = 0.$$

The moment of a vector **v** with respect to an axis E passing through a point O will be null if the mixed product $\mathbf{p} \times \mathbf{v} \cdot \mathbf{n}$ is null [see Eqs. (2.1) and (2.3)]. This will happen if **v** is null or **p** and **v** are parallel — and in this case O belongs to the straight line r, the line of action of **v**, so the axis and straight line are concurrent — or, also, if **v** and **n** are parallel, meaning that the axis and straight line are parallel (see Fig. 2.2). In short, the moment of a nonnull vector with respect to an axis will be null whenever the vector's line of action and axis are *coplanar*. (There is also another trivial case where the moment vector with respect to an axis vanishes. Which is that?)

The moment of a sliding vector $\mathbf{v}$ with respect to point O is equal to the vector sum of the moments of the vector with respect to three mutually orthogonal axes that intercept at O. Thus, if $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are orthonormal vectors, parallel to the axes x_1, x_2, x_3, passing through O, then $\mathbf{M}^{\mathbf{v}/x_j} = \mathbf{M}^{\mathbf{v}/O} \cdot \mathbf{n}_j\, \mathbf{n}_j,\ j = 1, 2, 3$, therefore (see Appendix A),

$$\mathbf{M}^{\mathbf{v}/O} = \sum_{j=1}^{3} \mathbf{M}^{\mathbf{v}/x_j}. \tag{2.4}$$

Example 2.2 Returning to the preceding example (see Fig. 2.3), the moment of vector $\mathbf{G}$ with respect to axis Y is null because the $\mathbf{G}$ line of action intercepts Y at point Q. The moments of vector $\mathbf{F}$ with respect to axes X, Y, or Z are null because the line of action of $\mathbf{F}$ intercepts the axes at O. If the angle θ is null, the moment of vector $\mathbf{G}$ with respect to axis X will also be null since the $\mathbf{G}$ line of action and axis X will be parallel. In fact, for $\theta = 0$, $\mathbf{M}^{\mathbf{G}/O} = -mvy_0\mathbf{n}_z$ and $\mathbf{M}^{\mathbf{G}/O} \cdot \mathbf{n}_x = 0$. It is also easy to see, looking at the results of the above example, that $\mathbf{M}^{\mathbf{G}/O} = \mathbf{M}^{\mathbf{G}/X} + \mathbf{M}^{\mathbf{G}/Y} + \mathbf{M}^{\mathbf{G}/Z}$, as Eq. (2.4) establishes for any θ value.

2.3 Vector Systems

Consider a set consisting of n sliding vectors of the same physical dimension, $\mathbf{v}_i$, associated with the line of actions $r_i,\ i = 1, 2, \ldots, n$, respectively, and m free vectors $\mathbf{M}_j,\ j = 1, 2, \ldots, m$, all with the dimension of a moment of a vector from the vector category $\mathbf{v}_i$. A set of vectors defined as such will be called a *vector system*. (Let us not forget that bound vectors are a particular case of sliding vectors and that, therefore, some or even all the vectors $\mathbf{v}_i$ above may consist of bound vectors.)

Example 3.1 The arm shown in Fig. 3.1 is hinged at its end A and can turn freely around the axis x_3. Force F is applied at end B, with components in the three coordinated directions; consider that the vertical force P, the weight of the arm, is applied at point O, mass center of the arm; assume the action of three force components, $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, on end A, as shown. Lastly, as the arm is free to turn exclusively around the axis x_3, two torque components, $\mathbf{T}_1$ and $\mathbf{T}_2$, shall be applied to it.

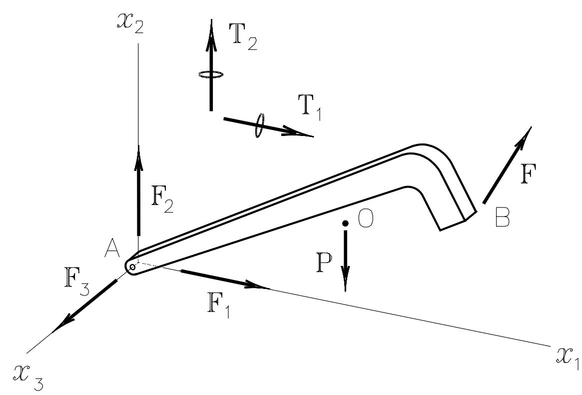

Figure 3.1

This group of seven vectors — five sliding vectors (the forces) and two free vectors (the torques) — forms a vector system, with $n = 5$ and $m = 2$. (If the reader did not clearly understand why these vectors and not others are involved, do not worry: The recognition of the forces and torques applied to a rigid body will be discussed later in this chapter. What matters for now is to recognize that this is a vector system.)

If $\mathcal{V}$ is a vector system consisting of n sliding vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, n$ and m free vectors $\mathbf{M}_j$, $j = 1, 2, \ldots, m$, the vector sum of the n sliding vectors is called *resultant* of the system, that is,

$$\mathbf{R}(\mathcal{V}) \rightleftharpoons \sum_{i=1}^{n} \mathbf{v}_i. \tag{3.1}$$

It is never too late to insist that the resultant of a system, obtained from a usual vector sum, is a free vector, not associated, therefore, with any line of action and, as such, not having defined its moment with respect to any point in the space.

Example 3.2 Figure 3.2 illustrates a system of vectors $\mathcal{V}$ associated to a cube with an edge with a length of 2 m. The vector $\mathbf{v}_1$ with magnitude $5u$ is associated with the axis x_3; the vector $\mathbf{v}_2$ with magnitude $10u$ is associated with the straight line containing A and B; and the vector $\mathbf{v}_3$, with magnitude $15u$, is associated with the straight line containing B and C, with the directions shown, u being a certain physical unit. The system also consists of the free vectors $\mathbf{M}_1$, parallel to axis x_1, with magnitude $20u$m, and $\mathbf{M}_2$, parallel to axis E, with magnitude $30\sqrt{2}u$m, with the directions indicated. The resultant $\mathbf{R}$ of this system will be the free vector

$$\mathbf{R} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 5u(-3\mathbf{n}_1 + 2\mathbf{n}_2 + \mathbf{n}_3).$$

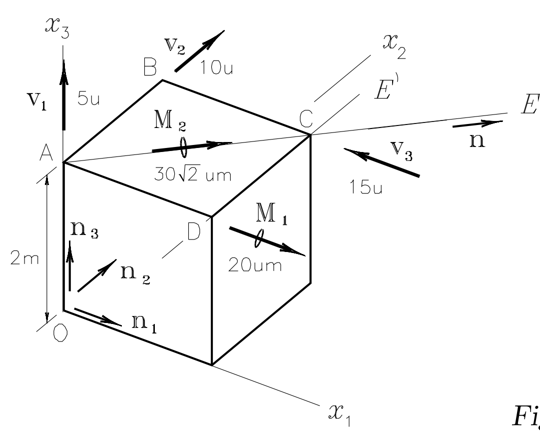

Figure 3.2

The *resultant moment* of a vector system $\mathcal{V}$ with respect to a point O is defined as the vector sum of the moments with respect to O of the sliding vectors of $\mathcal{V}$ with the free vectors of $\mathcal{V}$, that is,

$$\mathbf{M}^{\mathcal{V}/O} \rightleftharpoons \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/O} + \sum_{j=1}^{m} \mathbf{M}_j. \tag{3.2}$$

It is clear that the resultant moment of a system $\mathcal{V}$ is also a free vector.

The resultant moment of a vector system $\mathcal{V}$ with respect to an axis E, passing through a point O and parallel to an adimensional unit vector $\mathbf{n}$, is defined as the component of the resultant moment of the system at the point, in direction of the axis, that is,

$$\mathbf{M}^{\mathcal{V}/E} \rightleftharpoons \mathbf{M}^{\mathcal{V}/O} \cdot \mathbf{n}\,\mathbf{n}. \tag{3.3}$$

Example 3.3 Returning to the previous example (see Fig. 3.2), the resultant moment of the system with respect to the point O is

$$\begin{aligned}\mathbf{M}^{\mathcal{V}/O} &= \mathbf{M}^{\mathbf{v}_1/O} + \mathbf{M}^{\mathbf{v}_2/O} + \mathbf{M}^{\mathbf{v}_3/O} + \mathbf{M}_1 + \mathbf{M}_2\\ &= 0 + 2\mathrm{m}\,\mathbf{n}_3 \times 10u\,\mathbf{n}_2 + 2\mathrm{m}\,(\mathbf{n}_2 + \mathbf{n}_3) \times (-15u)\,\mathbf{n}_1\\ &\quad + 20u\mathrm{m}\,\mathbf{n}_1 + 30u\mathrm{m}\,(\mathbf{n}_1 + \mathbf{n}_2)\\ &= 30u\mathrm{m}\,(\mathbf{n}_1 + \mathbf{n}_3).\end{aligned}$$

The resultant moment with respect to the axis x_1 will be $\mathbf{M}^{\mathcal{V}/x_1} = 30u\mathrm{m}\mathbf{n}_1$ and the resultant moment with respect to the axis x_2 will be null. The resultant moment of this system with respect to the axis E, which contains A and C vertices, can be directly computed by

$$\mathbf{M}^{\mathcal{V}/E} = \mathbf{M}_1 \cdot \mathbf{n}\,\mathbf{n} + \mathbf{M}_2 = 40\sqrt{2}u\mathrm{m}\,\mathbf{n} = 40u\mathrm{m}\,(\mathbf{n}_1 + \mathbf{n}_2),$$

since the lines of action of the sliding vectors of the system intercept all on axis E.

Once the resultant and resultant moment with respect to any given point O of a vector system $\mathcal{V}$ are known, the resultant moment of a system is determined with respect to any other point. This is guaranteed by a very simple and extremely useful relationship established on what has usually been called the

> **Moments Transport Theorem.** *The resultant moment of a vector system $\mathcal{V}$ with respect to any point O is equal to the vector sum of the resultant moment of the system with respect to a given point O' with the moment, with respect to O, of a sliding vector vectorially equal to the resultant $\mathbf{R}$ of $\mathcal{V}$ and associated with a straight line passing through O', that is,*

$$\mathbf{M}^{\mathcal{V}/O} = \mathbf{M}^{\mathcal{V}/O'} + \mathbf{p}^{O'/O} \times \mathbf{R}. \tag{3.4}$$

The derivation of the theorem is simple, by basing it on the definitions of the resultant moment of a vector system with respect to one point, Eq. (3.2), and resultant of a system, Eq. (3.1). In fact

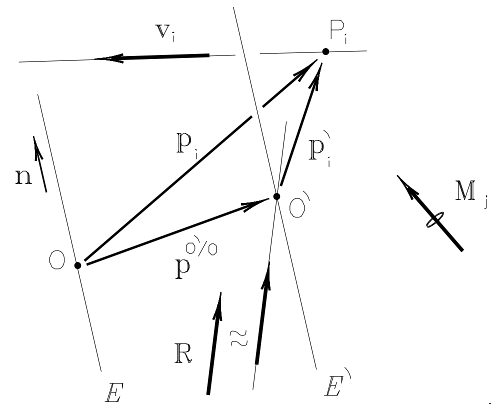

Figure 3.3

(see Fig. 3.3),

$$
\begin{aligned}
\mathbf{M}^{\mathcal{V}/O} &= \sum_{i=1}^{n} \mathbf{p}_i \times \mathbf{v}_i + \sum_{j=1}^{m} \mathbf{M}_j \\
&= \sum_{i=1}^{n} \mathbf{p}'_i \times \mathbf{v}_i + \sum_{i=1}^{n} \mathbf{p}^{O'/O} \times \mathbf{v}_i + \sum_{j=1}^{m} \mathbf{M}_j \\
&= \left(\sum_{i=1}^{n} \mathbf{p}'_i \times \mathbf{v}_i + \sum_{j=1}^{m} \mathbf{M}_j \right) + \mathbf{p}^{O'/O} \times \sum_{j=1}^{m} \mathbf{v}_i \\
&= \mathbf{M}^{\mathcal{V}/O'} + \mathbf{p}^{O'/O} \times \mathbf{R}.
\end{aligned}
$$

∎

This result indicates that two free vectors — the resultant, invariant with the point, and a resultant moment, dependent on the chosen point — fully characterize a vector system consisting of an arbitrary number of sliding (or bound) and free vectors.

Equation (3.4) can be extended to resultant moments of a system with respect to different axes. So, if **n** is an adimensional unitary vector, defining a direction in space, the resultant moments of a vector system $\mathcal{V}$, with respect to two axes parallel to **n**, passing through the points O and O′ (see Fig. 3.3) are related by

$$\mathbf{M}^{\mathcal{V}/E} = \mathbf{M}^{\mathcal{V}/E'} + \left(\mathbf{p}^{O'/O} \times \mathbf{R} \right) \cdot \mathbf{n}\, \mathbf{n}. \tag{3.5}$$

Equation (3.5) is the result of projecting Eq. (3.4) in the direction **n**. The second term on the right can be interpreted as the moment with respect to the axis E of a sliding vector vectorially equal to the resultant of $\mathcal{V}$, whose line of action passes through O′. This result is also known as the *parallel axes theorem*.

Example 3.4 Returning to Example 3.2 (see Fig. 3.2), the resultant moment of the system with respect to point A can be obtained, through Eq. (3.4), from

$$\begin{aligned}\mathbf{M}^{\mathcal{V}/A} &= \mathbf{M}^{\mathcal{V}/O} + \mathbf{p}^{O/A} \times \mathbf{R}\\ &= 30u\mathrm{m}\,(\mathbf{n}_1 + \mathbf{n}_3) + (-2\mathrm{m})\,\mathbf{n}_3 \times 5u\,(-3\mathbf{n}_1 + 2\mathbf{n}_2 + \mathbf{n}_3)\\ &= 10u\mathrm{m}\,(5\,\mathbf{n}_1 + 3\mathbf{n}_2 + 3\,\mathbf{n}_3).\end{aligned}$$

The resultant moment of the system with respect to the horizontal axis E', which passes through C and D, is, according to Eq. (3.5),

$$\begin{aligned}\mathbf{M}^{\mathcal{V}/E'} &= \mathbf{M}^{\mathcal{V}/x_2} + \big(\mathbf{p}^{O/D} \times \mathbf{R}\big) \cdot \mathbf{n}_2\,\mathbf{n}_2\\ &= 0 + \big((-2\mathrm{m})(\mathbf{n}_1 + \mathbf{n}_3) \times 5u\,(-3\,\mathbf{n}_1 + 2\,\mathbf{n}_2 + \mathbf{n}_3)\big) \cdot \mathbf{n}_2\,\mathbf{n}_2\\ &= 40u\mathrm{m}\,\mathbf{n}_2.\end{aligned}$$

When a vector system consists exclusively of sliding (or bound) vectors, it is called a *simple system*. For a simple system, therefore $m = 0$ and the resultant moment of the system with respect to a point or axis will be the sum of the moments of the sliding vectors comprising the system, with respect to the point or axis.

