2

Discussion of Maxwell's Equations

2.1 Introduction

The history of electromagnetism, and in particular that part which lead in the 19^{th} century to the formulation of the equations governing the electromagnetic fields, is studded with the names of the leading scientists of the time. Starting with Gauss, there was a more or less ongoing effort to understand the relationships and to model their interactions, and included efforts by B. Riemann, W. Thompson, M. Faraday, as well as Neumann, Kirchhoff and Weber and Helmholtz. Building on all this work, Maxwell's great insight was the introduction of the notion of the so-called displacement current, $\boldsymbol{D}$, a generalization of Faraday's idea of charge polarization or displacement. Using Faraday's work and the ideas of elastic continua, Maxwell developed his famous equations, and noting the close agreement between the electric ratio c and the velocity of light, asserted the coincidence of the two phenomena.[1]

His, and his contemporaries' assumption was that it was necessary to postulate the existence of some medium (the ether) through which the electromagnetic waves would be propagated, and idea finally put to rest by the famous experiment of Michaelson and Merely.

In this chapter, we will present Maxwell's equations and some of the related theory of electromagnetic potentials.

2.2 Geometry of the Radiating Structure

We will consider a prescribed radiating structure S as some subset of the usual three-dimensional Euclidean space $\mathbb{R}^3$ which represents a physical body

[1] It is interesting to realize that much of the development of the theory by Thompson, Maxwell, and others, which culminated in the model described by what are now called Maxwell's Equations, was made with the fluid dynamical model firmly in mind. We refer the interested reader to the interesting historical essay on the work leading to this system of equations in [143].

capable of supporting a flow of electric current. The following shapes are of particular interest:

(a) S consists of finitely many points with position vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m \in \mathbb{R}^3$. Such a configuration will be called an *antenna array*. The points could lie on a line (linear array) or in a plane (plane array) or be more generally distributed within a three dimensional volume.
(b) S is a curve of finite length in $\mathbb{R}^3$ (a wire) or a collection of such curves. This case could also be considered as the limiting case of (a) where the number m of points tends to infinity and the distances between them tend to zero.
(c) S is a connected part of the boundary ∂D of some open and bounded subset D of $\mathbb{R}^3$. This class includes, as particular cases, both *reflector-* and *slot antennas* and, more generally, so called *conformal antennas*.
(d) S is an infinite cylinder with axis in some direction (e.g. in the x_3-direction) with constant cross section S' which could be considered as a subset of the two dimensional Euclidean space $\mathbb{R}^2$. S' could be a disc, an annulus, a curve, or even a more complicated domain.

The first chapter was devoted to an elementary discussion of examples which fall under the first category. In this chapter, we will give a discussion of the equations governing electromagnetic radiation from structures of a general type which will include all the cases enumerated above.

2.3 Maxwell's Equations in Integral Form

Electromagnetic wave propagation is described by particular equations relating five vector fields $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{J}}$ and the scalar field ρ, where $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{D}}$ denote the **electric field** (in V/m) and **electric induction** (in As/m^2) respectively, while $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{B}}$ denote the **magnetic field** (in A/m) and **magnetic induction** (in $Vs/m^2 = T$ =Tesla). Likewise, $\boldsymbol{\mathcal{J}}$ and ρ denote the **current** (in A/m^2) and **charge distribution** (in As/m^3) of the medium. Here and throughout the book we use the **rationalized MKS-system**, i.e. V, A, m and s (see [130], section 1.8). All fields will be assumed to depend both on the space variable $\boldsymbol{x} \in \mathbb{R}^3$ and on the time variable $t \in \mathbb{R}$.

The actual equations that govern the behavior of the electromagnetic field, first completely formulated by Maxwell, may be expressed easily in integral form. Such a formulation, which has the advantage of being closely connected to the physical situation, has been used to effectively by a number of authors, in particular by Sommerfeld [125] and by Müller [105]. The more familiar differential form of Maxwell's equations can be derived very easily from the integral relations as we will see below in Section 2.5.

In order to write these integral relations, we begin by letting S be a connected smooth surface with boundary ∂S in the interior of a region D where electromagnetic waves propagate. In particular, we require that the unit normal vector $\boldsymbol{n}(\boldsymbol{x})$ for $\boldsymbol{x} \in S$ be continuous and directed always into "one side" of S, which we call the positive side of S. By $\boldsymbol{t}(\boldsymbol{x})$ we denote the unit vector tangent to the boundary of S at $\boldsymbol{x} \in \partial S$. This vector, lying in the tangent plane of S together with a vector $\boldsymbol{\nu}(\boldsymbol{x})$, $\boldsymbol{x} \in \partial S$, normal to ∂S is oriented so as to form a mathematically positive system (i.e. $\boldsymbol{t}$ is directed counterclockwise when we sit on the positive side of S). Furthermore, let $\Omega \in \mathbb{R}^3$ be an open set with boundary $\partial\Omega$ and outer unit normal vector $\boldsymbol{n}(\boldsymbol{x})$ at $\boldsymbol{x} \in \partial\Omega$. Then Maxwell's equations in integral form state:

$$\int\limits_{\partial S} \boldsymbol{\mathcal{H}} \cdot \boldsymbol{t} \, d\ell = \frac{\partial}{\partial t} \int\limits_{S} \boldsymbol{\mathcal{D}} \cdot \boldsymbol{n} \, dS \; + \; \int\limits_{S} \boldsymbol{\mathcal{J}} \cdot \boldsymbol{n} \, dS \quad \text{(Ampère's Law)} \tag{2.1a}$$

$$\int\limits_{\partial S} \boldsymbol{\mathcal{E}} \cdot \boldsymbol{t} \, d\ell = -\frac{\partial}{\partial t} \int\limits_{S} \boldsymbol{\mathcal{B}} \cdot \boldsymbol{n} \, dS \quad \text{(Law of Induction)} \tag{2.1b}$$

$$\int\limits_{\partial\Omega} \boldsymbol{\mathcal{D}} \cdot \boldsymbol{n} \, dS = \int\limits_{\Omega} \rho \, dV \qquad \text{(Gauss' Electric Law)} \tag{2.1c}$$

$$\int\limits_{\partial\Omega} \boldsymbol{\mathcal{B}} \cdot \boldsymbol{n} \, dS = 0 \quad \text{(Gauss' Magnetic Law)} \tag{2.1d}$$

The initial goal of using such equations to model the electromagnetic field is to enable us to determine uniquely the five field quantities which result from a given distribution of currents and charges. From this point of view, the four equations are incomplete and must be supplemented by equations which describe the interaction between the fields and the medium through which the fields propagate. These **constitutive relations**, characteristic of the medium, may be either linear or nonlinear. In this book we will deal exclusively with linear constitutive relations which we describe in the next section.

2.4 The Constitutive Relations

In light of the preceding comments, we will consider electromagnetic wave propagation in **linear, isotropic** media. This means, first, that there exist linear relationships (the **constitutive relations**) between $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{H}}$, and $\boldsymbol{\mathcal{B}}$:

$$\boldsymbol{\mathcal{D}} \; = \; \epsilon \, \boldsymbol{\mathcal{E}} \,, \quad \text{and} \quad \boldsymbol{\mathcal{B}} \; = \; \mu \, \boldsymbol{\mathcal{H}} \,. \tag{2.2}$$

In general, the quantities ϵ and μ may be space dependent, but we assume that they are independent of time and of direction and are therefore scalar (as opposed to tensor) quantities. Hence the term *isotropic*.

The **permittivity** or **dielectric constant**, ϵ, has a unit As/Vm, and is related to the ability of the medium to sustain an electric charge. Its value, ϵ_0, in a vacuum has been experimentally determined and is approximately

$8.854 \cdot 10^{-12}\, As/Vm$ while that, say, in fused quartz it is approximately $3.545 \cdot 10^{-11}\, As/Vm$.

The **magnetic permeability** for most substances, μ, is close to its value in vacuo $\mu_0 = 4\pi \cdot 10^{-7}\, Vs/Am$. Those substances for which μ is significantly different from this value are called magnetic, either **paramagnetic** or **diamagnetic** if $\mu > \mu_0$ or $\mu < \mu_0$, respectively. In the following, however, we always will assume that $\mu = \mu_0$.

Usually ϵ and μ are independent of the field strength although in some important situations this is not the case. As we will mention below, one concomitant effect of attempting to synthesize a highly focused beam, is the storage of power close to the antenna itself, which may degrade performance because of dramatic alterations in these constitutive parameters of the atmosphere.

The quantity $c_0 := 1/\sqrt{\epsilon_0\mu_0}$ has the dimension of velocity. It is a consequence of the field equations that this quantity is the velocity of propagation of the electromagnetic field disturbance through free space. Experimental measurements have shown that, in vacuo, this velocity is the same as that of light and hence $c_0 \approx 2.9979 \cdot 10^8\, m/s$.

Two special cases will be considered in the following: media in which the constitutive parameters vary smoothly, and media in which there are manifolds of discontinuity (interfaces) of these parameters. In a medium where ϵ and μ vary smoothly, Maxwell's equations are equivalent to a system of partial differential equations. In the second case where an interface exists, the behaviour of the constitutive parameters together with the correct choice of S and Ω lead to boundary conditions for these equations.

2.5 Maxwell's Equations in Differential Form

First, we consider a region D where μ and ϵ are constant (*homogeneous medium*) or at least continuous. In regions where the vector fields are smooth functions we can apply the Stokes and Gauss theorems for surfaces S and solids Ω lying completely in D:

$$\int\limits_S \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \int\limits_{\partial S} \boldsymbol{F} \cdot \boldsymbol{t}\, d\ell \quad \text{(Stokes)}, \tag{2.3}$$

$$\int\limits_\Omega \operatorname{div} \boldsymbol{F}\, dV = \int\limits_{\partial \Omega} \boldsymbol{F} \cdot \boldsymbol{n}\, dS \quad \text{(Gauss)}, \tag{2.4}$$

where $\boldsymbol{F}$ denotes one of the fields $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{B}}$ or $\boldsymbol{\mathcal{D}}$. With these formulas we can eliminate the boundary integrals in (2.1a-2.1d). We then use the fact that we can vary the surface S and the solid Ω in D arbitrarily. By equating the integrands we are led to Maxwell's equations in **differential form** so that Ampère's Law, the Law of Induction and Gauss' Electric and Magnetic Laws, respectively, become:

$$\operatorname{curl} \boldsymbol{\mathcal{H}} = \frac{\partial}{\partial t}\boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{J}} \tag{2.5a}$$

$$\operatorname{curl} \boldsymbol{\mathcal{E}} = -\frac{\partial}{\partial t}\boldsymbol{\mathcal{B}} \tag{2.5b}$$

$$\operatorname{div} \boldsymbol{\mathcal{D}} = \rho \tag{2.5c}$$

$$\operatorname{div} \boldsymbol{\mathcal{B}} = \boldsymbol{o} \tag{2.5d}$$

Taking the divergence of (2.5a), using (2.5c), and noting that div curl $= 0$ we derive an equation relating the current and charge densities:

$$\operatorname{div} \boldsymbol{\mathcal{J}} + \frac{\partial}{\partial t}\rho = 0. \tag{2.6}$$

We may consider (2.6), as analogous to the continuity or conservation equation in fluid dynamics. It expresses the fact that charge is conserved in the neighborhood of any point.

The current density $\boldsymbol{\mathcal{J}}$ commonly consists of two terms: one, $\boldsymbol{\mathcal{J}}_e$, associated with external sources of electromagnetic disturbances and the other, $\boldsymbol{\mathcal{J}}_c$, associated with conduction currents produced as a result of the electric field. In many cases we will be considering source free regions for which $\boldsymbol{\mathcal{J}}_e = \boldsymbol{o}$.

To the linear constitutive relations, we add a third, namely **Ohm's Law**, which relates the quantities $\boldsymbol{\mathcal{J}}_c$ and $\boldsymbol{\mathcal{E}}$ by a linear relation,

$$\boldsymbol{\mathcal{J}}_c = \sigma \boldsymbol{\mathcal{E}}. \tag{2.7}$$

The scalar function σ which is called the **conductivity** has units of $\frac{A}{Vm}$. Substances for which σ is not negligibly small are called **conductors**. Metals, for example, are good conductors as is brine. In general, the conductivity in metals decreases with increasing temperature, but in the case of other materials, the **semiconductors**, conductivity increases with temperature over a wide range.

