CHAPTER 1

Differentiable Manifolds

In differential geometry, n-dimensional Euclidean space is replaced by a differentiable manifold. In essence, this is a set M constructed by gluing together pieces that are homeomorphic to $\mathbb{R}^n$, so that M looks locally, if not globally, like Euclidean space. The idea is that all *local* concepts, such as the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point, can be carried over to M by means of these identifications. A simple, yet useful example to keep in mind is that of the two-dimensional unit sphere S^2, where for any point $p \in S^2$, the neighborhood $S^2 \setminus \{-p\}$ of p is homeomorphic to $\mathbb{R}^2$.

1. Basic Definitions

Recall that the vector space $\mathbb{R}^n$ is the set $\{(p_1, \dots, p_n) \mid p_i \in \mathbb{R}\}$, together with coordinate-wise addition and scalar multiplication. The i-th projection is the map $u^i : \mathbb{R}^n \to \mathbb{R}$ given by $u^i(p_1, \dots, p_n) = p_i$, and the j-th standard basis vector $\mathbf{e}_j$ is defined by $u^i(\mathbf{e}_j) = \delta_{ij}$.

Let U be a subset of $\mathbb{R}^n$. Given a function $f : U \to \mathbb{R}$, $p \in U$, the *i-th partial derivative* of f at p is

$$D_i f(p) = \lim_{t \to 0} \frac{f(p + t\mathbf{e}_i) - f(p)}{t} = (f \circ c)'(0),$$

where c is the line $c(t) = p + t\mathbf{e}_i$ through p in direction $\mathbf{e}_i$. f is said to be *smooth* or *differentiable* on U if it has continuous partial derivatives of any order on U.

A map $f : U \to \mathbb{R}^k$ is said to be smooth if all the component functions $f^i := u^i \circ f : U \to \mathbb{R}$ of f are smooth. In this case, the *Jacobian matrix* of f at p is the $k \times n$ matrix $Df(p)$ whose (i, j)-th entry is $D_j f^i(p)$. The Jacobian will often be identified with the linear transformation $\mathbb{R}^n \to \mathbb{R}^k$ it determines.

Definition 1.1. A second countable Hausdorff topological space M is said to be a *topological n-dimensional manifold* if it is locally homeomorphic to $\mathbb{R}^n$; i.e., if for any $p \in M$ there exists a homeomorphism x of some neighborhood U of p with some open set in $\mathbb{R}^n$. (U, x) is called a *chart*, or *coordinate system*, and x a *coordinate map*.

Definition 1.2. A *differentiable atlas* on a topological n-dimensional manifold M is a collection $\mathcal{A}$ of charts of M such that

(1) the domains of the charts cover M, and
(2) if (U, x) and $(V, y) \in \mathcal{A}$, then $y \circ x^{-1} : x(U \cap V) \to \mathbb{R}^n$ is smooth.

The map $y \circ x^{-1}$ is often referred to as the *transition map* from the chart (U, x) to (V, y).

If $\mathcal{A}$ is an atlas on M, a chart (U, x) is said to be *compatible* with $\mathcal{A}$ if $\{(U, x)\} \cup \mathcal{A}$ is again an atlas on M. A *differentiable structure* on M is a maximal differentiable atlas $\mathcal{A}$: Any chart compatible with $\mathcal{A}$ belongs to the atlas. Alternatively—for those uncomfortable with the term "maximal"—given two atlases $\mathcal{A}$ and $\mathcal{A}'$, define $\mathcal{A} \sim \mathcal{A}'$ if for any charts $(U, x) \in \mathcal{A}$ and $(V, y) \in \mathcal{A}'$, $y \circ x^{-1}$ and $x \circ y^{-1}$ are differentiable. A differentiable structure is then an equivalence class of the relation $\sim$ defined above.

DEFINITION 1.3. A *differentiable n-dimensional manifold* is a topological n-dimensional manifold together with a differentiable structure.

From now on, the term *manifold* will always denote a differentiable manifold.

EXAMPLES AND REMARKS 1.1. (i) In order to specify a differentiable structure, it suffices to provide some atlas $\mathcal{A}$: This atlas then determines a differentiable structure $\mathcal{A}'$ which consists of all charts (U, x) such that $x \circ y^{-1}$ and $y \circ x^{-1}$ are smooth for any coordinate map y of $\mathcal{A}$.

(ii) The *standard differentiable structure* on $\mathbb{R}^n$ is the one determined (as in (i)) by the atlas consisting of the single chart $(\mathbb{R}^n, 1_{\mathbb{R}^n})$, where $1_{\mathbb{R}^n}$ denotes the identity map.

(iii) Let V denote an n-dimensional real vector space. The *standard differentiable structure* on V is the one induced by the atlas $\{(V, L)\}$, where $L : V \to \mathbb{R}^n$ is some isomorphism. Why is this structure independent of the choice of L?

(iv) Any open subset U of a manifold M inherits a natural differentiable structure (of the same dimension) from that of M: An atlas $\{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ of M induces an atlas $\{(U \cap U_\alpha, x_\alpha|_{U \cap U\alpha})\}_{\alpha \in A}$ of U. For example, the set $GL(n) \subset M_{n,n} \cong \mathbb{R}^{n^2}$ of all invertible $n \times n$ real matrices is an n^2-dimensional manifold.

(v) Let $r > 0$. The *n-sphere S_r^n of radius r* is the compact topological subspace of $\mathbb{R}^{n+1}$ consisting of all points at distance r from the origin. Let $p_N = (0, \dots, 0, r)$ and $p_S = (0, \dots, 0, -r)$ denote the north and south poles, respectively, and set $U_N = S_r^n \setminus \{p_N\}, U_S = S_r^n \setminus \{p_S\}$. Then the collection $\{(U_N, x_N), (U_S, x_S)\}$ is a differentiable atlas on the sphere, where x_N and x_S are the "stereographic projections"

$$x_N(p_1, \dots, p_{n+1}) = \frac{r}{r - p_{n+1}}(p_1, \dots, p_n),$$
$$x_S(p_1, \dots, p_{n+1}) = \frac{r}{r + p_{n+1}}(p_1, \dots, p_n).$$

In fact, the transition map is given by

$$x_N \circ x_S^{-1} = x_S \circ x_N^{-1} = \frac{r^2}{|1_{\mathbb{R}^n}|^2} 1_{\mathbb{R}^n} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$$

and is clearly differentiable.

The sphere is thus described by two charts, and can therefore be considered to be the simplest nontrivial example of a manifold.

(vi) Let $(M_i^{n_i}, \mathcal{A}_i)$ be manifolds of dimension n_i, $i = 1, 2$. The collection

$$\mathcal{A}_1 \times \mathcal{A}_2 := \{(U \times V, x \times y) \mid (U, x) \in \mathcal{A}_1, (V, y) \in \mathcal{A}_2\}$$

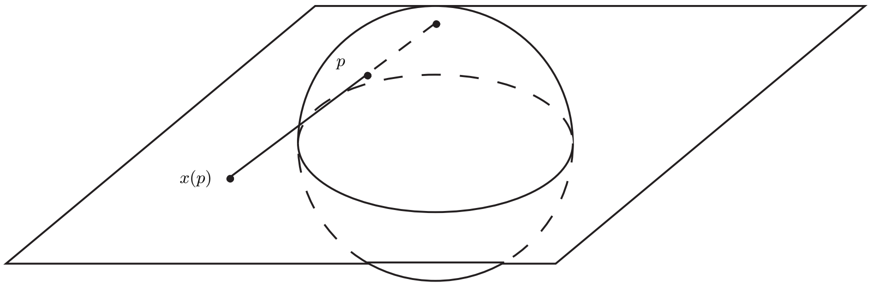

FIGURE 1. Stereographic projection from the north pole.

is an atlas on $M_1 \times M_2$. Here, $(x \times y)(p, q) = (x(p), y(q))$. The induced differentiable structure is called the *product manifold* $M_1 \times M_2$.

DEFINITION 1.4. A function $f : M \to \mathbb{R}$ is said to be *smooth* if $f \circ x^{-1} : x(U) \to \mathbb{R}$ is smooth for any chart (U, x) of M.

DEFINITION 1.5. A *partition of unity* on M is a collection $\{\phi_\alpha\}_{\alpha \in A}$ of smooth nonnegative functions ϕ_α on M such that

(1) $\{\operatorname{supp} \phi_\alpha\}_{\alpha \in A}$ is a locally finite cover of M. Recall that the support of a function is the closure of the set on which the function is nonzero. A collection of sets is locally finite if any point has a neighborhood that intersects at most finitely many of the sets.
(2) $\sum_\alpha \phi_\alpha \equiv 1$. (Why does this possibly infinite sum make sense?)

THEOREM 1.1. *Any open cover $\{U_\alpha\}_{\alpha \in A}$ of a manifold M admits a countable subordinate partition of unity $\{\phi_k\}_{k \in \mathbb{N}}$; i.e., for any integer k, there exists an $\alpha \in A$ such that* $\operatorname{supp} \phi_k \subset U_\alpha$.

There are several steps involved in the proof of Theorem 1.1. Given $\epsilon > 0$, $q \in \mathbb{R}^n$, $B_\epsilon(q)$ will denote the set of points at distance less than than ϵ from q.