Some simple systems consist of an infinite number of sliding vectors, each with an infinitesimal magnitude. Systems of this kind are called *distributed systems*. If $d\mathbf{v}$ is a vector of a distributed system $\mathcal{V}$ (see Fig. 3.4), its resultant $\mathbf{R}$ is defined as

$$\mathbf{R} \rightleftharpoons \int_{\mathcal{V}} d\mathbf{v}. \tag{3.6}$$

The resultant moment of a distributed simple system $\mathcal{V}$ with respect to a point O is defined as

$$\mathbf{M}^{\mathcal{V}/O} \rightleftharpoons \int_{\mathcal{V}} \mathbf{p} \times d\mathbf{v}, \tag{3.7}$$

where $\mathbf{p}$ is the position vector with respect to point O, of an arbitrary point on the line of action of $d\mathbf{v}$ (see Fig. 3.4).

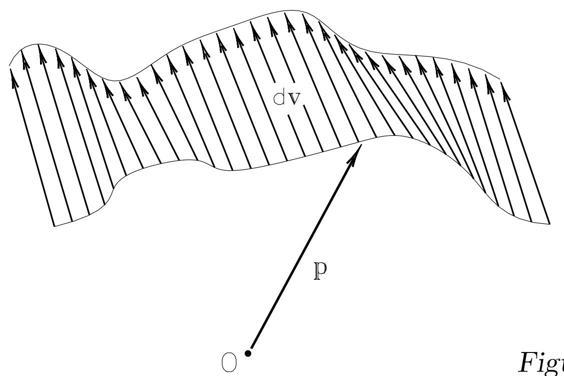

Figure 3.4

The resultant moment of a distributed system with respect to an axis E is defined as in Eq. (3.3), that is, it is the component, in the direction of the axis, of the resultant moment of the system with respect to any point on the same axis.

Example 3.5 Figure 3.5 shows the diagram of a vertical gate, with height a, which holds the water of a tank.

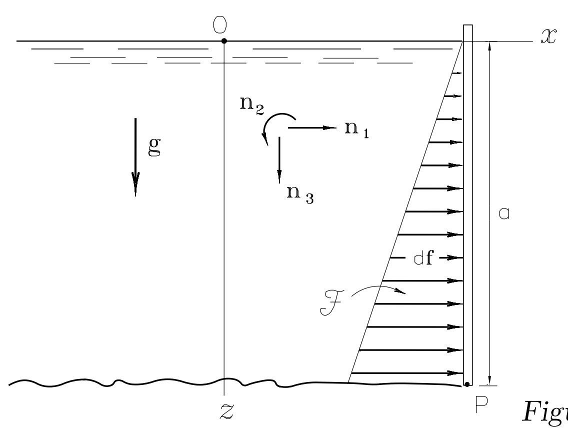

Figure 3.5

The pressure exerted by the fluid on the gate over the atmospheric pressure depends on the depth z, according to the hydrostatic relationship $p(z) = \rho g z$, where p is the pressure, ρ the density of the fluid, and g the gravitational acceleration magnitude. As the pressure varies exclusively with the vertical coordinate on each horizontal surface element, with an area $dA = l dz$, where l is the width (uniform) of the gate, an infinitesimal force $d\mathbf{f} = pdA\,\mathbf{n}_1 = \rho glzdz\,\mathbf{n}_1$ will be applied. So a distributed system $\mathcal{F}$, consisting of forces $d\mathbf{f}$ associated to horizontal lines of action (direction x), acts on the gate. The resultant force of the action of the fluid on the gate will be the resultant of this disributed system, given, according to Eq. (3.6), by (assuming ρ and g as constants)

$$\mathbf{R} = \int_0^a d\mathbf{f} = \rho gl \int_0^a zdz\,\mathbf{n}_1 = \frac{1}{2}\rho gla^2\,\mathbf{n}_1.$$

The resultant moment of this system with respect, say, to point P is, according to Eq. (3.7),

$$\mathbf{M}^{\mathcal{F}/P} = \int_0^a \mathbf{p} \times d\mathbf{f} = \rho gl \int_0^a -(a-z)\,\mathbf{n}_3 \times z\,dz\,\mathbf{n}_1 = -\frac{1}{6}\rho gla^3\,\mathbf{n}_2.$$

Example 3.6 Bar B, pivoted on one end at the fixed point O, moves on the plane of the figure with the angle θ varying with time according to the rate $\omega = d\theta/dt$ (see Fig. 3.6).

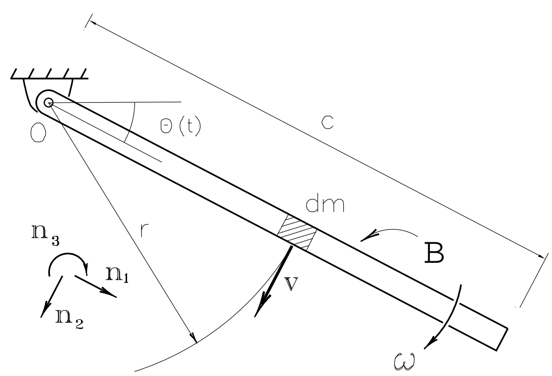

Figure 3.6

The bar is homogeneous with mass m and length c. Each element of B will have a mass $dm = \rho dr$, ρ being its density (mass per unit of length), and the velocity $v = r\omega$ (believe me), in the direction $\mathbf{n}_2$. The momentum vectors of the elements of B, $d\mathbf{G} = \mathbf{v}\,dm$, consist then of a simple distributed system $\mathcal{G}$, whose resultant,

$$\mathbf{R} = \int_B d\mathbf{G} = \int_0^c \rho\omega r\,dr\,\mathbf{n}_2 = \frac{1}{2}\rho\omega c^2\,\mathbf{n}_2 = \frac{1}{2}m\omega c\,\mathbf{n}_2,$$

is the momentum of the bar. Its angular momentum with respect to point O, is the resultant moment of the system $\mathcal{G}$ with respect to O, given by

$$\begin{aligned}\mathbf{M}^{\mathcal{V}/O} &= \int_0^c r\,\mathbf{n}_1 \times \rho\omega r\,dr\,\mathbf{n}_2 \\ &= \rho\omega \int_0^c r^2\,dr\,\mathbf{n}_3 \\ &= \frac{1}{3} m\omega c^2\,\mathbf{n}_3.\end{aligned}$$

The angular momentum vector of the body with respect to the axis, say, x_3 (axis passing through O, parallel to the unit vector $\mathbf{n}_3$) will be the component, in this direction, of the angular momentum vector with respect to the point O, that is,

$$\mathbf{M}^{\mathcal{V}/x_3} = \mathbf{M}^{\mathcal{V}/O} \cdot \mathbf{n}_3\,\mathbf{n}_3 = \frac{1}{3} m\omega c^2\,\mathbf{n}_3.$$

2.4 Equivalent Systems

Two vector systems $\mathcal{V}$ and $\mathcal{V}'$ are said to be *equivalent* if their resultants are equal and if their resultant moments are also equal with respect to some point O, that is,

$$\mathcal{V} \approx \mathcal{V}' \rightleftharpoons \begin{cases} \mathbf{R}(\mathcal{V}) = \mathbf{R}(\mathcal{V}'), \\ \mathbf{M}^{\mathcal{V}/O} = \mathbf{M}^{\mathcal{V}'/O}, \quad \text{for some O}, \end{cases} \tag{4.1}$$

where the symbol '$\approx$' means equivalence.

It is natural that the concept of equivalence is expected to be stronger, such as systems being equivalent with equal resultants and equal resultant moments for *any* point in space. It is easy to see, however, that this is exactly what will happen with systems that fulfill Eq. (4.1); otherwise, let us see: If $\mathcal{V}$ and $\mathcal{V}'$ are equivalent, from the moments transport theorem, Eq. (3.4), then, for any point O$'$

$$\mathbf{M}^{\mathcal{V}/O'} = \mathbf{M}^{\mathcal{V}/O} + \mathbf{p}^{O/O'} \times \mathbf{R}(\mathcal{V}) = \mathbf{M}^{\mathcal{V}'/O} + \mathbf{p}^{O/O'} \times \mathbf{R}(\mathcal{V}') = \mathbf{M}^{\mathcal{V}'/O'}, \tag{4.2}$$

as desired, that is, the resultants of the two systems being equal and their resultant moments also being equal for a given point, then the resultant moments will also be equal to each other for any other arbitrarily chosen point.

If $\mathcal{V}$ and $\mathcal{V}$' are equivalent systems, their moments with respect to any axis E will be equal. In fact, if O is a point on the axis, $\mathbf{M}^{\mathcal{V}/O} = \mathbf{M}^{\mathcal{V}'/O}$ and the component of this relation in the direction of the axis, given by the unit vector $\mathbf{n}$, will be $\mathbf{M}^{\mathcal{V}/O} \cdot \mathbf{n}\,\mathbf{n} = \mathbf{M}^{\mathcal{V}'/O} \cdot \mathbf{n}\,\mathbf{n}$; therefore,

$$\mathbf{M}^{\mathcal{V}/E} = \mathbf{M}^{\mathcal{V}'/E}, \quad \text{for every axis } E \qquad \text{if } \mathcal{V} \approx \mathcal{V}'. \tag{4.3}$$

Example 4.1 Consider the system $\mathcal{V}$, consisting of sliding vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, whose lines of action intercept point A, and by the free vector $\mathbf{M}_A$, orthogonal to the plane of the former (see Fig. 4.1). Also consider the vector system $\mathcal{V}'$, consisting of the sliding vector $\mathbf{v}$, whose line of action intercepts point B and is orthogonal to $\mathbf{n}_3$, with the direction shown, and by the free vector $\mathbf{M}_B$, parallel to $\mathbf{M}_A$. The magnitudes and directions are shown in the figure.

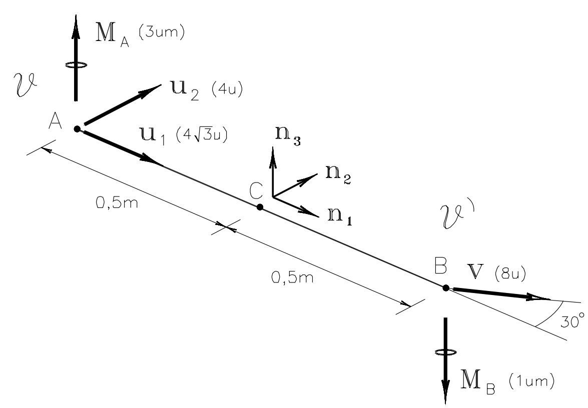

Figure 4.1

The resultant of system $\mathcal{V}$ can be expressed on the basis of $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ by

$$\mathbf{R} = \mathbf{u}_1 + \mathbf{u}_2 = 4u\,(\sqrt{3}\mathbf{n}_1 + \mathbf{n}_2),$$

and its resultant moment with respect, say, to point B is

$$\mathbf{M}^{\mathcal{V}/B} = \mathbf{p}^{A/B} \times \mathbf{u}_2 + \mathbf{M}_A = -u\mathrm{m}\,\mathbf{n}_3.$$

The resultant of system $\mathcal{V}'$ is

$$\mathbf{R}' = \mathbf{v} = 4u\,(\sqrt{3}\mathbf{n}_1 + \mathbf{n}_2),$$

and its resultant moment with respect to point B is

$$\mathbf{M}^{\mathcal{V}'/B} = \mathbf{M}_B = -u\mathrm{m}\,\mathbf{n}_3.$$

It results then that $\mathcal{V}$ and $\mathcal{V}'$ are equivalent. It is easy to see, for example, that both systems have a null resultant moment with respect to any axis passing through B and parallel to the plane defined by the directions of $\mathbf{n}_1$ and $\mathbf{n}_2$. (Choose any point and calculate the resultant moments of $\mathcal{V}$ and $\mathcal{V}'$ with respect to this point. What conclusion does one reach?)

Every vector system $\mathcal{V}$ has an infinite number of equivalent systems (does the reader agree?). The simplest of them will be, in general, systems consisting of a pair of vectors, one free and the other sliding. Now let us see: Taking a sliding vector equal to the resultant of $\mathcal{V}$ over a line of action passing through a given point Q, and a free vector equal to the resultant moment of $\mathcal{V}$ with respect to Q, we have a new system whose resultant, being equal to the single sliding vector that comprises it, is equal to the resultant of $\mathcal{V}$, and whose resultant moment with respect to point Q is also equal to the resultant moment of $\mathcal{V}$ with respect to Q. It is said, then, that *the system $\mathcal{V}$ was reduced to point* Q. Once a given system of vectors is reduced to point Q, as described in the above procedure, it can be easily reduced to any other point, using the moments transport theorem, Eq. (3.4). In fact, as the resultant is an invariant, one only needs to calculate the new resultant moment based on the previous one, using the theorem, to obtain the new reduction.

Example 4.2 Returning to the previous example (see Fig. 4.1), the system $\mathcal{V}'$ is a reduction of the system $\mathcal{V}$ at point B. The reduction of $\mathcal{V}$ at point C, intermediary between A and B, will consist of a vector equal to $\mathbf{R}$ applied to C and

$$\mathbf{M}^{\mathcal{V}/C} = \mathbf{p}^{A/C} \times \mathbf{u}_2 + \mathbf{M}_A = u\mathrm{m}\,\mathbf{n}_3.$$

Note that the same reduction would be obtained from $\mathcal{V}'$, that is,

$$\mathbf{M}^{\mathcal{V}'/C} = \mathbf{p}^{B/C} \times \mathbf{v} + \mathbf{M}_B = u\mathrm{m}\,\mathbf{n}_3.$$

In the more general case, as seen above, every vector system can be reduced to an arbitrary point, the reduction consisting of a pair of vectors: a sliding vector (equal to the resultant of the original system) and a free vector (equal to the resultant moment of the original system with respect to the point). Some systems, however, are even more easily reduced, as we will see ahead.

When a system $\mathcal{V}$ has a null resultant and nonnull resultant moment with respect to some point in space, it is called a *couple*. According to the moments transport theorem, Eq. (3.4), the resultant moment of a couple is the same for any point in the space, that is,

$$\mathbf{M}^{\mathcal{V}/O} = \mathbf{M}^{\mathcal{V}/O'} \quad \text{if} \quad \mathbf{R} = 0. \tag{4.4}$$

The *moment of the couple* is then an invariant that characterizes it fully. If $\mathcal{V}$ is a couple consisting of forces and torques, its resultant moment is called the *couple torque*.

Example 4.3 The mechanical system illustrated in Fig. 4.2 consists of a central element of mass $5m$, rigidly connected to four equally spaced spheres, two with mass m each and two with mass $2m$ each, in the configuration shown.

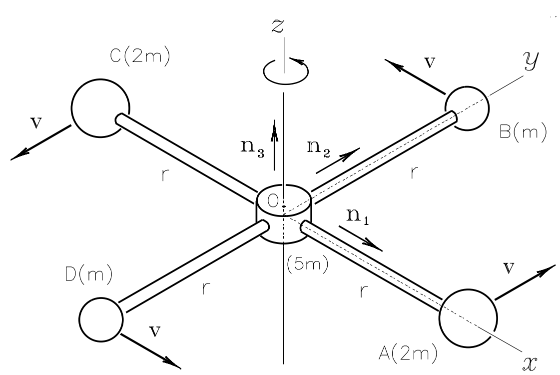

Figure 4.2

The system turns around the axis z at a constant rate so that each of the suspended masses has a velocity of magnitude v. Assuming dimensions so that all elements can be treated as particles, the set of momentum vectors forms a simple vector system with five elements, as follows: $\mathbf{G}_A = 2mv\mathbf{n}_2$, $\mathbf{G}_B = -mv\mathbf{n}_1$, $\mathbf{G}_C = -2mv\mathbf{n}_2$, $\mathbf{G}_D = mv\mathbf{n}_1$, and $\mathbf{G}_O = 0$. The resultant of this sytem is null and the vector system is, therefore, a couple. The resultant moment with respect to point O (the angular momentum of the set of particles with respect to O) is $\mathbf{M}^{\mathcal{V}/O} = 6mvr\mathbf{n}_3$. It is easy to see that the system's resultant moment is the same as for any other point in space.

