Preface

This book began life as a set of notes that I developed for a course at the University of Washington entitled *Introduction to Modern Algebra for Teachers*. Originally conceived as a text for future secondary-school mathematics teachers, it has developed into a book that could serve well as a text in an undergraduate course in abstract algebra or a course designed as an introduction to higher mathematics.

This book differs from many undergraduate algebra texts in fundamental ways; the reasons lie in the book's origin and the goals I set for the course. The course is a two-quarter sequence required of students intending to fulfill the requirements of the teacher preparation option for our B.A. degree in mathematics, or of the teacher preparation minor. It is required as well of those intending to matriculate in our university's Master's in Teaching program for secondary mathematics teachers. This is the principal course they take involving abstraction and proof, and they come to it with perhaps as little background as a year of calculus and a quarter of linear algebra. The mathematical ability of the students varies widely, as does their level of mathematical interest.

With such an audience, I have chosen to focus less on content and more on the doing of mathematics. Content matters, of course, but as much as a vehicle for mathematical insight as an end in itself. I wish for students to leave the course with an in-depth experience, perhaps their only one, in reading mathematics, speaking about mathematics, listening to others speak about mathematics, and writing mathematics. Surely, we hope that secondary mathematics teachers develop these skills in their own students; I want to ensure that they have the opportunity themselves. The course content becomes the raw material through which the students develop the ability to understand and communicate mathematics. I love algebra. I want my students to love algebra. But I also want them to master such skills as learning what a mathematical statement is, what a mathematical argument or proof is, how to present an argument orally, how to present an argument in writing, how to recognize a correct proof written or spoken by someone else, and how to converse effectively about mathematics.

In order to help my students achieve these goals, I strive to keep my lecturing to a minimum, and this has forced me to write notes that students can rely on as the primary source of the mathematical material. I expect my students to learn in large measure by reading the material in the book, working on the exercises, and, in groups of four or five students, discussing their findings with their colleagues. Through a combination of group discussion, individual attempts at discovering and writing solutions, further reading, and discussions with me and the course teaching assistant, the students ultimately master—to a greater or lesser extent—the material. Their understanding is expressed through their written proofs and oral explanations.

Learning by working in groups is natural in this course for two reasons. First, the intellectual processes of proof and mathematical communication are best learned by practice; no one practices when I lecture. Second, the course is part of the students' preparation to become mathematics teachers; as teachers, they will communicate mathematical ideas to others and listen as others do the same to them. By practicing in class, the students gain an appreciation of the difficulty and the importance of expressing mathematical ideas effectively.

An important facet of the course is writing. I insist that the students write their arguments in well-structured English prose. The teaching assistant and I provide detailed comments to help them learn how to do this. Some of the students are good writers, but they may not have realized previously that they can apply their writing skills to mathematics.

For this approach to work, the material has to be handed to the students in manageable portions, with small gaps between the portions for the student to fill in by doing the exercises, which, in turn, are themselves structured as sequences of questions with even smaller gaps between them. Whenever textbooks that I have used, or my own written materials, leave too large a gap, the students fall into the resulting chasm. For learning to occur, the steps must be the right size. The steps taken by standard undergraduate texts may suit other audiences (though not as many, in my experience, as one might suppose), but they are too large for this one, as I discovered when I first taught the course in the fall of 1996. Within a month, I began to supplement the chosen text with commentary that attempted to fill gaps in certain proofs. Then I began to write my own assignments, interweaving my material with passages in the text. Ultimately, I left the text behind, writing and rewriting assignments repeatedly as I taught the course until they evolved into this book.

One nonstandard feature of the book is that I prove only a few of the theorems in full. Most proofs are left as exercises, and these exercises form the heart of the course. Sometimes, the treatment of a single result is stretched out over several pages, as I ask the student to prove it in a sequence of cases, building up to the general case. An example of this is the treatment of Eisenstein's criterion in Section 11.3. Almost every exercise in which a student is asked to prove a theorem contains a detailed hint or outline of the proof. Indeed, to a mathematically experienced reader, some of my outlines may appear to be complete proofs themselves. Yet, for almost all students, the outlines are far from complete. Unwinding their meaning can be a significant challenge, and the unwinding process serves as the catalyst for learning. Students try to understand what is written; discuss their understandings with each other, the teaching assistant, and me; write drafts of proofs; use class time to show the drafts to each other and us; turn in the proofs; receive comments; and try again.