THEOREM 1.2. *If $\{U_\alpha\}$ is an open cover of M, then there is a countable differentiable atlas $\{(V_k, x_k)\}$ of M such that*

(1) *$\{V_k\}$ is a locally finite refinement of $\{U_\alpha\}$;*
(2) *$x_k(V_k) = B_3(0)$;*
(3) *the collection $\{W_k\}$, where $W_k = x_k^{-1}(B_1(0))$, is a cover of M.*

PROOF OF THEOREM 1.2. Since M is locally compact (i.e., every point has a neighborhood with compact closure), Hausdorff, and second countable, there exists a countable basis $\{Z_k\}$ for M with $\bar{Z}_k$ compact. Let $A_1 = \bar{Z}_1$. Given A_i compact, let j denote the smallest integer such that $A_i \subset Z_1 \cup \cdots \cup Z_j$; define $A_{i+1} = \bar{Z}_1 \cup \cdots \cup \bar{Z}_j \cup \bar{Z}_{i+1}$. Then $\{A_k\}$ is a sequence of compact sets with $A_k \subset \operatorname{int} A_{k+1}$, and $\cup_k A_k = M$. Define A_0 to be the empty set. Since $M = \cup_{i \geq 0}(A_{i+1} \setminus \operatorname{int} A_i)$, we may assume that for each $p \in M$, there exists a chart (V_p, x_p) sending p to 0, such that

$$x_p(V_p) = B_3(0), \quad V_p \subset U_\alpha \text{ for some } \alpha, \quad \text{and } V_p \subset (\operatorname{int} A_{i+2}) \setminus A_{i-1} \text{ for some } i.$$

Then $\{x_p^{-1}(B_1(0)\}_{p \in A_{i+1} \setminus \operatorname{int} A_i}$ is an open cover of the compact $A_{i+1} \setminus \operatorname{int} A_i$, and contains a finite subcover which we denote P_i. If $P = P_0 \cup P_1 \cup \cdots$, then P consists of a countable cover $\{V_k\}$ of M subordinate to $\{U_\alpha\}$. Each V_k is the domain of a chart $\{(V_k, x_k)\}$ with $x_k(V_k) = B_3(0)$, and the collection $\{x_k^{-1}(B_1(0))\}$ covers M.

It remains to show that $\{V_k\}$ is locally finite. Now, any $p \in M$ belongs to some $A_{i+1} \setminus \operatorname{int} A_i$. Then $W = (\operatorname{int} A_{i+2}) \setminus A_{i-1}$ is an open neighborhood of p that intersects at most finitely many V_k: Indeed, each V_k is contained in some set $(\operatorname{int} A_{j+2}) \setminus A_{j-1}$, so if V_k is to intersect W, then j cannot exceed $i+2$. Since there are only finitely many V_k in each crown $(\operatorname{int} A_{j+2}) \setminus A_{j-1}$, the statement follows. $\square$

Given $\epsilon > 0$, denote by $C_\epsilon(0)$ the open cube $(-\epsilon, \epsilon)^n$ in $\mathbb{R}^n$.

Lemma 1.1. *There exists a differentiable function $\phi : \mathbb{R}^n \to \mathbb{R}$ satisfying*

1. $\phi \equiv 1$ *on* $\bar{C}_1(0)$,
2. $0 < \phi < 1$ *on* $C_2(0) \setminus \bar{C}_1(0)$, *and*
3. $\phi \equiv 0$ *on* $\mathbb{R}^n \setminus C_2(0)$.

Proof of Lemma 1.1. Let $h : \mathbb{R} \to \mathbb{R}$ be given by

$$h(x) = \begin{cases} e^{-1/x}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and define

$$f(x) = \frac{h(2+x)h(2-x)}{h(2+x)h(2-x) + h(x-1) + h(-x-1)}.$$

This expression makes sense because $h(x-1) + h(-x-1)$ is nonnegative, and equals 0 only when $|x| \leq 1$, in which case $h(2+x)h(2-x) > 0$. Furthermore, $f(x) = 1$ if $|x| \leq 1$, $0 < f(x) < 1$ if $1 < |x| < 2$, and $f(x) = 0$ if $|x| \geq 2$. Now let $\phi(a_1, \ldots, a_n) = \Pi_{i=1}^n f(a_i)$. $\square$

Proof of Theorem 1.1. Let $\{(V_k, x_k)\}$ be a differentiable atlas as in Theorem 1.2, and ϕ the function from Lemma 1.1, where n equals the dimension of M. For each k define a function $\theta_k : M \to \mathbb{R}$ by

$$\theta_k(p) = \begin{cases} \phi \circ x_k(p), & \text{if } p \in V_k, \\ 0, & \text{otherwise.} \end{cases}$$

θ_k is differentiable on M, since it is differentiable on V_k, and is identically zero on the open neighborhood $M \setminus x_k^{-1}(\bar{C}_2(0))$ of $M \setminus V_k$. Any $p \in M$ belongs to $x_j^{-1}(B_1(0))$ for some j, so that $\theta_j(p) > 0$. Since $\{V_k\}$ is locally finite and $\operatorname{supp} \theta_k \subset V_k$, the collection $\{\operatorname{supp} \theta_k\}$ is a locally finite cover of M. This means that $\sum_k \theta_k(p)$ is finite for every $p \in M$; now set $\phi_k := \theta_k / (\sum_i \theta_i)$. $\square$

Exercise 1. Show that the transition maps for the atlas in Examples and Remarks 1.1(iv) are given by

$$x_N \circ x_S^{-1} = x_S \circ x_N^{-1} = \frac{r^2}{|1_{\mathbb{R}^n}|^2} 1_{\mathbb{R}^n} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\},$$

and deduce that $\{(U_N, x_N), (U_S, x_S)\}$ is indeed a differentiable atlas on the sphere.

(Notation: Given a manifold M, $1_M : M \to M$ denotes the identity map of M.)

Exercise 2. Let U be an open subset of M, V a set whose closure is contained in U. Show that there exists a smooth nonnegative $\phi : M \to \mathbb{R}$ that is identically 1 on the closure of V, and the support of which is contained in U.

2. Differentiable Maps

The superscript in the symbol M^n will refer to the dimension of the manifold M.

Definition 2.1. Let M^n, N^k denote manifolds, and suppose U is open in M. A map $f : U \to N$ is said to be *differentiable* or *smooth* if $y \circ f \circ x^{-1}$ is smooth as a map from $\mathbb{R}^n$ to $\mathbb{R}^k$ for any coordinate maps x of M and y of N.

If A is an arbitrary subset of M, $f : A \to N$ is said to be smooth if it can be extended to a smooth map $\bar{f} : U \to N$ for some open set U containing A.

Observe that the composition of differentiable maps is differentiable. $f : M \to N$ is said to be a *diffeomorphism* if it is bijective and both f and its inverse f^{-1} are smooth. The collection $\mathrm{Diff}(M)$ of all diffeomorphisms of M with itself is clearly a group under composition.

Examples and Remarks 2.1. (i) For a function $f : M \to \mathbb{R}$, the Definition 2.1 coincides with 1.4.

(ii) If (U, x) is a chart, then $x : U \to x(U) \subset \mathbb{R}^n$ is a diffeomorphism.

(iii) It is known that any two differentiable manifolds of dimension no larger than 3 which are homeomorphic are actually diffeomorphic. On the other hand, there exist "exotic" $\mathbb{R}^4$'s; i.e., manifolds that are homeomorphic but not diffeomorphic to $\mathbb{R}^4$ with the standard differentiable structure.

Given a subset A of M, let $\mathcal{F}(A)$ denote the set of all smooth functions $f : A \to \mathbb{R}$. $\mathcal{F}(A)$ is a real algebra (and in particular, both a ring and a vector space) under the operations

$$(f+g)(p) = f(p) + g(p), \quad (f \cdot g)(p) = f(p)g(p), \quad (\alpha f)(p) = \alpha f(p), \quad \alpha \in \mathbb{R}.$$

For example, if (U, x) is a chart, then $x^i \in \mathcal{F}(U)$, where $x^i := u^i \circ x$, $1 \le i \le \dim M$.

Definition 2.2. Let U be an open subset of M, $p \in U$, and set $\mathcal{F}_p^0(U) = \{f \in \mathcal{F}(U) \mid f \equiv 0 \text{ in a neighborhood of } p\}$. $\mathcal{F}_p^0(U)$ is an ideal in $\mathcal{F}(U)$, and the quotient algebra $\mathcal{F}_p = \mathcal{F}(U)/\mathcal{F}_p^0(U)$ is called the algebra of *germs of functions* at p.

Thus, a germ is an equivalence class of functions, with two functions being equivalent iff they agree on a neighborhood of the point. The reason we omitted U in the terminology for $\mathcal{F}_p = \mathcal{F}_p(U)$ is due to the fact that the map $\mathcal{F}(M) \to \mathcal{F}(U)$ given by $f \mapsto f \circ \imath$, where $\imath : U \to M$ denotes inclusion, induces an isomorphism $\mathcal{F}_p(M) \cong \mathcal{F}_p(U)$: This map is clearly injective; to see that it's surjective, let $f \in \mathcal{F}(U)$, and consider an open set V whose closure is contained in U. Let ϕ be the function from Exercise 2, and define a smooth function g on M by setting it equal to ϕf on U and 0 outside U. Since f and g coincide on V, the germ of g at p is mapped to the germ of f at p.