When a vector system $\mathcal{V}$ has a nonnull resultant and a null resultant moment with respect to a given point O in space, its reduction to that point consists exclusively of a sliding vector vectorially equal to its resultant associated with a line of action passing through the point. It is easy to see that, according to Eq. (3.4), for all points on this support, the resultant moment of the system will also vanish.

Example 4.4 Figure 4.3 illustrates a cylinder floating on a fluid at rest. The system of forces exerted by the fluid on the cylindrical shell is a distributed simple system that, for a vertical cross section, will have the indicated aspect, with the force magnitude varying with depth and its direction always orthogonal to the surface of the cylinder. The lines of action of all components of this system intercept; then the symmetry axis of the cylinder and its resultant moment with respect to this axis will therefore be null. The geometry of the body also guarantees the symmetry of this system of forces in the longitudinal direction, with the consequence that the resultant moment of the system with respect to point O is also null.

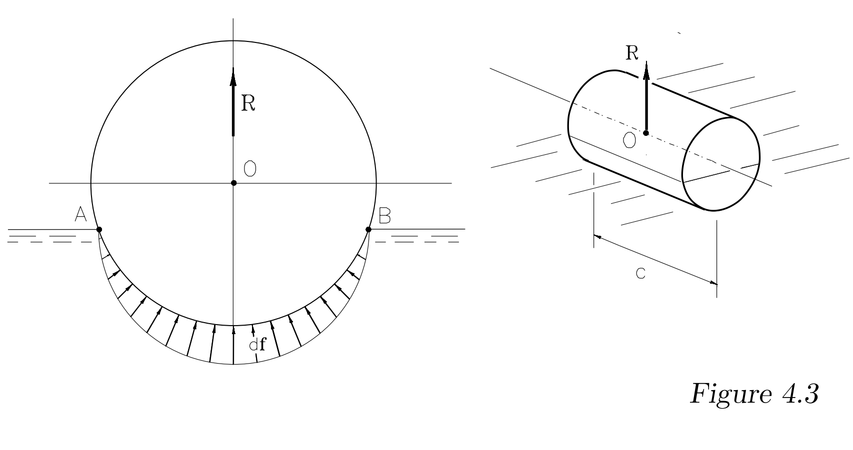

Figure 4.3

The resultant of the system,

$$\mathbf{R} = \int_{-c/2}^{c/2} \int_{A}^{B} d\mathbf{f},$$

will therefore be a vertical vector. The reduction of the system at point O will then consist exclusively of the vector $\mathbf{R}$ associated with the vertical line passing through O, as shown, constituting the *thrust* exerted by the fluid. (Note that the fluid also exerts a distributed force on the cylinder bases, but the symmetry guarantees that these forces cancel each other out and do not contribute to the thrust.)

When a system $\mathcal{V}$ has a null resultant and a null resultant moment with respect to a given point O, it is called a *null system.* In fact, also according to Eq. (3.4), the resultant moment of a null system will be null for all points, and, consequently, for all axes in the space. As Example 4.4 illustrates, the system of all forces acting on the cylinder bases will constitute a null system.

Example 4.5 Consider the vector system $\mathcal{F}$ consisting of three forces and one torque, applied on a disk, described as follows: two forces, $\mathbf{F}$ and $\mathbf{F}'$, both of a magnitude equal to 5N, exerted by the two ropes, fixed at points B and B′ respectively; the weight $\mathbf{P}$ of the disk, vertical and applied on its center, of magnitude 8 N; and the torque $\mathbf{T}$, vertical, applied to the disk, of a magnitude equal to $9\sqrt{3}$ N cm, in the direction indicated (see Fig. 4.4).

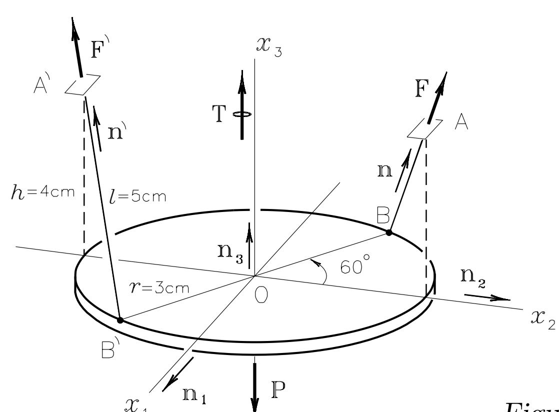

Figure 4.4

Adopting the basis of orthonormal vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, the unitary vectors in the directions of the ropes are

$$\mathbf{n} = \frac{1}{10}(3\sqrt{3}\mathbf{n}_1 + 3\mathbf{n}_2 + 8\mathbf{n}_3); \quad \mathbf{n}' = \frac{1}{10}(-3\sqrt{3}\mathbf{n}_1 - 3\mathbf{n}_2 + 8\mathbf{n}_3)$$

and the forces and torques applied to the disk, expressed on the same basis, are

$$\begin{aligned}
\mathbf{F} &= \frac{1}{2}(3\sqrt{3}\mathbf{n}_1 + 3\mathbf{n}_2 + 8\mathbf{n}_3)\,\mathrm{N},\\
\mathbf{F}' &= \frac{1}{2}(-3\sqrt{3}\mathbf{n}_1 - 3\mathbf{n}_2 + 8\mathbf{n}_3)\,\mathrm{N},\\
\mathbf{P} &= -8\mathbf{n}_3\,\mathrm{N},\\
\mathbf{T} &= 9\sqrt{3}\mathbf{n}_3\,\mathrm{N\,cm}.
\end{aligned}$$

The resultant of the system is

$$\mathbf{R} = \mathbf{F} + \mathbf{F}' + \mathbf{P} = 0.$$

The moment of the force $\mathbf{F}$ with respect to point O can be obtained from

$$\mathbf{M}^{\mathbf{F}/O} = \mathbf{p}^{B/O} \times \mathbf{F} = \frac{3}{2}(4\mathbf{n}_1 + 4\sqrt{3}\mathbf{n}_2 - 3\sqrt{3}\mathbf{n}_3)\,\mathrm{N\,cm}.$$

The moment of force $\mathbf{F}'$ with respect to point O is, likewise,

$$\mathbf{M}^{\mathbf{F}'/O} = \mathbf{p}^{B'/O} \times \mathbf{F}' = \frac{3}{2}(-4\mathbf{n}_1 - 4\sqrt{3}\mathbf{n}_2 - 3\sqrt{3}\mathbf{n}_3)\,\mathrm{N\,cm}.$$

The moment of the weight with respect to O is null, of course, due to the symmetry of the disk, and the resultant moment of the system with respect to the same point is

$$\mathbf{M}^{\mathcal{F}/O} = \mathbf{M}^{\mathbf{F}/O} + \mathbf{M}^{\mathbf{F}'/O} + \mathbf{T} = 0.$$

This is, therefore, a null system. It is easy to see that the resultant moment is null with respect to any other point or with respect to any chosen axis.

2.5 Central Axis

The resultant moments of a vector system $\mathcal{V}$ with respect to all points of a line parallel to its resultant are equal to each other. In fact, if O and O$'$ are two points on a line parallel to the resultant $\mathbf{R}$ (see Fig. 5.1), the product $\mathbf{p}^{O/O'} \times \mathbf{R}$ is null, so, from Eq. (3.4), $\mathbf{M}^{\mathcal{V}/O} = \mathbf{M}^{\mathcal{V}/O'}$.

Hence, moving one point parallel to the resultant of the system the resultant moment does not alter. The resultant moment does change, however, when moving the point in an arbitrary direction.

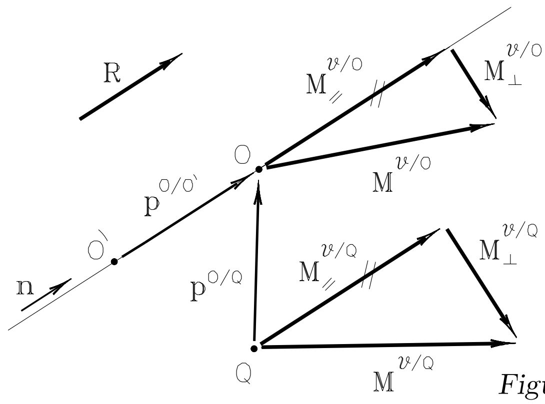

Figure 5.1

Now, if when we move from one point to another we obtain a new resultant moment, it would be useful to consider if there is any particular point for which the resultant moment vanishes. To answer this question, let us take the resultant moment of an arbitrary system $\mathcal{V}$ with respect to a given point O, $\mathbf{M}^{\mathcal{V}/O}$, and let us break it down in the direction of the resultant $\mathbf{R}$ of the system, that is (see Appendix A),

$$\mathbf{M}^{\mathcal{V}/O} = \mathbf{M}_{/\!/}^{\mathcal{V}/O} + \mathbf{M}_{\perp}^{\mathcal{V}/O}, \tag{5.1}$$

where the component of the resultant moment parallel to the resultant is

$$\mathbf{M}_{/\!/}^{\mathcal{V}/O} = \frac{1}{R^2}\,\mathbf{M}^{\mathcal{V}/O}\cdot\mathbf{R}\,\mathbf{R} \tag{5.2}$$

and the component of the resultant moment orthogonal to the resultant is (see Appendix A)

$$\mathbf{M}_{\perp}^{\mathcal{V}/O} = \frac{1}{R^2}\left(\mathbf{R}\times\mathbf{M}^{\mathcal{V}/O}\right)\times\mathbf{R}. \tag{5.3}$$

If Q is an arbitrary point, the vector difference between the resultant moment of $\mathcal{V}$ with respect to Q and O is, according to the moments transport theorem, given by $\mathbf{p}^{O/Q}\times\mathbf{R}$, a vector orthogonal to $\mathbf{R}$. It is then found that, by changing the point, only the orthogonal component, $\mathbf{M}_{\perp}^{\mathcal{V}/O}$, varies, while the parallel component remains invariant, that is,

$$\mathbf{M}_{/\!/}^{\mathcal{V}/O} = \mathbf{M}_{/\!/}^{\mathcal{V}/Q} = \mathbf{M}_{/\!/}^{\mathcal{V}}. \tag{5.4}$$

The conclusion is that the parallel moment is, like the resultant, an *invariant* of the system $\mathcal{V}$. Therefore, if for a given point P the component of the resultant moment of the system parallel to its resultant is different from zero, there will be no other point in the space with respect to which the resultant moment is null. The answer to the question asked previously is, therefore, negative, that is, it is *untrue*, in the most general case, that there is always a point in the space with respect to which the resultant moment of the system is null. Of course, if for a given point P the parallel moment of the system is null, it will be so for any other point.

As seen above, the parallel moment does not depend on the point, but the orthogonal moment varies with it. It would, then, be worth investigating if there is a point P for which the orthogonal moment vanishes, that is, if there is P so that

$$\mathbf{M}^{\mathcal{V}/P} = \mathbf{M}^{\mathcal{V}}_{/\!/}. \tag{5.5}$$

With this objective, basing ourselves on Eq. (3.4) and substituting Eqs. (5.1), (5.5), (5.4), and (5.3) in succession, we have

$$\mathbf{M}^{\mathcal{V}/O} = \mathbf{M}^{\mathcal{V}/P} + \mathbf{p}^{P/O} \times \mathbf{R};$$

hence,

$$\mathbf{M}^{\mathcal{V}/O}_{/\!/} + \mathbf{M}^{\mathcal{V}/O}_{\perp} = \mathbf{M}^{\mathcal{V}}_{/\!/} + \mathbf{p}^{P/O} \times \mathbf{R};$$

therefore,

$$\mathbf{M}^{\mathcal{V}}_{/\!/} + \frac{1}{R^2}\left(\mathbf{R} \times \mathbf{M}^{\mathcal{V}/O}\right) \times \mathbf{R} = \mathbf{M}^{\mathcal{V}}_{/\!/} + \mathbf{p}^{P/O} \times \mathbf{R}. \tag{5.6}$$

Now note that, after the term $\mathbf{M}^{\mathcal{V}}_{/\!/}$, present in both members, is simplified, we obtain a vector equation that is satisfied for all position vectors $\mathbf{p}^{P/O}$, so that

$$\mathbf{p}^{P/O} = \frac{1}{R^2}\mathbf{R} \times \mathbf{M}^{\mathcal{V}/O} + \lambda\mathbf{R}, \tag{5.7}$$

where λ is an arbitrary real number of dimension $\big[\mathrm{L}/\,\mathrm{Dim}[\mathbf{R}]\big]$.

Equation (5.7) describes a straight line parallel to the resultant passing through the point P*, whose position with respect to point O is given by the vector (see Fig. 5.2)

$$\mathbf{p}^{*} = \frac{1}{R^2}\mathbf{R} \times \mathbf{M}^{\mathcal{V}/O}. \tag{5.8}$$

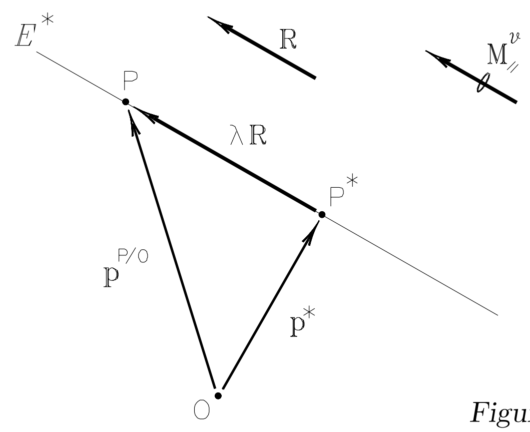

Figure 5.2

This line, the geometric place of the points with respect to which the resultant moment of the system is reduced to the parallel moment, is called the *central axis* of the system. Note that the vector $\mathbf{p}^*$ will exist whenever the resultant of the system is different from zero, that is, the central axis exists for any system that is not a couple or a null system.

The parallel moment, whose magnitude is the least possible among the resultant moments of the system with respect to any point in space, is, for this reason, also called the *minimum moment* of the system and, when the resultant moment with respect to any point O is known, is determined by Eq. (5.2). If the vector system is such that, for any given point O, the resultant moment and resultant of the system are orthogonal, the minimum moment of this system will be null.

Note that P* is the point of the central axis closest to point O. In fact, vector $\mathbf{p}^*$, being orthogonal to $\mathbf{R}$, is perpendicular to the central axis and P* will be the orthogonal projection of O on the axis (see Fig. 5.2).