By way of contrast, substances for which σ is negligibly small are called **dielectrics** or **insulators**. For such substances, their electromagnetic properties are completely determined by the other constitutive parameters ϵ and μ. For the purposes of analysis, it is often convenient to approximate good conductors by **perfect conductors**, characterized by $\sigma = \infty$, and good dielectrics by **perfect dielectrics** characterized by $\sigma = 0$. Examples of conductors are given in the following table:

The constitutive relations (2.2) and Ohm's law (2.7) allow us to eliminate $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{J}}_c$ from Maxwell's equations. Thus in a linear, isotropic, conducting medium we see that the propagation of the electromagnetic field is described by

Material	**Conductivity** in siemens/meter at 20^oC
Copper, annealed	$5.8005 \cdot 10^7$
Gold	$4.10 \cdot 10^7$
Steel	$0.5 - 1.0 \cdot 10^7$
Nickel	$1.28 \cdot 10^7$
Silver	$6.139 \cdot 10^7$
Tin	$0.869 \cdot 10^7$
Glass, ordinary	10^{-12}
Mica	$10^{-11} - 10^{-15}$
Porcelain	$3 \cdot 10^{-13}$
Quartz, fused	$< 2 \cdot 10^{-17}$
Methyl Alcohol	$7.1 \cdot 10^{-4}$
Water, distilled (18^oC)	$2 \cdot 10^{-4}$
Sea Water	$3 - 5$

Table 2.1. Table of Conductivities

$$\operatorname{curl} \boldsymbol{\mathcal{H}} = \epsilon \frac{\partial}{\partial t} \boldsymbol{\mathcal{E}} + \sigma \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{J}}_e \,, \tag{2.8a}$$

$$\operatorname{curl} \boldsymbol{\mathcal{E}} = -\mu \frac{\partial}{\partial t} \boldsymbol{\mathcal{H}} \,, \tag{2.8b}$$

$$\operatorname{div} (\epsilon \boldsymbol{\mathcal{E}}) = \rho \,, \tag{2.8c}$$

$$\operatorname{div} (\mu \boldsymbol{\mathcal{H}}) = 0 \,. \tag{2.8d}$$

Another remarkable equation which holds in isotropic *homogeneous* conductors in source free regions follows directly from these equations. Indeed observing that in this situation $\boldsymbol{\mathcal{J}}_e = \boldsymbol{o}$, taking the divergence of the equation (2.8a), and differentiating (2.8c) with respect to time, we find the two equations

$$0 = \epsilon \operatorname{div} \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} + \sigma \operatorname{div} \boldsymbol{\mathcal{E}}$$

and

$$\epsilon \,\mathrm{div}\, \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \;=\; \frac{\partial \rho}{\partial t}\,.$$

In this manner we arrive at the differential equation

$$\frac{\sigma}{\epsilon}\,\rho \;+\; \frac{\partial}{\partial t}\rho \;=\; 0\,,$$

which (since σ and ϵ are constant) has the unique solution for $t > 0$:

$$\rho(x,t) \;=\; \rho(x,0)\,\mathrm{e}^{-\sigma\, t/\epsilon}\,.$$

We can interpret σ/ϵ as an angular frequency characteristic of the medium, and call the reciprocal ϵ/σ the **relaxation time**. For copper this is approximately $1.5{\cdot}10^{-19}s$. Thus, for an electric disturbance incident from the exterior of a conductor, the electric charge density falls off exponentially with time. From this analysis it is clear that a metallic conductor does not support a charge and it is a reasonable approximation to replace (2.8c) with $\mathrm{div}(\epsilon\boldsymbol{\mathcal{E}}) = 0$.

2.6 Energy Flow and the Poynting Vector

The description of the performance of antennas, which is the central theme of this book, often involves numerical measures which depend for their definition on the notion of power contained in the field. The power in the electromagnetic field is most often described using the **Poynting vector** $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}$, and we are interested next in understanding how it arises. From the vector identity

$$\mathrm{div}\,(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}) \;=\; \boldsymbol{\mathcal{H}} \cdot \mathrm{curl}\,\boldsymbol{\mathcal{E}} \;-\; \boldsymbol{\mathcal{E}} \cdot \mathrm{curl}\,\boldsymbol{\mathcal{H}}$$

and Maxwell's equations (2.5a), (2.5b) we get immediately

$$\mathrm{div}\,(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}) \;=\; -\boldsymbol{\mathcal{H}} \cdot \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \;-\; \boldsymbol{\mathcal{E}} \cdot \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} \;-\; \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}}\,, \tag{2.9}$$

which is valid in any medium. In light of the constitutive relations, the terms involving time derivatives in (2.9) lead to

$$\begin{aligned}
\boldsymbol{\mathcal{H}} \cdot \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \;+\; \boldsymbol{\mathcal{E}} \cdot \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} &= \mu\, \boldsymbol{\mathcal{H}} \cdot \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} \;+\; \epsilon\, \boldsymbol{\mathcal{E}} \cdot \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \\
&= \mu\, \frac{1}{2}\,\frac{\partial}{\partial t}(\boldsymbol{\mathcal{H}} \cdot \boldsymbol{\mathcal{H}}) \;+\; \epsilon\, \frac{1}{2}\,\frac{\partial}{\partial t}(\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}}) \\
&= \frac{1}{2}\,\frac{\partial}{\partial t}\,(\boldsymbol{\mathcal{B}} \cdot \boldsymbol{\mathcal{H}} \;+\; \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\mathcal{E}})\,.
\end{aligned}$$

It can be shown (see e.g., [86]) that this equation expresses conservation of energy. In particular the right hand side of this equation represents the rates of increase of electric and magnetic internal energies $U_e := \frac{1}{2}\boldsymbol{\mathcal{D}} \cdot \boldsymbol{\mathcal{E}}$ and $U_m :=$

$\frac{1}{2}\boldsymbol{\mathcal{H}} \cdot \boldsymbol{\mathcal{B}}$, respectively, per unit volume. Using the Poynting vector $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}$ we can rewrite (2.9) as

$$\frac{\partial}{\partial t}(U_e + U_m) \; + \; \operatorname{div} \boldsymbol{\mathcal{S}} \; = \; -\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}} \,. \tag{2.10}$$

The term $\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}}$ is the rate per unit volume at which the electric field is doing work. In the absence of external currents this will represent heat dissipation. We note that this equation takes the form of a conservation law.

Using Gauss' Theorem (2.4) in (2.10), we find that, for any volume Ω with smooth surface $\partial\Omega$,

$$\frac{\partial}{\partial t} \int\limits_{\Omega} (U_e + U_m)\, dV \; + \; \int\limits_{\Omega} \boldsymbol{\mathcal{J}} \cdot \boldsymbol{\mathcal{E}}\, dV \; + \; \int\limits_{\partial\Omega} \boldsymbol{\mathcal{S}} \cdot \boldsymbol{n}\, dS \; = \; 0 \,. \tag{2.11}$$

This equation is sometimes called **Poynting's Theorem** or the energy balance equation. Setting $W := \int_\Omega (U_e + U_m) dV$ and $Q = \int_\Omega \boldsymbol{\mathcal{J}} \cdot \boldsymbol{\mathcal{E}}\, dV$, then W and Q represent, respectively, the total energy and the resistive dissipation of energy, called **Joule heat** in the conductor. There is a further decrease of energy if the field extends to the bounding surface of the volume, so the surface integral

$$\mathcal{P} \; = \int\limits_{\partial\Omega} \boldsymbol{\mathcal{S}} \cdot \boldsymbol{n}\, dS \; = \int\limits_{\partial\Omega} (\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}) \cdot \boldsymbol{n}\, dS \tag{2.12}$$

in (2.11) must represent the flow of energy across the boundary. Therefore, we may think of $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}$ as representing the amount of energy crossing the boundary per second, per unit area.

It is important to understand, however, that the vector $\boldsymbol{\mathcal{S}}$ is a *construct*; the actual quantity of importance in the energy balance equation is $\boldsymbol{\mathcal{S}} \cdot \boldsymbol{n}$. In light of Gauss' theorem, we can add the curl of an arbitrary field without changing the value of the integral in (2.11) and so the choice of the vector $\boldsymbol{\mathcal{S}}$ is not unique.

From this analysis, one may conclude that the conductivity of a medium is connected to the appearance of Joule heat. Thermodynamically irreversible, this process transforms the electromagnetic energy into heat and, consequently, the wave is attenuated as it penetrates the conductor. This effect is particularly pronounced in metals with high conductivity. This leads to the so-called **skin effect** which we will make more precise below in our discussion of time-harmonic fields.

2.7 Time Harmonic Fields

From now on we assume that all fields vary periodically in time with the same angular frequency $\omega = 2\pi/T$ and period T. This could be insured by assuming periodic time dependence of the applied external currents or fields.

It is very convenient to use the complex representation of the fields in the form

$$\begin{aligned} \boldsymbol{\mathcal{E}}(\boldsymbol{x},t) &= \operatorname{Re}\left(\boldsymbol{E}(\boldsymbol{x})\,\mathrm{e}^{-i\omega t}\right), & \boldsymbol{\mathcal{D}}(\boldsymbol{x},t) &= \operatorname{Re}\left(\boldsymbol{D}(\boldsymbol{x})\,\mathrm{e}^{-i\omega t}\right), \\ \boldsymbol{\mathcal{H}}(\boldsymbol{x},t) &= \operatorname{Re}\left(\boldsymbol{H}(\boldsymbol{x})\,\mathrm{e}^{-i\omega t}\right), & \boldsymbol{\mathcal{B}}(\boldsymbol{x},t) &= \operatorname{Re}\left(\boldsymbol{B}(\boldsymbol{x})\,\mathrm{e}^{-i\omega t}\right), \\ \boldsymbol{\mathcal{J}}(\boldsymbol{x},t) &= \operatorname{Re}\left(\boldsymbol{J}(\boldsymbol{x})\,\mathrm{e}^{-i\omega t}\right), \end{aligned}$$

as well as for ρ. Here, $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$ and $\boldsymbol{J}$ are now space dependent complex vector fields. By using these formulas the derivative with respect to time transforms into multiplication by $-i\omega$. Thus, Maxwell's equations (2.8a)–(2.8d) in conducting and isotropic media read for the space dependent parts

$$\operatorname{curl}\boldsymbol{H} = (-i\omega\epsilon + \sigma)\,\boldsymbol{E} \;+\; \boldsymbol{J}_e\,, \tag{2.13a}$$

$$\operatorname{curl}\boldsymbol{E} = i\omega\mu\,\boldsymbol{H}\,, \tag{2.13b}$$

$$\operatorname{div}(\epsilon\boldsymbol{E}) = \rho\,, \tag{2.13c}$$

$$\operatorname{div}\boldsymbol{H} = 0\,. \tag{2.13d}$$

We assume, for the following analysis, that $\mu = \mu_0$ is the magnetic permeability of vacuo. We remark that in this case (2.13d) follows directly from (2.13b) while for homogeneous media equation (2.13c) follows from (2.13a) (with $\rho = \operatorname{div}\boldsymbol{J}_e/(i\omega - \sigma/\epsilon)$). In these cases, both of them can be omitted from the system. In source free media in particular, $\operatorname{div}\boldsymbol{E} = 0$ and therefore no distributed charge ρ exists.

By taking the curl again we can eliminate either $\boldsymbol{E}$ or $\boldsymbol{H}$ from the system:

$$-\operatorname{curl}^2\boldsymbol{E} \;+\; i\omega\,\mu_0\,(\sigma - i\omega\,\epsilon)\,\boldsymbol{E} \;=\; -i\omega\mu_0\,\boldsymbol{J}_e\,, \quad \operatorname{div}(\epsilon\boldsymbol{E}) \;=\; \rho\,, \tag{2.14a}$$

$$-\operatorname{curl}\left(\frac{1}{\sigma - i\omega\epsilon}\operatorname{curl}\boldsymbol{H}\right) \;+\; i\omega\mu_0\,\boldsymbol{H} \;=\; -\operatorname{curl}\left(\frac{1}{\sigma - i\omega\epsilon}\,\boldsymbol{J}_e\right). \tag{2.14b}$$

With $\boldsymbol{E}$ or $\boldsymbol{H}$ from (2.14a) or (2.14b), respectively, one has to compute $\boldsymbol{H}$ or $\boldsymbol{E}$ by formulas (2.13b) or (2.13a), respectively.

It is convenient to introduce the complex **wave number** $k \in \mathbb{C}$ by

$$k^2 \;=\; i\omega\,\mu_0\,(\sigma - i\omega\,\epsilon) \;=\; \omega^2\mu_0\,\epsilon \;+\; i\omega\,\mu_0\,\sigma\,. \tag{2.15}$$

Since only k^2 occurs we can choose that branch of the square root with $\operatorname{Re}k \geq 0$ and also $\operatorname{Im}k \geq 0$.

