CHAPTER 2

Fiber Bundles

1. Basic Definitions and Examples

We have already encountered examples of manifolds that possess some additional structure, such as the tangent bundle TM of an n-dimensional manifold M. In this case, each point of TM has a neighborhood diffeomorphic to a product $U \times \mathbb{R}^n$, where U is an open set in M. Of course, TM itself need not be diffeomorphic to $M \times \mathbb{R}^n$. In most of the sequel, we will be concerned with manifolds that, roughly speaking, look locally like products.

As usual, all maps are assumed to be differentiable.

Definition 1.1. Let F, M, B denote manifolds, G a Lie group acting effectively on F (i.e., if $g(p) = p$ for all $p \in F$, then $g = e$). A *coordinate bundle over the base space B with total space M, fiber F, and structure group G* is a surjective map $\pi : M \to B$, called the *bundle projection*, together with a *bundle atlas* $\mathcal{A} = \{(\pi^{-1}(U_\alpha), (\pi, \phi_\alpha)\}_{\alpha \in A}$ on M; i.e.,

1. $\{U_\alpha\}_{\alpha \in A}$ is an open cover of B.
2. $(\pi, \phi_\alpha) : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism, called a *bundle chart*. Notice that for $p \in U_\alpha$, $\phi_{\alpha|\pi^{-1}(p)} : \pi^{-1}(p) \to F$ is a diffeomorphism. If p also belongs to U_β, then $\phi_{\beta|\pi^{-1}(p)} : \pi^{-1}(p) \to F$ need not coincide with ϕ_α; however, they must differ by the operation of some element in G. To be specific:
3. For $\alpha, \beta \in A$, there is a smooth map $f_{\alpha,\beta} : U_\alpha \cap U_\beta \to G$, called the *transition function from ϕ_α to ϕ_β* given by $f_{\alpha,\beta}(p) = \phi_\beta \circ (\phi_{\alpha|\pi^{-1}(p)})^{-1} : F \to F$; equivalently, $\phi_{\beta|\pi^{-1}(U_\alpha \cap U_\beta)} = (f_{\alpha,\beta} \circ \pi) \cdot \phi_{\alpha|\pi^{-1}(U_\alpha \cap U_\beta)}$.

Statement (3) says that the diagram

$$\begin{array}{ccc}
\pi^{-1}(U_\alpha \cap U_\beta) & = & \pi^{-1}(U_\alpha \cap U_\beta) \\
{\scriptstyle (\pi,\phi_\alpha)}\downarrow & & \downarrow{\scriptstyle (\pi,\phi_\beta)} \\
(U_\alpha \cap U_\beta) \times F & \longrightarrow & (U_\alpha \cap U_\beta) \times F \\
(p, m) & \longrightarrow & (p, f_{\alpha,\beta}(p)\, m)
\end{array}$$

commutes. Roughly speaking, the total space M consists of a collection $\cup U_\alpha \times F$, where the U_α's cover B, and copies of F belonging to intersecting U_α's are identified by means of elements of G. The projection π is a submersion by (2). The set $\pi^{-1}(p)$ is called the *fiber over* p. Notice that (3) implies that $f_{\alpha,\alpha} = e$, $(f_{\alpha,\beta}(p))^{-1} = f_{\beta,\alpha}(p)$, and $f_{\alpha,\gamma}(p) = f_{\beta,\gamma}(p) \cdot f_{\alpha,\beta}(p)$.

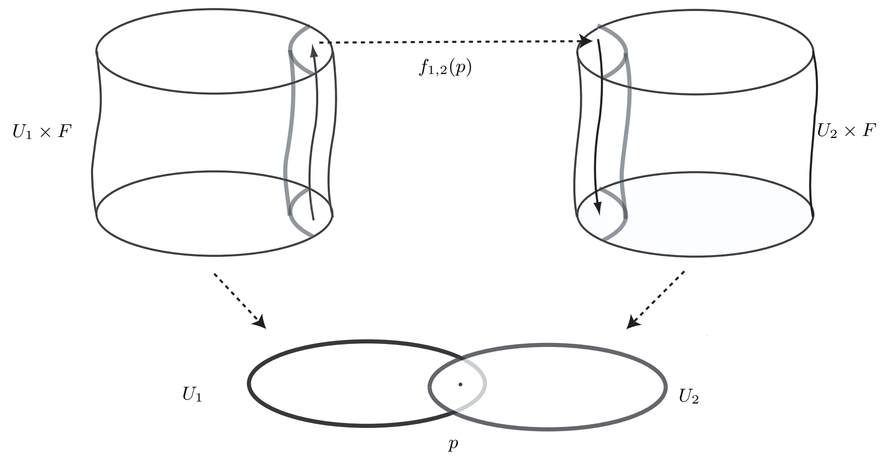

FIGURE 1

DEFINITION 1.2. A (real) *coordinate vector bundle of rank n* is a coordinate bundle with fiber $\mathbb{R}^n$ and structure group $GL(n)$ (or a subgroup of $GL(n)$).

We will often use Greek letters such as ξ to denote a bundle $\pi : M \to B$. If ξ is a rank n vector bundle, then each fiber $\pi^{-1}(b)$, $b \in B$, is a vector space: Given a bundle chart $(\pi^{-1}(U), (\pi, \phi))$ with $b \in U$, define the vector space operations on the fiber over b so that $\phi_{|\pi^{-1}(b)} : \pi^{-1}(b) \to \mathbb{R}^n$ becomes an isomorphism. The vector space structure is independent of the chosen chart because any transition function $f_{\phi,\psi}(p)$ at p is an isomorphism of $\mathbb{R}^n$.

EXAMPLES AND REMARKS 1.1. (i) The *trivial bundle* with base space B and fiber F is the projection $\pi : B \times F \to B$ onto the first factor. The structure group is $\{1_F\}$. In general, the size of the structure group measures how twisted the bundle is.

(ii) The tangent bundle TM of an n-dimensional manifold M is the total space of a rank n vector bundle over M with the bundle projection $\pi : TM \to M$ from Chapter 1: If $\{(U_\alpha, x_\alpha)\}$ is an atlas on M, then $\{(U_\alpha, (\pi, \phi_\alpha)\}$ is a bundle atlas on TM, where

$$\phi_\alpha = (dx_\alpha^1, \dots, dx_\alpha^n) : \pi^{-1}(U_\alpha) \to \mathbb{R}^n.$$

The transition function from ϕ_α to ϕ_β is $f_{\alpha,\beta} = D(x_\beta \circ x_\alpha^{-1}) \circ x_\alpha : U_\alpha \cap U_\beta \to GL(n)$. A similar argument applies to the tensor bundles $T_{r,s}(M)$ and the exterior bundles $\Lambda_k^*(M)$. The tangent bundle of M will be denoted τM to distinguish it from its total space TM (which, to confuse things further, is traditionally also referred to by the same name).

(iii) The *Hopf fibration* (see also Chapter 1, Examples and Remarks 9.1(i)): View $S^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ as the set of all $(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1}$ such that $\sum |z_i|^2 = 1$, and consider the free action of $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ on S^{2n+1} given by $z(z_1, \dots, z_{n+1}) = (z_1 z, \dots, z_{n+1} z)$. Since S^1 is compact, this action is proper, and by Chapter 1, Theorem 14.2, there exists a unique differentiable structure on the quotient S^{2n+1}/S^1 for which the projection becomes a submersion. The quotient is called *complex projective n-space* $\mathbb{C}P^n$ of dimension $2n$.

We claim that $\pi : S^{2n+1} \to \mathbb{C}P^n$ is the projection of a bundle with fiber and group S^1, called the *Hopf fibration*. In order to establish this, we exhibit an atlas of bundle charts satisfying Definition 1.1: for $i = 1, \dots, n+1$, define $\hat{U}_i = \{(z_1, \dots, z_{n+1}) \in S^{2n+1} \mid z_i \neq 0\}$, and $U_i = \pi(\hat{U}_i) \subset \mathbb{C}P^n$. It is easily checked that $\hat{U}_i = \pi^{-1}(U_i)$, so that $\{U_i\}$ is an open cover of $\mathbb{C}P^n$. Define $\phi_i : \hat{U}_i \to S^1$ by $\phi_i(z_1, \dots, z_{n+1}) = z_i/|z_i|$. Then $(\pi, \phi_i) : \hat{U}_i \to U_i \times S^1$ is a diffeomorphism with inverse $(\pi(w_1, \dots, w_{n+1}), z) \mapsto (z|w_i|/w_i)(w_1, \dots, w_{n+1})$, and the transition function $f_{i,j} : U_i \cap U_j \to S^1$ is given by $f_{i,j}(\pi(z_1, \dots, z_{n+1})) = z_j z_i^{-1}|z_i||z_j|^{-1}$.

Definition 1.3. Let $\pi_i : M_i \to B_i$ be two coordinate bundles with fiber F and group G. A differentiable map $h : M_1 \to M_2$ is said to be a *bundle map* if

(1) h maps each fiber $\pi_1^{-1}(p_1)$ diffeomorphically onto a fiber $\pi_2^{-1}(p_2)$, thereby inducing a differentiable map $\bar{h} : B_1 \to B_2$ such that $\pi_2 \circ h = \bar{h} \circ \pi_1$; and

(2) for any bundle charts $(\pi_1^{-1}(U_\alpha), (\pi_1, \phi_\alpha))$ and $(\pi_2^{-1}(V_\beta), (\pi_2, \psi_\beta))$ of π_1 and π_2, respectively, $p \in U_\alpha \cap \bar{h}^{-1}(V_\beta)$, the map $\psi_\beta \circ h \circ (\phi_{\alpha|\pi_1^{-1}(p)})^{-1}$ from F to F coincides with the operation of an element of G, and the resulting map
$$\begin{aligned} f_{\alpha,\beta} : U_\alpha \cap \bar{h}^{-1}(V_\beta) &\to G, \\ p &\mapsto \psi_\beta \circ h \circ (\phi_{\alpha|\pi_1^{-1}(p)})^{-1} \end{aligned}$$
is differentiable.

The two coordinate bundles are said to be *equivalent* if $B_1 = B_2$ and the induced map is the identity on the base. A *fiber bundle* is then defined to be an equivalence class of coordinate bundles. Alternatively, one could define it to be a coordinate bundle with a maximal atlas.