Example 5.1 Figure 5.3 reproduces the system analyzed in Example 3.2. The parallel moment of this system is, according to Eq. (5.2),

$$\begin{aligned}\mathbf{M}_{/\!/} &= \frac{1}{350u^2}\,[30u\mathrm{m}(\mathbf{n}_1 + \mathbf{n}_3)] \\ &\quad \cdot [5u(-3\mathbf{n}_1 + 2\mathbf{n}_2 + \mathbf{n}_3)]\ [5u(-3\mathbf{n}_1 + 2\mathbf{n}_2 + \mathbf{n}_3)] \\ &= -\frac{30}{7}u\mathrm{m}(-3\mathbf{n}_1 + 2\mathbf{n}_2 + \mathbf{n}_3).\end{aligned}$$

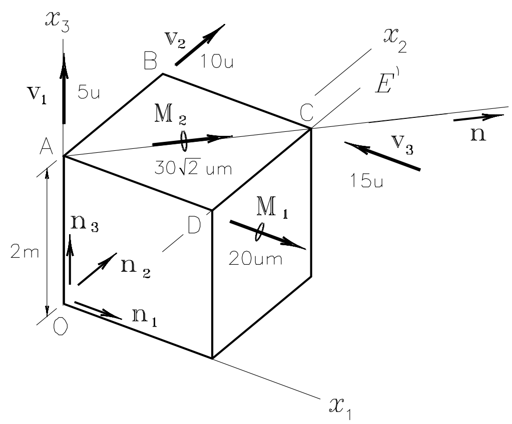

Figure 5.3

The position of the point of the central axis closest to point O is given, according to Eq. (5.8), by the position vector

$$\begin{aligned}\mathbf{p}^* &= \frac{1}{350u^2}\,[5u(-3\mathbf{n}_1 + 2\mathbf{n}_2 + \mathbf{n}_3)] \times [30u\mathrm{m}(\mathbf{n}_1 + \mathbf{n}_3)] \\ &= \frac{6}{7}(\mathbf{n}_1 + 2\mathbf{n}_2 - \mathbf{n}_3)\,\mathrm{m}.\end{aligned}$$

The central axis of the system is then the straight line given by the equation

$$\mathbf{p} = \frac{6}{7}(\mathbf{n}_1 + 2\mathbf{n}_2 - \mathbf{n}_3)\,\mathrm{m} + \lambda(-3\mathbf{n}_1 + 2\mathbf{n}_2 + \mathbf{n}_3),$$

where $\mathbf{p}$ is the position vector, with respect to point O, of an arbitrary point of the axis and λ is a real number, with dimension [L], that parametrizes the straight line.

Every vector system $\mathcal{V}$ with a nonnull resultant can be reduced to a pair of parallel vectors as follows: A sliding vector equal to the resultant of $\mathcal{V}$, associated to the central axis of the system, and a free vector equal to the parallel moment of the system. In other words, the reduction of any system to an arbitrary point on its central axis consists of exactly two of the system's invariants. When $\mathcal{V}$ is a system of forces, its reduction to an arbitrary point on the central axis forms a *wrench*, the name given to a system formed by a force and a torque parallel to each other. An everyday example of a wrench is the action of a screwdriver. In fact, the action of this tool on a screw consists of a force and a torque, both parallel to the axis of the screw. A wrench is said to be *direct* when the force and torque have the same direction (tightening the screw) and to be *inverse* when the directions are opposite (loosening the screw).

Vector systems whose parallel moment is null can be reduced to a single sliding vector, equal to its resultant and associated to the central axis of the system. This is the case of some particular simple systems, as we will see ahead.

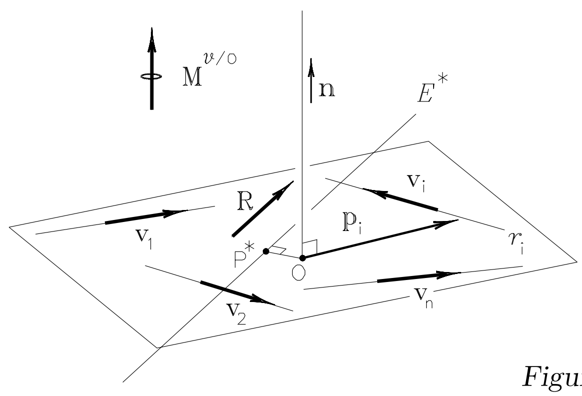

Figure 5.4

A simple system of vectors is called *coplanar* when its lines of action are all contained on the same plane (see Fig. 5.4). On the one hand, the resultant moment of such a system with respect to a point of the plane is necessarily orthogonal to it, since the moment of any of the system's component vectors with respect to a point of the plane is perpendicular to this plane. The resultant of the system, on the other hand, is parallel to the plane, so the parallel moment is null, while the central axis is contained in the plane. The system can, therefore, be reduced to a sliding vector equal to the resultant of the system, associated to the central axis.

Example 5.2 A broad-rimmed hat is laid on a smooth horizontal table. Three lines, fixed to the crown of the hat at points A, B, and C, are pulled horizontally with forces of the same magnitude F, in the directions indicated, skimming the crown of the hat (see Fig. 5.5). We wish to determine a point on the hat rim where a nail must be stuck, so that it does not move. The nail, once it is in place, will prevent the displacement of the point, letting the hat rotate freely around it. The problem is, therefore, to find a point on the rim where the system can be reduced to a single force, with a null resultant moment, thus preventing the hat from rotating.

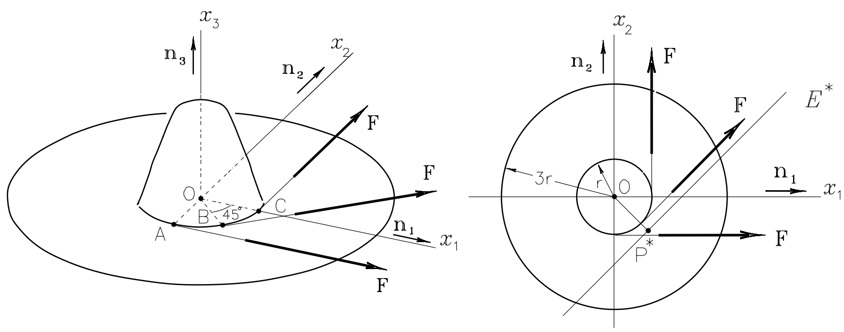

Figure 5.5

The system of forces $\mathcal{F}$ is coplanar and can be reduced to its resultant

$$\mathbf{R} = \frac{2+\sqrt{2}}{2} F(\mathbf{n}_1 + \mathbf{n}_2)$$

applied to a point on the central axis. The resultant moment at point O is

$$\mathbf{M}^{\mathcal{F}/O} = 3Fr\mathbf{n}_3,$$

and a point P* on the central axis can be given by the position vector [see Eq. (5.8)]

$$\begin{aligned}\mathbf{p}^* &= \frac{1}{R^2}\frac{2+\sqrt{2}}{2} F(\mathbf{n}_1+\mathbf{n}_2) \times 3Fr\mathbf{n}_3\\ &= \frac{3}{2+\sqrt{2}}\, r(\mathbf{n}_1 - \mathbf{n}_2)\\ &= 0.879\, r(\mathbf{n}_1 - \mathbf{n}_2).\end{aligned}$$

The central axis will, therefore, be a straight line parallel to **R** passing through P*, as shown in the figure. As the resultant moment with respect to any point of E^* is null, the nail, when fixed at any point on this axis, will react with a horizontal force equal to $-\mathbf{R}$, immobilizing the hat.

A simple vector system is called *parallel* when formed by sliding vectors whose line of actions are all parallel to a given straight line. If **n** is a unit vector characterizing the direction of the system, its resultant is necessarily parallel to **n** and the moment of any of its vectors with respect to an arbitrary point O is orthogonal to $\mathbf{n}$ (see Fig. 5.6). It then follows that the resultant moment and resultant of a parallel system are always orthogonal, independent of the chosen point, so the minimum moment is null and the system can be reduced to a sliding vector equal to the resultant, having the central axis of the system as line of action.

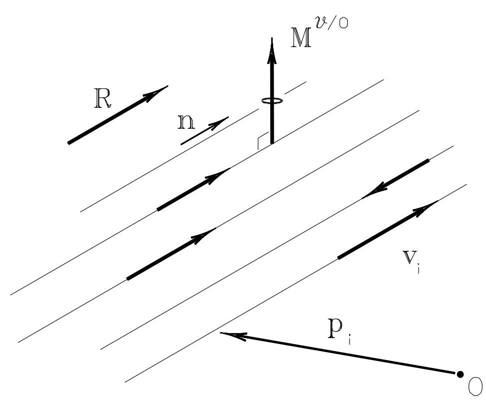

Figure 5.6

Example 5.3 The gravitational force exerted by the earth on a body C close to its surface can, because of the proportions involved, be considered as a parallel distributed system of forces $\mathcal{F}$. The resultant of this system is the weight of the body,

$$\mathbf{P} = \int_C d\mathbf{P} = \int_C \rho g \mathbf{n}\, dV = mg\,\mathbf{n},$$

where ρ is the field of the body's density, g is the magnitude of the gravitational acceleration, $\mathbf{n}$ is the vertical unitary, pointing to the surface, V is the volume, and m is the mass of the body (see Fig. 5.7). The resultant moment of this system with respect to an arbitrary point O is

$$\mathbf{M}^{\mathcal{F}/O} = \int_C (\mathbf{r} \times \rho g \mathbf{n})\, dV = \int_C \rho \mathbf{r}\, dV \times g\,\mathbf{n},$$

where $\mathbf{r}$ is the position vector, with respect to point O, of a generic point C. The central axis of this system will be a vertical straight line described, according to Eq. (5.7), by the position vector with respect to point O:

$$\begin{aligned}\mathbf{p} &= \frac{1}{P^2}\mathbf{P}\times\mathbf{M}^{\mathcal{F}/O} + \lambda\mathbf{P}\\ &= \frac{1}{m}\mathbf{n}\times\left(\int_C \rho\mathbf{r}dV\times\mathbf{n}\right) + \lambda mg\mathbf{n}\\ &= \frac{1}{m}\int_C \rho\mathbf{r}dV - \frac{1}{m}\int_C \rho\mathbf{r}dV\cdot\mathbf{n}\,\mathbf{n} + \lambda mg\mathbf{n}.\end{aligned}$$

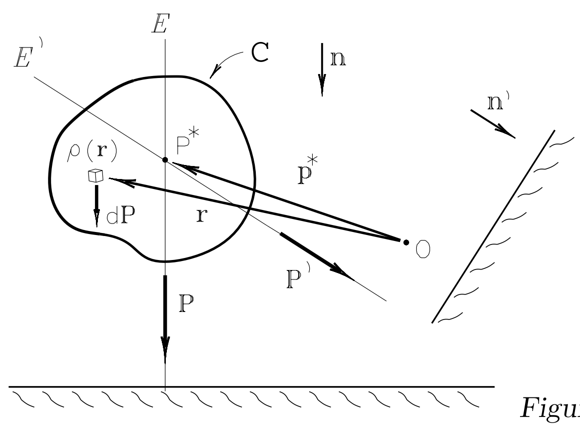

Figure 5.7

Note that the first term of the last line expresses nothing more than the position vector, with respect to O, of the mass center of the body (see Section 1.6),

$$\mathbf{p}^* = \frac{1}{m}\int_C \rho\mathbf{r}dV,$$

while the other two terms are vectors parallel to $\mathbf{n}$ and may be grouped in the form $\beta\mathbf{n}$, where β is an arbitrary scalar. The conclusion, then, is that the central axis of the system of gravitational forces on a body close to the earth's surface is a vertical line that passes through the mass center of the body. Now, modifying the orientation of the body in relation to the earth, only the orientation of the unitary $\mathbf{n}$ ($\mathbf{n}'$) in relation to the body is modified, with the new central axis parallel to $\mathbf{n}'$, passing through the mass center of C (see Fig. 5.7). Now, as the orientation given to the body was arbitrary, the result is that the central axes of all possible configurations will cross each other in the mass center of the body, by which we can, in any case, reduce the gravitational action of the earth on a small body close to its surface, to its weight applied to the mass center of the body.

When the lines of action of a simple vector system all converge at one point, we have a *concurrent system*. The resultant moment of the system with respect to the concurrence point will, naturally, be null, and the central axis of the system will then necessarily pass through the point. Every concurrent system can, therefore, be reduced to a sliding vector equal to its resultant associated to a line of action passing through the concurrence point. (This result is known as *Varignon's theorem.*)

Example 5.4 Consider the system of gravitational forces exerted by a particle P, of mass M, on a homogeneous bar AB, of mass m and length c, in the configuration shown in Fig. 5.8. This is a distributed simple system consisting of the forces of attraction $d\mathbf{F}$ between P and each element of mass $dm = \frac{m}{c}dy$, all with a support passing through P.

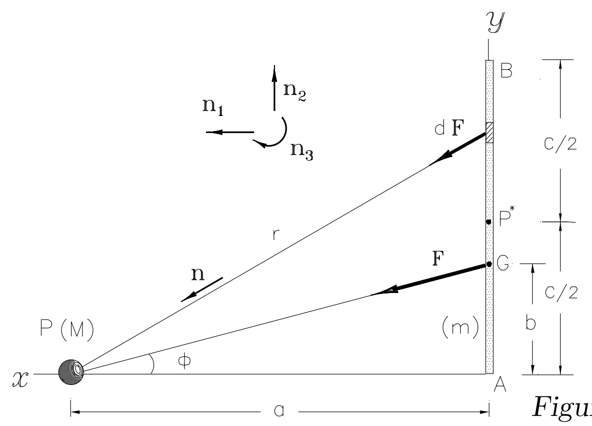

Figure 5.8

Each element of force is, according to the universal gravitational principle, Eq. (1.3.5),

$$\begin{aligned} d\mathbf{F} &= \frac{GMdm}{r^2}\,\mathbf{n} \\ &= \frac{GMm\,dy}{c(a^2+y^2)^{3/2}}\,(a\mathbf{n}_1 - y\mathbf{n}_2). \end{aligned}$$

The resultant gravitational force is then

$$\mathbf{F} = \int_0^c d\mathbf{F} = \frac{GMm}{a(a^2+c^2)^{1/2}}\left[\mathbf{n}_1 + \left(\frac{a}{c} - \left(1+\frac{a^2}{c^2}\right)^{1/2}\right)\mathbf{n}_2\right].$$

As the system is concurrent at P, it can be reduced to a gravitational force equal to the resultant calculated above, passing through P. The line of action of this force intercepts the bar at point G, center of gravity of the bar for the gravitational field exerted by particle P. The distance b from this point to the end A of the bar is

$$b = a\tan\phi = a\left[\left(1+\frac{a^2}{c^2}\right)^{1/2} - \frac{a}{c}\right] = \frac{a^2}{c}\left[\left(1+\frac{c^2}{a^2}\right)^{1/2} - 1\right].$$

One can see that point G lies between A and P*, mass center of the bar, that is, that $b < c/2$, which is equivalent to

$$\frac{a^2}{c}\left[\left(1+\frac{c^2}{a^2}\right)^{1/2} - 1\right] < \frac{c}{2},$$

or

$$1+\frac{c^2}{a^2} < \frac{c^4}{4a^4} + 1 + \frac{c^2}{a^2};$$

therefore,

$$\frac{c^4}{4a^4} > 0,$$

which is always true. The result, then, is that the center of gravity is situated below the mass center of the bar. This result clearly shows that the center of gravity and the mass center of a body are different concepts. The latter depends exclusively on the distribution of the body mass while the former depends also on the nature of the present gravitational field. Of course, as we saw in Example 5.3, both coincide in the case of the earth's gravitational attraction on a body of small dimensions close to its surface. One can also easily see that, in the case under study, G becames closer to P* when the a/c ratio increases. The reduction of the gravitational field at the mass center of the bar will consist of the gravitational force $\mathbf{F}$ applied to P* and a gravitational torque equal to the resultant moment of the system with respect to P*. Using Eq. (3.4), this torque is

$$\mathbf{M}^{\mathcal{F}/P^*} = \mathbf{p}^{G/P^*} \times \mathbf{F} = \frac{GMm(c-2b)}{2a(a^2+c^2)^{1/2}}\,\mathbf{n}_3.$$

2.6 Forces and Torques

The first step to be taken to establish the equations that govern the motion of a mechanical system — whether it is a simple particle moving on a plane or a mechanism with multiple interconnected bodies in three-dimensional motion — is to identify the set of forces and torques acting on it. For the sake of simplicity, we will call the system of vectors consisting of forces and torques acting on a mechanical system a *force system*. Once the forces and torques acting on the subject of interest are identified, it is necessary to choose a point to reduce the system; the choice of this point depends on the nature of the encountered force system itself, as well as on kinematic and inertia properties of the body or bodies under study. The general guidelines for choosing the most suitable point for reducing a system, therefore, will not be discussed in this chapter; the matter will be duly discussed later.