I have found that the material in this book consistently challenges all the students in the course. The few students at the top of each class ultimately succeed in meeting the challenge, while the large majority of students complete the course with some gaps in understanding. They all leave with a much firmer appreciation of the mathematical enterprise. A few students may fall by the wayside. Perhaps this is just as well. Not everyone is intended to be a secondary mathematics teacher, and if this course provides a few with the opportunity to discover this, it has served a useful purpose.

As for the content of the course, the book has three parts: "Integers," "Polynomials," and "All Together Now." In Part I, some fundamental ideas of algebra are introduced in the concrete context of integers, with rings brought in only in Chapter 6 as a way of organizing some of the ideas. The high point of Part I is Chapter 7, in which Fermat's and Euler's theorems on congruence are proved and RSA encryption is discussed. Part II treats polynomials with coefficients chosen from the integers or various fields. Again, the treatment is concrete at first, but ultimately makes contact with abstract ideas of ring theory. In Part III, the parallels we have seen between rings of integers and rings of polynomials are placed in the broader setting of Euclidean rings, for which some general theorems are proved and applied to the ring of Gaussian integers. The irreducible Gaussian integers are determined, and simultaneously we determine, as Fermat first did, which prime numbers are sums of two integer squares.

The choice of content—rings of integers and rings of polynomials—is a natural one for a course intended for future secondary teachers, who will go on to teach these topics in some form themselves. In the course, the topics are studied more deeply and more abstractly than at the high-school level, especially with the introduction of rings and fields. This provides the students with the opportunity to acquire a more advanced viewpoint on material that is at the core of secondary mathematics. The material can work equally well for a much wider range of mathematics students, and may be well suited for self-study. An important theme in the book is a familiar one in undergraduate algebra courses: Ideas introduced initially for the ring of integers make sense as well for rings of polynomials over fields and more generally still for Euclidean rings. The reader is taken through these settings, then returns from the abstraction of Euclidean rings to the concrete example of the Gaussian integers. There is nothing novel about this choice of topics. Almost every standard junior–senior algebra text includes many of them, perhaps even from the same perspective. Moreover, most standard texts include material on many other topics as well. This one does not. I have not aimed for comprehensiveness. Rather, I have aimed to provide a limited amount of material, with the goal of having students come to grips with, or even master, a small number of serious mathematical ideas.

The most significant omission from this book is group theory. There are a few arguments over the course of the book that might lend themselves to a group-theoretic perspective, in particular, those having to do with the group of units of a ring, but my sense is that introducing groups in this setting would do an injustice to the subject. I prefer to introduce groups in a geometric context, as symmetry groups of regular polygons and regular polyhedra, for example. Such a treatment did not seem to fit comfortably into this book, so I have chosen to omit groups altogether. The result is a book lacking the flexibility to be a comprehensive introduction to abstract algebra, but such a book is not what I have aimed to write. On this point, I wish to acknowledge my debt to Lindsay Childs, whose *A Concrete Introduction to Higher Algebra* provided me with the model of an algebra text that has integers and polynomials as its focus and yet is full of beautiful mathematics suitable for an undergraduate course.

The book contains more material than I can cover in our two-quarter course. The best of mathematics majors might be able to move quickly, but they may also be better served by a more traditional groups–rings–fields-style undergraduate algebra course. I do not aim to move quickly, and moreover, since I intend for much of the learning to be done by the students without lecture, I must allow them extra time. Let me describe what I envision as the core of the course, and what I might choose to cover myself within the constraints of two quarters.

In the first quarter, I would do all of Chapters 1 to 7. This is essentially a self-contained course on integers, induction, and some topics in elementary number theory, culminating in a description of RSA encryption. I would not have time for Chapter 8, and would leave this as recommended reading. I introduce the inclusion–exclusion principle in Chapter 7 and use it to provide a means of calculating the Euler phi function. The treatment of binomial coefficients in Chapter 8 allows us to give a proof of inclusion–exclusion, but the ideas are not used again later in the book, and this topic can be safely omitted. In a two-semester course, I would happily include Chapter 8 in the first semester.