Notice that if h is a bundle map, then by the second condition above, the coordinate bundle over B_1 with bundle charts of the form $(\pi_1^{-1}(\bar{h}^{-1}(U)), (\pi_1, \phi \circ h))$, where $(\pi_2^{-1}(U), (\pi_2, \phi))$ is a bundle chart of π_2, is equivalent to π_1. Its transition functions $f_{\phi\circ h, \psi\circ h}$ are equal to $f^2_{\phi,\psi} \circ \bar{h}$, where $f^2_{\phi,\psi}$ are the transition functions of π_2.

Exercise 47. (a) Show that the functions $f_{\alpha,\beta}$ from Definition 1.3 satisfy $f_{\alpha,\gamma} = f_{\beta,\gamma} \cdot f^1_{\alpha,\beta}$ and $f_{\alpha,\gamma} = f^2_{\beta,\gamma} \cdot f_{\alpha,\beta}$, where $f^1_{\alpha,\beta}$ and $f^2_{\beta,\gamma}$ are transition functions for π_1 and π_2, respectively.

(b) Conversely, suppose $\pi_i : M_i \to B$ are two coordinate bundles over B with fiber F and group G. Show that if there is a collection of maps $f_{\alpha,\beta}$ as in Definition 1.3 satisfying the identities in (a), then the bundles are equivalent.

Exercise 48. Identify $S^{4n+3} \subset \mathbb{R}^{4n+4} = \mathbb{H}^{n+1}$ with the set of all $(n+1)$-tuples of quaternions $(q_1, \dots, q_{n+1})$ such that $\sum |q_i|^2 = 1$ (see Chapter 1, Example 8.1(iii)). Replace complex numbers by quaternions in Examples and remarks 1.1 (iii) to construct *quaternionic projective space* $\mathbb{H}P^n$ and a fiber bundle $\pi : S^{4n+3} \to \mathbb{H}P^n$ over $\mathbb{H}P^n$ with fiber and group S^3. This bundle is often called a *generalized Hopf fibration*.

EXERCISE 49. (a) Show that $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere via

$$\begin{aligned} \mathbb{C}P^1 &\longrightarrow S^2, \\ [z_1, z_2] &\longmapsto \frac{1}{|z_1|^2 + |z_2|^2}(2z_1\bar{z}_2, |z_1|^2 - |z_2|^2). \end{aligned}$$

(b) Show that $\mathbb{H}P^1$ from Exercise 48 may similarly be identified with S^4. Thus, for $n = 1$, the Hopf fibrations become $S^3 \to S^2$ with fiber S^1, and $S^7 \to S^4$ with fiber S^3.

2. Principal and Associated Bundles

The Hopf fibration discussed in the previous section is a prime example of the following key concept:

DEFINITION 2.1. A fiber bundle $\pi : P \to B$ with fiber and group G is called a *principal G-bundle* if there exists a free right action of G on P and an atlas such that for each bundle chart $(\pi^{-1}(U), (\pi, \phi))$, the map $\phi : \pi^{-1}(U) \to G$ is G-equivariant; i.e.,

$$(\pi, \phi)(pg) = (\pi(p), \phi(p)g), \qquad p \in \pi^{-1}(U), \quad g \in G.$$

It follows that B is the quotient space P/G: Since $\pi(pg) = \pi(p)$, the orbit $G(p) = \{pg \mid g \in G\}$ of p is contained in $\pi^{-1}(\pi(p))$; conversely, if (π, ϕ) is a bundle chart around p, then for $q \in \pi^{-1}(\pi(p))$,

$$\begin{aligned} q &= (\pi, \phi)^{-1}(\pi(q), \phi(q)) = (\pi, \phi)^{-1}(\pi(p), \phi(p)\phi(p)^{-1}\phi(q)) \\ &= (\pi, \phi)^{-1}(\pi(p), \phi(p)g) = pg, \end{aligned}$$

where $g = \phi(p)^{-1}\phi(q) \in G$. Furthermore, the structure group is G acting on itself by left translations: for $p \in P$,

$$f_{\phi,\psi}(\pi(p)) = \psi(p)\phi(p)^{-1},$$

where the choice of the element $p \in \pi^{-1}(\pi(p))$ is irrelevant because

$$\psi(pg)\phi(pg)^{-1} = \psi(p)g(\phi(p)g)^{-1} = \psi(p)gg^{-1}\phi(p)^{-1} = \psi(p)\phi(p)^{-1}.$$

EXAMPLES AND REMARKS 2.1. (i) The Hopf fibrations $S^{2n+1} \to \mathbb{C}P^n$ and $S^{4n+3} \to \mathbb{H}P^n$ are principal S^1 and S^3 bundles.

(ii) The *trivial principal G-bundle over B* is the projection $B \times G \to B$ onto the first factor. The action of G is by right multiplication $(b, g_1)g = (b, g_1 g)$ on the second factor.

(iii) Let G be a Lie group, H a closed subgroup of G, and denote by B the homogeneous space G/H. We first show that the quotient space G^n/H^k admits a (unique) differentiable structure of dimension $n - k$ for which the projection $\pi : G \to G/H$ becomes a submersion. This actually follows from Theorem 14.2 in Chapter 1, but we provide an independent argument, since that theorem won't be proved until Chapter 5. Observe that π is an open map for the quotient topology on G/H: If U is open in G, then so is $\pi(U)$ (in G/H), because $\pi^{-1}(\pi(U)) = \cup_{h \in H} R_h(U)$ is open in G. Furthermore, the quotient space is Hausdorff: If $\pi(a) \neq \pi(b)$, so that $a^{-1}b \notin H$, there exists a neighborhood of $a^{-1}b$ that does not intersect H. Such a neighborhood always contains an open set of the form $U \cdot a^{-1}b \cdot U$, where U is a neighborhood of the identity with $U = U^{-1}$. Then $Ua^{-1}bU \cap H = \emptyset$, which implies that $bUH \cap aUH = \emptyset$. Thus, $\pi \circ L_b(U)$ and $\pi \circ L_a(U)$ are disjoint open sets containing $\pi(b)$ and $\pi(a)$, respectively.

In order to exhibit a manifold structure on G/H, recall that Frobenius' theorem applied to the distribution $L_{g*}H_e$, $g \in G$, guarantees the existence of a chart (U, x) around e, with $x(U) = (0,1)^n$, such that each slice

$$\{g \in U \mid x^{k+1}(g) = a_1, \dots, x^n(g) = a_{n-k}\}$$

is contained in a left coset of H. If S denotes the slice containing e, there exists a neighborhood V of e such that $V \cap S = V \cap H$ (since H is a submanifold of G), and $V = V^{-1}$, $V \cdot V \subset U$. For the sake of simplicity, denote V by U again. Let $N = (\pi_1 \circ x)^{-1}(a)$, where $\pi_1 : \mathbb{R}^n \to \mathbb{R}^k \times 0$ denotes projection, and $a := \pi_1 \circ x(e)$. We claim that π is one-to-one when restricted to N: Indeed, if $\pi(a) = \pi(b)$, then $a^{-1}b \in U \cap H = U \cap S$, so that b belongs to $L_a(U \cap S)$. The latter set, being connected, is contained in a single slice. Since it also contains a, a and b lie in the same slice, so that $x(a) = x(b)$; i.e, $a = b$.

It follows that $\pi_{|N} : N \to W := \pi(N)$ is an open, bijective map, hence a homeomorphism. So is $\tilde{x} := \pi_2 \circ x \circ (\pi_{|N})^{-1} : W \to \tilde{x}(W) \subset 0 \times \mathbb{R}^{n-k}$, where $\pi_2 : \mathbb{R}^n \to 0 \times \mathbb{R}^{n-k}$ denotes the projection onto the other factor. We may then take $(W, \tilde{x})$ as a chart around $\pi(e)$. In order to produce a chart around $\pi(a)$, consider the homeomorphism $\mathbb{L}_a$ of G/H induced by left-multiplication by a in G, $\mathbb{L}_a(\pi(g)) := \pi(ag)$. The desired chart is then given by $(\mathbb{L}_a(W), \tilde{x} \circ \mathbb{L}_{a^{-1}})$. Given $b \in G$, the corresponding transition function is $\pi_2 \circ x_{|N} \circ L_{a^{-1}b} \circ (\pi_2 \circ x_{|N})^{-1}$, so that the collection $\{(\mathbb{L}_a(W), \tilde{x} \circ \mathbb{L}_{a^{-1}}) \mid a \in G\}$ induces a differentiable structure on G/H.

It remains to check that π is differentiable at $g \in G$. Using the charts $(L_g(U), x \circ L_{g^{-1}})$ around g and $(\mathbb{L}_g(W), \tilde{x} \circ \mathbb{L}_{g^{-1}})$ around $\pi(g)$, we have

$$\tilde{x} \circ \mathbb{L}_{g^{-1}} \circ \pi \circ (x \circ L_{g^{-1}})^{-1} = \tilde{x} \circ \mathbb{L}_{g^{-1}} \circ \pi \circ L_g \circ x^{-1} = \tilde{x} \circ \pi \circ x^{-1} = \pi_2,$$

which establishes the claim.

Finally, we show that $\pi : G \to G/H$ is a principal H-bundle: Notice that for any $[g] := \pi(g)$, there exists a neighborhood $U = \mathbb{L}_g(W)$ of $[g]$ on which π has a right inverse s_U. In fact, taking $s_U = L_g \circ (\pi_{|N})^{-1} \circ \mathbb{L}_{g^{-1}|U}$, we have $\pi \circ s_U = 1_U$. Then the map

$$\begin{aligned} U \times H &\to \pi^{-1}(U), \\ ([g], h) &\mapsto (s_U[g]) \cdot h \end{aligned}$$

is a diffeomorphism. Its inverse is of the form (π, ϕ_U), where $\phi_U : \pi^{-1}(U) \to H$ is H-equivariant, since $\phi_U(g) = s_U(\pi(g))^{-1}g$, so that

$$\phi_U(gh) = s_U(\pi(gh))^{-1}gh = (s_U(\pi(g))^{-1} \cdot g)h = \phi_U(g)h.$$

Thus, the collection of such maps (π, ϕ_U) forms a principal bundle atlas on G over B.