Interactions between mechanical elements occur through forces and torques. As discussed in Chapter 1, the concept of force is assumed as primitive in mechanics, the same as in the case of the torque concept. Even though the moment of a force applied with respect to a given point is a torque, applied torques can be considered separately from the existence of force systems that consist of couples with those torques. This is, indeed, the general treatment adopted in Section 2.3 to define vector systems.

The interaction between two particles occurs by means of a force. Thus, given two particles P and Q, Newton's third law states that P exerts on Q a force $\mathbf{F}_{QP}$, associated to the line of action passing through Q and P, while Q exerts on P a force $\mathbf{F}_{PQ}$, also associated to the same line of action (see Fig. 6.1), satisfying the relationship

$$\mathbf{F}_{QP} = -\mathbf{F}_{PQ}. \tag{6.1}$$

It is convenient to classify the interaction forces in two categories, as follows: *field forces*, or distance action, occurring when the particles are not in contact; and *contact forces*, which are those from direct contact, which only occur when the relative position vector between the particles is null. The former includes the forces of gravita-

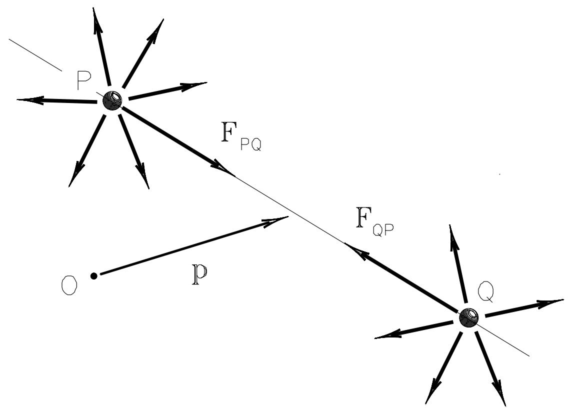

Figure 6.1

tional attraction and electromagnetic fields, among others; in the latter, as examples, are collision and friction forces.

In the mechanical model of a particle, the force system acting on a given particle will always be a concurrent simple system (see Fig. 6.1). Such a system is equivalent, as stated, to a force equal to its resultant applied on the concurrence point. In the case of a particle, therefore, the generally most suitable point for reducing the system is the particle itself.

Given an arbitrary point O, it is trivial that the moments with respect to O of the interaction forces between two particles P and Q satisfy the relationship

$$\mathbf{M}^{\mathbf{F}_{QP}/O} = -\mathbf{M}^{\mathbf{F}_{PQ}/O}, \tag{6.2}$$

that is, the moments with respect to any point are, like the forces, equal and contrary. In fact, the moments result from vectorial products of the same position vector $\mathbf{p}$ (see Fig. 6.1) with equal and opposite force vectors.

Example 6.1 Consider the set of four small spheres of different masses, at rest, laid on a smooth horizontal plane and interconnected by four wires, as shown in Fig. 6.2a. When the horizontal force $\mathbf{F}$ is applied, tractions occur on the wires, each sphere undergoing the forces indicated in Fig. 6.2b. On analyzing it a little more closely, one finds that, besides the forces parallel to the plane, each sphere also undergoes vertical forces, as shown in Fig. 6.2c, for sphere D.

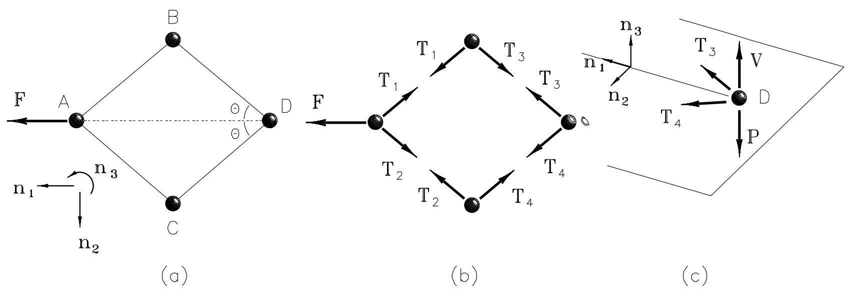

Figure 6.2

The force system acting on D consists of its vertical weight **P**, a field force; the vertical force V, exerted by the smooth plane, a contact force; horizontal tractions $\mathbf{T}_3$ and $\mathbf{T}_4$, exerted by the wires, that can be interpreted as contact forces if we include them as elements without mass, but belonging to the system or modeled as field forces exerted by spheres B and C, respectively. The resultant of the force system acting on D is

$$\mathbf{R} = (T_3 + T_4)\cos\theta\mathbf{n}_1 + (T_4 - T_3)\sin\theta\mathbf{n}_2 + (V - P)\mathbf{n}_3.$$

The system is equivalent, therefore, to **R** applied on D. The reader should not find it hard to analyze the force systems acting on each of the other spheres.

Interaction between two bodies (rigid or otherwise) occurs by means of forces and torques. It therefore requires more careful handling than the interaction between particles, generally involving nonsimple vector systems. When there is interaction without mutual contact, we have a *distance action system.* For example, the gravitational field established between two bodies of arbitrary geometry and whose dimensions are around the same size as the distance between their centers is not generally reducible to a single force. Thus, contrary to what is seen in Example 5.3 — where the system is parallel — and Example 5.4 — where the system is concurrent — the reduction at any point of a body of the gravitational field exerted by another consists of a gravitational force *and* a gravitational torque.

When two bodies have a point or region of their surfaces touching each other, one has a *contact system.* If there is a single point of mutual contact, that is, a single point P of a body C coinciding with point P′ of another body C' (see Fig. 6.3), one has, as in the model of a particle, a concurrent simple system at the point of contact. There is not, therefore, in this case, application of a torque between the bodies (although, of course, there could be a resultant moment with respect to a point in space other than the point of contact).

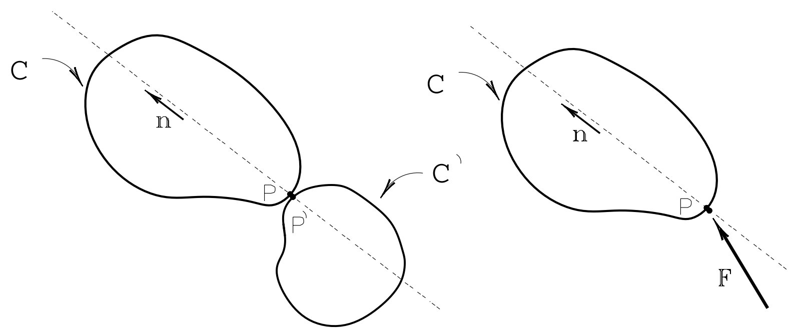

Figure 6.3

When two bodies have a line or region of their surfaces in contact, the interaction occurs through a more general force system. It is always convenient to model this interaction by reducing this system to a point P representing this contact (see Fig. 6.4). Reduction will consist, in the most general case, of a force $\mathbf{F}$ (equal to the resultant of the system) applied to the chosen point and a torque $\mathbf{T}$ (equal to the resultant moment of the system with respect to the point) that, merely for convenience and clarity, is also represented as if applied to the point. By adopting, as usual, an orthonormal basis $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ for the decomposition of the vectors, let us say that the contact interaction is modeled by three mutually orthogonal forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$, and three also mutually orthogonal torques, $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ in the directions of the chosen basis, as shown in Fig. 6.4.

The contact between two bodies is also called a *link*. The nature of the link will be given by the present force system. When, at the contact between two bodies, the three force components and the three torque components are different from zero for an arbitrary orientation of the base, there is a *rigid link*. This is what happens in the case of welding or fixing. (Although there may be local deformations in the link, we

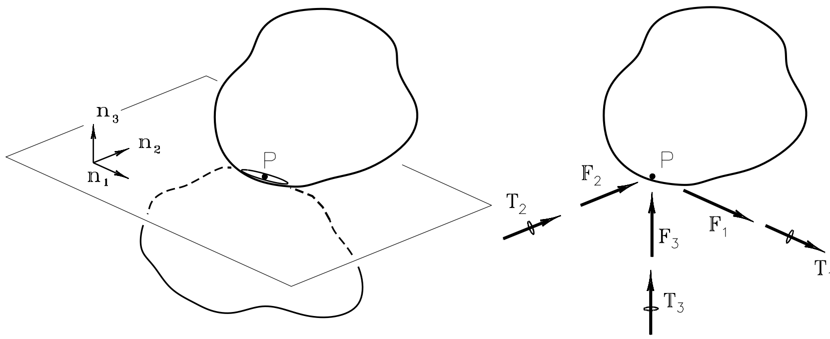

Figure 6.4

will keep the name *rigid link* characterizing the presence of forces and torques in the three directions.) The nature of a link depends on the *kinematic constraints* that the link imposes. We understand a kinematic constraint to be a displacement or a rotation that the presence of the link prevents. Every link can, therefore, be modeled by the displacements and rotations that it does not admit. The rigid link does not admit any relative displacement or rotation between the bodies in contact; hence it is modeled by a resultant force of an arbitrary direction (three components) and a resultant torque also of an arbitrary direction. For each free displacement admitted by a link, the component of the resultant force in the direction of the free displacement is null; for each free rotation admitted by a link, the resultant torque component in the direction of the free rotation is null. Of course, the reduction of the number of force or torque components will depend on the right choice of coordinated directions, that is, if a given direction of movement is free, in order to suppress the respective force or torque component, it is necessary that the direction corresponds to one of the chosen coordinated directions.

Appendix B provides a table of the models usually adopted for the more commonly found links, indicating the respective nonnull components to be considered, at least in principle. Only practice will give the reader confidence to properly identify the relevant components in each case. As a general rule, it is recommended to start by considering *all* six components, then duly eliminating those that correspond to the free displacements and rotations that the link admits. The configurations in Appendix B, far from including all cases, only give the main models adopted. Combinations of these are common; in this case, the components to be considered are the intersection of the sets of components of each link. For example, a ball and socket joint, which has only three force components, mounted on a rectangular slide consisting of two force and three torque components (see Appendix B), results in a link modeled by only two force components.

When one wishes to study the motion of a body C that is bound to other mechanical elements, we start, as already mentioned, by identifying the force system acting on C. Each link must, therefore, be *substituted* by the forces and torques that characterize it. This procedure is called *body isolation*, and the geometric representation of the reductions at the points representing the links is called a *free body diagram*.

Example 6.2 Bar B is linked to the guide A by means of a mechanism that includes a pivot and a runner (see Fig. 6.5a).

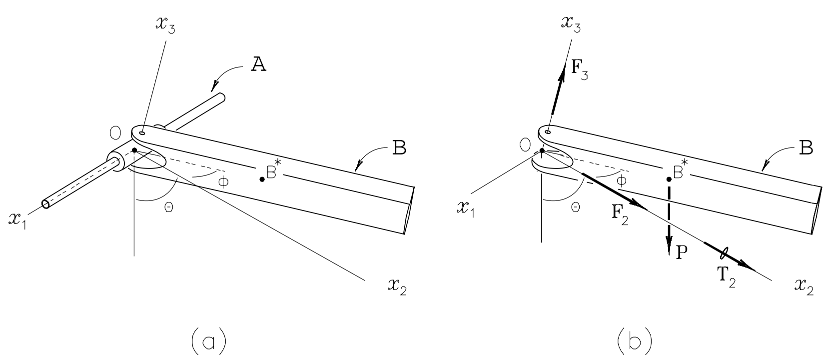

Figure 6.5

Point O, the intersection of the axes of the bar and guide, can be chosen to represent the contact and origin of a system of coordinates for the decomposition of the force and torque vectors involved. This is a compound link; the runner permits free displacement in the direction x_1 and free rotation in the same direction; the pivot admits a free rotation in the direction x_3 (see Appendix B). The force component in the direction x_1 and torque components in the directions x_1 and x_3 will vanish. Taking B, then, as the subject for study, its link with the guide is modeled by a system of forces whose reduction at point O comprises a force applied on O, with components $\mathbf{F}_2$ and $\mathbf{F}_3$ and one torque, $\mathbf{T}_2$. Figure 6.5b illustrates the free body diagram of the bar in which the gravitational action $\mathbf{P}$ is included. Note that the orientation of the axes was chosen to bring to the fore the suppression of the null components ($\mathbf{F}_1$, $\mathbf{T}_1$, and $\mathbf{T}_3$).

2.7 Friction

When two bodies touching at a single point have as a bound force only one component orthogonal to the tangent plane common to their surfaces at the point of contact, this force is called *normal force*, $\mathbf{N}$, and the contact surfaces are said to be *smooth.* When, on the contrary, there are components parallel to the tangent plane, the contact is said to be *with friction* and these components, added up vectorially, form the *friction force*, $\mathbf{F}_a$ (see Fig. 7.1).

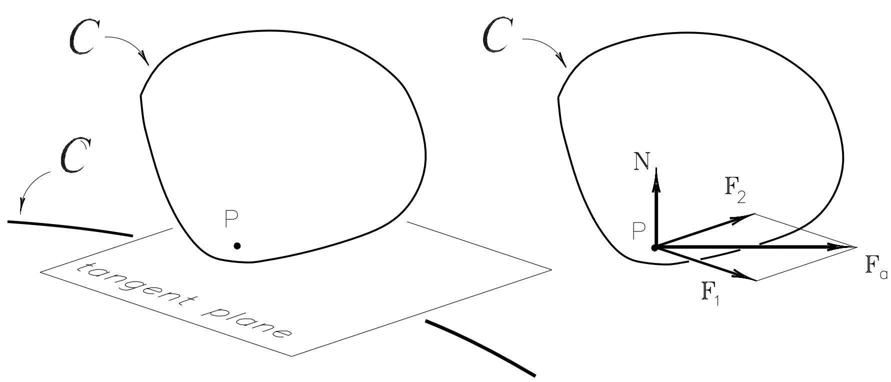

Figure 7.1

When two solid bodies have surfaces touching each other without the presence of a fluid, it is then said that *dry friction* or *Coulomb friction* is present. In this model, the friction component acting on one of the bodies is always a force whose direction is opposite to the relative motion between the surfaces, when sliding occurs, or on its *tendency* of motion where there is none. We understand that this motion tendency is the one to be expected if the contact were smooth, which sometimes is not easy to identify. In order to establish the direction of a tendency to the slide, it is necessary, in most cases, to analyze other links involved, as we will see in the following examples. When the direction of the friction force is unknown, the alternative is to consider two mutually orthogonal force components parallel to the plane tangent to the surface.