EXERCISE 3. Consider $\mathbb{R}$ with the two atlases $\{1_{\mathbb{R}}\}$ and $\{\phi\}$, where $\phi(t) = t^3$.

(a) Show that these atlases are not compatible; i.e., they determine different differentiable structures on $\mathbb{R}$.

(b) Show that the two differentiable manifolds from (a) are diffeomorphic.

EXERCISE 4. (a) Show that $f : S^n_r \to \mathbb{R}$, where $f(p_1, \dots, p_{n+1}) = \sum_i p_i$, is smooth.

(b) Show that $f : S^n_1 \to S^n_r$, where $f(p) = -rp$, is a diffeomorphism.

3. Tangent Vectors

A vector v in $\mathbb{R}^n$ acts on differentiable functions in a natural way, by assigning to $f : \mathbb{R}^n \to \mathbb{R}$ the derivative $D_v f(p) := Df(p) \cdot v$ of f in direction v. This assignment depends of course on the point p at which the derivative is evaluated; furthermore, it is linear, and satisfies the product rule $D_v(fg)(p) = f(p)D_v(g)(p) + g(p)D_v(f)(p)$. This is essentially the motivation behind the following:

DEFINITION 3.1. Let $p \in M$. A *tangent vector* v at p is a map $v : \mathcal{F}_p(M) \to \mathbb{R}$ satisfying

(1) $v(\alpha f + \beta g) = \alpha v(f) + \beta v(g)$; and
(2) $v(fg) = f(p)v(g) + g(p)v(f)$

for $\alpha, \beta \in \mathbb{R}$, $f, g \in \mathcal{F}_p(M)$.

In the above definition, we have used the same letter to denote both a germ and a function belonging to that germ: If U is a neighborhood of p, then a tangent vector v at p induces a map $\mathcal{F}(U) \to \mathbb{R}$ given by $v(f) := v([f])$. The point p is called the *footpoint* of v, and the set M_p of all tangent vectors at p is called the *tangent space* of M at p. It is a real vector space under the operations $(v + w)(f) = v(f) + w(f)$, $(\alpha v)(f) = \alpha v(f)$.

In the familiar context of Euclidean space, one can think of a tangent vector at p as simply being a vector v whose origin has been translated to p, denoted (p, v). Then $(p, v)(f) = D_v f(p)$. Notice that one recovers v from the way (p, v) acts on functions: $v = ((p, v)(u^1), \dots, (p, v)(u^n))$.

The first condition in Definition 3.1 says that a tangent vector is a linear operator on (germs of) functions, and the second that it is a *derivation*.

Let x be a coordinate map around p (that is, p belongs to the domain of x), and as usual, let $x^i = u^i \circ x$. The *coordinate vector fields* at p are the tangent vectors $\partial/\partial x^i(p) \in M_p$ given by

$$\frac{\partial}{\partial x^i}(p)(f) := D_i(f \circ x^{-1})(x(p)), \qquad f \in \mathcal{F}(M), \quad 1 \le i \le n. \tag{3.1}$$

One often denotes the left side of (3.1) by $\partial f/\partial x^i(p)$. For example, in $\mathbb{R}^n$, the standard coordinate vector fields at p are $\partial/\partial u^i(p)$, where $\partial f/\partial u^i(p) = D_i f(p)$. We will often denote them simply by D_i. When $n = 1$, we write D instead of $\partial/\partial u$, so that $Df(a) = f'(a)$.

The coordinate vector fields actually form a basis for the tangent space at a point. In order to show this, we need the following:

LEMMA 3.1. *Let U denote a star-shaped neighborhood of $0 \in \mathbb{R}^n$ — that is, the line segment connecting the origin to any point of U is also contained inside U. Given $f \in \mathcal{F}U$, there exist n functions $\psi_i \in \mathcal{F}U$, with $\psi_i(0) = D_i f(0)$, such that*

$$f = f(0) + \sum_i u^i \psi_i.$$

PROOF. For any fixed $p \in U$, consider the line segment $c(t) = tp$, and set $\phi = f \circ c$. ϕ is a differentiable function on $[0,1]$, and $\phi'(t) = \sum_i p_i D_i f(tp)$. Thus,

$$f(p) - f(0) = \phi(1) - \phi(0) = \int_0^1 \phi' = \sum_i p_i \int_0^1 D_i f(tp)\, dt.$$

The claim then follows by setting $\psi_i(p) := \int_0^1 D_i f(tp)\, dt$. □

PROPOSITION 3.1. *Let (U, x) be a chart around p. Then any tangent vector $v \in M_p$ can be uniquely written as a linear combination $v = \sum_i \alpha_i \partial/\partial x^i(p)$. In fact, $\alpha_i = v(x^i)$.*

Thus, M_p^n is an n-dimensional vector space with basis $\{\partial/\partial x^i(p)\}_{1 \le i \le n}$.

PROOF. We may assume without loss of generality that $x(p) = 0$, and that $x(U)$ is star-shaped. By Lemma 3.1, any $f \in \mathcal{F}M$ satisfies $f \circ x^{-1} = f(p) + \sum u^i \psi_i$, with $\psi_i(0) = \partial/\partial x^i(p)(f)$. Thus, $f|_U = f(p) + \sum_i x^i (\psi_i \circ x)|_U$, and

$$v(f) = v(f(p)) + \sum_i [v(x^i) \cdot \psi_i(0) + x^i(p) \cdot v(\psi_i \circ x)] = \sum_i v(x^i) \frac{\partial}{\partial x^i}(p)(f),$$

where we have used the result of Exercise 5 below. It remains to show that the $\partial/\partial x^i(p)$ are linearly independent; observe that

$$\frac{\partial}{\partial x^i}(p)(x^j) = D_i(x^j \circ x^{-1})(0) = D_i(u^j)(0) = \delta_{ij}.$$

Thus, if $\sum \alpha_i \partial/\partial x^i(p) = 0$, then $0 = \sum \alpha_i \partial/\partial x^i(p)(x^j) = \alpha_j$. □

Notice that if x and y are two coordinate systems at p, then taking $v = \partial/\partial y^i(p)$ in Proposition 3.1 yields

$$\frac{\partial}{\partial y^i}(p) = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i}(p) \frac{\partial}{\partial x^j}(p) = \sum_{j=1}^n D_i(u^j \circ x \circ y^{-1})(y(p)) \frac{\partial}{\partial x^j}(p) \tag{3.2}$$

for $1 \le i \le n$. This means that the transition matrix from the basis $\{\partial/\partial x^i(p)\}$ to the basis $\{\partial/\partial y^i(p)\}$ is the Jacobian matrix of $x \circ y^{-1}$ at $y(p)$.

EXERCISE 5. Let $c \in \mathbb{R}$. Show that if $c \in \mathcal{F}M$ denotes the constant function $c(p) :\equiv c$ for all $p \in M$, then $v(c) = 0$ for any tangent vector v at any point of M.

EXERCISE 6. Write down (3.2) explicitly for the n-sphere of radius r, if x and y denote stereographic projections.

4. The Derivative

In calculus, one usually thinks of the Jacobian $Df(p)$ of $f : \mathbb{R}^n \to \mathbb{R}^k$ as the derivative of f at p. It is therefore natural, when seeking a meaningful generalization of this concept for a map $f : M \to N$ between manifolds M and N, to look for a linear transformation. In view of the previous section, where we defined vector spaces at each point of a manifold, this suggests a linear transformation $f_{*p} : M_p \to N_{f(p)}$ between the respective tangent spaces. We would of course like f_{*p} to correspond to $Df(p)$ when $M = \mathbb{R}^n$ and $N = \mathbb{R}^k$, if $\mathbb{R}^n_p$ is identified with the set of pairs (p, v), $v \in \mathbb{R}^n$; i.e, we require that $f_{*p}(p, v) = (f(p), Df(p)v)$ for all $v \in \mathbb{R}^n$. Now, if $\phi : \mathbb{R}^k \to \mathbb{R}$ is differentiable, then by the Chain rule,

$$\begin{aligned} f_{*p}(p,v)(\phi) &= (f(p), Df(p)v)(\phi) = D_{Df(p)v}\phi(f(p)) = D\phi(f(p))\,Df(p)v \\ &= D_v(\phi \circ f)(p) = (p,v)(\phi \circ f). \end{aligned}$$

This motivates the following:

DEFINITION 4.1. Let M and N denote differentiable manifolds of dimensions n and k respectively, $f : U \to N$ a differentiable map, where U is open in M, and $p \in U$. The *derivative of f at p* is the map $f_{*p} : M_p \to N_{f(p)}$ given by

$$(f_{*p}v)(\phi) := v(\phi \circ f), \qquad \phi \in \mathcal{F}(N), \quad v \in M_p.$$

It is clear from the definition that f_{*p} is a linear transformation.

PROPOSITION 4.1. *With notation as in Definition 4.1, let x be a coordinate map around $p \in U$, y a coordinate map around $f(p) \in N$. Then the matrix of f_{*p} with respect to the bases $\{\partial/\partial x^i(p)\}$ and $\{\partial/\partial y^j((f(p))\}$ is the Jacobian matrix of $y \circ f \circ x^{-1}$ at $x(p)$.*

PROOF.

$$\begin{aligned} f_{*p}\frac{\partial}{\partial x^j}(p) &= \sum_i f_{*p}\frac{\partial}{\partial x^j}(p)(y^i)\frac{\partial}{\partial y^i}(f(p)) = \sum_i \frac{\partial}{\partial x^j}(p)(y^i \circ f)\frac{\partial}{\partial y^i}(f(p)) \\ &= \sum_i D_j(u^i \circ (y \circ f \circ x^{-1}))(x(p))\,\frac{\partial}{\partial y^i}(f(p)). \end{aligned}$$

□

EXAMPLES AND REMARKS 4.1. (i) It follows from Definition 4.1 that the identity map 1_M of M has as derivative at $p \in M$ the identity map 1_{M_p} of M_p.