We have seen that given a fiber bundle $\pi : M \to B$ with fiber F and group G, a bundle atlas $\{(\pi^{-1}(U_\alpha), (\pi, \phi_\alpha))\}$ determines a family of transition functions $f_{\alpha,\beta} : U_\alpha \cap U_\beta \to G$ which satisfy $f_{\alpha,\gamma} = f_{\beta,\gamma} \cdot f_{\alpha,\beta}$. It turns out that the bundle may be reconstructed from these transition functions. More generally, one has the following:

PROPOSITION 2.1. *Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of a manifold B, and G a Lie group acting effectively on a manifold F. Suppose there is a collection of maps $f_{\alpha,\beta} : U_\alpha \cap U_\beta \to G$ such that*

$$f_{\alpha,\gamma}(p) = f_{\beta,\gamma}(p) \cdot f_{\alpha,\beta}(p), \qquad p \in U_\alpha \cap U_\beta \cap U_\gamma, \quad \alpha, \beta, \gamma \in A. \tag{2.1}$$

Then there exists a fiber bundle $\pi : M \to B$ with fiber F, structure group G, and a bundle atlas whose transition functions are the given collection $\{f_{\alpha,\beta}\}$. Furthermore, if $F = G$ and G acts on itself by left translations, then the atlas is a principal bundle atlas.

Notice that taking $\alpha = \beta = \gamma$ in (2.1) implies that $f_{\alpha,\alpha} \equiv e$. Taking $\alpha = \gamma$ then yields $f_{\alpha,\beta}^{-1} = f_{\beta,\alpha}$.

PROOF. Consider the disjoint union $\cup_{\alpha\in A}(U_\alpha \times F)$, and the quotient space M under the equivalence relation:

$$(p, q_1) \sim (p, q_2) \text{ iff } q_2 = f_{\alpha,\beta}(p)q_1 \text{ for some } \alpha, \beta \in A.$$

If $\rho : \cup_\alpha(U_\alpha \times F) \to M$ denotes the projection, then each restriction $\rho : U_\alpha \times F \to \rho(U_\alpha \times F)$ is a homeomorphism, and its inverse (π, ϕ_α) may be taken as a bundle chart. By construction, the transition functions of this atlas are the $f_{\alpha,\beta}$. $\square$

As a simple application, consider the group $G = \{\pm 1\}$ acting on $\mathbb{R}$ by multiplication. The circle $B = S^1$ of unit complex numbers admits $U_1 = S^1 \setminus \{-i\}$ and $U_2 = S^1 \setminus \{i\}$ as open cover. Then the map

$$f_{1,2} : U_1 \cap U_2 \to G,$$
$$z \mapsto \begin{cases} 1, & \text{if } \operatorname{Re} z > 0, \\ -1, & \text{otherwise}, \end{cases}$$

determines a rank 1 vector bundle over the circle, called a *Moebius band*.

DEFINITION 2.2. Let $\pi : M \to B$ be a fiber bundle with fiber F and group G. The principal G-bundle obtained as in Proposition 2.1 from the transition functions of π is called the *principal bundle associated to* π.

Thus, a fiber bundle with group G induces an associated principal G-bundle. One can recover the original bundle from the principal one: More generally, let $\pi_P : P \to B$ be a principal G-bundle, F a manifold on which G acts effectively on the left. Define an equivalence relation $\sim$ on the space $P \times F$ by setting $(p, m) \sim (pg, g^{-1}m)$, and denote the quotient space $(P \times F)/\sim$ by $P \times_G F$. There is a well-defined map $\pi : P \times_G F \to B$ given by $\pi[p, m] = \pi_P(p)$.

THEOREM 2.1. *Let $\pi_P : P \to B$ be a principal G-bundle, F a manifold on which G acts on the left. Then the map $\pi : P \times_G F \to B$ constructed above is a fiber bundle over B with fiber F and structure group G, called* the fiber

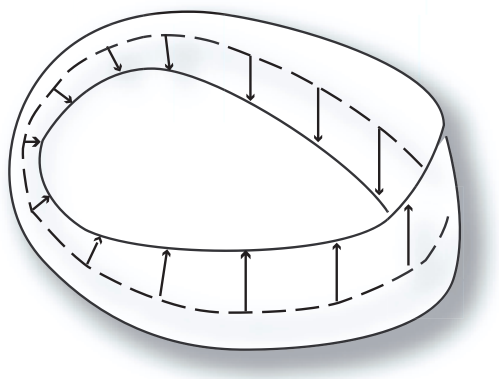

FIGURE 2. A Moebius band.

bundle with fiber F associated to the principal bundle $\pi_P : P \to B$ and the given action of G. *Furthermore, the principal G-bundle associated to π is π_P.*

PROOF. If $(\pi_P, \phi) : \pi_P^{-1}(U) \to U \times G$ is a principal bundle chart, define $\bar{\phi} : \pi^{-1}(U) \to U \times F$ by $\bar{\phi}[p, m] = \phi(p)m$. We claim that $(\pi, \bar{\phi})$ is a candidate for a bundle chart; i.e., it is invertible. Indeed, define $s : U \to \pi_P^{-1}(U)$ by $s(b) = (\pi_P, \phi)^{-1}(b, e)$; then $f : U \times F \to \pi^{-1}(U)$, where $f(b, m) = [s(b), m]$, is the inverse of $(\pi, \bar{\phi})$: On the one hand,

$$(\pi, \bar{\phi}) \circ f(b, m) = (\pi, \bar{\phi})[s(b), m] = (\pi_P(s(b)), \phi(s(b))m) = (b, m);$$

on the other, given $p \in \pi_P^{-1}(U)$, we have $s(\pi_P(p)) = p\phi(p)^{-1}$, so that

$$\begin{aligned} f \circ (\pi, \bar{\phi})[p, m] &= f(\pi_P(p), \phi(p)m) = [s(\pi_P(p)), \phi(p)m] \\ &= [p\phi(p)^{-1}, \phi(p)m] = [p, m]. \end{aligned}$$

This establishes the claim. Since both $(\pi, \bar{\phi})$ and f are continuous, they are homeomorphisms. Now, let (π_P, ϕ) and (π_P, ψ) be two principal bundle charts with overlapping domains. Given b in the projection of their intersection and $m \in F$, the transition function of the (candidates for) associated bundle charts at b is given by

$$\begin{aligned} f_{\bar{\phi}, \bar{\psi}}(b)m &= \bar{\psi} \circ (\pi, \bar{\phi})^{-1}(b, m) = \bar{\psi}[(\pi_P, \phi)^{-1}(b, e), m] \\ &= (\psi \circ (\pi_P, \phi)^{-1}(b, e))m = (f_{\phi,\psi}(b)e)m \\ &= f_{\phi,\psi}(b)m. \end{aligned}$$

The collection of charts therefore induces a differentiable structure on $P \times_G F$ and satisfies the requirements for a bundle atlas. Since the transition functions of the bundle coincide with those of π_P, π_P is the principal G-bundle associated to π. $\square$

EXAMPLE 2.1 (The Frame Bundle of a Vector Bundle). Let $\pi : E \to B$ denote a rank n vector bundle over B. We shall construct a principal $GL(n)$-bundle $\pi_P : Fr(E) \to B$, called the *frame bundle of E*, with the same transition functions. It will then follow from Proposition 2.1 and Theorem 2.1 that the frame bundle of E is the principal $GL(n)$-bundle associated to π, and that $E \to B$ is equivalent to $Fr(E) \times_{GL(n)} \mathbb{R}^n \to B$. Denote by E_b the fiber $\pi^{-1}(b)$ over $b \in B$, and let $Fr(E_b)$ be the collection of all *frames* of the vector space E_b; i.e., the collection of ordered bases $p = (v_1, \dots, v_n)$ of E_b. Each such frame can be viewed as an isomorphism $\mathbb{R}^n \to E_b$ mapping $\mathbf{e}_i$ to v_i for $1 \le i \le n$. Given two frames $p_i : \mathbb{R}^n \to E_b$, there exists a unique $g \in GL(n)$ such that $p_1 = p_2 g$. Identifying any single frame p with $e \in GL(n)$ yields a bijective map $Fr(E_b) \leftrightarrow GL(n)$.

Let $Fr(E) := \cup_{b \in B} Fr(E_b)$, $\pi_P : Fr(E) \to B$ the map that assigns the point b to a frame of E_b. If $(\pi, \bar{\phi}) : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a vector bundle chart for E, define

$$\begin{aligned} \phi : \pi_P^{-1}(U) &\to G = GL(n), \\ p &\mapsto \bar{\phi}_{|E_{\pi_P(p)}} \circ p. \end{aligned}$$

ϕ is G-equivariant by construction, and $(\pi_P, \phi) : \pi_P^{-1}(U) \to U \times G$ is therefore bijective. Given another vector bundle chart $(\pi, \bar{\psi})$ over U, we have

$$(\pi_P, \psi) \circ (\pi_P, \phi)^{-1}(b, g) = (\pi_P, \psi)(\bar{\phi}_{|E_b}^{-1} g) = (b, \bar{\psi} \circ (\bar{\phi}_{|E_b}^{-1}) g).$$

The collection of maps (π_P, ϕ) therefore induces a differentiable structure on $Fr(E)$ and forms a principal bundle atlas with transition functions $f_{\phi,\psi} = f_{\bar{\phi},\bar{\psi}}$. This establishes the claim.

Notice that there is an explicit equivalence between $Fr(E) \times_{GL(n)} \mathbb{R}^n \to B$ and π: if $p = (v_1, \dots, v_n)$ is a frame of E_b, the equivalence maps $[p, (\alpha_1, \dots, \alpha_n)] \in Fr(E) \times_G \mathbb{R}^n$ to $\sum \alpha_i v_i \in E_b$.

Exercise 50. Consider a principal G-bundle $\pi_P : P \to B$ and an associated bundle $\pi : P \times_G F \to B$.

(a) Show that $\rho : P \times F \to P \times_G F$, where $\rho(p, q) = [p, q]$, is a principal G-bundle, and that the projection $\pi_1 : P \times F \to P$ onto the first factor is a G-equivariant map inducing π on the base spaces.

$$\begin{array}{ccc} P \times F & \xrightarrow{\pi_1} & P \\ {\scriptstyle \rho}\big\downarrow & & \big\downarrow{\scriptstyle \pi_P} \\ P \times_G F & \xrightarrow[\pi]{} & B \end{array}$$

(b) Show that for any $p \in P$, the map $F \to \pi^{-1}(\pi(p))$ given by $q \mapsto \rho(p, q)$ is a diffeomorphism.

(c) If F is a vector space and G acts linearly on F, show that π is a vector bundle.