In the dry friction model without relative sliding, it is considered that the friction force magnitude has an upper limit depending linearly on the normal component present in the contact, a condition expressed by the inequality

$$|\mathbf{F}_a| \leq \mu|\mathbf{N}|, \tag{7.1}$$

where $\mathbf{F}_a$ is the friction force, $\mathbf{N}$ is the normal force, and the adimensional constant μ, called the *friction coefficient*, expresses, in a simplified form, the complex interaction between two rough surfaces, depending on the material and surface finishing of the bodies in contact. Equation (7.1), consisting of an inequality, merely establishes an upper limit value for the magnitude F_a of the component of the contact force parallel to the tangent plane, a function of the magnitude N of the component orthogonal to the plane. The effective value of the friction force, nonetheless, may only be determined from the dynamic solution of the problem.

Example 7.1 The isosceles triangular plate is at rest, with its vertices A, B, and C lying on the sloping plane π, also having its vertex A pivoted on the fixed pin P, as illustrated in Fig. 7.2a. The action of the pin on the plate consists of the force components $\mathbf{A}_1$ and $\mathbf{A}_2$, parallel to the plane, and there are no torque components (why?). As the pin prevents the vertex A from moving, that is, it inhibits any tendency to sliding, the action of the plane on this support is reduced to the normal $\mathbf{N}_A$. Assuming that the plane is not smooth, the contact on the other two vertices will include normal and friction components. Due to pivoting, the tendency to sliding of the support at B is orthogonal to the edge AB, hence the arbitrated direction for the friction force $\mathbf{F}_B$ (see Fig. 7.2c). Similarly, the friction force $\mathbf{F}_C$ will be orthogonal to the edge AC. Figure 7.2b shows a similar situation, differing, however, by the link at the pin, which is no longer a pivot but a simple contact that we will consider smooth. The action applied by it on the plate can be reduced, then, to force N, parallel to the plane and orthogonal to edge CA, applied at vertex A. The action of the plane on the vertices can be modeled in this way: On vertex A there is no tendency to sliding in the direction orthogonal to the edge AC, due to the

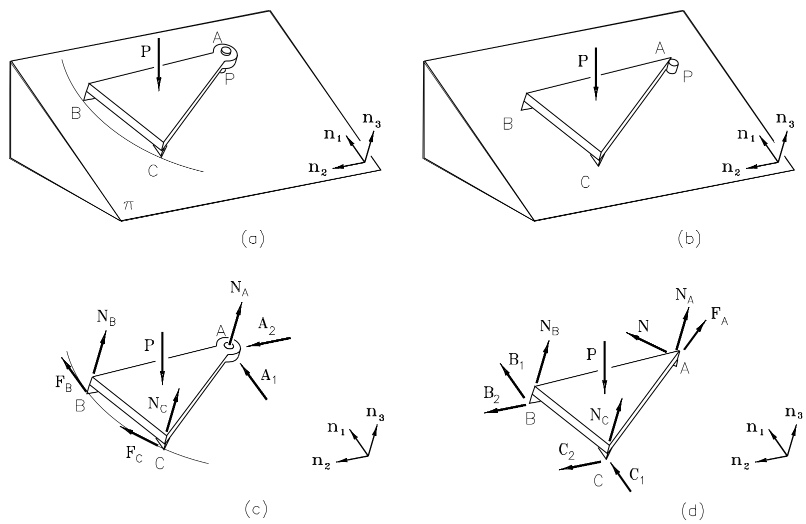

Figure 7.2

presence of the pin, with the result that the contact of the plane is reduced to the normal $\mathbf{N}_A$ and to the friction force $\mathbf{F}_A$, parallel to that edge (see Fig. 7.2d); at vertex B there is a sliding tendency in an unknown direction (at least in principle) and the contact must be modeled with three components (normal, $\mathbf{N}_B$, and friction, $\mathbf{B}_1$ and $\mathbf{B}_2$); at vertex C, as with B, we have the normal, $\mathbf{N}_C$, and friction components, $\mathbf{C}_1$ and $\mathbf{C}_2$.

Also with regard to the above example, it is worth noting that, in both situations studied, the signs chosen for the friction components are arbitrary. In other words, only the directions to be considered in each case are part of the links modeling; magnitudes and signs may only be determined — and not always fully — after a dynamic analysis of the problem. (Of course, there are situations, such as the weight of a body or a normal exerted by a simple support, where the sign is known.) The reader can always infer, using strictly personal guidelines (common sense, experience, etc.), the direction of an unknown link force; the final result of the dynamic analysis will indicate, by the sign, if the choice is right or not. It is recommended, when there is no clear indication which sign is correct, to infer components in the positive direction of the Cartesian axes adopted; anyhow, the choice will not entail any error.

In the preceding example, it is assumed that there is no sliding of the plate. In principle, therefore, there is not necessarily any relation between the normal and friction components at each link. When the touching surfaces between two bodies have relative motion, it is said that there is dynamic friction; the model, in this case, assumes a linear relationship between the friction and normal components, in the form

$$|\mathbf{F}_a| = \mu'|\mathbf{N}|, \tag{7.2}$$

where μ' is a constant, called the *dynamic friction coefficient*, and is generally dependent on the material and surface finishing of both bodies in contact. If there is sliding, the friction force will always be opposite to the relative motion.

Example 7.2 The homogenous bar B relies on the inclined plane π and pivoted on the pin P, fixed on the plane (see Fig. 7.3). B moves over the plane, turning around P, under the action of gravity. This is reducible, as we have already seen, to its weight $\mathbf{P}$, applied to B^*, the mass center of B. As the bar is homogenous and is fully supported by the plane, the normal force exerted by it can be considered uniformly distributed, as shown in Fig. 7.3b. The resultant of this distribution will be

$$\mathbf{N} = \int_0^c d\mathbf{N}.$$

The friction component exerted by the plane will also be distributed along the bar, having as a resultant

$$\mathbf{F}_a = \int_0^c d\mathbf{F}_a.$$

As there is sliding, each friction force element will have an opposite direction to the motion of the respective point in relation to the plane, resulting in a parallel distribution; on the other hand, sliding guarantees that $|d\mathbf{F}_a| = \mu'|d\mathbf{N}|$, with the result that the distribution, besides parallel, is uniform, as shown in Fig. 7.3b. It is easy to see that the central axis of this distributed system passes through B^*; the action of the plane on the bar can, then, be reduced to $\mathbf{F}_a = \mu' N\mathbf{n}_1$ and $\mathbf{N} = N\mathbf{n}_3$ applied at B^*, as indicated in Fig. 7.3c.

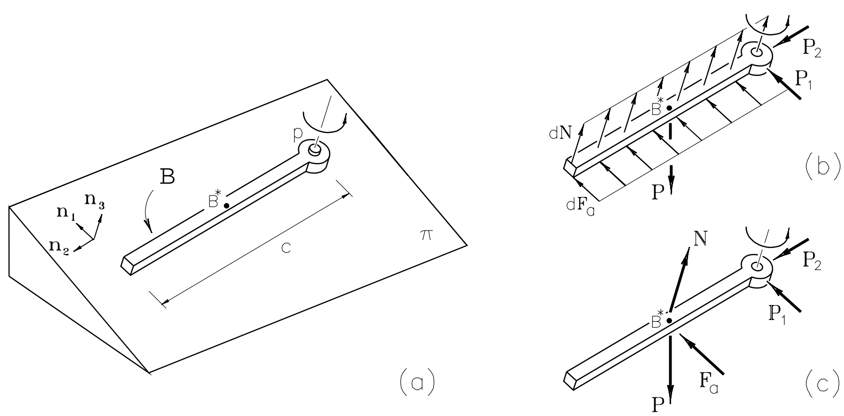

Figure 7.3

When a fluid intervenes in the contact between the bodies (typical case of contact with a lubricant), one then has *viscous friction.* In this model, much more complex than that of dry friction, the friction force depends, essentially, on the relative velocity between the surfaces in contact and the viscosity and thickness of the fluid film between the surfaces. The viscous friction will not be studied here. The following example, illustrating quite a simple case, intends to give only a general idea of the difference in treatment given to the viscous friction model compared to that of dry friction.

Example 7.3 Two flat plates are displaced at a constant relative velocity v, as illustrated in Fig. 7.4, with an oil film of uniform thickness e completely filling the region of mutual contact over area A.
The shearing stress inside the fluid (the dragging force per unit of area exerted by a layer of fluid on its neighboring layer) is given by the ratio

$$\tau = \mu \frac{\partial v_x}{\partial y},$$

where τ is the shearing stress, in the direction of the relative motion, with dimension $[\mathrm{ML}^{-1}\mathrm{T}^{-2}]$, μ is the viscosity of the fluid, with dimension $[\mathrm{ML}^{-1}\mathrm{T}^{-1}]$, and $\partial v_x/\partial y$ is the gradient, in the direction normal to the motion, of the velocity of the fluid. This model of linear relationship between the shearing stress and the velocity gradient is attributed to Newton, and the fluids that satisfy this hypothesis are called *Newtonian fluids.* For

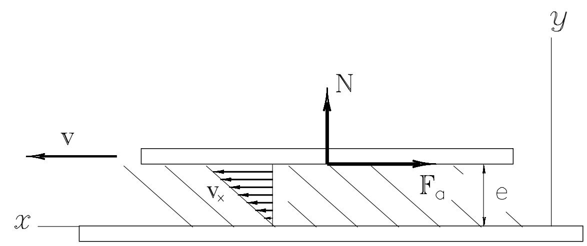

Figure 7.4

a fine film (small e) moving on a steady state (constant v), the velocity profile in the fluid can be assumed to be linear, as shown, and

$$\frac{\partial v_x}{\partial y} = \frac{v}{e}.$$

The magnitude of the friction force applied on the upper plate will then be

$$F_a = \tau A = \mu A \frac{\partial v_x}{\partial y} = \frac{\mu A v}{e}.$$

Note that, in this model, there is no ratio established between the magnitude of the friction force, F_a, and the normal force, N, present between the surfaces in contact.

As mentioned at the beginning of the previous section, the correct modeling of the *force system* acting on an element or a mechanical system whose motion we wish to study is the starting point for a successful solution. If a force or torque component is not considered at this initial stage of analysis, this will give a wrong result; if a component is unduly included, although this would not introduce an error, it will hinder or even make the solution unfeasible, unnecessarily increasing the number of unknown quantities to be determined. It is, therefore, desirable to take special care when modeling the links and distance action systems. A careful study of Appendix B may be of value to the reader.

Example 7.4 Figure 7.5a illustrates a disk welded to a horizontal axis, moving around a second vertical axis in the indicated direction. The link between the axes consists of a joint. The disk relies on the horizontal plane, rolling over it under the action of the torque $\mathbf{T}$, parallel to x_2, as shown, with its center B describing a circular path around point A. Figure 7.5b shows the diagram of the corresponding free body.

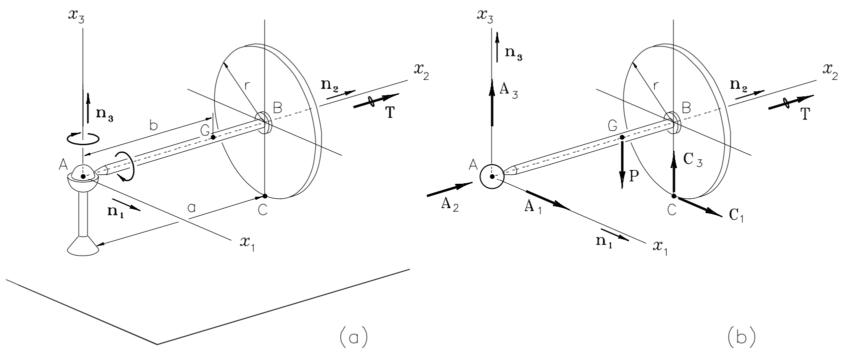

Figure 7.5

The forces $\mathbf{C}_3$ (normal) and $\mathbf{C}_1$ (friction) act on the point at which the disk touches the plane. Note that a friction component was not included in the direction of x_2 because, if the contact were smooth, point C would describe a circular path, parallel to the center of the disk. Vertical weight $\mathbf{P}$ acts on point G, the mass center of the set. Three force components, $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$, are applied to point A. As there are no other efforts to be considered, we have a system acting on the set consisting of six forces and one torque, whose resultant is

$$\mathbf{R} = (A_1 + C_1)\mathbf{n}_1 + A_2\mathbf{n}_2 + (A_3 + C_3 - P)\mathbf{n}_3$$

and whose resultant moment with respect, say, to point A is

$$\begin{aligned}\mathbf{M}^{\mathcal{F}/A} &= \mathbf{p}^{C/A} \times (\mathbf{C}_1 + \mathbf{C}_3) + \mathbf{p}^{G/A} \times \mathbf{P} + \mathbf{T}\\ &= (aC_3 - bP)\mathbf{n}_1 + (T - rC_1)\mathbf{n}_2 - aC_1\mathbf{n}_3.\end{aligned}$$

Example 7.5 Plate P, with mass m and mass center P*, is pivoted on O on the fork G that, in its turn, can revolve around the fixed bearing M (see Fig. 7.6a). A light rope, with one end fixed at point A, is being stretched as shown. The Cartesian axes $\{x_1, x_2, x_3\}$ have origin in O, with x_2 aligned with the bearing axis and x_3 orthogonal to the plate.

The orthonormal basis $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ is parallel to the chosen coordinated axes. Adopting this basis for the decomposition of the vectors, the force system applied to P will consist of three force components, $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, exerted by the pivot on O; vertical weight $\mathbf{P}$, applied to P*, traction $\mathbf{F}$, in the direction of the rope, applied to A; and the torque $\mathbf{T}_1$, exerted by the link on O, which permits free rotations in the directions $\mathbf{n}_2$ and $\mathbf{n}_3$ (see Fig. 7.6b).

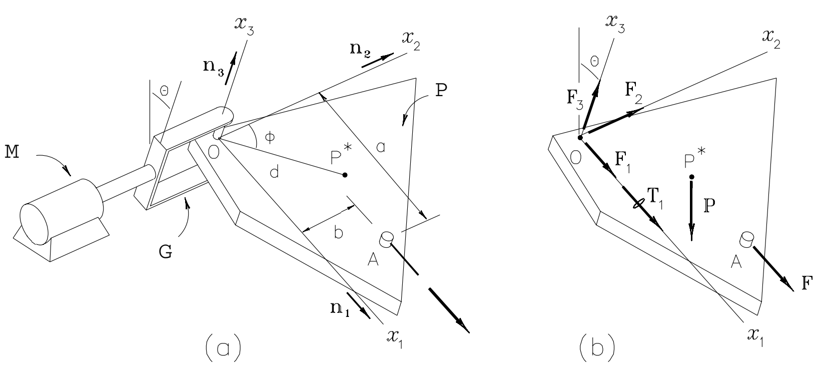

Figure 7.6

The resultant of this system is

$$\mathbf{R} = (F + F_1 + mg\sin\theta)\mathbf{n}_1 + F_2\mathbf{n}_2 + (F_3 - mg\cos\theta)\mathbf{n}_3.$$

Its resultant moment with respect to point O is

$$\begin{aligned}\mathbf{M}^{\mathcal{F}/O} &= \mathbf{p}^{P^*/O} \times \mathbf{P} + \mathbf{p}^{A/O} \times \mathbf{F} + \mathbf{T}_1 \\ &= (T_1 - mgd\cos\phi\cos\theta)\mathbf{n}_1 + mgd\sin\phi\cos\theta\mathbf{n}_2 \\ &\quad - (Fb + mgd\cos\phi\sin\theta)\mathbf{n}_3.\end{aligned}$$

When a particle is fixed, that is, at rest, it is said that it is in *equilibrium.* The condition for the equilibrium of a particle P is that the resultant of the system of forces acting on P is null, that is,

$$\mathbf{R} = 0 \qquad \text{if P is in equilibrium.} \tag{7.3}$$

In fact, every force system acting on a particle is necessarily a concurrent simple system and, therefore, equivalent to a force equal to its resultant, applied to the particle. Now, Newton's first law establishes exactly that if the resultant of the forces applied to a particle is null, then it will remain at rest or in uniform motion. (Strictly speaking, it will remain at rest or in uniform motion on an inertial reference frame, but this subject involving of the reference frames will be discussed later.) Although both conditions characterize the equilibrium, the equilibrium is usually understood as the condition of rest.