In *homogeneous* media ϵ and σ are constant. In this case we note that $\operatorname{curl}^2 = \operatorname{grad}\operatorname{div} - \Delta$ and arrive at the inhomogeneous (vector-) **Helmholtz equations**

$$\Delta\boldsymbol{E} \;+\; k^2\,\boldsymbol{E} \;=\; \nabla\rho/\epsilon \;-\; i\omega\mu_0\,\boldsymbol{J}_e\,, \quad \Delta\boldsymbol{H} \;+\; k^2\,\boldsymbol{H} \;=\; -\operatorname{curl}\boldsymbol{J}_e\,, \tag{2.16}$$

$\operatorname{div}\boldsymbol{E} = \rho/\epsilon$, and $\operatorname{div}\boldsymbol{H} = 0$.

Writing $\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) = \frac{1}{2}[\boldsymbol{E}(\boldsymbol{x})\exp(-i\omega t) + \overline{\boldsymbol{E}(\boldsymbol{x})}\exp(i\omega t)]$ and analogously for $\boldsymbol{\mathcal{H}}$ we have for the Poynting vector $\boldsymbol{\mathcal{S}}$ from Section 2.7 (after a short calculation)

$$\begin{aligned}\boldsymbol{\mathcal{S}}(\boldsymbol{x},t) &= \boldsymbol{\mathcal{E}}(\boldsymbol{x},t) \times \boldsymbol{\mathcal{H}}(\boldsymbol{x},t) \\ &= \frac{1}{2}\operatorname{Re}\left[\boldsymbol{E}(\boldsymbol{x}) \times \overline{\boldsymbol{H}(\boldsymbol{x})}\right] + \frac{1}{2}\operatorname{Re}\left[\boldsymbol{E}(\boldsymbol{x}) \times \boldsymbol{H}(\boldsymbol{x})\, \mathrm{e}^{-2i\omega t}\right].\end{aligned}$$

The first term $\frac{1}{2}\operatorname{Re}[\boldsymbol{E} \times \overline{\boldsymbol{H}}]$ is real and constant with respect to the time variable t. The second term is also real but varying in time with frequency 2ω, so that its time average is zero. Hence the time average of $\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}$ is equal to the real part of $\boldsymbol{S}$ where

$$\boldsymbol{S} := \frac{1}{2}[\boldsymbol{E} \times \overline{\boldsymbol{H}}] \tag{2.17}$$

denotes the **complex Poynting vector**.

The time average of the power flux from the volume into the region outside is then given by (cf. (2.12))

$$P = \operatorname{Re} \int_{\partial\Omega} \boldsymbol{S} \cdot \boldsymbol{n}\, dS\,. \tag{2.18}$$

2.8 Vector Potentials

The advantage of the vector Helmholtz equations (2.16) over the original Maxwell system (2.13a)–(2.13d) is that every Cartesian component u of the fields satisfies the scalar Helmholtz equation $\Delta u + k^2 u = f$ where f denotes the corresponding component of the right hand side. However, since the condition on the divergence couples the components again, it is not an easy task to construct solutions of (2.13a)- (2.13d) or, equivalently, (2.16) directly. This is the main reason why it is very convenient to introduce **vector potentials $\boldsymbol{A}$**. In this section, we assume that the medium is homogeneous, i.e. ϵ, μ and σ are constant.

It is well known (see, e.g., [94]) that in regions Ω without interior boundaries the condition $\operatorname{div} \boldsymbol{H} = 0$ is equivalent to the existence of a vector field $\boldsymbol{A}$ such that $\boldsymbol{H} = \operatorname{curl} \boldsymbol{A}$. Vector fields $\boldsymbol{A}$ with this property are called *vector potentials* for the field $\boldsymbol{H}$. Note that they are not unique. Indeed, with $\boldsymbol{A}$ also $\boldsymbol{A} + \nabla\varphi$ for any differentiable function φ is also a vector potential for $\boldsymbol{H}$. Substitution of $\boldsymbol{H} = \operatorname{curl} \boldsymbol{A}$ into the second equation of (2.16) yields

$$\operatorname{curl}\left(\Delta \boldsymbol{A} + k^2 \boldsymbol{A}\right) = -\operatorname{curl} \boldsymbol{J}_e$$

which is certainly satisfied if

$$\Delta \boldsymbol{A} + k^2 \boldsymbol{A} = -\boldsymbol{J}_e + \nabla\varphi \tag{2.19}$$

where φ is any differentiable scalar function.

On the other hand, if $\boldsymbol{A}$ satisfies (2.19) then

$$\boldsymbol{H} \;=\; \operatorname{curl} \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} \;=\; i\omega\mu_0\, \boldsymbol{A} \;+\; \frac{1}{\sigma - i\omega\epsilon}\, \nabla(\operatorname{div} \boldsymbol{A} - \varphi) \tag{2.20}$$

satisfies the Maxwell system (2.13a)–(2.13d) with $\rho = \operatorname{div} \boldsymbol{J}_e/(i\omega - \sigma/\epsilon)$. The vector potential $\boldsymbol{A}$ used to express the magnetic field $\boldsymbol{H}$ is also called **magnetic Hertz potential**. The following example is of particular importance:

Example 2.1. (TM-mode)

Let $\boldsymbol{A}$ be a solution of (2.19) of the form $\boldsymbol{A}(\boldsymbol{x}) = u(\boldsymbol{x})\, \boldsymbol{p}(\boldsymbol{x})$ with scalar field u and vector field $\boldsymbol{p}$ such that $\operatorname{curl} \boldsymbol{p} = \boldsymbol{o}$. This situation is called *TM-mode* (transverse-magnetic mode) since $\boldsymbol{H} = \operatorname{curl}(u\, \boldsymbol{p}) = \nabla u \times \boldsymbol{p}$ has no component in $\boldsymbol{p}-$direction.

As a particular example we take $\boldsymbol{p}$ being constant, without loss of generality $\boldsymbol{p} = \hat{\boldsymbol{e}}_3$, the unit vector in x_3-direction, and $\boldsymbol{J}_e = g\, \hat{\boldsymbol{e}}_3$. If u is a solution of the **three dimensional scalar Helmholtz equation**

$$\Delta u \;+\; k^2 u \;=\; -g \tag{2.21}$$

we have

$$\boldsymbol{H} = \operatorname{curl}(u\, \hat{\boldsymbol{e}}_3) \;=\; \left(\frac{\partial u}{\partial x_2}\,,\, -\frac{\partial u}{\partial x_1}\,,\, 0 \right)^{\top}, \tag{2.22a}$$

$$\boldsymbol{E} = i\omega\mu_0\, u\, \hat{\boldsymbol{e}}_3 \;+\; \frac{1}{\sigma - i\omega\epsilon}\, \nabla\big(\partial u/\partial x_3\big)\,. \tag{2.22b}$$

If we choose g and u to be constant with respect to x_3 then $\boldsymbol{E}$ has only a x_3-component. This mode is also called **E-mode**. In this case equation (2.21) reduces to the two dimensional scalar Helmholtz equation for u.

Analogously, we can introduce **electric Hertz potentials**. Indeed, if $\boldsymbol{J}_e = \boldsymbol{o}$ then $\operatorname{div} \boldsymbol{E} = 0$ and we substitute the ansatz $\boldsymbol{E} = \operatorname{curl} \boldsymbol{A}$ into the first equation of (2.16). This yields

$$\operatorname{curl}\,\big(\Delta \boldsymbol{A} \;+\; k^2 \boldsymbol{A}\big) \;=\; \boldsymbol{o}$$

which is certainly satisfied if

$$\Delta \boldsymbol{A} \;+\; k^2 \boldsymbol{A} \;=\; \nabla\varphi \tag{2.23}$$

where again φ is any differentiable scalar function. If, on the other hand, $\boldsymbol{A}$ satisfies equation (2.23) then

$$\boldsymbol{E} \;=\; \operatorname{curl} \boldsymbol{A} \quad \text{and} \quad \boldsymbol{H} \;=\; (\sigma - i\omega\epsilon)\, \boldsymbol{A} \;+\; \frac{1}{i\omega\mu_0}\, \nabla(\operatorname{div} \boldsymbol{A} - \varphi) \tag{2.24}$$

satisfies the Maxwell system (2.13a)–(2.13d). Analogously to above we consider the following example:

Example 2.2. (TE-mode)

Let $\boldsymbol{J}_e = \boldsymbol{o}$ and $\boldsymbol{A}$ be a solution of (2.23) of the form $\boldsymbol{A} = u\,\boldsymbol{p}$ with $\operatorname{curl}\boldsymbol{p} = \boldsymbol{o}$. This situation describes the **TE-mode** since now $\boldsymbol{E}$ has no component in $\boldsymbol{p}-$direction. With the particular choice $\boldsymbol{p} = \hat{\boldsymbol{e}}_3$ and a scalar solution u of the three dimensional Helmholtz equation $\Delta u + k^2 u = 0$ we have

$$\boldsymbol{E} = \operatorname{curl}(\hat{\boldsymbol{e}}_3 u) \;=\; \left(\frac{\partial u}{\partial x_2}\,,\,-\frac{\partial u}{\partial x_1}\,,\,0\right)^{\top}, \tag{2.25a}$$

$$\boldsymbol{H} = (\sigma - i\omega\epsilon)\,u\,\hat{\boldsymbol{e}}_3 \;+\; \frac{1}{i\omega\mu_0}\,\nabla\big(\partial u/\partial x_3\big)\,. \tag{2.25b}$$

The case where u is independent of x_3 is also called **H-mode**.

We observe that in both, the H- and the E-mode the electric and magnetic fields are perpendicular to each other. This is not true, in general, for the TE or TM mode or even for arbitrary solutions of Maxwell's equations (except in the far field, cf. (2.32)).

2.9 Radiation Condition, Far Field Pattern

We will see that solutions of Maxwell's equations decay or increase exponentially for conducting media, i.e. when $\sigma > 0$. For $\sigma = 0$, however, every solution must decay as $1/r$ for $r \to \infty$. To illustrate this let us consider one of the simplest possible magnetic Hertz potentials, namely those which are radially symmetric. Therefore, let us assume that $\mu = \mu_0$, ϵ, and σ are constant, $\boldsymbol{A} = \boldsymbol{A}(r)$, $r > 0$, and $\boldsymbol{J}_e = \boldsymbol{o}$, $\varphi = 0$. The Helmholtz equation (2.19) in spherical coordinates reduces to the ordinary differential equation

$$\frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial \boldsymbol{A}(r)}{\partial r}\right) \;+\; k^2\,\boldsymbol{A}(r) \;=\; \boldsymbol{o}\,, \quad r > 0\,, \tag{2.26}$$

which has the two linearly independent solutions

$$\boldsymbol{A}(r) \;=\; \boldsymbol{p}\,\frac{\mathrm{e}^{ikr}}{r} \quad \text{and} \quad \boldsymbol{A}(r) \;=\; \boldsymbol{p}\,\frac{\mathrm{e}^{-ikr}}{r}$$

as it is readily seen. The corresponding magnetic- and electric fields are given by (2.20), i.e.