Exercise 51. Let H be a Lie subgroup of G, $\pi_G : P_G \to B$, $\pi_H : P_H \to B$ two principal G and H-bundles over B respectively with $P_H \subset P_G$. $P_H \to B$ is said to be a *principal subbundle* of $P_G \to B$ if for any $b \in B$, there exists a neighborhood U of b and principal bundle charts (π_H, ϕ), (π_G, ψ) of π_H, π_G over U such that

$$(\pi_G, \psi) \circ (\pi_H, \phi)^{-1} : U \times H \to U \times G$$

is the inclusion map. Show that in this case, given an action of G on a manifold F, the total spaces $P_H \times_H F$ and $P_G \times_G F$ of the associated F-bundles are diffeomorphic via a fiber preserving map. We say the structure group G of $P_G \times_G F \to B$ is *reducible* to H.

EXERCISE 52. Prove that the structure group G of a bundle is reducible to H (see Exercise 51) iff the bundle admits an atlas with H-valued transition functions.

3. The Tangent Bundle of S^n

In this section, we apply some of the concepts introduced above to discuss a basic example, that of the tangent bundle of the n-sphere.

The standard action of $SO(n+1)$ on S^n yields a map $SO(n+1) \to S^n$ that sends $g \in SO(n+1)$ to $g(\mathbf{e}_1)$. The subgroup of $SO(n+1)$ acting trivially on $\mathbf{e}_1$ may be identified with $SO(n)$, and one has an induced map

$$\begin{aligned} SO(n+1)/SO(n) &\to S^n, \\ [g] &\mapsto g\mathbf{e}_1, \end{aligned}$$

which is one-to-one by construction. It is also onto since $SO(n+1)$ acts transitively on the unit sphere. This map is a homeomorphism ($SO(n+1)/SO(n)$ being compact) which is easily checked to be a diffeomorphism. By Examples and Remarks 2.1(iii), $SO(n+1) \to S^n$ is a principal $SO(n)$-bundle.

On the other hand, the tangent bundle of S^n is a vector bundle with group $GL(n)$. For $p \in S^n$, the derivative of the inclusion map $S^n \hookrightarrow \mathbb{R}^{n+1}$ induces an inner product on the tangent space of S^n at p. By requiring the second component ϕ of each bundle chart (π, ϕ) to be a linear isometry $\phi_{|S^n_p} : S^n_p \to \mathbb{R}^n$, we obtain a reduction of the structure group to $O(n)$; cf. Exercise 52. Since the sphere is orientable, the group may further be reduced to $SO(n)$. (More generally, we will see in the next section that any vector bundle admits a reduction of its structure group $GL(n)$ to $O(n)$. The bundle is said to be *orientable* if its structure group is further reducible to $SO(n)$. The Moebius band from the preceding section is an example of a nonorientable bundle.)

In terms of principal bundles, we are reducing the frame bundle $Fr(TS^n)$ of Example 2.1 to the $SO(n)$-subbundle $SO(TS^n) \to S^n$ of *oriented orthonormal frames* whose fiber over $p \in S^n$ consists of all positively oriented orthonormal frames of S^n_p.

We claim that $SO(TS^n) \to S^n$ is equivalent to $SO(n+1) \to S^n$: In fact, the map $f : SO(n+1) \to SO(TS^n)$ which sends $g \in SO(n+1)$ to the ordered orthonormal frame $(\mathcal{J}_{g\mathbf{e}_1} g\mathbf{e}_2, \dots, \mathcal{J}_{g\mathbf{e}_1} g\mathbf{e}_{n+1})$ of $S^n_{g\mathbf{e}_1}$ induces the identity on S^n. Its inverse maps an orthonormal frame $v_1, \dots, v_n$ of S^n_p to the element $g \in SO(n+1)$ defined by $g\mathbf{e}_1 = p$, $g\mathbf{e}_{i+1} = \mathcal{J}_p^{-1} v_i$, $1 \le i \le n$. Since f is $SO(n)$-equivariant, the claim now follows from the following theorem:

THEOREM 3.1. *Let $\pi_i : P_i \to B$, $i = 1, 2$, be two principal G-bundles over B. If $h : P_1 \to P_2$ is a G-equivariant map inducing the identity on B, then the two bundles are equivalent.*

Proof. If $(\pi_1^{-1}(U_\alpha), (\pi_1, \phi_\alpha))$ and $(\pi_2^{-1}(V_\beta), (\pi_2, \psi_\beta))$ are bundle charts of π_1 and π_2, there are smooth maps $\pi_1^{-1}(U_\alpha) \to G$ and $\pi_1^{-1}(V_\beta) \to G$ given by $p \mapsto \phi_\alpha(p)$ and $q \mapsto (\psi_\beta \circ h)(q)$ respectively. Thus, the assignment

$$\begin{aligned} f_{\alpha,\beta} : U_\alpha \cap V_\beta \to G,& \\ b \mapsto (\psi_\beta \circ h)(p)\phi_\alpha(p)^{-1},& \end{aligned}$$

where p is any element of the fiber over b, is a well-defined smooth map. The bundles are then equivalent by Definition 1.3, since $f_{\alpha,\beta}(b) = \psi_\beta \circ h \circ (\phi_{\alpha|\pi_1^{-1}(b)})$: Indeed, let $g \in G$, $a := \phi_\alpha(p)$; then $g = aa^{-1}g = \phi_\alpha(p)a^{-1}g = \phi_\alpha(pa^{-1}g)$, so that

$$\begin{aligned} \psi_\beta \circ h \circ (\phi_{\alpha|\pi_1^{-1}(b)})^{-1}(g) &= \psi_\beta \circ h(pa^{-1}g) = \psi_\beta(h(p)) \cdot a^{-1}g \\ &= (\psi_\beta \circ h)(p) \cdot \phi_\alpha(p)^{-1} \cdot g. \end{aligned}$$

□

Corollary 3.1. *Let $\pi_i : E_i \to B$, $i = 1, 2$, be two rank n vector bundles over B. If $h : E_1 \to E_2$ is a diffeomorphism mapping each fiber $\pi_1^{-1}(b)$ linearly onto $\pi_2^{-1}(b)$, then the bundles are equivalent.*

Proof. Define $f : Fr(E_1) \to Fr(E_2)$ by $f(p) = h \circ p$, where $p : \mathbb{R}^n \to \pi_1^{-1}(b)$ is a frame of E_1. By Theorem 3.1, the two frame bundles are equivalent, and therefore so are the associated vector bundles $E_i \to B$. □

Corollary 3.1 provides another approach to the tangent bundle of the sphere, or more generally, to the tangent bundle TM of any homogeneous space $M = G/H$: Let $p = eH \in M$, so that H is the *isotropy group* at p of the action; i.e., $H = \{g \in G \mid gp = p\}$. The *linear isotropy representation* at p is the homomorphism $\rho : H \to GL(M_p)$ given by $\rho(h) = h_{*p}$. It is not difficult to show that if M is connected and G acts effectively on M, then ρ is one-to-one; in this case, ρ induces an effective linear action of H on M_p.

Proposition 3.1. *If G acts effectively on the homogeneous space $M = G/H$, then the tangent bundle of M is equivalent to the bundle $G \times_H M_p \to M$, where H acts on M_p via the linear isotropy representation at p.*

Proof. Consider the map $f : G \times_H M_p \to TM$ defined by $f[g, u] = g_* u$, which is clearly smooth, and linear on each fiber. Its inverse is given as follows: if $v \in M_q$, then by transitivity of the action of G, there exists some $g \in G$ such that $gp = q$. Then $f^{-1}(v) = [g, g_{*p}^{-1}v]$. This is well-defined, for if $q = \bar{g}p = gp$, then $g^{-1}\bar{g} \in H$, so that $\bar{g} = gh$ for some $h \in H$, and

$$[g, g_{*p}^{-1}v] = [gh, \rho(h)^{-1}g_{*p}^{-1}v] = [gh, h_{*p}^{-1}g_{*p}^{-1}v] = [gh, (gh)_{*p}^{-1}v] = [\bar{g}, \bar{g}_{*p}^{-1}v].$$

□

The hypothesis that G act effectively on M in Proposition 3.1 is not restrictive: Exercise 54 shows that M can always be realized as $\bar{G}/\bar{H}$, where $\bar{G}$ acts effectively on M.

Exercise 53. Let S^3 denote the group of quaternions of norm 1. Identify $SO(3)$ with the special orthogonal group of $\text{span}\{i, j, k\} = \mathbb{R}^3$, and define $\rho : S^3 \to SO(3)$ by $\rho(p)q = pqp^{-1}$ (quaternion multiplication).

(a) Show that ρ is a homomorphism with kernel $\{\pm 1\}$. It is not hard to see that ρ is onto, so that $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$ and ρ is the standard double covering.

(b) Consider the principal $SO(2)$-bundle $\pi : SO(3) \to S^2$. Prove that $\pi \circ \rho : S^3 \to S^2$ is equivalent to the Hopf fibration.

EXERCISE 54. Let $M = G/H$ be a homogeneous space.

(a) Show that the subgroup of G which acts trivially on M is the largest normal subgroup $N(H)$ of G which lies in H.

(b) Show that $\bar{G} = G/N(H)$ acts effectively on M, and that $M = \bar{G}/\bar{H}$, where $\bar{H} = H/N(H)$.

4. Cross-Sections of Bundles

A trivial bundle $B \times F \to B$ has the property that through any point $(b, m) \in B \times F$, there is a copy $B \times \{m\}$ of B; alternatively, the map $s : B \to B \times F$ given by $s(b) = (b, m)$ is a lift of the identity 1_B (in the sense that $\pi \circ s = 1_B$) through (b, m). It is by no means clear that such lifts exist in general, and they have a special name:

DEFINITION 4.1. Let $\xi = \pi : M \to B$ be a fiber bundle. A map $s : B \to M$ is said to be a *cross-section of* ξ if $\pi \circ s = 1_B$.

For example, a vector field on a manifold M is a cross-section of the tangent bundle of M; a differential k-form on M is a cross-section of the bundle $\Lambda_k^*(M) \to M$. It is common practice to abbreviate cross-section by *section*. Before looking at further examples, we point out that one can construct from a given vector bundle ξ many other vector bundles whose structure is induced by that of ξ. We illustrate the procedure in detail for the dual ξ^* of a vector bundle ξ. It is convenient to denote the fiber $\pi^{-1}(b)$ of a vector bundle $\pi : E \to B$ over b by E_b, and we will often do so.