The force system acting on a rigid body is not, as already mentioned, necessarily simple or concurrent. It so happens that stricter conditions must be imposed to characterize its equilibrium. In fact, it is said that a rigid body C is in equilibrium when three of its noncolinear points are fixed. The condition for such a situation to occur is that the force system $\mathcal{F}$ acting on the body is a null system. In other words, a rigid body will be in equilibrium, that is, all its points will be at rest, when the resultant force *and* the resultant moment with respect to some point are both null.

$$\mathbf{R} = 0 \qquad \text{and} \qquad \mathbf{M}^{\mathcal{F}/O} = 0 \qquad \text{if } C \text{ is in equilibrium,} \tag{7.4}$$

where O is an arbitrary point. This result may only be strictly shown further in the text, in Chapter 7, when the dynamics of the rigid body will be studied, but we can assume it as true for this section's requirements.

Example 7.6 Figure 7.7 reproduces the situation studied in Example 4.5

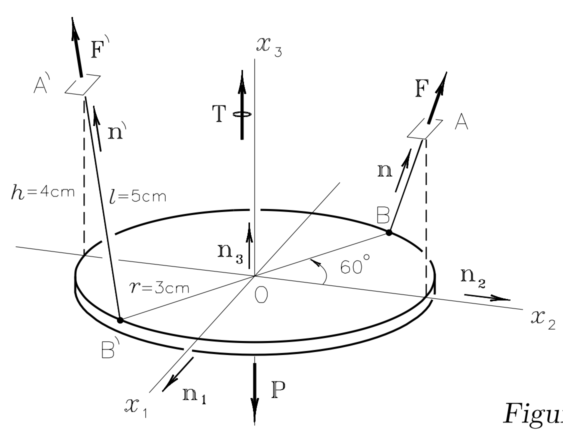

Figure 7.7

(take another look at it). The disk undergoes a force system $\mathcal{F}$, consisting of the forces $\mathbf{P}$, $\mathbf{F}$, and $\mathbf{F}'$ and torque $\mathbf{T}$. As determined in the example, the resultant force $\mathbf{R}$ is null and the resultant moment with respect to point O, $\mathbf{M}^{\mathcal{F}/O}$, is also null. The force system $\mathcal{F}$, therefore, is a null system and the disk is in equilibrium. In fact, the disk would be at rest under the action of the weight and vertical tractions on the ropes if torque $\mathbf{T}$ was not there. By applying the torque, the resultant moment with respect to point O will be compensated by the moment produced by the horizontal components of the tractions on the ropes, while the weight will be counterbalanced by the vertical components of the tractions on the ropes, resulting in a null system.

Example 7.7 Returning now to Example 5.2 (take another look at it), the force system consisting of the three forces of magnitude F applied to the crown of the hat, as illustrated in Fig. 7.8, plus the force

$$\mathbf{P} = -\frac{2+\sqrt{2}}{2}F(\mathbf{n}_1 + \mathbf{n}_2)$$

exerted by the nail stuck in point $\mathrm{P}^* = 0.879\,r\,(1, -1, 0)$ will be a system with both a null resultant and resultant moment, thus guaranteeing the hat's equilibrium.

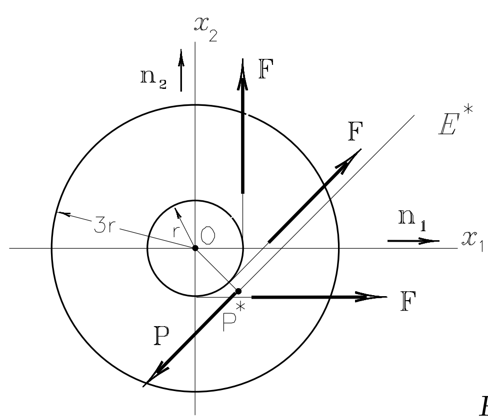

Figure 7.8

Exercise Series #2 (Sections 2.1 to 2.6)

P2.1 A horizontal force of 40N is applied to the arm of the torquimeter, as shown. Determine the magnitude of the moment produced by this force at point O. What is the component of this moment that loosens the screw?

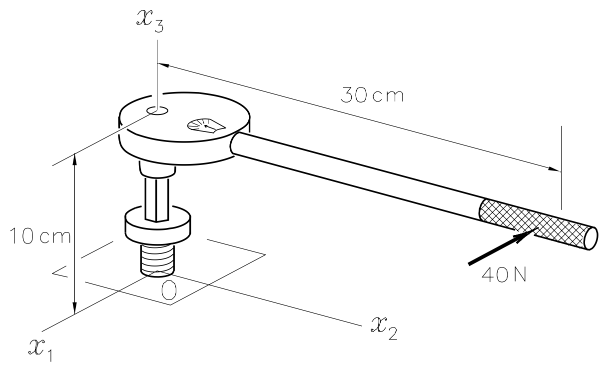

Figure P 2.1

P2.2 In order to drill a wall with the help of a bit, a vertical force of 90N is applied to the arm B and a horizontal force of 60N on handle C, as shown. What is the vertical force that must be applied to the handle so as not to bend the bit at A? What is the reduction of this new system to point A?

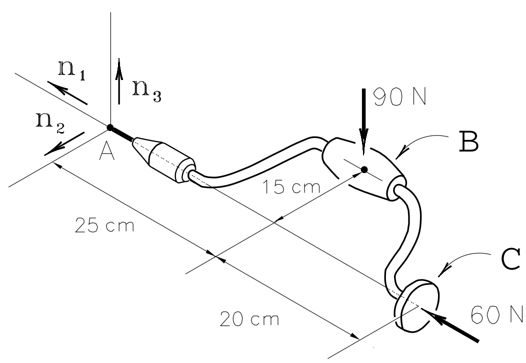

Figure P 2.2

P2.3 A system of forces and torques, consisting of the weight P, the forces on supports A and B and the torques at the input and output axes, equal to 20 N m and 50 N m, respectively, is applied to the gear box. For the box to stay motionless, this system must be null. Determine the forces on supports. If the input torque is increased to 25 N m, it is necessary to screw down support B. Calculate the effort on this screw. (Assuming that there are no losses, the input and output powers always remain equal to each other.)

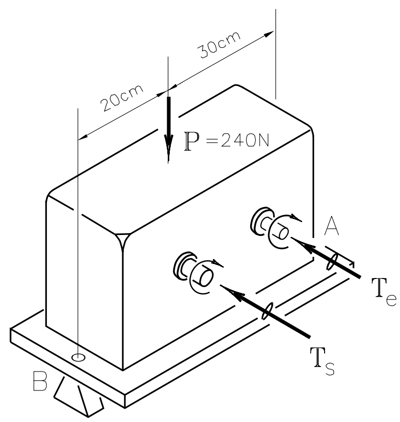

Figure P 2.3

P2.4 A system $\mathcal{S}$ consists of four bound vectors, whose components and respective points of application on a given system of Cartesian axes are $\mathbf{v}_1 = (2, 0, 0)$ in $(0, 1, 2)$; $\mathbf{v}_2 = (0, 1, 1)$ in $(0, 0, 3)$; $\mathbf{v}_3 = (3, -2, 1)$ in $(1, 2, 0)$; $\mathbf{v}_4 = (0, 0, 4)$ in $(-1, 0, 2)$. Determine the magnitude of its resultant, its resultant moment with respect to the origin, and its minimum moment.

P2.5 Referring to the preceding exercise, determine the coordinates of the point closest to the origin where the system can be reduced to a wrench.

P2.6 Consider the system made up of the sliding vectors $\mathbf{v}_i$, $i = 1, \ldots, 4$, associated to the indicated lines of action, and the free vector $\mathbf{M}$. Determine the vector $\mathbf{v}_5$, bound to the vertex A of the cube, with an edge equal to 1 m, whose inclusion in the system permits its reduction to a single nonnull vector. Is there more than one solution?

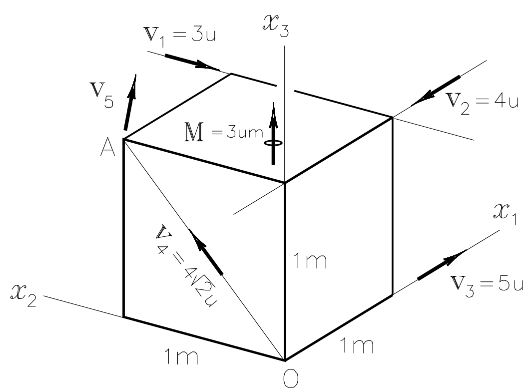

Figure P 2.6

P2.7 Acting on a solid cube with an edge $a = 2\,\text{m}$ are forces $\mathbf{f}_1$, associated to the edge AB, and of 30N magnitude, $\mathbf{f}_2$, associated to the line of action s and of 50N magnitude, $\mathbf{f}_3$, associated to diagonal CF, and $\mathbf{f}_4$, applied to vertex E. Determine the $\mathbf{f}_3$ magnitude and the $\mathbf{f}_4$ components so that the system can be substituted by the action of a screwdriver applied to point O, in the direction of the straight line s. Also determine the efforts to be applied on the screwdriver.

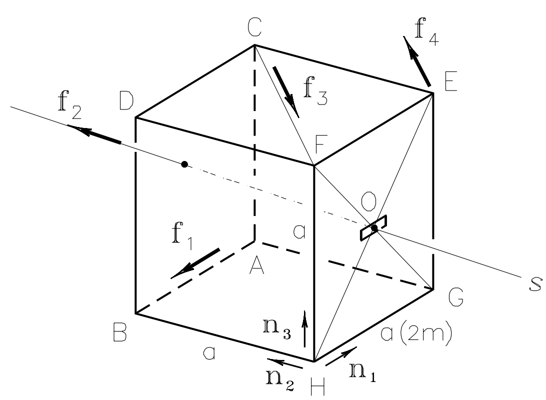

Figure P 2.7

P2.8 The figure illustrates a child's swing in a public square. Among the forces acting on the homogeneous horizontal bar B, consider the subsystem $\mathcal{S}$, comprising the tractions on the four cables and the weight of the bar (200 N). Show that the central axis of this sytem intercepts the axis x_1. If, at a given time, $\mathbf{F}_1 = \mathbf{F}_2 = 100\,\text{N}$, $\mathbf{F}_3 = \mathbf{F}_4 = 120\,\text{N}$, $\beta_1 = 60^0$, $\beta_2 = 30^0$ (children are playing there), determine the x_1 coordinate of the intersecting point.

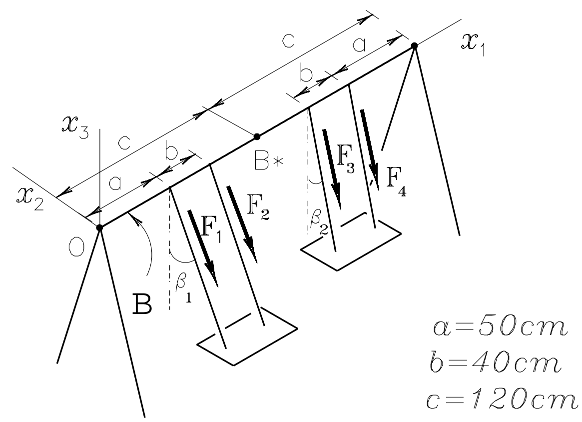

Figure P 2.8

P2.9 A rectangular channel, with width 1, contains water of density ρ, trapped by a cylinder with radius r and mass m, that lies on the rough bottom surface, being kept in position by a horizontal cable, as shown. Determine the traction on the cable to keep the cylinder at rest.

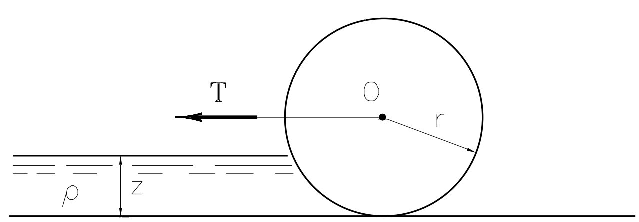

Figure P 2.9

P2.10 Two segments were cut and shaped from a wire with linear density ρ: one rectilinear, with length a, and another in the form of a semicircle with radius r, both fixed in the illustrated configuration. Analyze the gravitational action exerted by the rectilinear segment on the curvilinear one, determining the gravitational torque with respect to point O. Can this system of vectors be reduced to a single force?

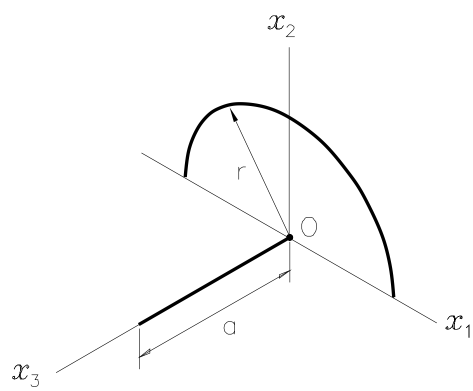

Figure P 2.10

P2.11 Two particles P_1 and P_2, of masses $2m$ and m, respectively, are interconnected by two cords, one with length a and the other $8a$, the latter suspended around a cylinder with a diameter of $2a$, set horizontally on bearings without friction. The system is in equilibrium in the illustrated configuration. Determine the angle θ_1.

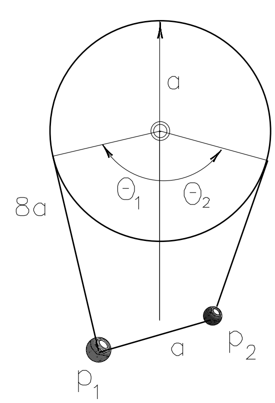

Figure P 2.11

P2.12 The telescopic antenna, of mass m, is pivoted on the point O at the support S and can turn freely around the axis x_1. The antenna is at rest in the vertical position (the friction existing between the elements is sufficient to support its own weight), when a force $\mathbf{F}$, with components F_1, F_2, F_3, is applied to the end P, on the Cartesian basis of the figure. Reduce to point G, the antenna's mass center, the system of external forces acting on the antenna at this instant.