(ii) If $g : N \to Q$ is differentiable, then $g \circ f$ is differentiable, and $(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$. In particular, if $f : M \to N$ is a diffeomorphism, then by (i), f_{*p} is an isomorphism with inverse $(f^{-1})_{*f(p)}$. Furthermore, given coordinate maps x and y of M and N respectively, the diagram

$$\begin{tikzcd} M_p \arrow[r, "f_{*p}"] \arrow[d, "x_{*p}"'] & N_{f(p)} \arrow[d, "y_{*f(p)}"] \\ \mathbb{R}^n_{x(p)} \arrow[r, "(y\circ f\circ x^{-1})_{*x(p)}"] & \mathbb{R}^k_{(y\circ f)(p)} \end{tikzcd}$$

commutes. Observe that $x_{*p}\partial/\partial x^i(p) = \partial/\partial u^i(x(p))$, since $x_*\partial/\partial x^i(u^j) = \partial/\partial x^i(u^j \circ x) = \partial/\partial x^i(x^j) = \delta_{ij}$.

(iii) A (smooth) curve in M is a (smooth) map $c : I \to M$, where I is an interval of real numbers. The *tangent vector* to c at t is $\dot{c}(t) := c_{*t}D(t)$. Thus, given $\phi \in \mathcal{F}(M)$,

$$\dot{c}(t)(\phi) = c_{*t}D(t)(\phi) = D(t)(\phi \circ c) = (\phi \circ c)'(t).$$

(iv) Let E be an n-dimensional real vector space with its canonical differentiable structure, cf. Examples and Remarks 1.1(iii). For any $v \in E$, E may be naturally identified with its tangent space E_v at v by "parallel translation" $\mathcal{J}_v : E \to E_v$, defined as follows: Given $w \in E$, let $\gamma(t) = v + tw$, and set $\mathcal{J}_v w := \dot{\gamma}(0)$. If $x : E \to \mathbb{R}^n$ is any isomorphism, then

$$\begin{aligned}\mathcal{J}_v w = \dot{\gamma}(0) &= \sum_i \dot{\gamma}(0)(x^i)\frac{\partial}{\partial x^i}(v) = \sum_i D(0)(x^i \circ \gamma)\frac{\partial}{\partial x^i}(v) \\ &= \sum_i x^i(w)\frac{\partial}{\partial x^i}(v),\end{aligned}$$

so that $\mathcal{J}_v$, being linear and one-to-one, is an isomorphism.

Notice that for $E = \mathbb{R}^n$ and $x = 1_{\mathbb{R}^n}$, we obtain $\mathcal{J}_v \mathbf{e}_i = \partial/\partial u^i(v)$. This formalizes our heuristic description of the tangent space of $\mathbb{R}^n$ at v from the previous section, since the map

$$\begin{aligned}\{v\} \times \mathbb{R}^n &\to \mathbb{R}^n_v, \\ (v, w) &\mapsto \mathcal{J}_v w\end{aligned}$$

is an isomorphism that preserves the action on $\mathcal{F}(\mathbb{R}^n)$.

Consider, for example, a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^k$. By Proposition 4.1, the matrix of L_{*v} with respect to the standard coordinate vector fields bases is that of the Jacobian of L. But since L is linear,

$$D_i(u^j \circ L)(v) = \lim_{t \to 0} \frac{(u^j \circ L)(v + t\mathbf{e}_i) - (u^j \circ L(v))}{t} = (u^j \circ L)(\mathbf{e}_i),$$

so that the Jacobian matrix of L is just the matrix of L in the standard basis. Thus, the following diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{L} & \mathbb{R}^k \\ {\scriptstyle \mathcal{J}_v}\downarrow & & \downarrow{\scriptstyle \mathcal{J}_{Lv}} \\ \mathbb{R}^n_v & \xrightarrow{L_{*v}} & \mathbb{R}^k_{Lv} \end{array}$$

commutes.

(v) Let U be an open set in M, $f \in \mathcal{F}U$, $p \in U$. The *differential* of f at p is the element $df(p)$ of the dual space M_p^* (i.e., $df(p) : M_p \to \mathbb{R}$ is linear) defined by

$$df(p)(v) := v(f), \qquad v \in M_p.$$

Thus, for example, $\{dx^i(p)\}$ is the basis dual to $\{\partial/\partial x^i(p)\}$. Notice also that the diagram

$$\begin{tikzcd} M_p \arrow[r, equal] \arrow[d, "df(p)"'] & M_p \arrow[d, "f_{*p}"] \\ \mathbb{R} \arrow[r, "\mathcal{J}_{f(p)}"] & \mathbb{R}_{f(p)} \end{tikzcd}$$

commutes:

$$\mathcal{J}_{f(p)} df(p)(v) = \mathcal{J}_{f(p)} v(f) = v(f) D_{f(p)} = f_{*p} v.$$

DEFINITION 4.2. The *tangent bundle* (resp. *cotangent bundle*) of M is the set $TM = \cup_{p\in M} M_p$ (resp. $T^*M = \cup_{p\in M} M_p^*$). The *bundle projections* are the maps $\pi : TM \to M$ and $\tilde{\pi} : T^*M \to M$ which map a tangent or cotangent vector to its footpoint.

PROPOSITION 4.2. *The differentiable structure $\mathcal{D}$ on M^n induces in a natural way $2n$-dimensional differentiable structures on the tangent and cotangent bundles of M.*

PROOF. For each chart (U, x) of M, define a chart $(\pi^{-1}(U), \bar{x})$ of TM, where $\bar{x} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ is given by

$$\bar{x}(v) = (x \circ \pi(v), dx^1(\pi(v))v, \dots, dx^n(\pi(v))v).$$

Similarly, define $\tilde{x} : \tilde{\pi}^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\tilde{x}(\alpha) = (x \circ \tilde{\pi}(\alpha), \alpha(\partial/\partial x^1(\tilde{\pi}(\alpha))), \dots, \alpha(\partial/\partial x^n(\tilde{\pi}(\alpha)))).$$

One checks that the collection $\{\bar{x}^{-1}(V) \mid (U, x) \in \mathcal{D}, V \text{ open in } \mathbb{R}^{2n}\}$ forms a basis for a second countable Hausdorff topology on TM. A similar argument, using $\tilde{x}$ instead of $\bar{x}$, works for T^*M.

Let $\mathcal{A} = \{(\pi^{-1}(U), \bar{x}) \mid (U, x) \in \mathcal{D}\}$. We claim that $\mathcal{A}$ is an atlas for TM: clearly, each $\bar{x} : \pi^{-1}(U) \to x(U) \times \mathbb{R}^n$ is a homeomorphism. Furthermore, if (V, y) is another chart of M, and $(a, b) \in x(U \cap V) \times \mathbb{R}^n$, then

$$\bar{y} \circ \bar{x}^{-1}(a, b) = (y \circ x^{-1}(a), D(y \circ x^{-1})(a)(b)).$$

To see this, write $b = \sum b_i \mathbf{e}_i$; then

$$\bar{x}^{-1}(a, b) = \sum_i b_i \frac{\partial}{\partial x^i}(x^{-1}(a)) = \sum_{i,j} b_i \frac{\partial y^j}{\partial x^i}(x^{-1}(a)) \frac{\partial}{\partial y^j}(x^{-1}(a)),$$

so that

$$\begin{aligned} (\bar{y} \circ \bar{x}^{-1})(a, b) &= (y \circ x^{-1}(a), \sum_{i,j} b_i \frac{\partial y^j}{\partial x^i}(x^{-1}(a)) \mathbf{e}_j) \\ &= (y \circ x^{-1}(a), D(y \circ x^{-1})(a)(b)). \end{aligned}$$

$\square$

For example, the bundle projection $\pi : TM \to M$ is differentiable, since for any pair (U, x), $(\pi^{-1}(U), \bar{x})$ of related charts, $x \circ \pi \circ \bar{x}^{-1} : x(U) \times \mathbb{R}^n \to x(U)$ is the projection onto the first factor.

Any $f : M \to N$ induces a differentiable map $f_* : TM \to TN$, called the *derivative* of f: For $v \in M_p$, set $f_*v := f_{*p}v$. Differentiability follows from the easily checked identity:

$$(\bar{y} \circ f_* \circ \bar{x}^{-1})(a, b) = (y \circ f \circ x^{-1}(a), D(y \circ f \circ x^{-1})(a)b).$$

EXERCISE 7. Show that if M is connected, then any two points of M can be joined by a smooth curve.

EXERCISE 8. (a) Prove that $\mathcal{J}_v : \mathbb{R}^n \to (\mathbb{R}^n)_v$ from Examples and Remarks 4.1(iv) satisfies $\mathcal{J}_v w(f) = D_w f(v) = (f \circ c)'(0)$, where c is any curve with $c(0) = v$, $c'(0) = w$.

(b) Show that any $v \in TM$ equals $\dot{c}(0)$ for some curve c in M.