PROPOSITION 4.1. *Let $\xi = \pi : E \to B$ be a rank n vector bundle, and define $E^* = \cup_{b \in B} E_b^*$. For $\alpha \in E_b^*$, let $\pi^*(\alpha) = b$. There exists a natural rank n vector bundle structure on $\xi^* = \pi^* : E^* \to B$ induced by ξ. ξ^* is called the* dual bundle *of ξ.*

PROOF. Let (π, ϕ) be a bundle chart of ξ over $U \subset B$. Since $\phi_{|E_b} : E_b \to \mathbb{R}^n$ is an isomorphism for each $b \in U$, so is $\bar{\phi}_{|E_b^*} : E_b^* \to \mathbb{R}^{n*} \cong \mathbb{R}^n$, where $\bar{\phi}_{|E_b^*} := (\phi_{|E_b})^{-1*}$ (recall that the transpose of a linear transformation $L : V \to W$ is the linear map $L^* : W^* \to V^*$ given by $(L^*\alpha)v = \alpha(Lv)$ for $\alpha \in W^*$, $v \in V$). Then $(\pi^*, \bar{\phi}) : \pi^{*-1}(U) \to U \times \mathbb{R}^n$ is one-to-one, onto, and its restriction to each E_b^* is linear; if (π, ψ) is another bundle chart, then

$$(\pi^*, \bar{\psi}) \circ (\pi^*, \bar{\phi})^{-1}(b, \alpha) = (b, \bar{\psi} \circ (\bar{\phi}_{|E_b^*})^{-1}\alpha) = (p, (\phi \circ \psi_{|E_b}^{-1})^*\alpha).$$

Thus, there exist unique topological and differentiable structures on E^* for which the maps $(\pi^*, \bar{\phi})$ become local diffeomorphisms. These maps form a bundle atlas, since the transition functions are given by $f_{\bar{\phi}, \bar{\psi}}(b) = f_{\psi, \phi}(p)^*$. $\square$

Given two vector bundles $\xi_i = \pi_i : E_i \to B$, one defines in a similar fashion the *tensor product bundle* $\xi_1 \otimes \xi_2$ with fiber $E_{1b} \otimes E_{2b}$ over b, the *homomorphism bundle* $\mathrm{Hom}(\xi_1, \xi_2)$ whose fiber over b consists of all linear transformations $E_{1b} \to E_{2b}$, etc. The isomorphism $\mathrm{Hom}(E_{1b}, E_{2b}) \cong E_{1b}^* \otimes E_{2b}$ induces an equivalence $\mathrm{Hom}(\xi_1, \xi_2) \cong \xi_1^* \otimes \xi_2$.

Definition 4.2. A *Euclidean metric* on a vector bundle $\xi = \pi : E \to B$ is a section s of the bundle $(\xi \otimes \xi)^*$ such that $s(b)$ is an inner product on E_b for each $b \in B$. A Euclidean metric on the tangent bundle of a manifold M is called a *Riemannian metric* on M.

Loosely translated, a Euclidean metric on ξ is just an inner product on the fibers that varies smoothly with the base point.

Theorem 4.1. *Every vector bundle $\xi = \pi : E \to B$ admits a Euclidean metric.*

Proof. Consider a locally finite cover of B by sets $\{U_\alpha\}$ whose preimages are the domains of bundle charts $\{(\pi, \phi_\alpha)\}$. Define a Euclidean metric s_α on each $\pi^{-1}(U_\alpha)$ so that ϕ_α becomes a linear isometry: $s_\alpha(u, v) = \langle \phi_\alpha u, \phi_\alpha v \rangle$, where $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, and extend s_α to all of B by setting $\bar{s}_\alpha(b) = \psi_\alpha(b) s_\alpha(b)$ if $b \in U_\alpha$ and $\bar{s}_\alpha(b) = 0$ otherwise. Then $s = \sum_\alpha \bar{s}_\alpha$ is a Euclidean metric on ξ. $\square$

Theorem 4.1 implies that every rank n vector bundle admits a reduction of its structure group to $O(n)$, by requiring that charts be linear isometries when restricted to each fiber.

Notice that a vector bundle always admits a section, namely *the zero section* given by $s(b) = 0 \in E_b$. Principal bundles, on the other hand, do not, in general, admit sections:

Theorem 4.2. *A principal G-bundle $\pi : P \to B$ admits a section iff it is trivial.*

Proof. If $\pi : B \times G \to B$ is trivial, then for any fixed $g \in G$, the map $s(b) := (b, g)$ defines a section of π. Conversely, suppose $s : B \to P$ is a section. Since $p \in P$ and $s(\pi(p))$ belong to the same fiber, there is a well-defined equivariant map $\phi : P \to G$ such that $p = s(\pi(p))\phi(p)$. $(\pi, \phi) : P \to B \times G$ is then an equivalence by Theorem 3.1. $\square$

Example 4.1. Recall from Section 3 that the principal $SO(n)$-bundle over S^n associated to the tangent bundle of S^n is $\pi : SO(n+1) \to SO(n+1)/SO(n) = S^n$. When $n = 3$, S^3 is identified with the group of quaternions of norm 1, and $\mathbf{e}_1 = 1 \in \mathbb{H}$.

Consider the map $s : S^3 \to SO(4)$ given by $s(q)u = qu$, for $q \in S^3 \subset \mathbb{H}$, $u \in \mathbb{H} = \mathbb{R}^4$. Then $(\pi \circ s)(q) = s(q)1 = q$; i.e., s is a section of $\pi : SO(4) \to S^3$, and $SO(4)$ is diffeomorphic to $S^3 \times SO(3)$ (although not isomorphic, as a group, to the direct product $S^3 \times SO(3)$). Since π is trivial, so is the associated tangent bundle $TS^3 \to S^3$. We saw in Chapter 1 that even-dimensional spheres do not admit a nowhere-zero vector field; i.e., their tangent bundle does not admit a nowhere-zero section, and is therefore nontrivial. Thus, none of the bundles $SO(n+1) \to S^n$ admit sections when n is even.

EXERCISE 55. Consider the map

$$\phi : S^3 \times S^3 \to SO(4), \qquad \phi(q_1, q_2)u = q_1 u q_2^{-1}, \qquad q_i \in S^3, \quad u \in \mathbb{H} = \mathbb{R}^4.$$

Show that ϕ is a Lie group homomorphism, and determine its kernel. It is not hard to see that ϕ is onto, so that $S^3 \times S^3$ is the two-fold covering group of $SO(4)$, denoted $Spin(4)$. Notice that if $\imath, \Delta : S^3 \to S^3 \times S^3$ are the imbedding-homomorphisms given by $\imath(q) = (q, e)$ and $\Delta(q) = (q, q)$, then $\phi \circ \imath$ is the section from Example 4.1, and $\phi \circ \Delta$ is the two-fold covering from Exercise 53.

EXERCISE 56. Let $\xi_i = \pi_i : E_i \to B$ be vector bundles over B, $i = 1, 2$, and denote by $\Gamma\xi_i$ the collection of sections of ξ_i.

(a) Show that $\Gamma\xi_i$ is a module over the ring of smooth functions $B \to \mathbb{R}$.

(b) Show that $\Gamma \operatorname{Hom}(\xi_1, \xi_2)$ and $\operatorname{Hom}(\Gamma\xi_1, \Gamma\xi_2)$ are naturally isomorphic as modules.

EXERCISE 57. A *complex vector bundle* is a bundle with fiber $\mathbb{C}^n$ whose transition functions are complex linear. Show that a real rank $2n$ vector bundle ξ admits a complex vector bundle structure iff there exists a section J of the bundle $\operatorname{Hom}(\xi, \xi)$ such that J^2 equals minus the identity on the total space. J is called a *complex structure* on ξ.

5. Pullback and Normal Bundles

Let $\xi = \pi : M \to B$ denote a fiber bundle with fiber F and group G. Given a manifold $\bar{B}$ and a map $f : \bar{B} \to B$, one can construct in a natural way a bundle over $\bar{B}$ with the same fiber and group: Consider the subset

$$f^*M = \{(b, m) \in \bar{B} \times M \mid \pi(m) = f(b)\}$$

together with the subspace topology from $\bar{B} \times M$, and denote by $\pi_1 : f^*M \to \bar{B}$, $\pi_2 : f^*M \to M$ the projections.

PROPOSITION 5.1. *$f^*\xi = \pi_1 : f^*M \to \bar{B}$ is a fiber bundle with fiber F and group G, called the* pullback bundle of ξ via f, *and $\pi_2 : f^*M \to M$ is a bundle map covering f. Furthermore, $f^*\xi$ is uniquely characterized by the property that $\pi \circ \pi_2 = f \circ \pi_1$;*

$$\begin{array}{ccc} f^*M & \xrightarrow{\pi_2} & M \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \bar{B} & \xrightarrow[f]{} & B \end{array}$$

i.e., if $\bar{\xi} = \bar{\pi} : \bar{M} \to \bar{B}$ is a fiber bundle with fiber F and group G, and there exists a bundle map $\bar{f} : \bar{\xi} \to \xi$ covering $f : \bar{B} \to B$, then $\bar{\xi} \cong f^*\xi$.

PROOF. A bundle chart (π, ϕ) of ξ over $U \subset B$ induces a chart $(\pi_1, \bar{\phi})$ of $f^*\xi$ over $f^{-1}(U)$, where $\bar{\phi} = \phi \circ \pi_2$. It is easily checked that the transition functions satisfy $f_{\bar{\phi}, \bar{\psi}} = f_{\phi, \psi} \circ f$, so that $f^*\xi$ is a bundle as claimed, and π_2 is a bundle map by definition. For the uniqueness part, let $\bar{\xi}$ be a bundle as in the statement. By the remark following Definition 1.3, the coordinate bundle over $\bar{B}$ with bundle charts of the form $(\bar{\pi}^{-1}(f^{-1}(U)), (\bar{\pi}, \phi \circ \bar{f}))$, where $(\pi^{-1}(U), (\pi, \phi))$ is a bundle chart of π, is equivalent to $\bar{\xi}$. Since it has the same transition functions as $f^*\xi$, $f^*\xi$ is equivalent to $\bar{\xi}$. □

Observe that the structure group of $f^*\xi$ may very well be smaller than G, since its transition functions are those of ξ composed with f.

If $\xi_i = \pi_i : E_i \to B$ are two vector bundles of rank n_i over B, then $\xi_1 \times \xi_2 = \pi_1 \times \pi_2 : E_1 \times E_2 \to B \times B$ is a vector bundle of rank $n_1 + n_2$. Consider the diagonal imbedding $\Delta : B \to B \times B$, $\Delta(b) = (b, b)$.