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{x}) = \operatorname{curl}\boldsymbol{A}(\boldsymbol{x}) \;&=\; \nabla\frac{\mathrm{e}^{\pm ikr}}{r}\times\boldsymbol{p}\,,\\ \boldsymbol{E}(\boldsymbol{x}) = i\omega\mu_0\,\boldsymbol{p}\,\frac{\mathrm{e}^{\pm ikr}}{r} \;&+\; \frac{1}{\sigma - i\omega\epsilon}\,\nabla\left(\boldsymbol{p}\cdot\nabla\frac{\mathrm{e}^{\pm ikr}}{r}\right).\end{aligned}$$

From the asymptotic behaviour

$$\nabla \frac{e^{\pm ikr}}{r} = \pm ik \frac{e^{\pm ikr}}{r} \left[\hat{\boldsymbol{x}} + \mathcal{O}(1/r)\right] \quad \text{as } r \to \infty\,,$$

$$\frac{\partial^2}{\partial x_j \partial x_\ell} \frac{e^{\pm ikr}}{r} = -k^2 \frac{e^{\pm ikr}}{r} \left[\frac{x_j}{r}\frac{x_\ell}{r} + \mathcal{O}(1/r)\right] \quad \text{as } r \to \infty\,,$$

uniformly in $\hat{\boldsymbol{x}} \in S^2$, we observe that

$$\boldsymbol{H}(\boldsymbol{x}) = \pm ik \frac{e^{\pm ikr}}{r} \left[\hat{\boldsymbol{x}} \times \boldsymbol{p} + \mathcal{O}(1/r)\right], \tag{2.27a}$$

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}) &= i\omega\mu_0 \frac{e^{\pm ikr}}{r} \left[\boldsymbol{p} - \hat{\boldsymbol{x}}(\hat{\boldsymbol{x}} \cdot \boldsymbol{p}) + \mathcal{O}(1/r)\right] \\ &= i\omega\mu_0 \frac{e^{\pm ikr}}{r} \left[\hat{\boldsymbol{x}} \times (\boldsymbol{p} \times \hat{\boldsymbol{x}}) + \mathcal{O}(1/r)\right], \end{aligned} \tag{2.27b}$$

$$\boldsymbol{S}(\boldsymbol{x}) = \frac{1}{2} \boldsymbol{E}(\boldsymbol{x}) \times \overline{\boldsymbol{H}(\boldsymbol{x})} \tag{2.27c}$$

$$= \pm\frac{1}{2}\omega\mu_0 \overline{k} \frac{e^{\mp 2r \operatorname{Im} k}}{r^2} \left[\left(|\boldsymbol{p}|^2 - |\hat{\boldsymbol{x}} \cdot \boldsymbol{p}|^2\right)\hat{\boldsymbol{x}} + \mathcal{O}(1/r)\right]. \tag{2.27d}$$

Here we clearly see the asymptotic behavior as $r \to \infty$: for $\sigma > 0$ we have that $\operatorname{Im} k > 0$, i.e. $\boldsymbol{H}$, $\boldsymbol{E}$, and $\boldsymbol{S}$ are exponentially decreasing or increasing, respectively, depending on the sign in the exponential term. If $\sigma = 0$, however, k is real valued and the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ decay as $1/r$ while $\boldsymbol{S}$ decays as $1/r^2$. This is different from the static case where it is well known that the fields could decay more rapidly (see also Section 5.2).

We now formulate radiation conditions on $\boldsymbol{E}$ and $\boldsymbol{H}$ which are independent of the special example for $\boldsymbol{A}$ and distinguish between the two possible solutions.

If the medium is conducting i.e. if $\operatorname{Im} k > 0$, then, from conservation of energy, the radiated power cannot increase with r, thus we must take the positive sign in the exponential terms of $\boldsymbol{A}$, $\boldsymbol{E}$ and $\boldsymbol{H}$. Formulated in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$ it is sufficient to require that

$$\boldsymbol{E} \quad \text{and} \quad \boldsymbol{H} \quad \text{are bounded.} \tag{2.28}$$

In vacuo, $\epsilon = \epsilon_0$ and $\sigma = 0$, i.e. k is real valued and positive. We observe that, by using the Cauchy-Schwarz inequality, the Poynting vector

$$\boldsymbol{S}(\boldsymbol{x}) = \frac{1}{2}\omega\mu_0 k \frac{1}{r^2} \left[\left(|\boldsymbol{p}|^2 - |\hat{\boldsymbol{x}} \cdot \boldsymbol{p}|^2\right)\hat{\boldsymbol{x}} + \mathcal{O}(1/r)\right]$$

is directed into the direction $\hat{\boldsymbol{x}}$ which represents outgoing rather than incoming fields. Therefore, we also choose the positive sign in the exponential terms of $\boldsymbol{A}$, $\boldsymbol{E}$ and $\boldsymbol{H}$. Then, $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy the **Silver-Müller radiation conditions**:

$$\boldsymbol{E}(\boldsymbol{x}) \times \hat{\boldsymbol{x}} + \frac{1}{Y_0} \boldsymbol{H}(\boldsymbol{x}) = \mathcal{O}\left(1/r^2\right), \tag{2.29a}$$

$$\boldsymbol{H}(\boldsymbol{x}) \times \hat{\boldsymbol{x}} - Y_0 \boldsymbol{E}(\boldsymbol{x}) = \mathcal{O}\left(1/r^2\right) \tag{2.29b}$$

as $r \to \infty$ uniformly with respect to $\hat{\boldsymbol{x}} \in S^2$. Here,

$$Y_0 \;:=\; \sqrt{\frac{\epsilon_0}{\mu_0}} \;=\; \frac{k}{\omega\mu_0} \;=\; \frac{\omega\epsilon_0}{k} \tag{2.30}$$

denotes the *admittance* in non-conductive media. In vacuo it is $Y_0 \approx 2.654 \cdot 10^{-3}\, A/V$.

It turns out that these radiation conditions describe the correct asymptotic behavior of electromagnetic waves generated by sources lying in a *compact set*. It can be shown (see [29]) that these conditions are equivalent (i.e. any solution $(\boldsymbol{E}, \boldsymbol{H})$ of the time harmonic Maxwell's equations which satisfies one of (2.29a), (2.29b) also satisfies the other one) and that they imply the following asymptotic behavior of $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{\mathrm{e}^{ikr}}{r}\left[\boldsymbol{E}_\infty(\hat{\boldsymbol{x}}) \;+\; \mathcal{O}(1/r)\right], \tag{2.31a}$$

$$\boldsymbol{H}(x) = Y_0\,\frac{\mathrm{e}^{ikr}}{r}\left[\boldsymbol{H}_\infty(\hat{\boldsymbol{x}}) \;+\; \mathcal{O}(1/r)\right] \tag{2.31b}$$

as $r \to \infty$ uniformly with respect to $\hat{\boldsymbol{x}} \in S^2$. The vector fields $\boldsymbol{E}_\infty$ and $\boldsymbol{H}_\infty$ are defined on the unit sphere S^2 and are called *far field pattern*. In the particular example above the far field patterns are given by

$$\boldsymbol{H}_\infty(\hat{\boldsymbol{x}}) \;=\; \frac{ik}{Y_0}\,\hat{\boldsymbol{x}} \times \boldsymbol{p} \;=\; i\omega\mu_0\,\hat{\boldsymbol{x}} \times \boldsymbol{p}\,, \qquad \boldsymbol{E}_\infty(\hat{\boldsymbol{x}}) \;=\; i\omega\mu_0\,\hat{\boldsymbol{x}} \times (\boldsymbol{p} \times \hat{\boldsymbol{x}})\,.$$

In general, they enjoy the following properties (cf. [29]):

$$\boldsymbol{H}_\infty(\hat{\boldsymbol{x}}) \;=\; \hat{\boldsymbol{x}} \times \boldsymbol{E}_\infty(\hat{\boldsymbol{x}}), \quad \hat{\boldsymbol{x}} \cdot \boldsymbol{E}_\infty(\hat{\boldsymbol{x}}) \;=\; \hat{\boldsymbol{x}} \cdot \boldsymbol{H}_\infty(\hat{\boldsymbol{x}}) \;=\; 0 \quad \text{for } \hat{\boldsymbol{x}} \in S^2. \tag{2.32}$$

To explain the physical meaning of the far field pattern we consider the energy distribution which is given by the complex Poynting vector $\frac{1}{2}[\boldsymbol{E} \times \overline{\boldsymbol{H}}]$ (cf. (2.17)) in any nonconducting medium, i.e. we assume k and Y_0 are both real valued and positive. The time averaged power radiated through the sphere S_a of radius a can be written as (cf. (2.12)):

$$P_a \;=\; \mathrm{Re}\left[\int\limits_{S_a} \boldsymbol{n} \cdot \boldsymbol{S}\, dS\right] \;=\; \frac{1}{2}\,\mathrm{Re}\left[\int\limits_{S_a} \hat{\boldsymbol{x}} \cdot (\boldsymbol{E} \times \overline{\boldsymbol{H}})\, dS\right], \tag{2.33}$$

and so the power radiated into the far field is given by

$$P_\infty \;:=\; \frac{1}{2}\,\mathrm{Re}\left[\lim_{a\to\infty} \int\limits_{S_a} \hat{\boldsymbol{x}} \cdot (\boldsymbol{E} \times \overline{\boldsymbol{H}})\, dS\right]. \tag{2.34}$$

Using the definitions of the far field patterns (2.31a), (2.31b) and the properties (2.32) we can express P_∞ in terms of $\boldsymbol{E}_\infty$ alone:

$$P_\infty \;=\; \frac{Y_0}{2} \int\limits_{S^2} |\boldsymbol{E}_\infty|^2\, dS\,. \tag{2.35}$$

2.10 Radiating Dipoles and Line Sources

The construction of solutions of the Maxwell equations (2.13a)–(2.13d) by introducing vector potentials is a purely mathematical approach. In this section we will briefly connect this construction with the physical electromagnetic fields radiated by infinitesimal or finite linear current elements.

To derive the fields for an electric dipole we start with a small volume element $\Delta V(\boldsymbol{z})$ centered at $\boldsymbol{z}$ which e.g., can be, but doesn't have to be, a ball with center $\boldsymbol{z}$ and radius ϵ. Assuming the total current to be $I\,\hat{\boldsymbol{a}}$ for some unit vector $\hat{\boldsymbol{a}}$ we define the current density $\boldsymbol{J}_e$ by $\boldsymbol{J}_e(\boldsymbol{x}) = j(\boldsymbol{x})\,\hat{\boldsymbol{a}}$ where $j \in C^1(\mathbb{R}^3)$ is any function with the properties that $j(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin \Delta V(\boldsymbol{z})$ and $\int_{\Delta V(\boldsymbol{z})} j(\boldsymbol{y})\,d\boldsymbol{y} = I$.

We solve Maxwell's equations (2.13a)–(2.13d) for this particular current distribution by introducing a magnetic Hertz potential $\boldsymbol{A} = u\,\hat{\boldsymbol{a}}$. Then u must satisfy the Helmholtz equation

$$\Delta u \;+\; k^2 u \;=\; -j \quad \text{in } \mathbb{R}^3\,. \tag{2.36}$$

The following theorem is well known (see, e.g., [41])

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and*

$$\Phi(\boldsymbol{x},\boldsymbol{y}) \;:=\; \frac{\mathrm{e}^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi\,|\boldsymbol{x}-\boldsymbol{y}|}\,, \quad \boldsymbol{x} \neq \boldsymbol{y}\,, \tag{2.37}$$

denote the fundamental solution *of the Helmholtz equation in $\mathbb{R}^3$. Then, for every $j \in C^1(\overline{\Omega})$, the volume potential*

$$u(\boldsymbol{x}) \;=\; \int\limits_{\Omega} \Phi(\boldsymbol{x},\boldsymbol{y})\,j(\boldsymbol{y})\,d\boldsymbol{y}\,, \quad \boldsymbol{x} \in \mathbb{R}^3\,, \tag{2.38}$$

with density j is two times continuously differentiable in Ω and in $\mathbb{R}^3 \setminus \overline{\Omega}$ and

$$\Delta u \;+\; k^2 u \;=\; -j \;\; in\;\Omega\,, \quad \Delta u \;+\; k^2 u \;=\; 0 \;\; in\;\mathbb{R}^3 \setminus \overline{\Omega}\,.$$

Furthermore, u satisfies the **Sommerfeld radiation condition**

$$\frac{\partial}{\partial r}u(\boldsymbol{x}) \;-\; i\,k\,u(\boldsymbol{x}) \;=\; \mathcal{O}\big(1/r^2\big) \quad as\; r \to \infty\,, \tag{2.39}$$

uniformly with respect to $\hat{\boldsymbol{x}} = \boldsymbol{x}/\,|\boldsymbol{x}| \in S^2$.

Applying this result to the current distribution yields that u, defined by

$$\begin{aligned} u(\boldsymbol{x}) &= \int\limits_{\Delta V(\boldsymbol{z})} \Phi(\boldsymbol{x},\boldsymbol{y})\,j(\boldsymbol{y})\,d\boldsymbol{y} \\ &= I\,\Phi(\boldsymbol{x},\boldsymbol{z}) \;+ \int\limits_{\Delta V(\boldsymbol{z})} j(\boldsymbol{y})\,\big[\Phi(\boldsymbol{x},\boldsymbol{y}) - \Phi(\boldsymbol{x},\boldsymbol{z})\big]\,d\boldsymbol{y}\,, \end{aligned}$$

solves (2.36). For fixed $\boldsymbol{x} \neq \boldsymbol{z}$ we let the region $\Delta V(\boldsymbol{z})$ shrink to the point $\boldsymbol{z}$ while keeping the total current I fixed. Then $u(\boldsymbol{x})$ converges to $u(\boldsymbol{x}) = I\,\Phi(\boldsymbol{x},\boldsymbol{z})$.