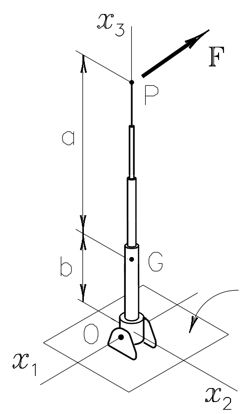

Figure P 2.12

P2.13 The balloon has volume V and specific mass $\rho_0/3$, where ρ_0 is the density of the atmosphere. The AO rope is flexible (does not resist bending), with length a and mass m. Determine the height b of the balloon, the dragging force of the wind, and the force at point O, knowing that the slope at this point is θ_0.

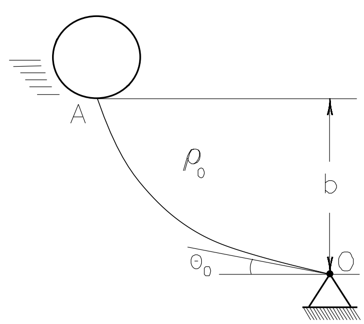

Figure P 2.13

P2.14 Particle P, with a mass of m, is r away from the mass center B^* of the homogeneous bar B, with mass M and length $2a$. By reducing the system of gravitational forces exerted by P on B at point B^*, this results in a gravitational force and a gravitational torque. Calculate the limit of this pair when $a/r \longrightarrow 0$.

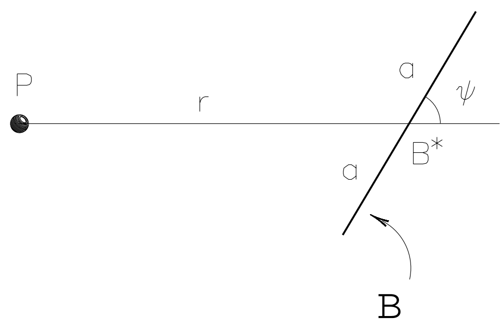

Figure P 2.14

P2.15 To turn the axis, the monkey wrench weighing 30 N would be applied to the hexagonal end, a horizontal force of 40 N being applied at its end B, as illustrated. When ascertaining, however, that the monkey wrench was slightly smaller than necessary, someone suggested that the same result would be obtained by applying a screwdriver (a wrench, therefore) at some point on groove A. Determine the r-coordinate of this point.

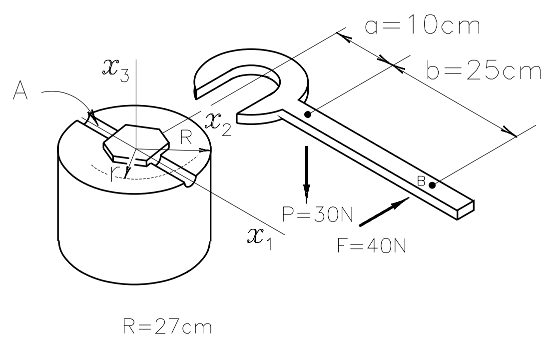

Figure P 2.15

P2.16 The figure is a sketch of a ceiling fan. The rotor/blades set (H) revolves around the symmetry axis, in relation to the motor cover (C), which is fixed on the support (S) by a ball connection that can be tightened to a greater or lesser degree, using a butterfly screw (B). Point G is the mass center of the suspended set. The fan weighs 50 N and is switched off, with its symmetry axis on the horizontal ($\theta = 0$). Tightening the butterfly screw permits the joint to support torques to the maximum limit of 8 N m, guaranteeing, therefore, immobility of the set. When the fan is switched on, the system of forces exerted by the air in motion on the blades is equivalent to a wrench applied at point Q, with resultant $\mathbf{F} = -30\mathbf{n}_1$ N and a resultant torque of $\mathbf{T} = -6\mathbf{n}_1$ N m. Does the cover stay still? Assuming that the tightening moment on the butterfly is proportional to the maximum resistant torque on the joint, how much extra percentage in tightening the butterfly is needed to keep immobility (or, if one concluded that the first tightening is sufficient, what is the percentage of loosening that would still keep this mobility)?

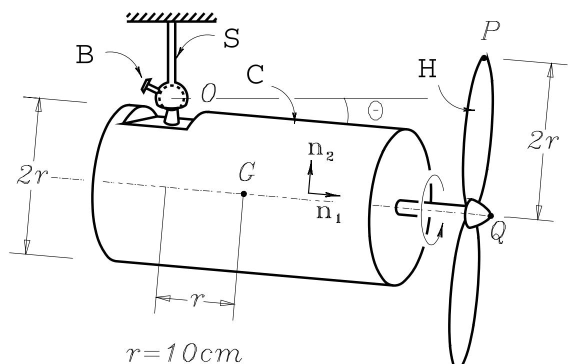

Figure P 2.16

P2.17 The system consists of the sliding vectors $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$, associated to the straight lines indicated and of magnitudes equal to 5 N, 3 N, $10\sqrt{2}$ N and 2 N, respectively, and the free vectors $\mathbf{M}_1$, of magnitude 10 N m, and $\mathbf{M}_2$, of magnitude 8 N m. The cube in the figure has edges with a length of 2 m. Determine the position vector, with respect to A, of the point of the central axis closest to it.

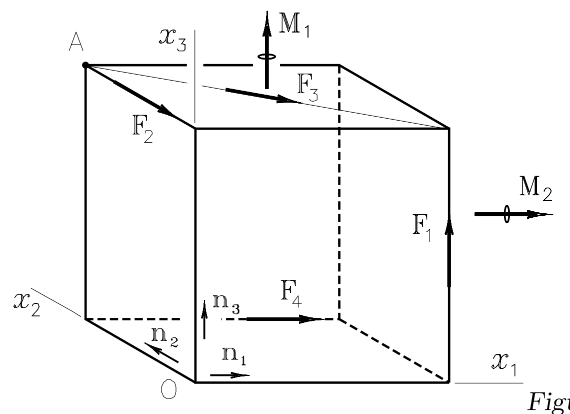

Figure P 2.17

P2.18 Consider the system illustrated in the figure, consisting of two small disks P and Q, of the same mass m each, joined by two rigid light bars of the same length r, at a vertical axis that is revolving in relation to the reference frame $\mathcal{R}$, as indicated. The axes $\{x_1, x_2, x_3\}$ are fixed on the vertical axis and the articulations at O and Q are pivots revolving around the x_3-axis the z-and axis respectively, so that the bars always stay on the plane x_1x_2. Indicate the system of external forces acting on the mechanical system.

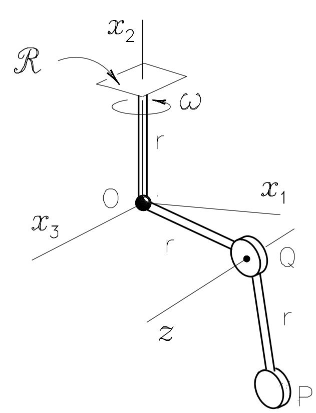

Figure P 2.18

P2.19 Consider the lid, shown in the figure, consisting of a homogeneous rectangular plate of mass m, joined at the support by two hinges, being opened using a cable, passing through a pulley, to which a force of magnitude F is applied. Draw a free body diagram of the plate, indicating all the force and torque components applied. Reduce this system to vertex B.

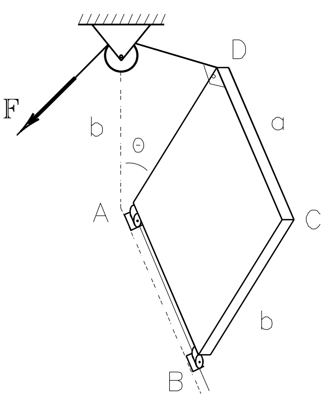

Figure P 2.19

P2.20 A gate, in the shape of a fourth of a cylinder with radius r, is being designed to regulate the level of a canal. The gate shall be able to rotate freely around the axis x, thanks to the pivots at A and B. Specify the relative density d of the building material of the gate, so that it opens when the water level reaches the height $a = \frac{2}{3}r$.

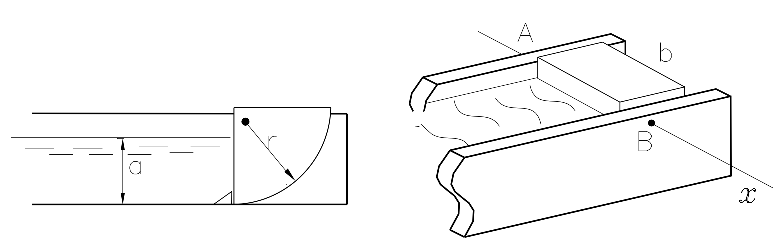

Figure P 2.20

P2.21 The homogeneous bar B, of mass M, is partly lying on the fixed circular base A being pivoted at its end O to a vertical pin fixed on the base, as illustrated. At the other end, Q, a light wire, with length a, is fixed, with a small sphere P, of mass m hanging from it. Consider the mechanical system consisting of B and P and draw a diagram of the external forces applied to it (there is friction between B and A, but the friction on the pin is negligible).

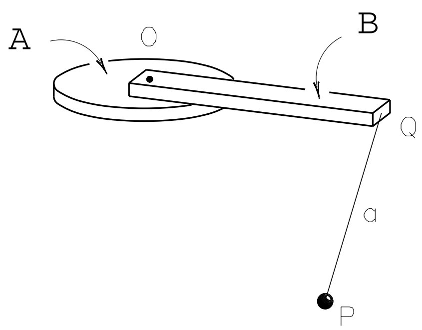

Figure P 2.21

P2.22 The light beam is supported at its ends, subject to the vertical loading $q(x) = Q(1 - \frac{x}{c})$ and to torque $T = \frac{1}{6}Qc^2$, in the direction indicated. Determine the forces V_1 and V_2 on the supports, knowing that the beam is at rest. Now, removing the right support, the force on the left support assumes a new value V_1' and the system is no longer a null system (the beam will move). Can this new system be reduced to a single force applied at one point?

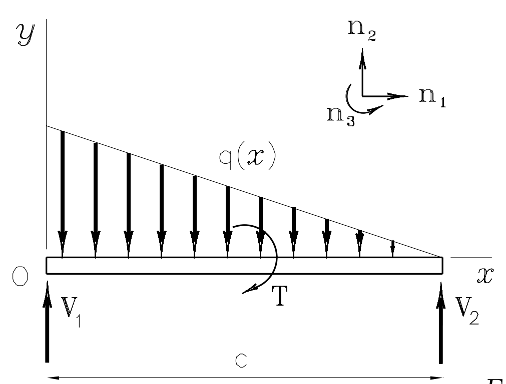

Figure P 2.22

P2.23 Bar B is connected to the guide rod G by a compound link, described below. The axes $\{x_1, x_2, x_3\}$ are fixed to the bar, with x_1 orthogonal to the cursor C and x_2 parallel to its longitudinal axis. The bar can turn freely in relation to the cursor around the axis x_3, while the latter can freely slide with respect to the guide rod, as shown. The axis x_3, however, is built to make a constant angle α, with the guide rod axis. How many force and torque components does this link offer, and how many degrees of freedom does it permit?

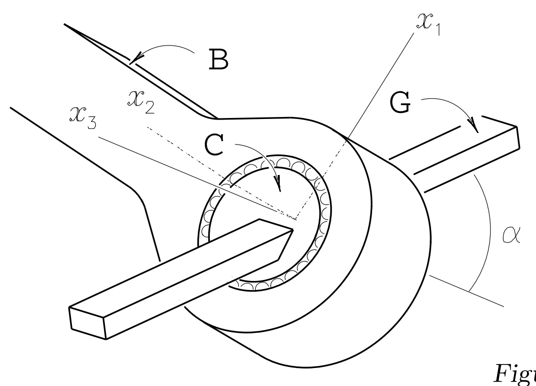

Figure P 2.23

P2.24 A series of n identical blades, with length $2a$ each, are piled on top of each other, as illustrated. What is the maximum balance of each to guarantee the set's equilibrium?

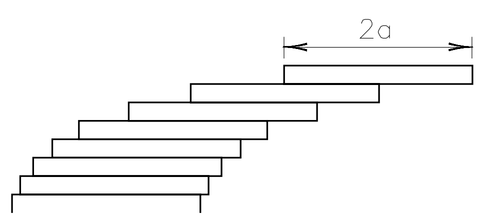

Figure P 2.24

P2.25 Two cylinders with radius r are lying at rest on a horizontal surface, kept together by a rope with length $2r$, joining their centers, and supporting a third homogeneous cylinder, with mass m and radius R, as shown. Assuming that the frictions are negligible, calculate the stress on the rope.

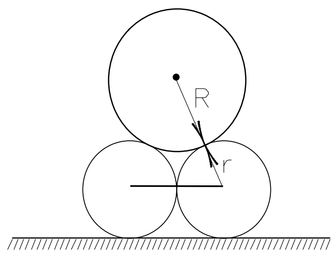

Figure P 2.25

P2.26 A uniform bar with length c can slide inside a cylindrical surface with a radius of r. Determine the maximum angle θ that guarantees the equilibrium of the bar if the friction coefficient at the points of contact is μ.

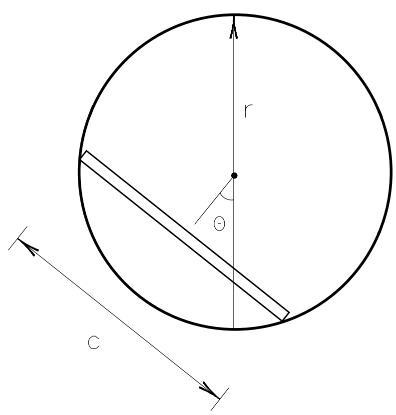

Figure P 2.26

P2.27 Three homogeneous identical balls are lying at rest on a horizontal surface, touching each other and kept together by a rope passing through the common equatorial plane. A fourth ball, with mass of $m = 10$ kg, lies on top of the other three. Calculate the stress on the rope, not considering all frictions.

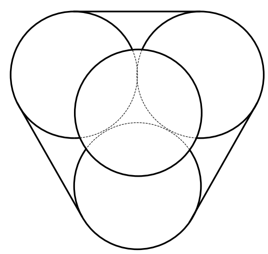

Figure P 2.27

P2.28 A prismatic and homogeneous block of concrete, with density equal to double that of the water, is at rest lying on the bottom of a canal of width a, damming water to a height of $a/2$, as illustrated. The distributed forces exerted by the bottom of the canal on the block have horizontal components (whose resultant is known as *friction force*) and vertical components (whose resultant is called *normal force*). The vertical distributed force is equivalent to a force equal to its resultant applied at a certain point. Determine the coordinate b of this point.

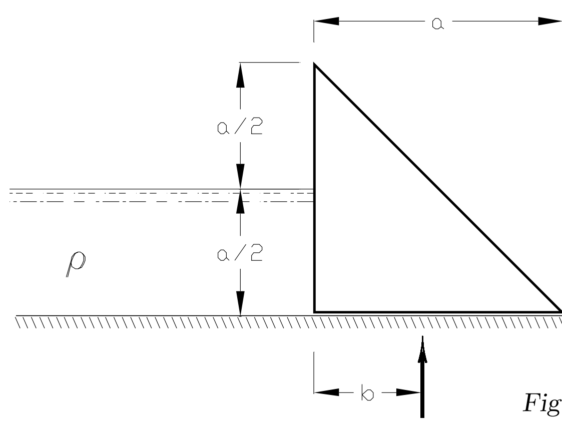

Figure P 2.28

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