EXERCISE 9. For positive ρ, σ, consider the helix $c : \mathbb{R} \to \mathbb{R}^3$, given by $c(t) = (\rho \cos t, \rho \sin t, \sigma t)$. Express $\dot{c}(t)$ in terms of the standard basis of $\mathbb{R}^3_{c(t)}$.

EXERCISE 10. Let M be connected, $f : M \to N$ a differentiable map. Show that if $f_{*p} = 0$ for all p in M, then f is a constant map.

EXERCISE 11. Fill in the details of the argument for the cotangent bundle in the proof of Proposition 4.2.

5. The Inverse and Implicit Function Theorems

Let U be an open set in M, $f : U \to N$ a differentiable map. The *rank* of f at $p \in U$ is the rank of the linear map $f_{*p} : M_p \to N_{f(p)}$, that is, the dimension of the space $f_*(M_p)$. Recall the following theorem from calculus:

THEOREM 5.1 (Inverse Function Theorem). *Let U be an open set in $\mathbb{R}^n$, $f : U \to \mathbb{R}^n$ a differentiable map. If f has maximal rank (=n) at $p \in U$, then there exists a neighborhood V of p such that the restriction $f : V \to f(V)$ is a diffeomorphism.*

The inverse function theorem immediately generalizes to manifolds:

THEOREM 5.2 (Inverse Function Theorem for Manifolds). *Let M and N be manifolds of dimension n, and $f : U \to N$ a smooth map, where U is open in M. If f has maximal rank at $p \in U$, then there exists a neighborhood V of p such that the restriction $f : V \to f(V)$ is a diffeomorphism.*

PROOF. Consider coordinate maps x at p, y at $f(p)$, and apply Theorem 5.1 to $y \circ f \circ x^{-1}$. Conclude by observing that x and y are diffeomorphisms. $\square$

We now use the inverse function theorem to derive the Euclidean version of one of the essential tools in differential geometry:

THEOREM 5.3 (Implicit Function Theorem). *Let U be a neighborhood of 0 in $\mathbb{R}^n$, $f : U \to \mathbb{R}^k$ a smooth map with $f(0) = 0$. For $n \leq k$, let $\imath : \mathbb{R}^n \to \mathbb{R}^k$ denote the inclusion $\imath(a_1, \dots, a_n) = (a_1, \dots, a_n, 0, \dots, 0)$, and for $n \geq k$, let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ denote the projection $\pi(a_1, \dots, a_k, \dots, a_n) = (a_1, \dots, a_k)$.*

(1) *If $n \leq k$ and f has maximal rank ($= n$) at 0, then there exists a coordinate map g of $\mathbb{R}^k$ around 0 such that $g \circ f = \imath$ in a neighborhood of $0 \in \mathbb{R}^n$.*

(2) *If $n \geq k$ and f has maximal rank ($= k$) at 0, then there exists a coordinate map h of $\mathbb{R}^n$ around 0 such that $f \circ h = \pi$ in a neighborhood of $0 \in \mathbb{R}^n$.*

PROOF. In order to prove (1), observe that the $k \times n$ matrix $(D_j f^i(0))$ has rank n. By rearranging the component functions f^i of f if necessary (which amounts to composing f with an invertible transformation, hence a diffeomorphism of $\mathbb{R}^k$), we may assume that the $n \times n$ submatrix $(D_j f^i(0))_{1 \leq i,j \leq n}$ is invertible. Define $F : U \times \mathbb{R}^{k-n} \to \mathbb{R}^k$ by

$$F(a_1, \dots, a_n, a_{n+1}, \dots, a_k) := f(a_1, \dots, a_n) + (0, \dots, 0, a_{n+1}, \dots, a_k).$$

Then $F \circ \imath = f$, and the Jacobian matrix of F at 0 is

$$\begin{pmatrix} (D_j f^i(0))_{1 \leq i \leq n} & 0 \\ (D_j f^i(0))_{n+1 \leq i \leq k} & 1_{\mathbb{R}^{k-n}} \end{pmatrix},$$

which has nonzero determinant. Consequently, F has a local inverse g, and $g \circ f = g \circ F \circ \imath = \imath$. This establishes (1). Similarly, in (2), we may assume that the $k \times k$ submatrix $(D_j f^i(0))_{1 \leq i,j \leq k}$ is invertible. Define $F : U \to \mathbb{R}^n$ by

$$F(a_1, \dots, a_n) := (f(a_1, \dots, a_n), a_{k+1}, \dots, a_n).$$

Then $f = \pi \circ F$, and the Jacobian of F at 0 is

$$\begin{pmatrix} (D_j f^i(0))_{1 \leq j \leq k} & (D_j f^i(0))_{k+1 \leq j \leq n} \\ 0 & 1_{\mathbb{R}^{n-k}} \end{pmatrix},$$

which is invertible. Thus, F has a local inverse h, and $f \circ h = \pi \circ F \circ h = \pi$. $\square$

6. Submanifolds

The implicit function theorem enables us to construct new examples of manifolds. Before doing so, however, there are certain "nice" maps, such as the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$, that deserve special recognition:

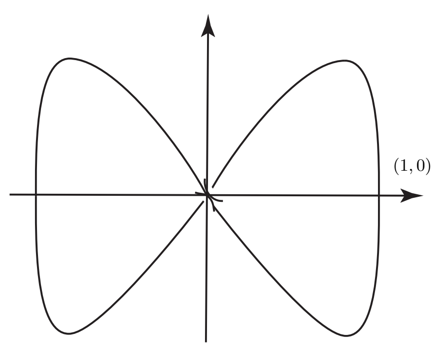

FIGURE 2. The lemniscate $c|_{(0,2\pi)}$.

DEFINITION 6.1. A map $f : M^n \to N^k$ is said to be an *immersion* if for every $p \in M$ the linear map $f_{*p} : M_p \to N_{f(p)}$ is one-to-one (so that $n \leq k$). If in addition f maps M homeomorphically onto $f(M)$ (where $f(M)$ is endowed with the subspace topology), then f is called an *imbedding*.

Notice that if M is compact, then an injective immersion is an imbedding. This is not true in general: For example, the curve $c : \mathbb{R} \to \mathbb{R}^2$ which parametrizes a lemniscate, $c(t) = (\sin t, \sin 2t)$, is an immersion; its restriction to $(0, 2\pi)$ is a one-to-one immersion, but not an imbedding, although $c_{|(0,\pi)}$ is. In fact, an immersion is always *locally* an imbedding:

PROPOSITION 6.1. *If $f : M^n \to N^k$ is an immersion, then for any $p \in M$, there exists a neighborhood U of p, and a coordinate map y defined on some neighborhood V of $f(p)$ such that*

(1) *A point q belongs to $f(U) \cap V$ iff $y^{n+1}(q) = \cdots = y^k(q) = 0$, i.e., $y(f(U) \cap V) = (\mathbb{R}^n \times \{0\}) \cap y(V)$;*
(2) *$f|_U$ is an imbedding.*

PROOF. Consider the inclusion $\imath : \mathbb{R}^n \to \mathbb{R}^k$, and let x be a coordinate map around p with $x(p) = 0$, $\tilde{y}$ a coordinate map around $f(p)$ with $(\tilde{y} \circ f)(p) = 0$. Since $\tilde{y} \circ f \circ x^{-1}$ has maximal rank at 0, there exists by the implicit function theorem a chart g of $\mathbb{R}^k$ around 0, and a neighborhood W of $0 \in \mathbb{R}^n$ such that $g \circ \tilde{y} \circ f \circ x^{-1}|_W = \imath|_W$. Set $U = x^{-1}(W)$, $y = g \circ \tilde{y}$; by restricting the domain of g if necessary, (1) clearly holds. (2) follows from the fact that $f|_U = y^{-1} \circ \imath \circ x|_U$ is a composition of imbeddings. $\square$

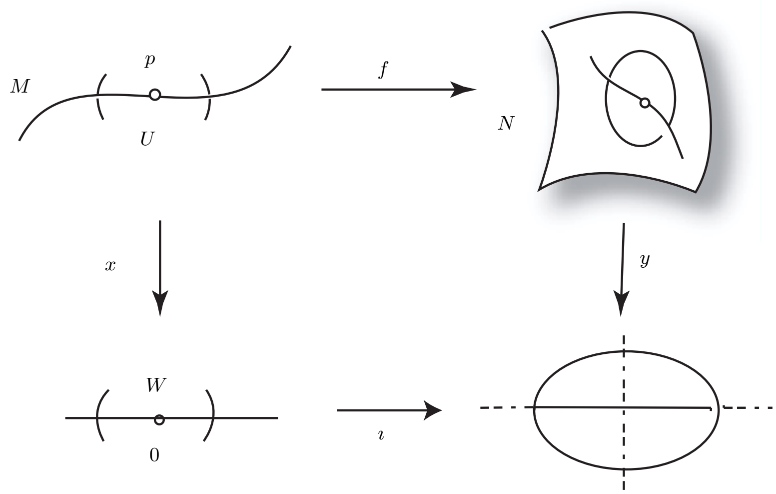

FIGURE 3

REMARK 6.1. When f in Proposition 6.1 is an imbedding, then $f(U)$ equals $f(M) \cap W$ for some open set W in N. Thus, in this case, (1) reads

$$f(M) \cap V = \{q \in V \mid y^{n+1}(q) = \cdots = y^k(q) = 0\}.$$

DEFINITION 6.2. Let M, N be manifolds with $M \subset N$. M is said to be a *submanifold* of N (respectively an *immersed submanifold* of N) if the inclusion map $\imath : M \hookrightarrow N$ is an imbedding (respectively an immersion).