Remarks:

- Actually, the function $\Phi_-(\boldsymbol{x},\boldsymbol{y}) := \frac{\exp(-ik|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$ is also a fundamental solution. As we have made clear in the previous section, however, the fields based on potentials with Φ_- do not satisfy the radiation condition and are therefore physically not relevant.
- By this "shrinking process" the current density j has to tend to infinity since the total current I is kept fixed. The limit of these currents j cannot be a function in the ordinary sense. Therefore, by this limiting process, we extend the concept of a function to the wider class of " distributions". We actually take $j(\boldsymbol{x}) = I\,\delta(\boldsymbol{x}-\boldsymbol{z})$ where δ denotes Dirac's delta-distribution introduced formally by the property $\int_{\mathbb{R}^3} \delta(\boldsymbol{y})g(\boldsymbol{y})\,d\boldsymbol{y} = g(0)$ for every $g \in C(\mathbb{R}^3)$. We can write formally

$$\Delta_{\boldsymbol{x}}\Phi(\boldsymbol{x},\boldsymbol{z}) \;+\; k^2\Phi(\boldsymbol{x},\boldsymbol{z}) \;=\; -\delta(\boldsymbol{x}-\boldsymbol{z}).$$

  This formulation can be made mathematically rigorous by using the theory of distributions (see [136]).

The magnetic and electric fields corresponding to the potential $\boldsymbol{A}(\boldsymbol{x}) = I\,\Phi(\boldsymbol{x},\boldsymbol{z})\,\hat{\boldsymbol{a}}$ are given by (2.20), i.e.

$$\boldsymbol{H}(\boldsymbol{x}) = I\,\mathrm{curl}\left(\Phi(\boldsymbol{x},\boldsymbol{z})\,\hat{\boldsymbol{a}}\right) \;=\; I\,\nabla_x\Phi(\boldsymbol{x},\boldsymbol{z}) \times \hat{\boldsymbol{a}} \tag{2.40a}$$

$$\boldsymbol{E}(\boldsymbol{x}) = i\omega\mu_0\,I\,\Phi(\boldsymbol{x},\boldsymbol{z})\,\hat{\boldsymbol{a}} \;+\; \frac{I}{\sigma - i\omega\epsilon}\,\nabla_x\left(\nabla_x\Phi(\boldsymbol{x},\boldsymbol{z})\cdot\hat{\boldsymbol{a}}\right). \tag{2.40b}$$

These are the fields of an **electric dipole** with dipole moment $I\hat{\boldsymbol{a}}$.

We now want to derive the fields of a **magnetic dipole**. Let again $\Delta V(\boldsymbol{z})$ be a volume element with center $\boldsymbol{z}$ and $j_m(\boldsymbol{x})$ as before an approximation of $M\,\delta(\boldsymbol{x})$. The vector field $\boldsymbol{J}_e(\boldsymbol{x}) = \mathrm{curl}\,(j_m(\boldsymbol{x})\,\hat{\boldsymbol{a}}) = \nabla j_m(\boldsymbol{x}) \times \hat{\boldsymbol{a}}$ describes the current distribution of a small *circular wire*. We set $\boldsymbol{J}_m := j_m(\boldsymbol{x})\,\hat{\boldsymbol{a}}$ and call this auxiliary quantity a **magnetic current distribution**. It is our aim to solve Maxwell's equations (2.13a)- (2.13d) with this choice of $\boldsymbol{J}_e = \mathrm{curl}\,\boldsymbol{J}_m$. Instead of going through the details again we make use of a mathematical trick:

We define the purely mathematical vector fields

$$\tilde{\boldsymbol{E}} \;:=\; \boldsymbol{H} \;-\; \boldsymbol{J}_m \quad \text{and} \quad \tilde{\boldsymbol{H}} \;:=\; \boldsymbol{E}\,.$$

Then, since $\mathrm{div}\,\boldsymbol{J}_e = 0$, the fields $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{H}}$ solve the system

$$\mathrm{curl}\,\tilde{\boldsymbol{E}} = (\sigma - i\omega\epsilon)\,\tilde{\boldsymbol{H}}\,, \quad \mathrm{curl}\,\tilde{\boldsymbol{H}} = i\omega\mu_0\,\tilde{\boldsymbol{E}} + i\omega\mu_0\boldsymbol{J}_m\,,$$

$$\operatorname{div}\tilde{\boldsymbol{E}} = -\operatorname{div}\boldsymbol{J}_m\,, \quad \operatorname{div}\tilde{\boldsymbol{H}} = 0\,.$$

Formally, this looks like a Maxwell system with the roles of $\sigma - i\omega\epsilon$ and $i\omega\mu_0$ interchanged. The current density in this case is $i\omega\mu_0\boldsymbol{J}_m$. Therefore, we solve this system as in the case of an electric dipole and arrive at the potential $\boldsymbol{A}(\boldsymbol{x}) = i\omega\mu_0 M\,\Phi(\boldsymbol{x},\boldsymbol{z})\,\hat{\boldsymbol{a}}$ and thus:

$$\boldsymbol{E}(\boldsymbol{x}) = \operatorname{curl}\boldsymbol{A}(\boldsymbol{x}) \;=\; i\omega\mu_0\,M\,\nabla_x\Phi(\boldsymbol{x},\boldsymbol{z})\times\hat{\boldsymbol{a}}\,, \tag{2.41a}$$

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{x}) &= \frac{1}{i\omega\,\mu}\operatorname{curl}\boldsymbol{E}(\boldsymbol{x}) \;=\; M\,\big(-\Delta + \nabla\operatorname{div}\big)\big(\Phi(\boldsymbol{x},\boldsymbol{z})\hat{\boldsymbol{a}}\big)\\ &= k^2 M\,\Phi(\boldsymbol{x},\boldsymbol{z})\,\hat{\boldsymbol{a}} \;+\; M\,\nabla_x\bigg(\nabla_x\Phi(\boldsymbol{x},\boldsymbol{z})\cdot\hat{\boldsymbol{a}}\bigg)\,.\end{aligned} \tag{2.41b}$$

These are the fields of a **magnetic dipole** with dipole moment $M\hat{\boldsymbol{a}}$.

To find the asymptotic behaviour of these fields and the corresponding far field patterns we have to study the fundamental solution $\Phi(\boldsymbol{x},\boldsymbol{z})$ as $r = |\boldsymbol{x}|$ tends to infinity. From the representation

$$|\boldsymbol{x}-\boldsymbol{z}| \;=\; |\boldsymbol{x}| \;-\; \frac{\boldsymbol{x}\cdot\boldsymbol{z}}{|\boldsymbol{x}|} \;+\; a(\boldsymbol{x},\boldsymbol{z}) \quad \text{with} \quad |a(\boldsymbol{x},\boldsymbol{z})| \;\le\; 4\,\frac{|\boldsymbol{z}|^2}{|\boldsymbol{x}|}$$

for all $\boldsymbol{x},\boldsymbol{z}\in\mathbb{R}^3$ with $\boldsymbol{x}\neq\boldsymbol{o}$, $|\boldsymbol{z}|\le\frac{1}{2}\,|\boldsymbol{x}|$, we derive the asymptotic representation of the fundamental solution Φ of the Helmholtz equation in the form (using polar coordinates r, θ and ϕ with respect to the origin):

$$\Phi(\boldsymbol{x},\boldsymbol{z}) = \frac{\mathrm{e}^{ikr}}{4\pi r}\,\big[\mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}} \;+\; \mathcal{O}(1/r)\big] \quad \text{as } r = |\boldsymbol{x}|\to\infty, \tag{2.42a}$$

$$\nabla_x\Phi(\boldsymbol{x},\boldsymbol{z}) = ik\,\frac{\mathrm{e}^{ikr}}{4\pi r}\,\big[\mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}}\,\hat{\boldsymbol{x}} \;+\; \mathcal{O}(1/r)\big] \quad \text{as } r\to\infty, \tag{2.42b}$$

$$\frac{\partial^2\Phi(\boldsymbol{x},\boldsymbol{z})}{\partial x_j\partial x_\ell} = -k^2\,\frac{\mathrm{e}^{ikr}}{4\pi r}\,\Big[\mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}}\,\frac{x_j}{r}\,\frac{x_\ell}{r} \;+\; \mathcal{O}(1/r)\Big] \quad \text{as } r\to\infty\,, \tag{2.42c}$$

uniformly in θ, ϕ and $\boldsymbol{z}$ in any compact subset of $\mathbb{R}^3$. Again, we have set $\hat{\boldsymbol{x}} = \boldsymbol{x}/r$. From this we see that the fields generated by an **electric dipole** or **magnetic dipole** with moment $I\hat{\boldsymbol{a}}$ and $M\hat{\boldsymbol{a}}$, respectively, satisfy

$$\boldsymbol{H}(\boldsymbol{x}) = ik\,I\,\frac{\mathrm{e}^{ikr}}{4\pi r}\,\big[\mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}}(\hat{\boldsymbol{x}}\times\hat{\boldsymbol{a}}) \;+\; \mathcal{O}(1/r)\big]\,, \tag{2.43a}$$

$$\boldsymbol{E}(\boldsymbol{x}) = i\omega\mu_0\,I\,\frac{\mathrm{e}^{ikr}}{4\pi r}\,\big[\mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}}[\hat{\boldsymbol{x}}\times(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{x}})] \;+\; \mathcal{O}(1/r)\big] \tag{2.43b}$$

and

$$\boldsymbol{E}(\boldsymbol{x}) = -k\omega\mu_0\,M\,\frac{\mathrm{e}^{ikr}}{4\pi r}\,\big[\mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}}(\hat{\boldsymbol{x}}\times\hat{\boldsymbol{a}}) \;+\; \mathcal{O}(1/r)\big]\,, \tag{2.43c}$$

$$\boldsymbol{H}(\boldsymbol{x}) = k^2\,M\,\frac{\mathrm{e}^{ikr}}{4\pi r}\,\big[\mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}}[\hat{\boldsymbol{x}}\times(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{x}})] \;+\; \mathcal{O}(1/r)\big]\,, \tag{2.43d}$$

respectively, as $r \to \infty$.

The far field patterns of an electric dipole have been computed in Section 2.9 already. (This was the motivation for the Silver-Müller radiation condition). We repeat that the far field patterns generated by an electric or magnetic dipole are given by

$$\boldsymbol{H}_\infty(\hat{\boldsymbol{x}}) = \frac{ikI}{4\pi}\, \mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}}\,(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{a}})\,, \tag{2.44a}$$

$$\boldsymbol{E}_\infty(\hat{\boldsymbol{x}}) = \frac{i\omega\mu_0 I}{4\pi}\, \mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}}\,[\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{x}})]\,. \tag{2.44b}$$

and

$$\boldsymbol{E}_\infty(\hat{\boldsymbol{x}}) = \frac{-k\omega\mu_0 M}{4\pi}\, \mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}}\,(\hat{\boldsymbol{a}} \times \hat{\boldsymbol{x}})\,, \tag{2.44c}$$

$$\boldsymbol{H}_\infty(\hat{\boldsymbol{x}}) = \frac{k^2 M}{4\pi}\, \mathrm{e}^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{z}}\,[\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{x}})]\,. \tag{2.44d}$$

Example 2.4. As a special example we consider the case $\hat{\boldsymbol{a}} = \hat{\boldsymbol{e}}_3$ (the unit vector in x_3-direction). Using spherical polar coordinates (r, θ, ϕ) of $\boldsymbol{x}$ with respect to $\boldsymbol{z}$ and coordinate vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ we compute

$$\nabla_x \Phi(\boldsymbol{x}, \boldsymbol{z}) = \left(ik - \frac{1}{r}\right)\frac{\mathrm{e}^{ikr}}{4\pi r}\,\hat{\boldsymbol{x}}, \tag{2.45a}$$

$$\nabla_x \Phi(\boldsymbol{x}, \boldsymbol{z}) \times \hat{\boldsymbol{e}}_3 = -\left(ik - \frac{1}{r}\right)\frac{\mathrm{e}^{ikr}}{4\pi r}\,\sin\theta\,\hat{\boldsymbol{\phi}} \tag{2.45b}$$