Definition 5.1. The *Whitney sum* $\xi_1 \oplus \xi_2$ is the rank $(n_1 + n_2)$ vector bundle $\Delta^*(\xi_1 \times \xi_2)$.

The fiber of $\xi_1 \oplus \xi_2$ over $b \in B$ is $E_{1b} \oplus E_{2b}$.

Definition 5.2. Let $\xi_i = \pi_i : E_i \to B$ be two vector bundles over B. A map $h : E_1 \to E_2$ is said to be a *homomorphism* if it maps each fiber E_{1b} linearly into E_{2b}.

Thus, a homomorphism $h : E_1 \to E_2$ is just another word for a section s of the bundle $\mathrm{Hom}(\xi_1, \xi_2)$: We can go from one to the other via $s(b) = h_{|E_{1b}}$. By Corollary 3.1, if h is an isomorphism on each fiber, then h is an equivalence. Conversely, an equivalence is a homomorphism (and a bundle map). More generally:

Proposition 5.2. *Let $\xi_i = \pi_i : E_i \to B_i$ be vector bundles over B_i, $i = 1, 2$. If $h : E_1 \to E_2$ maps each fiber $\pi_1^{-1}(b_1)$ linearly into a fiber $\pi_2^{-1}(b_2)$, then $h = f \circ g$, where g is a homomorphism and f a bundle map.*

Proof. Consider the pullback bundle $\bar{h}^*\xi_2$, where $\bar{h} : B_1 \to B_2$ is the map induced by h. If $pr_2 : \bar{h}^*E_2 \to E_2$ is the bundle map given by projection onto the second factor, then $h = pr_2 \circ g$, where $g : E_1 \to \bar{h}^*E_2$ is the homomorphism $g(u) = (\pi_1(u), h(u))$.

$$\begin{tikzcd}
E_1 \arrow[r, "g"] \arrow[d, "\pi_1"'] & \bar{h}^*E_2 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & E_2 \arrow[d, "\pi_2"] \\
B_1 \arrow[r, "1_{B_1}"'] & B_1 \arrow[r, "\bar{h}"'] & B_2
\end{tikzcd}$$

□

Theorem 5.1. *Let $\xi_i = \pi_i : E_i \to B$ denote vector bundles over B, $h : E_1 \to E_2$ a homomorphism.*

(1) *If h is one-to-one (on each fiber), then* $\operatorname{coker} h = \xi_2/h(\xi_1)$ *is a vector bundle over B.*
(2) *If h is onto, then* $\ker h$ *is a vector bundle over B.*

Proof. (1) Suppose that for each $b \in B$, the restriction $h : E_{1b} \to E_{2b}$ is injective. If ξ_1 has rank n and ξ_2 rank $n + k$, then the vector space $E_{2b}/h(E_{1b})$ has dimension k. We construct a bundle atlas for $\operatorname{coker} h = \cup_{b \in B} E_{2b}/h(E_{1b})$:

Let $b \in B$, (π_1, ϕ) and (π_2, ψ) be bundle charts on $\pi_i^{-1}(U)$, where U is a neighborhood of b. Consider the map $g : U \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^{n+k})$ given by

$$g(p) = \psi \circ h \circ (\phi_{|E_{1p}})^{-1}.$$

$g(p)$ has rank n for all $p \in U$, and we may assume, by reordering coordinates if necessary, that the first n rows of the matrix $M(b)$ of $g(b)$ with respect to the standard bases are linearly independent; i.e., that $pr_1 \circ g(b) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, where $pr_1 : \mathbb{R}^{n+k} \to \mathbb{R}^n$ is the projection. By continuity, this holds for all p in a neighborhood (which we also call U) of b. It follows that for each $p \in U$, the map

$$\begin{aligned}\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k &\to \mathbb{R}^{n+k},\\ (u, v) &\mapsto g(p)u + (0, v)\end{aligned}$$

is an isomorphism, and $f : U \times \mathbb{R}^n \times \mathbb{R}^k \to U \times \mathbb{R}^{n+k}$, where $f(p, u, v) = (p, g(p)u + (0, v))$, is an equivalence of trivial bundles. Thus, $(\pi_2, \Psi) := f^{-1} \circ (\pi_2, \psi)$ is a bundle chart for ξ_2. By construction, $v \in E_{2p}$ belongs to $h(E_{1p})$ iff $(\pi_2, \Psi)(v) \in p \times \mathbb{R}^n \times 0 \subset U \times \mathbb{R}^n \times \mathbb{R}^k$. Therefore, if $\pi : \operatorname{coker} h \to B$ is the natural projection and $pr_2 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ the projection onto the second factor, then the bundle chart (π_2, Ψ) of ξ_2 induces a diffeomorphism

$$\begin{aligned}\pi^{-1}(U) &\to U \times \mathbb{R}^k,\\ w + h(E_{1p}) &\mapsto (p, (pr_2 \circ \Psi)w),\end{aligned}$$

which is linear on each fiber. This yields a bundle atlas on $\operatorname{coker} h \to B$: smoothness of the transition functions follows from smoothness of h and of the transition functions of ξ_i.

(2) Suppose $h : E_{1b} \to E_{2b}$ is onto for each $b \in B$, with $n+k$ and n denoting the ranks of ξ_1 and ξ_2 respectively. Let U, $g : U \to \operatorname{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^n)$ be as in (1). We may assume that $g(p)\mathbf{e}_1, \dots, g(p)\mathbf{e}_n$ are independent for each p in U. Define a bundle equivalence

$$\begin{aligned}f : U \times \mathbb{R}^{n+k} &\to U \times \mathbb{R}^n \times \mathbb{R}^k,\\ (p, a) &\mapsto (p, g(p)a, a_{n+1}, \dots, a_{n+k}),\end{aligned}$$

so that $(\pi_1, \Phi) := f \circ (\pi_1, \phi)$ is a bundle chart for ξ_1 over U. By construction, $h(v) = 0$ for $v \in \pi_1^{-1}(U)$ iff $(\pi_1, \Phi)(v) \in U \times 0 \times \mathbb{R}^k$. $(\pi_1, pr_2 \circ \Phi)$ is therefore a bundle chart for $\ker h$. $\square$

If $h : \xi_1 \to \xi_2$ is a one-to-one homomorphism, then $h(\xi_1)$ is a subbundle of ξ_2 equivalent to ξ_1. An *exact sequence of bundle homomorphisms* is a sequence of homomorphisms

$$0 \longrightarrow \xi_1 \xrightarrow{h} \xi_2 \xrightarrow{f} \xi_3 \longrightarrow 0$$

such that the kernel of each map equals the image of the preceding one; thus, h is one-to-one, f is onto, and $h(\xi_1) = \ker f$.

Proposition 5.3. *If* $0 \longrightarrow \xi_1 \xrightarrow{h} \xi_2 \xrightarrow{f} \xi_3 \longrightarrow 0$ *is an exact sequence of homomorphisms, then there exists an equivalence* $g : \xi_2 \to \xi_1 \oplus \xi_3$ *with* $g \circ h : \xi_1 \to \xi_1 \oplus \xi_3$ *being the inclusion, and* $f \circ g^{-1} : \xi_1 \oplus \xi_3 \to \xi_3$ *the projection.*

PROOF. Consider a Euclidean metric $\langle , \rangle$ on ξ_2 (cf. Theorem 4.1). Since the metric is smooth as a section, the orthogonal projection $\pi : \xi_2 \to h(\xi_1)$ is a bundle homomorphism. Being onto, its kernel $h(\xi_1)^\perp$ is a bundle, and the map

$$L : h(\xi_1) \oplus h(\xi_1)^\perp \to \xi_2,$$
$$(v, w) \mapsto v + w$$

is an equivalence. The restriction $\bar{h} : \xi_1 \to h(\xi_1)$ of h is also an equivalence. Furthermore, the restriction $\bar{f} : h(\xi_1)^\perp \to \xi_3$ of f is a one-to-one homomorphism because $\ker f = h(\xi_1)$, so that by rank considerations, it is an equivalence. Thus, $g := (\bar{h}^{-1} \oplus \bar{f}) \circ L^{-1} : \xi_2 \to \xi_1 \oplus \xi_3$ has the required properties. $\square$

EXAMPLE 5.1. Let $\xi = \pi : E \to B$ be a vector bundle over B, and denote by τ_E, τ_B the tangent bundles of E and B. Since $\pi_* : TE \to TB$ maps the fiber over $u \in E$ linearly onto the fiber over $\pi(u) \in B$, π_* induces an epimorphism $h : \tau_E \to \pi^* \tau_B$ by Proposition 5.2. Its kernel $\ker h = \ker \pi_*$ is therefore the total space of a bundle $\mathcal{V}\xi = \pi_{\mathcal{V}} : \mathcal{V}E \to E$ over E, called the *vertical bundle of* ξ. By Proposition 5.3,

$$\tau_E \cong \mathcal{V}\xi \oplus \pi^* \tau_B.$$

The fiber $\mathcal{V}E_u$ of $\mathcal{V}\xi$ over $u \in E$ can be described as follows: If $b = \pi(u)$, and $\imath : E_b = \pi^{-1}(b) \to E$ denotes inclusion, then $\mathcal{V}E_u = \imath_*(E_b)_u$ as an immediate consequence of Proposition 6.2 in Chapter 1 (here, $(E_b)_u$ is the tangent space of E_b at u).

Let $f : M \to N$ be an immersion. Since $f_* : TM \to TN$ is linear and one-to-one, it induces a monomorphism $h : \tau_M \to f^* \tau_N$.

DEFINITION 5.3. Let $f : M \to N$ be an immersion. The *normal bundle of* f is the bundle $\nu(f) = f^* \tau_N / h(\tau_M)$ over M.

Since $0 \to \tau_M \to f^* \tau_N \to \nu(f) \to 0$ is an exact sequence of homomorphisms, Proposition 5.3 implies that $f^* \tau_N \cong \tau_M \oplus \nu(f)$. In fact, given a Euclidean metric on $f^* \tau_N$ (for instance one induced by a Riemannian metric on N), $\nu(f)$ is equivalent to the orthogonal complement of $h(\tau_M)$.

EXAMPLE 5.2. Consider the inclusion $\imath : S^n \to \mathbb{R}^{n+1}$. By the remark following Proposition 5.1, the pullback of the trivial tangent bundle of $\mathbb{R}^{n+1}$ via $\imath$ is the trivial rank $(n + 1)$ bundle ϵ^{n+1} over S^n. The normal bundle of $\imath$ is also the trivial rank 1 bundle ϵ^1 over S^n: Indeed, the restriction of the position vector field $p \mapsto \mathcal{J}_p p$ to the sphere is a section of the frame bundle of $\tau_{S^n}^\perp$. Thus,

$$\tau_{S^n} \oplus \epsilon^1 \cong \epsilon^{n+1},$$

even though τ_{S^n} is not, in general, trivial.