By Remark 6.1, if M is an n-dimensional submanifold of N^k, then for any p in M, there exists a neighborhood V of p in N, and a chart (V,x) of N such that

$$M \cap V = \{q \in V \mid x^{n+1}(q) = \cdots = x^k(q) = 0\}.$$

When $f : M \to N$ is a one-to-one immersion (resp. imbedding), then M is diffeomorphic to an immersed submanifold (resp. submanifold) of N: namely $f(M)$, where $f(M)$ is endowed with the differentiable structure for which $f : M \to f(M)$ is a diffeomorphism. Clearly, $\imath : f(M) \to N$ is a one-to-one immersion (resp. imbedding). More generally, two immersions $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are said to be *equivalent* if there is a diffeomorphism $g : M_1 \to M_2$ such that $f_2 \circ g = f_1$. This defines an equivalence relation where each equivalence class contains a unique immersed submanifold of N.

Definition 6.3. Let $f : M^n \to N^k$ be differentiable. A point $p \in M$ is said to be a *regular point* of f if f_* has rank k at p; otherwise, p is called a *critical point*. $q \in N$ is said to be a *regular value* of f if its preimage $f^{-1}(q)$ contains no critical points (for example, if $q \notin f(M)$).

Theorem 6.1. *Let $f : M^n \to N^k$ be a smooth map, with $n \geq k$. If $q \in N$ is a regular value of f and if $A := f^{-1}(q) \neq \emptyset$, then A is a topological manifold of dimension $n-k$. Moreover, there exists a unique differentiable structure for which A becomes a differentiable submanifold of M.*

Proof. Let $y : V \to \mathbb{R}^k$ be a coordinate map around q with $y(q) = 0$; given $p \in A$, let $x : U \to \mathbb{R}^n$ be a coordinate map sending p to 0. Decompose $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, and denote by π_i, $i = 1, 2$, the projections of $\mathbb{R}^n$ onto the two factors; finally, let $\imath_2 : \mathbb{R}^{n-k} \to \mathbb{R}^n$ be the map given by $\imath_2(a_1, \dots, a_{n-k}) = (0, \dots, 0, a_1, \dots, a_{n-k})$.
Since $y \circ f \circ x^{-1}$ has maximal rank at $0 \in \mathbb{R}^n$, there exists, by Theorem 5.3(2), a chart (W,h) around 0 in $\mathbb{R}^n$ such that $y \circ f \circ x^{-1} \circ h = \pi_1|_W$. Set $\tilde{W} = \pi_2(W)$. $\tilde{W}$ is open in $\mathbb{R}^{n-k}$, and $y \circ f \circ x^{-1} \circ h \circ \imath_2|_{\tilde{W}} = \pi_1 \circ \imath_2|_{\tilde{W}} = 0$. Thus, if $z := x^{-1} \circ h \circ \imath_2|_{\tilde{W}}$, then $z(\tilde{W}) \subset A$. We claim that $z(\tilde{W}) = A \cap (x^{-1} \circ h)(W)$, so that z maps $\tilde{W}$ homeomorphically onto a neighborhood of p in A in the subspace topology. Clearly, $z(\tilde{W}) \subset A \cap (x^{-1} \circ h)(W)$, since $z(\tilde{W}) = (x^{-1} \circ h \circ \imath_2)(\tilde{W}) = (x^{-1} \circ h)(W \cap (0 \times \mathbb{R}^{n-k}))$. Conversely, if $\tilde{p} \in A \cap (x^{-1} \circ h)(W)$, then $\tilde{p} = (x^{-1} \circ h)(u)$ for a unique $u \in W$, and $0 = y \circ f(\tilde{p}) = (y \circ f \circ x^{-1} \circ h)(u) = \pi_1(u)$, so that $u = (0, a) \in 0 \times \tilde{W}$. Then $\tilde{p} = z(a) \in z(\tilde{W})$. It follows that the inclusion $\imath : A \hookrightarrow M$ is a topological imbedding.

Endow A with the differentiable structure induced by the charts $(z(\tilde{W}), z^{-1})$ as p ranges over A. Then $\imath : A \hookrightarrow M$ is smooth, since $x \circ \imath \circ (z^{-1})^{-1} = h \circ \imath_2$. $\square$

Examples and Remarks 6.1. (i) Let $r > 0$, and consider the map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $f(a) = |a|^2 - r^2$. Since $Df(a) = 2(a_1, \dots, a_{n+1})$, f has maximal rank 1 everywhere except at the origin. Thus, $S^n_r = f^{-1}(0)$ is a differentiable submanifold of $\mathbb{R}^{n+1}$. This differentiable structure coincides with the one introduced in Examples and Remarks 1.1: it is straightforward to check that the inclusion of the sphere into Euclidean space is smooth for the atlas introduced there; i.e., that $\imath \circ x^{-1} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is differentiable, if x denotes stereographic projection.

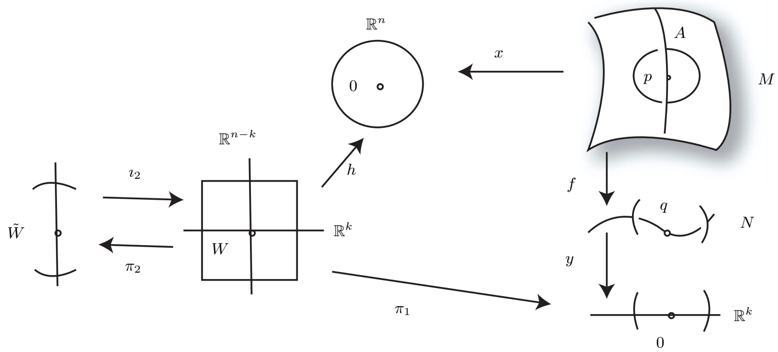

FIGURE 4

(ii) Let $f : M^n \to N^k$ be a differentiable map as in Definition 6.3. A point of N that is not a regular value is called a *critical value of* f. Sard proved that if U is an open set in $\mathbb{R}^n$, and $f : U \to \mathbb{R}^k$ is differentiable, then the set of critical values of f has measure zero; i.e., given any $\epsilon > 0$, there exists a sequence of k-dimensional cubes containing the set of critical values, whose total volume is less than ϵ. A proof of Sard's theorem can be found in [**25**]. As a consequence, the set of regular values of a map $f : M \to N$ between manifolds is dense in N, since its complement cannot contain an open nonempty set.

(iii) A surjective differentiable map $f : M^n \to N^k$ is said to be a *submersion* if every point of M is a regular point of f. In this case, f has no critical values, and each $p \in M$ belongs to the $(n-k)$-dimensional submanifold $f^{-1}(f(p))$.

Let $\imath : A \to M$ be an imbedding. For $p \in A$, $\imath_{*p}$ identifies the tangent space A_p with a subspace of M_p.

PROPOSITION 6.2. *Let q be a regular value of $f : M^n \to N^k$, where $n \geq k$, and suppose that $A := f^{-1}(q) \neq \emptyset$. Then for $p \in A$, $\imath_{*p}A_p = \ker f_{*p}$.*

PROOF. Since both subspaces have common dimension $n - k$, it suffices to check that $\imath_{*p}A_p \subset \ker f_{*p}$. Let $v \in A_p$. For $\phi \in \mathcal{F}N$, we have

$$(f_{*p}\imath_{*p}v)(\phi) = (f \circ \imath)_{*p}v(\phi) = v(\phi \circ f \circ \imath) = 0,$$

where the last identity follows from the fact that $f \circ \imath \equiv q$, so that $\phi \circ f \circ \imath$ is a constant function. This establishes the result. $\square$

EXAMPLE 6.1. Given manifolds M, N with $p \in M$, $q \in N$, define imbeddings $\imath_q : M \to M \times N$ and $\jmath_p : N \to M \times N$ by $\imath_q(p) = \jmath_p(q) = (p, q)$. If π_1, π_2 denote the projections of $M \times N$ onto M and N, then

$$\pi_1 \circ \imath_q = 1_M, \quad \pi_2 \circ \jmath_p = 1_N, \quad \pi_1 \circ \jmath_p = p, \quad \pi_2 \circ \imath_q = q,$$

where p is identified with the constant map $M \to M$ sending every point to p, and similarly for q. Thus,

$$\pi_{1*} \circ \imath_{q*p} = 1_{M_p}, \quad \pi_{2*} \circ \jmath_{p*q} = 1_{N_q}, \quad \pi_{1*} \circ \jmath_{p*q} = 0, \quad \pi_{2*} \circ \imath_{q*p} = 0.$$

This implies that the map

$$L : M_p \times N_q \to (M \times N)_{(p,q)},$$
$$(u, v) \mapsto \imath_{q*p} u + \jmath_{p*q} v$$

is an isomorphism with inverse $(\pi_{1*(p,q)}, \pi_{2*(p,q)})$: Both maps are linear, and by the above, $(\pi_{1*(p,q)}, \pi_{2*(p,q)}) \circ L = 1_{M_p \times N_q}$. The claim follows since both spaces have the same dimension.