(since $\hat{\boldsymbol{x}} \times \hat{\boldsymbol{e}}_3 = -\sin\theta\,\hat{\boldsymbol{\phi}}$), and

$$\nabla_x \frac{\partial}{\partial x_3}\Phi(\boldsymbol{x}, \boldsymbol{z}) = \left(\frac{3}{r^2} - \frac{3ik}{r} - k^2\right)\frac{\mathrm{e}^{ikr}}{4\pi r}\,\cos\theta\,\hat{\boldsymbol{x}} + \left(\frac{ik}{r} - \frac{1}{r^2}\right)\frac{\mathrm{e}^{ikr}}{4\pi r}\,\hat{\boldsymbol{e}}_3 \tag{2.45c}$$

and thus for the *electric dipole* by (2.40a), (2.40b):

$$\boldsymbol{H}(\boldsymbol{x}) = -I\left(ik - \frac{1}{r}\right)\frac{\mathrm{e}^{ikr}}{4\pi r}\,\sin\theta\,\hat{\boldsymbol{\phi}}, \tag{2.46a}$$

$$\boldsymbol{E}(\boldsymbol{x}) = i\omega\mu_0 I\,\Phi(\boldsymbol{x}, \boldsymbol{z})\hat{\boldsymbol{e}}_3 \;-\; \frac{I}{\sigma - i\omega\epsilon}\,\nabla\frac{\partial}{\partial x_3}\Phi(\boldsymbol{x}, \boldsymbol{z})$$

i.e., since $\hat{\boldsymbol{e}}_3 = \cos\theta\,\hat{\boldsymbol{x}} - \sin\theta\,\hat{\boldsymbol{\theta}}$,

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x}) = {} & \frac{2I}{\sigma - i\omega\epsilon}\left(\frac{1}{r} - ik\right)\frac{\mathrm{e}^{ikr}}{4\pi r^2}\,\cos\theta\,\hat{\boldsymbol{x}} \\ & - \frac{I}{\sigma - i\omega\epsilon}\left(k^2 + \frac{ik}{r} - \frac{1}{r^2}\right)\frac{\mathrm{e}^{ikr}}{4\pi r}\,\sin\theta\,\hat{\boldsymbol{\theta}}.\end{aligned} \tag{2.46b}$$

Analogously, we have for the *magnetic dipole*

$$\boldsymbol{E}(\boldsymbol{x}) = -i\omega\,\mu\,M\left(ik - \frac{1}{r}\right)\frac{\mathrm{e}^{ikr}}{4\pi r}\,\sin\theta\,\hat{\boldsymbol{\phi}}, \tag{2.47a}$$

$$\boldsymbol{H}(\boldsymbol{x}) = 2M\left(\frac{1}{r} - ik\right)\frac{\mathrm{e}^{ikr}}{4\pi r^2}\,\cos\theta\,\hat{\boldsymbol{x}} \;-\; I\left(k^2 + \frac{ik}{r} - \frac{1}{r^2}\right)\frac{\mathrm{e}^{ikr}}{4\pi r}\,\sin\theta\,\hat{\boldsymbol{\theta}} \tag{2.47b}$$

In the special case $\sigma = 0$ and $\boldsymbol{z} = \boldsymbol{o}$ the far field patterns are given by

$$\boldsymbol{E}_\infty(\theta,\phi) \;=\; -\frac{i\omega\mu_0 I}{4\pi}\,\sin\theta\,\hat{\boldsymbol{\theta}}\,, \qquad \boldsymbol{H}_\infty(\theta,\phi) \;=\; -\frac{i\omega\mu_0 I}{4\pi}\,\sin\theta\,\hat{\boldsymbol{\phi}} \tag{2.48a}$$

for the electric dipole and

$$\boldsymbol{E}_\infty(\theta,\phi) \;=\; \frac{k\omega\mu_0 M}{4\pi}\,\sin\theta\,\hat{\boldsymbol{\theta}}\,, \qquad \boldsymbol{H}_\infty(\theta,\phi) \;=\; -\frac{k\omega\mu_0 M}{4\pi}\,\sin\theta\,\hat{\boldsymbol{\phi}} \tag{2.48b}$$

for the magnetic dipole. The radiated power from (2.35) takes the form

$$P_\infty \;=\; \frac{\omega^2\mu_0^2 I^2}{12\pi}\,Y_0,, \qquad P_\infty \;=\; \frac{k^2\omega^2\mu_0^2 M^2}{12\pi}\,Y_0\,, \tag{2.49}$$

respectively.

We would like now to return to our example in Section 1.2. There we introduced the notion of an array by assuming a (finite) number of electric dipoles at locations $\boldsymbol{y}_n$, $n = -N,\dots,N$. Let $a_n\,\hat{\boldsymbol{p}}$ be the common dipole moment. Then, according to (2.44b), the n^{th} dipole generates the electric far field pattern

$$\boldsymbol{E}_{n,\infty}(\hat{\boldsymbol{x}}) \;=\; a_n\,\frac{i\omega\mu_0}{4\pi}\,\mathrm{e}^{-ik\boldsymbol{y}_n\cdot\hat{\boldsymbol{x}}}\,\bigl[\hat{\boldsymbol{x}} \times \bigl(\hat{\boldsymbol{p}} \times \hat{\boldsymbol{x}}\bigr)\bigr]\,.$$

The whole array generates the far field pattern by superposition, i.e.

$$\boldsymbol{E}_\infty(\hat{\boldsymbol{x}}) \;=\; \frac{i\omega\mu_0}{4\pi}\,\hat{\boldsymbol{x}} \times \bigl(\hat{\boldsymbol{p}} \times \hat{\boldsymbol{x}}\bigr)\sum_{n=-N}^{N} a_n\,\mathrm{e}^{-ik\boldsymbol{y}_n\cdot\hat{\boldsymbol{x}}}\,.$$

This formula coincides with (1.3).

The electromagnetic fields of a *finite line current* flowing along the straight line $\boldsymbol{x} = s\,\hat{\boldsymbol{e}}_3$, $s \in [-\ell,\ell]$, of length 2ℓ and direction $\hat{\boldsymbol{e}}_3$ can be modeled by the limiting process of an array, when the distance d between the elements tends to zeros and the number of elements to infinity. This leads to the determination of $\boldsymbol{H}$ and $\boldsymbol{E}$ from the potential

$$u(\boldsymbol{x}) \;=\; \int_{-\ell}^{\ell} I(s)\,\varPhi(\boldsymbol{x}, s\hat{\boldsymbol{e}}_3)\,ds \;=\; \int_{-\ell}^{\ell} I(s)\,\frac{\exp(ik\,|\boldsymbol{x} - s\hat{\boldsymbol{e}}_3|)}{4\pi\,|\boldsymbol{x} - s\hat{\boldsymbol{e}}_3|}\,ds \tag{2.50}$$

via

$$\boldsymbol{H}(\boldsymbol{x}) \;=\; \operatorname{curl}\left(u(\boldsymbol{x})\,\hat{\boldsymbol{e}}_3\right), \qquad \boldsymbol{E}(\boldsymbol{x}) \;=\; i\omega\mu_0\,u(\boldsymbol{x})\,\hat{\boldsymbol{e}}_3 + \frac{1}{\sigma - i\omega\epsilon}\,\nabla\big(\partial u(\boldsymbol{x})/\partial x_3\big)\,. \tag{2.51}$$

The far field patterns are computed, using (2.44a)–(2.44d), by

$$\begin{aligned}\boldsymbol{E}_\infty(\hat{\boldsymbol{x}}) &= \frac{i\omega\mu_0}{4\pi}\,[\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{e}}_3 \times \hat{\boldsymbol{x}})] \int\limits_{-\ell}^{\ell} I(s)\,\mathrm{e}^{-iks\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{e}}_3}\,ds \\ &= \frac{i\omega\mu_0}{4\pi}\,[\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{e}}_3 \times \hat{\boldsymbol{x}})] \int\limits_{-\ell}^{\ell} I(s)\,\mathrm{e}^{-iks\cos\theta}\,ds\,, \end{aligned} \tag{2.52a}$$

$$\begin{aligned}\boldsymbol{H}_\infty(\hat{\boldsymbol{x}}) &= \frac{ik}{4\pi}\,(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{e}}_3) \int\limits_{-\ell}^{\ell} I(s)\,\mathrm{e}^{-iks\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{e}}_3}\,ds \\ &= \frac{ik}{4\pi}\,(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{e}}_3) \int\limits_{-\ell}^{\ell} I(s)\,\mathrm{e}^{-iks\cos\theta}\,ds\,. \end{aligned} \tag{2.52b}$$

2.11 Boundary Conditions on Interfaces

If we consider a situation in which a surface S separates two homogeneous media from each other, the constitutive parameters ϵ, μ and σ are no longer continuous but piecewise continuous with finite jumps on S. While on both sides of S Maxwell's equations (2.5a)–(2.5d) hold, the presence of these jumps implies that the fields satisfy certain conditions on the surface.

To derive the mathematical form of this behaviour (the boundary conditions) we apply the law of induction (2.1b) to a narrow rectangle-like surface C, containing the normal n to the surface S and whose long sides C_+ and C_- are parallel to S and are on the opposite sides of it, cf. Figure 2.1.

When we let the height of the narrow sides, AA' and BB', approach zero C_+ and C_- approach a curve C on S, the surface integral $\frac{\partial}{\partial t}\int_R \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\nu}\,dS$ will vanish in the limit since the field remains finite (note, that the normal $\boldsymbol{\nu}$ is the normal to R lying in the tangential plane of S). Hence, the line integrals $\int_C \boldsymbol{\mathcal{E}}_+ \cdot \boldsymbol{t}\,d\ell$ and $\int_C \boldsymbol{\mathcal{E}}_- \cdot \boldsymbol{t}\,d\ell$ must be equal. Since the curve C is arbitrary the integrands $\boldsymbol{\mathcal{E}}_+ \cdot \boldsymbol{t}$ and $\boldsymbol{\mathcal{E}}_- \cdot \boldsymbol{t}$ coincide on every arc C, i.e.

$$\boldsymbol{n} \times \boldsymbol{\mathcal{E}}_+ \;-\; \boldsymbol{n} \times \boldsymbol{\mathcal{E}}_- \;=\; \boldsymbol{o} \quad \text{on } S. \tag{2.53}$$

A similar argument holds for the magnetic field in (2.1a) if the current distribution $\boldsymbol{\mathcal{J}} = \sigma\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{J}}_e$ remains finite. In this case, the same arguments lead to the boundary condition

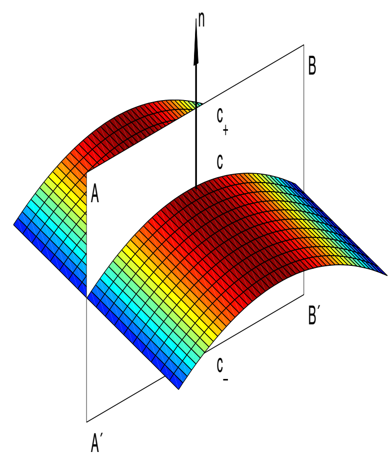

Fig. 2.1. The derivation of the boundary conditions

$$\boldsymbol{n} \times \boldsymbol{\mathcal{H}}_+ \; - \; \boldsymbol{n} \times \boldsymbol{\mathcal{H}}_- \; = \; \boldsymbol{o} \quad \text{on } S. \tag{2.54}$$

If, however, the external current distribution is a surface current, i.e. if $\boldsymbol{\mathcal{J}}_e$ is of the form $\boldsymbol{\mathcal{J}}_e(\boldsymbol{x} + \tau \boldsymbol{n}(\boldsymbol{x})) = \boldsymbol{\mathcal{J}}_s(\boldsymbol{x})\delta(\tau)$ for small τ and $\boldsymbol{x} \in S$ and with tangential surface field $\boldsymbol{\mathcal{J}}_s$ and σ is finite, then the surface integral $\int_R \boldsymbol{\mathcal{J}}_e \cdot \boldsymbol{\nu} \, dS$ will tend to $\int_C \boldsymbol{\mathcal{J}}_s \cdot \boldsymbol{\nu} \, d\ell$, and so the boundary condition is

$$\boldsymbol{n} \times \boldsymbol{\mathcal{H}}_+ \; - \; \boldsymbol{n} \times \boldsymbol{\mathcal{H}}_- \; = \; \boldsymbol{\mathcal{J}}_s \quad \text{on } S. \tag{2.55}$$

We will call (2.53) and (2.54) or (2.55) the **transmission boundary conditions**.