EXERCISE 58. Show that if ξ is a vector bundle, then $\xi \oplus \xi$ admits a complex structure, see Exercise 57. *Hint:* Let $J(u, v) = (-v, u)$.

EXERCISE 59. If $\xi = \pi : E \to B$ is a vector bundle, show that the vertical bundle of ξ is equivalent to the pullback $\pi^* \xi$. *Hint:* Recall the canonical isomorphism $\mathcal{J}_u$ of the vector space E_b with its tangent space $(E_b)_u$ at u. Show that $f : \pi^* E \to \mathcal{V}E$ is an equivalence, where $f(u, v) = \mathcal{J}_u v$.

EXERCISE 60. If $\xi = \pi : E \to B$ is a vector bundle, then by Example 5.1 and Exercise 59, $\tau_E \cong \pi^*\xi \oplus \pi^*\tau_B$. Prove that if s is the zero section of ξ, then

$$s^*\tau_E \cong \xi \oplus \tau_B.$$

Thus, the normal bundle of the zero section in ξ is ξ itself.

6. Fibrations and the Homotopy Lifting/Covering Properties

Although we have so far only considered bundles over manifolds, the definition used also makes sense for manifolds with boundary (and even for topological spaces—the traditional type of base in bundle theory— if we replace diffeomorphisms by homeomorphisms). Let B be a manifold, $I = [0,1]$, and for $t \in I$, denote by $\imath_t : B \to B \times I$ the imbedding $\imath_t(b) = (b,t)$. Recall that two maps $f, g : \bar{B} \to B$ are said to be *homotopic* if there exists $H : \bar{B} \times I \to B$ with $H \circ \imath_0 = f$ and $H \circ \imath_1 = g$. H is called a *homotopy of* f *into* g.

Homotopies play an essential role in the classification of bundles: In this section, we will see that if ξ is a bundle over B, then for any two homotopic maps $f, g : \bar{B} \to B$, the induced bundles $f^*\xi$ and $g^*\xi$ are equivalent.

We begin by introducing the notion of fibration, which is weaker than that of fiber bundle:

DEFINITION 6.1. A surjective map $\pi : M \to B$ is said to be a *fibration* if it has the *homotopy lifting property*: namely, given $f : \bar{B} \to M$, any homotopy $H : \bar{B} \times I \to B$ of $\pi \circ f$ can be lifted to a homotopy $\tilde{H} : \bar{B} \times I \to M$ of f; i.e., $\pi \circ \tilde{H} = H$, $\tilde{H} \circ \imath_0 = f$.

In order to show that a fiber bundle $\xi = \pi : M \to B$ is a fibration, we first rephrase the problem: Notice that a homotopy $H : \bar{B} \times I \to B$ can be lifted to $\tilde{H} : \bar{B} \times I \to M$ iff the pullback bundle $H^*M \to \bar{B} \times I$ admits a section. Indeed, if $\tilde{H}$ is a lift of H, then $(b,t) \mapsto (b,t,\tilde{H}(b,t))$ is a section. Conversely, if s is a section of $H^*M \to \bar{B} \times I$, then $\pi_2 \circ s$ is a lift of H, where $\pi_2 : H^*M \to M$ is the second factor projection. In other words, the homotopy lifting property may be paraphrased as saying that if ξ is a fiber bundle over $B \times I$, then any section of $\xi_{|B \times 0}$ can be extended to a section of ξ.

We begin with the following:

LEMMA 6.1. *Let ξ be a principal bundle over $B \times I$. Then any $b \in B$ has a neighborhood U such that the restriction $\xi_{|U \times I}$ is trivial.*

PROOF. By compactness of $b \times I$, there exist neighborhoods $V_1, \dots, V_k$ of b, and intervals $I_1, \dots, I_k$ such that $\{V_i \times I_i\}$ is a cover of $b \times I$, and each restriction $\xi_{|V_i \times I_i}$ is trivial. We claim that U may be taken to be $V_1 \cap \cdots \cap V_k$. The proof will be by induction on k.

The case $k = 1$ being trivial, assume the statement holds for $k-1$. Order the intervals I_j by their left endpoints, so that if $I_j = (t_j^0, t_j^1)$, then $t_j^0 < t_{j+1}^0$ (if $t_j^0 = t_{j+1}^0$, then either I_j or I_{j+1} can be discarded). We may also assume that $t_1^1 < t_2^1$ since otherwise I_2 may be discarded. If $t_0 \in (t_2^0, t_1^1)$, then by the induction hypothesis, ξ is trivial over $U_1 \times [0, t_0)$, and over $U_2 \times (t_0, 1]$, where $U_1 = V_1$ and $U_2 = V_2 \cap \cdots \cap V_k$. Let s_1 and s_2 be sections over these two sets. For each $(q,t) \in (U_1 \cap U_2) \times (t_2^0, t_1^1)$, there exists a unique $g(q,t)) \in G$ such that

$s_1(q,t) = s_2(q,t)g(q,t)$, and $g : (U_1 \cap U_2) \times (t_2^0, t_1^1) \to G$ is smooth because the sections are. Extend g to a differentiable map $g : (U_1 \cap U_2) \times (t_2^0, 1] \to G$. We then obtain a section s of ξ restricted to $(U_1 \cap U_2) \times I$ by defining

$$s(q,t) = \begin{cases} s_1(q,t), & \text{for } t \le t_0, \\ s_2(q,t)g(q,t), & \text{for } t \ge t_0. \end{cases}$$

□

THEOREM 6.1. *Let $\xi = \pi : P \to B \times I$ be a principal G-bundle, and consider the maps $p : B \times I \to B \times 1$, $p(b,t) = (b,1)$, and $j : B \times 1 \to B \times I$, $j(b,1) = (b,1)$. Then*

$$\xi \cong (j \circ p)^* \xi = p^* \xi_{|B \times 1}.$$

PROOF. Denote by $\pi_B : P \to B$ and $u : P \to I$ the maps obtained by composing π with the projections of $B \times I$ onto its two factors. We will construct a G-equivariant bundle map $f : P \to \pi^{-1}(B \times 1)$ covering p; the theorem will then follow from Theorem 3.1 and Proposition 5.1.

By Lemma 6.1, there exists a countable cover $\{U_n\}$ of B such that ξ is trivial over each $U_n \times I$. Let s_n denote a section of $\xi_{|U_n \times I}$, and $\{\phi_n\}$ a partition of unity subordinate to $\{U_n\}$. Since any element in $\pi^{-1}(b,t)$ with $b \in U_n$ can be written as $s_n(b,t)g$ for a unique $g \in G$, the assignment

$$\begin{aligned} f_n : \pi^{-1}(U_n \times I) &\to \pi^{-1}(U_n \times I), \\ s_n(b,t)g &\mapsto s_n(b, \min\{t + \phi_n(b), 1\})g \end{aligned}$$

is a G-equivariant bundle map. Furthermore, f_n is the identity on an open set containing $\pi^{-1}(\partial U_n \times I)$, and may therefore be continuously extended to all of P by defining $f_n(q) = q$ for $q \notin \pi^{-1}(U_n \times I)$. Finally, set $f = f_1 \circ f_2 \circ \cdots$. The composition makes sense because all but finitely many f_n are the identity on a neighborhood of any point. f is G-equivariant since each f_n is, and $u \circ f = \min\{u + (\sum \phi_n) \circ \pi_B, 1\}$, so that $u \circ f \equiv 1$. Thus, f maps into $\pi^{-1}(B \times 1)$, and furthermore, f is differentiable, because although $u \circ f_n$ is in general only continuous, $u \circ f$ is differentiable. This completes the proof. □

COROLLARY 6.1 (Homotopy Lifting Property). *A fiber bundle is a fibration.*

PROOF. As noted at the beginning of this section, what needs to be shown is that if ξ is a fiber bundle over $B \times I$ with group G and fiber F, then any section s of $\xi_{|B \times 1}$ can be extended to the whole bundle. With the notation of Theorem 6.1, if $\pi : P \to B \times I$ denotes the principal G-bundle associated to ξ, then there exists a bundle map $f : P \to \pi^{-1}(B \times 1)$ covering p. f then induces a bundle map $f : \xi \to \xi_{|B \times 1}$ between the associated bundles with fiber F. Thus, $f^{-1} \circ s \circ p$ is a section of ξ. Furthermore, the restriction of f to $\pi^{-1}(B \times 1)$ is the identity, so $f^{-1} \circ s \circ p$ is an extension of s. □

Recall that $\imath_t : B \to B \times I$ denotes the imbedding $\imath_t(b) = (b,t)$.

COROLLARY 6.2. *Let ξ be a fiber bundle over $B \times I$. Then $\imath_0^* \xi \cong \imath_1^* \xi$.*

PROOF. We may assume that ξ is a principal bundle. Let $p_1 : B \times I \to B$ denote the projection onto the first factor. With notation as in Theorem 6.1, $j \circ p = \imath_1 \circ p_1$, and $\xi \cong (j \circ p)^*\xi = p_1^*\imath_1^*\xi$. Thus,

$$\imath_0^*\xi \cong \imath_0^* p_1^* \imath_1^* \xi = (p_1 \circ \imath_0)^* \imath_1^* \xi = \imath_1^* \xi,$$

since $p_1 \circ \imath_0 = 1_B$. □

COROLLARY 6.3 (The Homotopy Covering Property). *Let ξ denote a fiber bundle over B. If $f, g : \bar{B} \to B$ are homotopic, then $f^*\xi \cong g^*\xi$.*

PROOF. Let $H : \bar{B} \times I \to B$ be a homotopy with $H \circ \imath_0 = f$ and $H \circ \imath_1 = g$. By Corollary 6.2,

$$f^*\xi = \imath_0^*(H^*\xi) \cong \imath_1^*(H^*\xi) = g^*\xi.$$

□

EXERCISE 61. Show that a bundle over a contractible space (one for which the identity map is homotopic to a constant map) is trivial.

EXERCISE 62. Suppose $M \to B$ is a fiber bundle, and let $\tilde{H} : \bar{B} \times I \to M$ be a lift (the existence of which is guaranteed by Corollary 6.1) of some homotopy $H : \bar{B} \times I \to B$. Show that $\tilde{H}$ may be chosen to be *stationary* with respect to H; i.e., if $H(b,t)$ is constant in t for some $b \in \bar{B}$, then so is $\tilde{H}(b,t)$.