EXERCISE 12. Let U be an open set in $\mathbb{R}^n$, $f \in \mathcal{F}U$. Show that $F : U \to \mathbb{R}^{n+1}$, where $F(a) = (a, f(a))$, is a differentiable imbedding. It follows that $F(U)$ is a differentiable n-submanifold of $\mathbb{R}^{n+1}$, called the *graph* of f. For example, if $U = \mathbb{R}^n$ and $f(a) = |a|^2$, the corresponding graph is called a paraboloid.

EXERCISE 13. Suppose $f : M \to N$ is differentiable, and let Q denote a submanifold of N. f is said to be *transverse regular* at $p \in f^{-1}(Q)$ if $f_{*p} M_p + Q_{f(p)} = N_{f(p)}$. Show that if f is transverse regular at every point of $f^{-1}(Q) \neq \emptyset$, then $f^{-1}(Q)$ is a submanifold of M of codimension equal to the codimension of Q in N. Theorem 6.1 is the special case when Q consists of a single point.

EXERCISE 14. For $p \in \mathbb{R}^{n+1}$, let $\mathcal{J}_p : \mathbb{R}^{n+1} \to (\mathbb{R}^{n+1})_p$ denote the canonical isomorphism. Use Proposition 6.2 to show that if $p \in S^n_r$, then

$$\imath_*(S^n_r)_p = \mathcal{J}_p(p^\perp),$$

where $p^\perp = \{a \in \mathbb{R}^{n+1} \mid \langle a, p \rangle = 0\}$ is the orthogonal complement of p.

EXERCISE 15. Prove that if M is compact, then $f : M^n \to \mathbb{R}^n$ cannot have maximal rank everywhere. Show by means of an example that such an f can nevertheless have maximal rank on a dense subset of M.

7. Vector Fields

In calculus, one defines a vector field on an open set $U \subset \mathbb{R}^n$ as a differentiable map $F = (f_1, \dots, f_n) : U \to \mathbb{R}^n$. When graphing a vector field on, say, $\mathbb{R}^2$, one draws the vector $F(p)$ with its origin at p, in order to distinguish it from the values of F at other points; in terms of tangent spaces, this means that $F(p)$ is considered to be a vector in the tangent space of $\mathbb{R}^n$ at p. It is now natural to generalize this concept to manifolds as follows:

DEFINITION 7.1. Let U be an open set of the differentiable manifold M^n. A (differentiable) *vector field* on U is a (differentiable) map $X : U \to TM$ such that $\pi \circ X = 1_U$. Here $\pi : TM \to M$ denotes the tangent bundle projection.

Thus, the value of X at p, which we often denote by X_p, is a vector in M_p. Any $f \in \mathcal{F}U$ determines a new function Xf on U by setting $Xf(p) := X_p(f)$. If (U, x) is a chart, the *coordinate vector fields* are the vector fields $\partial/\partial x^i$ whose value at $p \in U$ is $\partial/\partial x^i(p)$, cf. (3.1). Any vector field X on U can then be written as $X = \sum_i X(x^i)\partial/\partial x^i = \sum_i dx^i(X)\partial/\partial x^i$.

PROPOSITION 7.1. *Let $X : U \to TM$ be a map such that $\pi \circ X = 1_U$. The following statements are equivalent:*

(1) *X is a vector field on U (i.e., X, as a map, is differentiable).*
(2) *If (V, x) is a chart with $V \subset U$, then $Xx^i \in \mathcal{F}V$.*
(3) *If $f \in \mathcal{F}V$, then $Xf \in \mathcal{F}V$.*

PROOF. (1)⇒(2): Recall that (V, x) induces a coordinate map $\bar{x}$ on $\pi^{-1}(V)$, where $\bar{x}(v) = (x \circ \pi(v), v(x^1), \ldots, v(x^n))$. Since X is smooth, $\bar{x} \circ X|_V = (x \circ 1|_V, X|_V(x^1), \ldots, X|_V(x^n))$ also has that property. Thus, each component function Xx^i is differentiable on V.

(2)⇒(3): If each $X|_V(x^i) \in \mathcal{F}V$, then $X|_V(f) = \sum_i (X|_V x^i)\partial f/\partial x^i \in \mathcal{F}V$.

(3)⇒(1): $\bar{x} \circ X|_V = (x, X|_V(x^1), \ldots, X|_V(x^n))$ is smooth, and therefore so is $X|_V$. Since this is true for any chart (V, x) with $V \subset U$, X is differentiable. □

EXAMPLE 7.1. A vector field X on $\mathbb{R}^n$ induces a differentiable map $F = (f^1, \ldots, f^n) : \mathbb{R}^n \to \mathbb{R}^n$, where $f^i = du^i(X)$; conversely, any smooth map $F : U \to \mathbb{R}^n$ on an open subset U of $\mathbb{R}^n$ determines a vector field X on U, with $X(p) = \mathcal{J}_p F(p)$.

Let $\mathfrak{X}U$ denote the set of vector fields on U. $\mathfrak{X}U$ is a real vector space and a module over $\mathcal{F}U$ with the operations $(X + Y)_p = X_p + Y_p$, $(\phi X)_p = \phi(p)X_p$. If $f, g \in \mathcal{F}U$ and $\alpha, \beta \in \mathbb{R}$, then $X(\alpha f + \beta g) = \alpha(Xf) + \beta(Xg)$, and $X(fg) = (Xf)g + (Xg)f$.

We recall two theorems from the theory of ordinary differential equations:

THEOREM 7.1 (Existence of Solutions). *Let $F : U \to \mathbb{R}^n$ be a differentiable map, where U is open in $\mathbb{R}^n$. For any $a \in U$, there exists a neighborhood W of a, an interval I around 0, and a differentiable map $\psi : I \times W \to U$ such that*

(1) *$\psi(0, u) = u$, and*
(2) *$D\psi(t, u)\mathbf{e}_1 = F \circ \psi(t, u)$*

for $t \in I$ and $u \in W$.

Theorem 7.1 may be interpreted as follows: A curve $c : I \to U$ is called an *integral curve* of (the system of ordinary differential equations defined by) F if $c^{i\prime} = F^i \circ c$, $1 \le i \le n$; in this case, $Dc = F \circ c$, and the restriction of F to c is the "velocity field" of c. Thus, 7.1 asserts that integral curves $t \mapsto c(t) := \psi(t, u)$ exist for arbitrary initial conditions $c(0) = u$, that they depend smoothly on the initial conditions, and that at least locally, they can be defined on a fixed common interval. Also notice that in manifold notation, c is an integral curve of $F : \mathbb{R}^n \to \mathbb{R}^n$ iff $\dot{c} = X \circ c$, where $X = \mathcal{J}F$, cf. the example above.

THEOREM 7.2 (Uniqueness of Solutions). *If c, $\tilde{c} : I \to U$ are two integral curves of $F : U \to \mathbb{R}^n$ with $c(t_0) = \tilde{c}(t_0)$ for some $t_0 \in I$, then $c = \tilde{c}$.*

DEFINITION 7.2. Let M be a manifold, $X \in \mathfrak{X}M$, and I an interval. A curve $c : I \to M$ is called an *integral curve* of X if $\dot{c} = X \circ c$.

THEOREM 7.3. *Let M be a manifold, $X \in \mathfrak{X}M$. For any $q \in M$, there exists a neighborhood V of q, an interval I around 0, and a differentiable map $\Phi : I \times V \to M$ such that*

(1) *$\Phi(0, p) = p$, and*
(2) *$\Phi_* \frac{\partial}{\partial t}(t, p) = X \circ \Phi(t, p)$*

for all $t \in I$, $p \in V$. *Here,* $\partial/\partial t(t,p) := \imath_{p*}D(t)$ *for the injection* $\imath_p : I \to I \times V$ *which maps* t *to* (t,p).

Notice that

$$\Phi_* \frac{\partial}{\partial t}(t,p) = \dot{\overbrace{\Phi \circ \imath_p}}(t) = \dot{\Phi}_p(t),$$

where $\Phi_p(t) = \Phi(t,p)$. Theorem 7.3 asserts that for any $p \in V$, $\Phi_p : I \to M$ is an integral curve of X passing through p at $t = 0$. Φ is called a *local flow* of X.

Proof. Let (U,x) be a chart around q, and set $G := x(U)$, $a := x(q)$, and

$$F := (dx^1(X), \dots, dx^n(X)) \circ x^{-1} : G \to \mathbb{R}^n.$$

By Theorem 7.1, there exists a neighborhood W of a, an interval I around 0, and a map $\psi : I \times W \to G$ such that (1) and (2) of 7.1 hold. Let $V := x^{-1}(W)$, and $\Phi : I \times V \to M$ be given by $\Phi(t,p) = x^{-1} \circ \psi(t, x(p))$. $\square$

An argument similar to the one above generalizes the uniqueness theorem 7.2 to manifolds:

Theorem 7.4. *If* $c, \tilde{c} : I \to M$ *are two integral curves of* $X \in \mathfrak{X}M$ *with* $c(t_0) = \tilde{c}(t_0)$ *for some* $t_0 \in I$, *then* $c = \tilde{c}$.

For each $p \in M$, let I_p denote the maximal open interval around 0 on which the (unique by 7.4) integral curve $\Phi_p : I_p \to M$ of X with $\Phi_p(0) = p$ is defined.