In many applications it is also important to consider the case in which the interface S is covered by a thin layer of very high conductivity, i.e. $\sigma(\boldsymbol{x} + \tau \boldsymbol{n}(x)) = \sigma_s(\boldsymbol{x})\delta(\tau)$ for small τ and $\boldsymbol{x} \in S$ and with surface conductivity σ_s. If $\boldsymbol{\mathcal{J}}_e$ remains finite then the surface integral $\int_R \boldsymbol{\mathcal{J}} \cdot \boldsymbol{n} \, dS$ will tend to $\int_C \sigma_s \, \boldsymbol{\mathcal{E}} \cdot \boldsymbol{n} \, d\ell + \int_C \boldsymbol{\mathcal{J}}_s \cdot \boldsymbol{n} \, d\ell$, i.e.

$$\boldsymbol{n} \times \boldsymbol{\mathcal{H}}_+ \; - \; \boldsymbol{n} \times \boldsymbol{\mathcal{H}}_- \; = \; \sigma_s \, \boldsymbol{n} \times (\boldsymbol{\mathcal{E}} \times \boldsymbol{n}) \; + \; \boldsymbol{\mathcal{J}}_s \quad \text{on } S. \tag{2.56}$$

We will call this the **conductive boundary condition**. This condition has been used (see e.g. [121],[122]) to model the situation in which the field penetrates the object only to a small depth. Thus this condition is closely related to the transmission conditions as well as to the impedance, or Leontovich, condition which we mention below.

A special and very important case is that of a **perfectly conducting medium** with boundary S. Such a medium is characterized by the fact that the electric field vanishes inside this medium, and (2.53) reduces to

$$\boldsymbol{n} \times \boldsymbol{\mathcal{E}} \;=\; \boldsymbol{o} \quad \text{on } S \tag{2.57}$$

Another important case is the **impedance-** or **Leontovich boundary condition**

$$\boldsymbol{n} \times \boldsymbol{\mathcal{H}} \;=\; \lambda\, \boldsymbol{n} \times (\boldsymbol{\mathcal{E}} \times \boldsymbol{n}) \quad \text{on } S \tag{2.58}$$

which, under appropriate conditions, may be used as an approximation of the transmission conditions [120].

Finally, we specify the boundary conditions to the E- and H-modes derived Section 2.8. We assume that the surface S is an infinite cylinder in x_3-direction with constant cross section. Furthermore, we assume that the volume current density j vanishes near the boundary S and that the surface current densities take the form $\boldsymbol{J}_s = j_s \hat{\boldsymbol{e}}_3$ for the E-mode and $\boldsymbol{J}_s = j_s\big(\boldsymbol{n} \times \hat{\boldsymbol{e}}_3\big)$ for the H-mode. We use the notation $[v] := v|_+ - v|_-$ for the jump of the function v at the boundary. Also, we abbreviate (only for this table) $\sigma' = \sigma - i\omega\epsilon$. We list the boundary conditions in the following table.

Bound. cond.	E-mode	H-mode
transmission	$\big[k^2 u\big] = 0$ on S, $\big[\sigma' \frac{\partial u}{\partial n}\big] = -j_s$ on S,	$\big[\mu \frac{\partial u}{\partial n}\big] = 0$ on S, $\big[k^2 u\big] = j_s$ on S,
conductive	$\big[k^2 u\big] = 0$ on S, $-\big[\sigma' \frac{\partial u}{\partial n}\big] = \sigma_s k^2 u + j_s$,	$\big[\mu \frac{\partial u}{\partial n}\big] = 0$ on S, $\big[k^2 u\big] = \sigma_s i\omega\mu \frac{\partial u}{\partial n} + j_s$,
impedance	$\lambda\, k^2 u + \sigma' \frac{\partial u}{\partial n} = -j_s$ on S,	$k^2 u - \lambda\, i\omega\mu \frac{\partial u}{\partial n} = j_s$ on S,
perfect conductor	$u = 0$ on S,	$\frac{\partial u}{\partial n} = 0$ on S.

2.12 Hertz Potentials and Classes of Solutions

In this section we recall some of the most important classes of solutions of Maxwell's equations (2.13a)–(2.13d) in homogeneous, isotropic and source free media. We use the constructions with the Hertz potentials in the TM and TE modes described in Section 2.8.

(A) Plane waves:

First, we take $\boldsymbol{A}_e = \boldsymbol{o}$, $\boldsymbol{A}_m(\boldsymbol{x}) = -1/(k\omega\mu)\,\exp(ik\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{x})\,\boldsymbol{a}$ for some fixed vector $\boldsymbol{a}\in\mathbb{C}^3$ and unit vector $\hat{\boldsymbol{\alpha}}\in\mathbb{R}^3$. This results in **plane waves**:

$$\boldsymbol{E}(\boldsymbol{x}) = (\hat{\boldsymbol{\alpha}}\times\boldsymbol{a})\,\mathrm{e}^{ik\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{x}}\,, \tag{2.59a}$$

$$\boldsymbol{H}(\boldsymbol{x}) = \frac{k}{\omega\mu}\,\hat{\boldsymbol{\alpha}}\times(\hat{\boldsymbol{\alpha}}\times\boldsymbol{a})\,\mathrm{e}^{ik\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{x}}\,. \tag{2.59b}$$

The corresponding time dependent waves are

$$\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) = (\hat{\boldsymbol{\alpha}}\times\boldsymbol{a})\,\mathrm{e}^{-\mathrm{Im}\,k\,\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{x}}\,\mathrm{e}^{i\,\mathrm{Re}\,k\,\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{x}-i\omega t}\,,$$

$$\boldsymbol{\mathcal{H}}(\boldsymbol{x},t) = \frac{k}{\omega\mu}\,\hat{\boldsymbol{\alpha}}\times(\hat{\boldsymbol{\alpha}}\times\boldsymbol{a})\,\mathrm{e}^{-\mathrm{Im}\,k\,\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{x}}\,\mathrm{e}^{i\,\mathrm{Re}\,k\,\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{x}-i\omega t}\,.$$

The phase factor is constant on the planes $\mathrm{Re}\,k\,\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{x} = \omega t+\delta$ traveling with velocity $v=\omega/\mathrm{Re}\,k$ (**phase velocity**) in the direction $\hat{\boldsymbol{\alpha}}$. For non-conductive media $v=1/\sqrt{\mu\epsilon}$. We see that $\boldsymbol{H} = \frac{k}{\omega\mu}\,\hat{\boldsymbol{\alpha}}\times\boldsymbol{E} = Y_0\,\hat{\boldsymbol{\alpha}}\times\boldsymbol{E}$. The quantity

$$Y_0 \;:=\; \frac{k}{\omega\mu} \tag{2.60}$$

is called the **intrinsic admittance** of the medium which is equal to $\sqrt{\epsilon/\mu}$ for non-conductive media (see (2.30)).

(B) Spherical waves:

As a second class of solutions we take $\boldsymbol{A}_e = \boldsymbol{o}$ and $\boldsymbol{A}_m(\boldsymbol{x}) = Y_n^m(\theta,\phi)\,\lambda_n(kr)\,r\,\hat{\boldsymbol{x}}$ where r,θ,ϕ are the spherical polar coordinates of x and $\hat{\boldsymbol{x}}$ denotes the coordinate vector in $r-$direction. $Y_n^m(\theta,\phi) = P_n^m(\cos\theta)\exp(im\phi)$, $|m|\le n$, $n\in\mathbb{N}_0$, denote the spherical harmonics where we have denoted the associated Legendre function of order n and degree m by Y_n^m. By λ_n we denote either the **spherical Bessel function** j_n or the **spherical Hankel functions** $h_n^{(1)}$ or $h_n^{(2)}$ of the first and second kind, respectively, and order of n. We refer to [139, 50, 30] for an introduction into Bessel- and Hankel functions. Since $Y_n^m(\theta,\phi)\,\lambda_n(kr)$ are solutions of the Helmholtz equation (2.21) for $r>0$ it is easily seen that $\boldsymbol{A}_m$ satisfies the inhomogeneous vector Helmholtz equation

$$\Delta\boldsymbol{A}_m \;+\; k^2\boldsymbol{A}_m \;=\; 2\,\nabla\big[Y_n^m(\theta,\phi)\,\lambda_n(kr)\big] \quad \text{for } r>0.$$

The fields

$$\boldsymbol{E}(\boldsymbol{x}) = i\omega\,\mu\,\mathrm{curl}\,[Y_n^m(\theta,\phi)\,\lambda_n(kr)\,r\,\hat{\boldsymbol{x}}] \tag{2.61a}$$

$$= -i\omega\,\mu\,\hat{\boldsymbol{x}}\times\nabla\,[Y_n^m(\theta,\phi)\,\lambda_n(kr)\,r] \tag{2.61b}$$

$$\boldsymbol{H}(\boldsymbol{x}) = \frac{1}{i\omega\,\mu}\mathrm{curl}\,\boldsymbol{E}(\boldsymbol{x}) \;=\; \mathrm{curl}^{\,2}\,[Y_n^m(\theta,\phi)\,\lambda_n(kr)\,r\,\hat{\boldsymbol{x}}] \tag{2.61c}$$

are called **toroidal fields**. Analogously, $\boldsymbol{A}_m = \boldsymbol{o}$ and $\boldsymbol{A}_e(\boldsymbol{x}) = Y_n^m(\theta,\phi)\,\lambda_n(kr)\,r\,\hat{\boldsymbol{x}}$ lead to **spheroidal fields** of the form

$$\boldsymbol{H}(\boldsymbol{x}) = (\sigma - i\omega\,\epsilon)\,\mathrm{curl}\,\left[Y_n^m(\theta,\phi)\,\lambda_n(kr)\,r\,\hat{\boldsymbol{x}}\right], \tag{2.62a}$$

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{\sigma - i\omega\,\epsilon}\,\mathrm{curl}\,\boldsymbol{H}(\boldsymbol{x}) \;=\; \mathrm{curl}^{\,2}\left[Y_n^m(\theta,\phi)\,\lambda_n(kr)\,r\,\hat{\boldsymbol{x}}\right]. \tag{2.62b}$$

The fields with $\lambda_n = j_n$ are smooth at the origin while the fields with $\lambda_n = h_n^{(1),(2)}$ are singular at the origin.

(C) Cylindrical waves:

Electromagnetic waves in waveguides are described by using cylindrical coordinates ρ, ϕ, z. We set $\boldsymbol{A}_e = \boldsymbol{o}$ and $\boldsymbol{A}_m(x) = \boldsymbol{a}\,\exp(i\beta z)\,\Lambda_n(\kappa\rho)\,\exp(in\phi)$ with constant vector $\boldsymbol{a} \in \mathbb{C}^3$ and $\beta \in \mathbb{R}$ where $\kappa = \sqrt{k^2 - \beta^2}$ (with $\mathrm{Re}\,\kappa \geq 0$ and $\mathrm{Im}\,\kappa \geq 0$). Here, Λ_n denotes one of the **cylindrical Bessel functions** J_n or **Hankel functions** $H_n^{(1)}$, $H_n^{(2)}$ of the first and second kind, respectively, and of order n. Then $\Lambda_n(\kappa\rho)\,\exp(in\phi)$ solves the two dimensional Helmholtz equation $\Delta u + \kappa^2 u = 0$ for $\rho > 0$. We arrive at the fields

$$\boldsymbol{E}(\boldsymbol{x}) = -i\omega\,\mu\,\boldsymbol{a} \times \nabla\left[\exp(i\beta z)\,\Lambda_n(\kappa\rho)\,\exp(in\phi)\right], \tag{2.63a}$$

$$\boldsymbol{H}(\boldsymbol{x}) = \frac{1}{i\omega\,\mu}\mathrm{curl}\,\boldsymbol{E}(\boldsymbol{x}) \;=\; \mathrm{curl}^{\,2}\left[\exp(i\beta z)\,\Lambda_n(\kappa\rho)\,\exp(in\phi)\,\boldsymbol{a}\right] \tag{2.63b}$$

which are smooth at the line $\rho = 0$ only if $\Lambda_n = J_n$.

Now we check which of these special solutions of Maxwell's equations satisfy the radiation conditions (2.28) or (2.29a), (2.29b). We restrict ourselves to the case of k being real.

From (2.31a), (2.31b) we conclude that a necessary (but not sufficient!) condition for the Silver-Müller radiation condition (2.29a), (2.29b) to hold is that the fields decay as $1/r$ when r tends to infinity. From this we see that no plane or cylindrical wave satisfies the Silver-Müller radiation conditions (for the latter see Section 2.13!).