EXERCISE 63. Let $\pi : M \to B$ be a fibration, $b \in B$, $p \in \pi^{-1}(B)$.

(a) Show that any curve $c : I \to B$ with $c(0) = b$ may be lifted to a curve $\bar{c}$ in M with $\bar{c}(0) = p$.

(b) Let c_i, $i = 1, 2$, denote two curves in B from b to $\tilde{b}$, and H a homotopy of c_1 into c_2 with $H(0,s) = b$, $H(1,s) = \tilde{b}$ for all $s \in I$. Prove that, if $\bar{c}_i$ is a lift of c_i to M with $\bar{c}_i(0) = p$, then $\bar{c}_1$ is homotopic to $\bar{c}_2$, and the two curves have the same endpoint.

(c) Prove that the lift of c in (a) is unique.

7. Grassmannians and Universal Bundles

The collection $G_{n,k}$ of all n-dimensional subspaces (or *n-planes*) of $\mathbb{R}^{n+k}$ is called the *Grassmannian manifold* of n-planes of $\mathbb{R}^{n+k}$. Consider the map $\pi : O(n+k) \to G_{n,k}$ given by $\pi(L) = L(\mathbb{R}^n)$, where $\mathbb{R}^n$ denotes the subspace $\mathbb{R}^n \times 0 \subset \mathbb{R}^{n+k}$. π is onto, and $\pi(L) = \pi(T)$ iff $L^{-1} \circ T(\mathbb{R}^n) = \mathbb{R}^n$; i.e., iff $L^{-1} \circ T \in O(n) \times O(k) \subset O(n+k)$. π therefore induces a bijective correspondence

$$O(n+k)/O(n) \times O(k) \longleftrightarrow G_{n,k},$$

and we endow $G_{n,k}$ with the differentiable structure for which this correspondence becomes a diffeomorphism. $G_{n,k}$ is then a compact homogeneous space of dimension $\binom{n+k}{2} - \binom{n}{2} - \binom{k}{2} = nk$. $G_{1,k}$, for example, is just $\mathbb{R}P^k$.

One can explicitly describe a differentiable atlas for $G_{n,k}$: Given an n-plane P in $G_{n,k}$, decompose $\mathbb{R}^{n+k} = P \oplus P^\perp$, and denote by π_1, π_2 the projections of $\mathbb{R}^{n+k}$ onto P and $P^\perp$ respectively. Let U be the open neighborhood of P consisting of all n-planes V such that $\pi_{1|V} : V \to P$ is an isomorphism, and define $x : U \to \mathrm{Hom}(P, P^\perp)$ by $x(V) = \pi_2 \circ \pi_{1|V}^{-1}$. x is a homeomorphism with inverse $x^{-1}(L) = \{u + Lu \mid u \in P\}$. Since $\mathrm{Hom}(P, P^\perp) \cong P^* \otimes P^\perp$ is a vector space of dimension nk, x may be considered as a coordinate map $x : U \to \mathbb{R}^{nk}$. It is straightforward to check that the transition maps for the collection of all such charts are differentiable.

There is a canonical rank n vector bundle $\gamma_{n,k}$ over $G_{n,k}$: its total space is the subset $E(\gamma_{n,k})$ of $G_{n,k} \times \mathbb{R}^{n+k}$ consisting of all pairs (P, u) such that $u \in P$, and $\pi : E(\gamma_{n,k}) \to G_{n,k}$ is given by $\pi(P, u) = P$. Thus, the fiber over $P \in G_{n,k}$ is P itself. The differentiable atlas of $G_{n,k}$ described above induces a bundle atlas on $\gamma_{n,k}$: given $P \in G_{n,k}$, the orthogonal projection $p : \mathbb{R}^{n+k} \to P$, and $U = \{V \in G_{n,k} \mid p_{|V} \text{ is an isomorphism}\}$, let $\phi : \pi^{-1}(U) \to P \cong \mathbb{R}^n$ be given by $\phi(v) = p(v)$. Then $(\pi, \phi) : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a diffeomorphism which maps each fiber isomorphically onto $\mathbb{R}^n$.

$\gamma_{n,k}$ is called the *universal rank n bundle* over $G_{n,k}$, the reason being that *any* rank n vector bundle over a manifold B is equivalent to $f^*\gamma_{n,k}$ for sufficiently large k and some map $f : B \to G_{n,k}$. Recall that the pullback of a bundle is less twisted than the original, since the transition functions of the former equal those of the latter composed with the pullback map. Roughly speaking, the universal bundle is so twisted that any other bundle is a diluted version of it. Some more work is needed before we are in a position to prove this, but it can already be established in the case of a tangent bundle:

Example 7.1. A classical theorem in topology states that any n-manifold M can be immersed in Euclidean space $\mathbb{R}^{n+k}$, provided k is large enough. If f is such an immersion, then f_*M_p is an n-dimensional subspace of $\mathbb{R}^{n+k}_{f(p)}$ for each p in M, and $\mathcal{J}^{-1}_{f(p)} f_* M_p$ is an element of $G_{n,k}$. The map

$$\begin{aligned} \bar{h} : TM &\to E(\gamma_{n,k}), \\ v &\mapsto (\mathcal{J}^{-1}_{f(p)} f_* M_p, \mathcal{J}^{-1}_{f(p)} f_* v), \end{aligned}$$

for $v \in M_p$, is a bundle map covering $h : M \to G_{n,k}$, where $h(p) = \mathcal{J}^{-1}_{f(p)} f_* M_p$. Thus, the tangent bundle of M is equivalent to $h^*\gamma_{n,k}$ by Proposition 5.1.

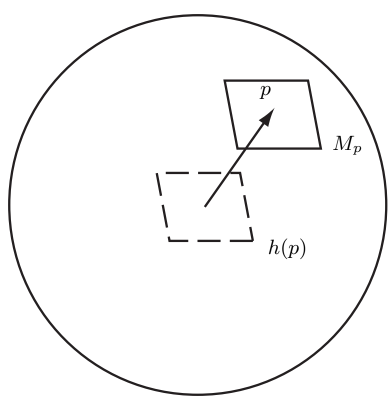

Figure 3. A classifying map $h : S^2 \to G_{2,1}$ for τS^2.

In order to deal with the general case, we will need the following:

LEMMA 7.1. *Let ξ denote a vector bundle over an n-dimensional manifold B. Then B can be covered by $n+1$ sets $U_0, \dots, U_n$, where each restriction $\xi_{|U_i}$ is trivial.*

PROOF. Choose an open cover of B such that ξ is trivial over each element. It is a well-known theorem in topology that this (and in fact any) cover of an n-dimensional manifold B admits a refinement $\{V_\alpha\}_{\alpha \in A}$ with the property that any point in B belongs to at most $n+1$ V_α's. Let $\{\phi_\alpha\}$ be a partition of unity subordinate to this cover, and denote by A_i the collection of all subsets of A with $i+1$ elements. Given $a = \{\alpha_0, \dots, \alpha_i\} \in A_i$, denote by W_a the set consisting of those $b \in B$ such that $\phi_\alpha(b) < \phi_{\alpha_0}(b), \dots, \phi_{\alpha_i}(b)$ for all $\alpha \neq \alpha_0, \dots, \alpha_i$. Then

(1) each W_a is open,
(2) $W_a \cap W_{a'} = \emptyset$ if $a \neq a'$, and
(3) $\xi_{|W_a}$ is trivial.

Statements (1) and (2) follow immediately from the definition of these sets; (3) holds because $W_a \subset \cap_{j=0}^{i} \operatorname{supp} \phi_{\alpha_j} \subset \cap_{j=0}^{i} V_{\alpha_j}$, and ξ is trivial over each V_α. Define $U_i = \cup_{a \in A_i} W_a$. By (1), U_i is open, and by (2) and (3), ξ is trivial over U_i.

It remains to show that $U_0, \dots, U_n$ cover B. For any fixed $b \in B$, consider the set $a = \{\alpha \in A \mid \phi_\alpha(b) > 0\}$. a is nonempty because $\phi_\alpha(b) > 0$ for some α, and $a \in A_j$ for some $j \leq n$ because at most $n+1$ of the sets V_α contain b, so that at most $n+1$ of the functions ϕ_α are positive at b. Then $b \in W_a \subset U_j$. $\square$

THEOREM 7.1. *Let ξ be a rank n vector bundle over B. For large enough l, there is a map $f : B \to G_{n,l}$ such that $\xi \cong f^*\gamma_{n,l}$.*

$G_{n,l}$ is then called a *classifying space* and f a *classifying map* for ξ.

PROOF OF THEOREM 7.1. By Lemma 7.1, there is an open cover $U_1, \dots, U_k$ of B with the restriction of ξ over each U_i being trivial, so that there exist bundle charts $(\pi, \phi_i) : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$. Let $\psi_1, \dots, \psi_k$ be a partition of unity subordinate to $U_1, \dots, U_k$, and define $\Phi_i : E(\xi) \to \mathbb{R}^n$ for each $i = 1, \dots, k$ by

$$\Phi_i(u) = \begin{cases} (\psi_i \circ \pi)(u)\phi_i(u), & u \in \pi^{-1}(U_i), \\ 0, & u \notin \pi^{-1}(U_i). \end{cases}$$

Φ_i is linear on each fiber of ξ, but not one-to-one in general. However, $\Phi = (\Phi_1, \dots, \Phi_k) : E(\xi) \to \mathbb{R}^{nk}$ is one-to-one: suppose $\Phi(u) = 0$; if $b = \pi(u)$, then $\psi_j(b) > 0$ for some j, and $b \in U_j$. Since $\Phi(u) = 0$, $\Phi_j(u)$ must also vanish. But Φ_j is an isomorphism on E_b, so that $u = 0$, and Φ is one-to-one. Then

$$\begin{aligned} \bar{f} : E(\xi) &\to E(\gamma_{n,n(k-1)}), \\ u &\mapsto (\Phi(\pi^{-1}(\pi(u))), \Phi(u)) \end{aligned}$$

is a bundle map covering

$$\begin{aligned} f : B &\to G_{n,n(k-1)}, \\ b &\mapsto \Phi(\pi^{-1}(b)), \end{aligned}$$

and $\xi \cong f^*\gamma_{n,n(k-1)}$ as claimed. $\square$

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