Theorem 7.5. *Given any* $X \in \mathfrak{X}M$, *there exists a unique open set* $W \subset \mathbb{R} \times M$ *and a unique differentiable map* $\Phi : W \to M$ *such that*

(1) $I_p \times \{p\} = W \cap (\mathbb{R} \times \{p\})$ *for all* $p \in M$, *and*
(2) $\Phi(t,p) = \Phi_p(t)$ *if* $(t,p) \in W$.

Φ is called the *maximal flow* of X. By (2), $\{0\} \times M \subset W$, and (1), (2) of Theorem 7.3 are satisfied.

Proof. (1) determines W uniquely, while (2) does the same for Φ. It thus remains to show that W is open, and that Φ is differentiable.

Fix $p \in M$, and let I denote the set of all $t \in I_p$ for which there exists a neighborhood of (t,p) contained in W on which Φ is differentiable. We will establish that I is nonempty, open and closed in I_p, so that $I = I_p$: I is nonempty because $0 \in I$ by Theorem 7.3, and is open by definition. To see that it is closed, consider $t_0 \in \bar{I}$; by 7.3, there exists a local flow $\Phi' : I' \times V' \to M$ with $0 \in I'$ and $\Phi_p(t_0) \in V'$. Let $t_1 \in I$ be small enough that $t_0 - t_1 \in I'$ (recall that t_0 belongs to the closure of I) and $\Phi_p(t_1) \in V'$ (by continuity of Φ_p). Choose an interval I_0 around t_0 such that $t - t_1 \in I'$ for $t \in I_0$. Finally, by continuity of Φ at (t_1, p), there exists a neighborhood V of p such that $\Phi(t_1 \times V) \subset V'$.

We claim that Φ is defined and differentiable on $I_0 \times V$, so that $t_0 \in I$: Indeed, if $t \in I_0$ and $q \in V$, then by definition of I_0 and V, $t - t_1 \in I'$ and $\Phi(t_1, q) \in V'$, so that $\Phi'(t - t_1, \Phi(t_1, q))$ is defined. The curve $s \mapsto \Phi'(s - t_1, \Phi(t_1, q))$ is an integral curve of X which equals $\Phi(t_1, q)$ at t_1. By uniqueness, $\Phi(t,q) = \Phi'(t - t_1, \Phi(t_1, q))$ is defined, and Φ is therefore differentiable at (t,q). $\square$

DEFINITION 7.3. Let $\Phi : \mathbb{R} \times M \to M$ be differentiable, and define $\Phi_t : M \to M$ by $\Phi_t(p) := \Phi(t,p)$. $\{\Phi_t\}_{t\in\mathbb{R}}$ is called a *one-parameter group of diffeomorphisms* of M if

(1) $\Phi_0 = 1_M$, and
(2) $\Phi_{t_1+t_2} = \Phi_{t_1} \circ \Phi_{t_2}, \quad t_1, t_2 \in \mathbb{R}$.

Observe that each Φ_t is indeed a diffeomorphism of M with inverse Φ_{-t}. If Φ is a one-parameter group of diffeomorphisms, then the vector field X defined by $X_p := \Phi_* \frac{\partial}{\partial t}|_{(0,p)}$ has Φ as maximal flow (since integral curves are defined for all time). Conversely, if $X \in \mathfrak{X}M$, then the maximal flow of X induces a one-parameter group of diffeomorphisms provided X is *complete*; i.e., provided integral curves are defined for all time. The exercises at the end of the section establish that vector fields on compact manifolds are always complete.

EXAMPLE 7.2. Consider the vector field $X \in \mathfrak{X}\mathbb{R}^2$ whose value at $a = (a_1, a_2)$ is given by $-a_2 D_1|_a + a_1 D_2|_a$. Fix $p = (p_1, p_2) \in \mathbb{R}^2$, and let $c : \mathbb{R} \to \mathbb{R}^2$ denote the curve

$$c(t) = ((\cos t)p_1 - (\sin t)p_2, (\sin t)p_1 + (\cos t)p_2).$$

Then

$$\dot c(t) = (-(\sin t)p_1 - (\cos t)p_2)D_1|_{c(t)} + ((\cos t)p_1 - (\sin t)p_2)D_2|_{c(t)} = X \circ c(t).$$

Thus, c is the integral curve of X with $c(0) = p$, and X is complete. The one-parameter group of X is the rotation group

$$\Phi_t(p_1, p_2) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

EXERCISE 16. Show explicitly that Φ in Theorem 7.3 satisfies (1) and (2).

EXERCISE 17. With notation as in Theorem 7.5,

(a) Show by means of an example that there need not exist an open interval I around 0 such that $I \times M \subset W$. *Hint:* Let $M = \mathbb{R}$, $X_t = -t^2 D_t$.

(b) Show that if such an interval exists, then it equals all of $\mathbb{R}$; i.e., $W = \mathbb{R} \times M$, and integral curves are defined for all time.

(c) Prove that if M is compact, then any vector field on M is complete.

EXERCISE 18. Let $\phi : [\alpha, \beta) \to M$ be an integral curve of $X \in \mathfrak{X}M$, and suppose that for some sequence $t_n \to \beta$, $\phi(t_n) \to p$ for some $p \in M$.

(a) Show that $\bar\phi : [\alpha, \beta] \to M$, where $\bar\phi|_{[\alpha,\beta)} = \phi$ and $\bar\phi(\beta) = p$, is continuous.

(b) Prove that if $c : I \to M$ is the maximal integral curve of X with $c(\beta) = p$, then $[\alpha, \beta] \subset I$, and $c_{|[\alpha,\beta]} = \bar\phi$.

(c) Use parts (a) and (b) to recover the result from Exercise 17 (c): Namely, if M is compact, then every integral curve of $X \in \mathfrak{X}M$ is defined on all of $\mathbb{R}$.

8. The Lie Bracket

Consider two vector fields X and Y on an open subset U of M, with flows Φ_s and Ψ_t, respectively. It may well happen that these flows commute; i.e., that $\Phi_s \circ \Psi_t = \Psi_t \circ \Phi_s$ for small s and t. This is the case for example when X and Y are coordinate vector fields, since the standard fields D_i and D_j in

Euclidean space have commuting flows. In general, the Lie bracket $[X, Y]$ of X and Y is a new vector field that detects noncommuting flows. This concept actually makes sense in the more general setting of an arbitrary vector space E:

Definition 8.1. A *Lie bracket* on a real vector space E is a map $[,] : E \times E \to E$ satisfying:

(1) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$,
(2) $[X, Y] = -[Y, X]$, and
(3) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

for all $X, Y, Z \in E$, $\alpha, \beta \in \mathbb{R}$. By (1) and (2), the Lie bracket is linear in the second component also. (3) is called the *Jacobi identity*. A vector space together with a Lie bracket is called a *Lie algebra*.

A trivial example of a Lie algebra is $\mathbb{R}^n$ with $[,] \equiv 0$. This is the so-called abelian n-dimensional Lie algebra. $\mathbb{R}^3$ is also a Lie algebra, if one takes the Lie bracket to be the classical cross-product of two vectors.

Let M be a differentiable manifold, p a point in an open set U of M, and $X, Y \in \mathfrak{X}U$. Define $X_pY : \mathcal{F}_pU \to \mathbb{R}$ by setting $(X_pY)f := X_p(Yf)$. X_pY is not a tangent vector at p, because although it is linear on functions, it is not a derivation. However, $X_pY - Y_pX$ is one:

$$\begin{aligned}(X_pY - Y_pX)(fg) &= X_p(Y(fg)) - Y_p(X(fg))\\ &= X_p(f(Yg) + g(Yf)) - Y_p(f(Xg) + g(Xf))\\ &= (X_pf)(Y_pg) + f(p)X_p(Yg) + (X_pg)(Y_pf) + g(p)X_p(Yf)\\ &\quad - (Y_pf)(X_pg) - f(p)Y_p(Xg) - (Y_pg)(X_pf)\\ &\quad - g(p)Y_p(Xf)\\ &= f(p)(X_pY - Y_pX)(g) + g(p)(X_pY - Y_pX)(f).\end{aligned}$$

Thus, $p \mapsto X_pY - Y_pX$ is a vector field on U.

Definition 8.2. Let $X, Y \in \mathfrak{X}U$, where U is open in M. The *Lie bracket* of X with Y is the vector field $[X, Y]$ on U defined by $[X, Y]_p := X_pY - Y_pX$.

It is straightforward to check that $\mathfrak{X}U$ with the above bracket is a Lie algebra. One often denotes $X(Yf)$ by XYf, so that one may write

$$[X, Y] = XY - YX.$$

Observe also that for $f \in \mathcal{F}U$, $[fX, Y] = f[X, Y] - (Yf)X$.

Proposition 8.1. *Let (U, x) denote a chart of M^n. Then $[\partial/\partial x^i, \partial/\partial x^j] \equiv 0$ for $1 \le i, j \le n$.*

Proof. For $\phi \in \mathcal{F}U$,

$$\begin{aligned}[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]\phi &= \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\phi - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\phi\\ &= D_i\left(\frac{\partial}{\partial x^j}\phi \circ x^{-1}\right) \circ x - D_j\left(\frac{\partial}{\partial x^i}\phi \circ x^{-1}\right) \circ x\\ &= D_i(D_j(\phi \circ x^{-1})) \circ x - D_j(D_i(\phi \circ x^{-1})) \circ x = 0.\end{aligned}$$

$\square$

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