From the asymptotic behaviour of the spherical Hankel functions

$$h_n^{(1),(2)}(t) = \frac{1}{t}\,\mathrm{e}^{\pm i\left(t-(n+1)\pi/2\right)}\left\{1 \;+\; \mathcal{O}(1/t)\right\} \quad \text{as } t \to \infty \tag{2.64a}$$

$$\frac{d}{dt}h_n^{(1),(2)}(t) = \frac{1}{t}\,\mathrm{e}^{\pm i\left(t-n\pi/2\right)}\left\{1 \;+\; \mathcal{O}(1/t)\right\} \quad \text{as } t \to \infty \tag{2.64b}$$

$$j_n(t) = \frac{1}{t}\,\cos\!\big(t-(n+1)\pi/2\big)\left\{1 \;+\; \mathcal{O}(1/t)\right\} \quad \text{as } t \to \infty \tag{2.64c}$$

$$\frac{d}{dt}j_n(t) = -\frac{1}{t}\,\sin\!\big(t-(n+1)\pi/2\big)\left\{1 \;+\; \mathcal{O}(1/t)\right\} \quad \text{as } t \to \infty. \tag{2.64d}$$

we conclude that only the spherical wave functions

$$\boldsymbol{E}(\boldsymbol{x}) = i\omega\,\mu\,\mathrm{curl}\,\left[Y_n^m(\theta,\phi)\,h_n^{(1)}(kr)\,r\,\hat{\boldsymbol{x}}\right]$$

$$\boldsymbol{H}(\boldsymbol{x}) = \frac{1}{i\omega\,\mu}\mathrm{curl}\,\boldsymbol{E}(\boldsymbol{x}) \;=\; \mathrm{curl}^{\,2}\left[Y_n^m(\theta,\phi)\,h_n^{(1)}(kr)\,r\,\hat{\boldsymbol{x}}\right]$$

and

$$\boldsymbol{H}(\boldsymbol{x}) = (\sigma - i\omega\,\epsilon)\,\mathrm{curl}\,\left[Y_n^m(\theta,\phi)\,h_n^{(1)}(kr)\,r\,\hat{\boldsymbol{x}}\right],$$

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{\sigma - i\omega\,\epsilon}\,\mathrm{curl}\,\boldsymbol{H}(\boldsymbol{x}) \;=\; \mathrm{curl}^{\,2}\left[Y_n^m(\theta,\phi)\,h_n^{(1)}(kr)\,r\,\hat{\boldsymbol{x}}\right]$$

satisfy the Silver-Müller radiation condition with far field patterns

$$\boldsymbol{E}_\infty(\theta,\phi) = \frac{i\omega\,\mu}{k}\,\mathrm{e}^{-\pi(n+1)i/2}\left[\frac{\partial}{\partial\theta}Y_n^m(\theta,\phi)\,\hat{\boldsymbol{\phi}} - \frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi}Y_n^m(\theta,\phi)\,\hat{\boldsymbol{\theta}}\right] \tag{2.65a}$$

$$\boldsymbol{H}_\infty(\theta,\phi) = \hat{\boldsymbol{x}} \times \boldsymbol{E}_\infty(\theta,\phi), \tag{2.65b}$$

and

$$\boldsymbol{H}_\infty(\theta,\phi) = \frac{\sigma - i\omega\,\epsilon}{k}\,\mathrm{e}^{-\pi(n+1)i/2}\left[\frac{\partial}{\partial\theta}Y_n^m(\theta,\phi)\,\hat{\boldsymbol{\phi}} - \frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi}Y_n^m(\theta,\phi)\,\hat{\boldsymbol{\theta}}\right]$$

$$\boldsymbol{E}_\infty(\theta,\phi) = \boldsymbol{H}_\infty(\theta,\phi) \times \hat{\boldsymbol{x}}\,,$$

respectively.

2.13 Radiation Problems in Two Dimensions

We have seen in the previous sections that the outgoing spherical waves and the fields generated by electric and magnetic dipoles satisfy the Silver-Müller radiation condition (2.29a), (2.29b) if k is real. As we mentioned above, this radiation condition describes the correct behaviour of the radiating fields generated by sources lying in a compact set. If, however, the sources are distributed on an unbounded set e.g., along an infinite line, the fields decay more slowly. First we look at the cylindrical waves again for the special case where k is real and positive, $\beta = 0$, $\boldsymbol{a} = \hat{\boldsymbol{e}}_3$ and $\Lambda_n = H_n^{(1)}$. With cylindrical coordinates ρ, ϕ, z we now have

$$\boldsymbol{E}(\boldsymbol{x}) = -i\omega\,\mu\,\hat{\boldsymbol{e}}_3 \times \nabla\left[H_n^{(1)}(\kappa\rho)\,\exp(in\phi)\right], \tag{2.66a}$$

$$\boldsymbol{H}(\boldsymbol{x}) = \frac{1}{i\omega\,\mu}\mathrm{curl}\,\boldsymbol{E}(\boldsymbol{x}) \;=\; \mathrm{curl}^{\,2}\left[H_n^{(1)}(\kappa\rho)\,\exp(in\phi)\,\hat{\boldsymbol{e}}_3\right] \tag{2.66b}$$

We have seen that $\boldsymbol{E}$ and $\boldsymbol{H}$ do not satisfy the Silver-Müller radiation condition (2.29a), (2.29b) since they are only bounded in x_3-direction but do not decay to zero. We will now show that they satisfy a weaker form of the radiation condition in the $(x_1, x_2)-$plane. Let us set $u(\rho,\phi) = H_n^{(1)}(\kappa\rho)\,\exp(in\phi)$. Then u satisfies the **two dimensional Helmholtz equation**

$$\Delta u \;+\; \kappa^2\,u \;=\; 0 \quad \text{in } \mathbb{R}^2 \setminus \{\boldsymbol{o}\}\,. \tag{2.67}$$

Furthermore, from the asymptotic behaviour of the Hankel functions

$$H_n^{(1)}(t) = \sqrt{\frac{2}{\pi t}}\, \mathrm{e}^{i\left(t - n\pi/2 - \pi/4\right)} \left\{1 \;+\; \mathcal{O}(1/t)\right\} \quad \text{as } t \to \infty \quad (2.68a)$$

$$\frac{d}{dt} H_n^{(1)}(t) = \sqrt{\frac{2}{\pi t}}\, \mathrm{e}^{i\left(t - n\pi/2 + \pi/4\right)} \left\{1 \;+\; \mathcal{O}(1/t)\right\} \quad \text{as } t \to \infty \quad (2.68b)$$

we conclude that u satisfies the **two dimensional Sommerfeld radiation condition**

$$\frac{\partial}{\partial r} u(x) \;-\; i\,\kappa\, u(x) \;=\; \mathcal{O}\big(1/\rho^{3/2}\big) \quad \text{as } \rho \to \infty \qquad (2.69)$$

uniformly with respect to $\phi \in [0, 2\pi]$.

Let u be any solution u of the Helmholtz equation (2.67) satisfying the radiation condition (2.69). Then it can be shown (see [29]) that u and ∇u decay as $\mathcal{O}(1/\sqrt{\rho})$. Furthermore, for

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}) &= -i\omega\,\mu\,\hat{\boldsymbol{e}}_3 \times \nabla u(\rho, \phi) \\ &= i\omega\,\mu \left[\frac{1}{\rho}\,\frac{\partial u(\rho,\phi)}{\partial \phi}\,\hat{\boldsymbol{\rho}} \;-\; \frac{\partial u(\rho,\phi)}{\partial \rho}\,\hat{\boldsymbol{\phi}}\right] \qquad (2.70a) \\ \boldsymbol{H}(\boldsymbol{x}) &= \frac{1}{i\omega\,\mu} \operatorname{curl} \boldsymbol{E}(\boldsymbol{x}) \;=\; \operatorname{curl}^2\big(u(\rho,\phi)\,\hat{\boldsymbol{e}}_3\big) \\ &= -\Delta u(\rho,\phi)\,\hat{\boldsymbol{e}}_3 \;=\; \kappa^2\, u(\rho,\phi)\,\hat{\boldsymbol{e}}_3 \qquad (2.70b) \end{aligned}$$

since $\operatorname{div}(\hat{\boldsymbol{e}}_3 u) = 0$. From this we conclude that the cylindrical waves $\boldsymbol{E}$ and $\boldsymbol{H}$ from (2.70a), (2.70b) satisfy the **cylindrical Silver-Müller radiation condition**

$$\boldsymbol{E}(\boldsymbol{x}) \times \hat{\boldsymbol{\rho}} \;+\; \frac{1}{Y_0}\,\boldsymbol{H}(\boldsymbol{x}) = \mathcal{O}\big(1/\rho^{3/2}\big)\,, \quad \rho \to \infty\,, \qquad (2.71a)$$

$$\boldsymbol{H}(\boldsymbol{x}) \times \hat{\boldsymbol{\rho}} \;-\; Y_0\,\boldsymbol{E}(\boldsymbol{x}) = \mathcal{O}\big(1/\rho^{3/2}\big)\,, \quad \rho \to \infty\,, \qquad (2.71b)$$

uniformly with respect to $\phi \in [0, \pi]$. Again, $Y_0 = \frac{k}{\omega\mu}$ denotes the admittance from (2.30).

The Sommerfeld radiation condition (2.69) implies that u and ∇u have the asymptotic forms

$$u(\rho,\phi) = \frac{\exp(i\kappa\rho)}{\sqrt{\rho}}\, u_\infty(\phi) \;+\; \mathcal{O}(1/\rho^{3/2})\,, \quad \rho \to \infty\,, \qquad (2.72a)$$

$$\begin{aligned} \frac{\partial u(\rho,\phi)}{\partial \rho} &= \frac{\partial}{\partial \rho}\frac{\exp(i\kappa\rho)}{\sqrt{\rho}}\, u_\infty(\phi) \;+\; \mathcal{O}(1/\rho^{3/2})\,, \quad \rho \to \infty\,, \\ &= i\kappa\, \frac{\exp(i\kappa\rho)}{\sqrt{\rho}}\, u_\infty(\phi) \;+\; \mathcal{O}(1/\rho^{3/2})\,, \quad \rho \to \infty\,, \qquad (2.72b) \end{aligned}$$

$$\frac{\partial u(\rho,\phi)}{\partial \phi} = \frac{\exp(i\kappa\rho)}{\sqrt{\rho}}\, \frac{d\, u_\infty(\phi)}{d\phi} \;+\; \mathcal{O}(1/\rho^{3/2})\,, \quad \rho \to \infty\,, \qquad (2.72c)$$

uniformly with respect to ϕ. Again we call the non-radial function u_∞ the **far field pattern** of the scalar potential u. This asymptotic form of u yields immediately the corresponding asymptotic behaviour

$$\boldsymbol{E}(\boldsymbol{x}) = \omega\mu\kappa \, \frac{\exp(i\kappa\rho)}{\sqrt{\rho}} \, u_\infty(\phi) \, \hat{\boldsymbol{\phi}} \; + \; \mathcal{O}(1/\rho^{3/2}) \, , \quad \rho \to \infty \, , \tag{2.73a}$$

$$\boldsymbol{H}(\boldsymbol{x}) = \kappa^2 \, \frac{\exp(i\kappa\rho)}{\sqrt{\rho}} \, u_\infty(\phi) \, \hat{\boldsymbol{e}}_3 \; + \; \mathcal{O}(1/\rho^{3/2}) \, , \quad \rho \to \infty \, , \tag{2.73b}$$

The vector fields

$$\boldsymbol{E}_\infty(\phi) \; := \; \omega\mu\kappa \, u_\infty(\phi) \, \hat{\boldsymbol{\phi}} \quad \text{and} \quad \boldsymbol{H}_\infty(\phi) \; := \; \omega\mu\kappa \, u_\infty(\phi) \, \hat{\boldsymbol{e}}_3$$

are called the **far field pattern** of the two dimensional vector fields $\boldsymbol{E}$ and $\boldsymbol{H}$. They satisfy also

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{\exp(i\kappa\rho)}{\sqrt{\rho}} \, \boldsymbol{E}_\infty(\phi) \; + \; \mathcal{O}(1/\rho^{3/2}) \, , \quad \rho \to \infty \, , \tag{2.74a}$$

$$\boldsymbol{H}(\boldsymbol{x}) = Y_0 \, \frac{\exp(i\kappa\rho)}{\sqrt{\rho}} \, \boldsymbol{H}_\infty(\phi) \; + \; \mathcal{O}(1/\rho^{3/2}) \, , \quad \rho \to \infty \, , \tag{2.74b}$$

and

$$\boldsymbol{H}_\infty(\phi) \; = \; \hat{\boldsymbol{\rho}} \times \boldsymbol{E}_\infty(\phi), \quad \hat{\boldsymbol{\rho}} \cdot \boldsymbol{E}_\infty(\phi) \; = \; \hat{\boldsymbol{\rho}} \cdot \boldsymbol{H}_\infty(\phi) \; = \; 0 \quad \text{for all } \phi \, , \tag{2.75}$$

compare with (2.32).

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