The core of the second quarter is Chapters 9 through 16, with Chapter 10 done lightly and with Chapter 17 left as supplementary reading. Chapter 10 contains an extended treatment of cubic and quartic polynomials with real coefficients. This represents a bit of a detour from the main flow of the book. I would certainly cover Sections 10.1 and 10.2, since this allows the student to put the quadratic formula in a broader context. For the same reason, I might wish to cover Sections 10.4 and 10.7 as well. In a one-year course, I would cover the entire chapter. The analysis of cubic and quartic polynomials is one of the great stories in the history of algebra, and represents the most important mathematics done in the sixteenth century. Every student of algebra should study this, especially future secondary teachers. Indeed, I have twice taught a course for preservice and in-service teachers on this material, and I have incorporated some of my notes for that course in Chapter 10. But, again, the chapter in its entirety takes us away from the flow of the book, since the main goal of Part II is for the student to see polynomials with coefficients in a fixed field as a natural analogue of integers. By Chapter 12, this idea should be apparent. Part II concludes, in Chapter 14, with the construction of the polynomial analogue of the ring of integers modulo a prime number, and this construction allows us to obtain the roots of a nonconstant polynomial f with coefficients in a field F through the construction of an extension field K of F that contains those roots. In Chapter 15, our work with rings of integers and rings of polynomials is placed in a broader and more abstract setting, applicable to other rings, and in Chapter 16, we focus on one such ring, the ring of Gaussian integers. Chapter 17 treats topics that have been considered throughout the book, but in less than the usual detail and without complete proofs. In a one-year course, I would include this material, and might even wish to say more about the missing proofs.

I do not have any experience with paring the material down to an amount suitable for a one-semester course. If I were to do so, I might cover Chapters 1 through 6, the first four sections of Chapter 9, the first two sections of Chapter 10, and Chapter 12.

Acknowledgments

I learned algebra from Michael Artin, my Ph.D. advisor, a great educator and great mathematician. In his undergraduate algebra course at M.I.T. and the book he lovingly developed from the course, he has introduced thousands of students to the beauty of algebra.

The Department of Mathematics at the University of Washington has provided me with a wonderful home for mathematics and the teaching of mathematics. I am grateful to many colleagues. I will mention just a few. I had the great privilege to work with and learn from Robert B. Warfield, Jr., in the years before his untimely death. He did everything well and with passion: research, teaching, department and university service, and above all, his role as husband and father. From him I learned that it is possible to teach all one's courses, from business algebra for five hundred to graduate algebra for five, with an eye toward active engagement of the students. My colleague Steve Monk offered our algebra course for teaching majors for many years before I did, setting the example I took as my starting point. He used a standard undergraduate algebra text, but he emphasized student understanding of what

it means to do mathematics. David Collingwood, James Morrow, and John Sullivan have inspired me as well, with their passion for teaching and for students. I have been fortunate to have colleagues willing to use portions of my notes in their own teaching: Ed Curtis, Julianne Harris, John Sullivan, Monty McGovern, and John Palmieri. I have also been fortunate to work with three extraordinary teaching assistants: Adam Nyman, Rebekah Hahn, and Joan Lind. And I have been the continuing beneficiary of Brooke Miller's wise counsel on students and much else.

As I have taught my students, I have learned from them how to teach mathematics and how to write mathematics. I am enormously in their debt. Among the many students who have inspired me are Christine Fraser, Melissa Johnson, Joy Scott, Leonard Barnett, Maria Cason, Leslie Chen, Cutts Peaslee, Lisa Behmer, and of course, Susan Sturms, my friend and constant prod.

That my course notes, idiosyncratic as they were, might be publishable as a book is an idea for which I must thank Ina Lindemann, of Springer-Verlag. Her appreciation of my vision, her encouragement, and her patience made this book a reality. But her greatest gift to me was David Kramer. Two years ago, shortly after she and I agreed to go ahead with publication of this book, I assumed administrative duties that took me away from teaching and from the book. I had hoped to give it a line-by-line reading in order to improve it both locally and globally and to add an index. However, this work was not getting done, and moreover, it might have been better done with the help of another reader. A year ago, I proposed to Ina that we might find such a reader. Ina granted my wish, and David appeared on my (electronic) doorstep. He is the reader of my dreams. We have collaborated off and on over the past eight months, finding no mathematical, grammatical, or stylistic detail too small for lively and stimulating debate. We have had fun, and we have made a better book. I could not list the ways, large and small, in which David has improved it, from sharpening an argument here to adding a problem there, but let me note at least that his amplification of certain passages has brought into the text his special warmth, wit, and humor. Working with David has been an unexpected and joyous privilege.

My first teachers of mathematics, and of much more, were my parents. They teach me still, about how to live a mentally and emotionally vibrant life well into one's ninth decade. I have also benefited from decades of support given to me by Annie McNab and my siblings, Jeff and Gail.

From my wife, Gail, I have learned much of what I know about grace, generosity, and love. She is a gifted teacher. If only I were a better student. Jessica and Joel have been the longest-suffering of my many students. Joel, you can't tease me anymore about never being able to finish a book. Here it is.

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