2 The Scalar Product

The physical quantities force and path are oriented quantities and are represented by the vectors **F** and **s**. The *mechanical work* W performed by a force **F** along a straight path **s** is

$$W = Fs\cos\varphi = |\mathbf{F}|\,|\mathbf{s}|\cos\varphi,$$

where φ is the angle enclosed by **F** and **s**. W by itself, although originating from two vectors, is a *scalar* quantity. With a view on physical applications of this kind, we therefore define:

The *scalar product* $\mathbf{a}\cdot\mathbf{b}$ of two vectors is understood as

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|\cdot\cos\varphi,$$

where φ is the angle enclosed by **a** and **b**. $\mathbf{a}\cdot\mathbf{b}$ is a *real number*. Expressed by words, the scalar product is defined as follows: $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|$ multiplied by the projection of **b** onto **a**, or vice versa.

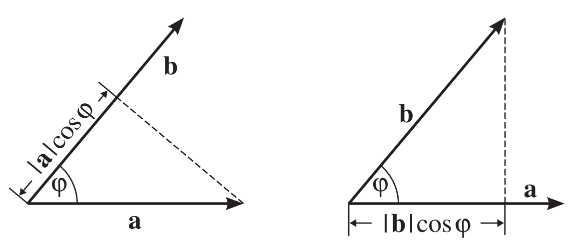


Illustration of the scalar product.

The visual meaning of the scalar product:

magnitude of the projection of **b** onto **a** multiplied by $|\mathbf{a}|$, or

magnitude of the projection of **a** onto **b** multiplied by $|\mathbf{b}|$.

Properties of the scalar product: $\mathbf{a} \cdot \mathbf{b}$ takes its maximum value for φ equal to zero ($\cos 0 = 1$, $\mathbf{a}$ parallel to $\mathbf{b}$)

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}|.$$

For $\varphi = \pi$ the scalar product takes its minimum value ($\cos \pi = -1$, $\mathbf{a}$ antiparallel to $\mathbf{b}$), namely

$$\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}| \cdot |\mathbf{b}|.$$

For $\varphi = \pi/2$, $\mathbf{a} \cdot \mathbf{b} = 0$ holds, even if $\mathbf{a}$ and $\mathbf{b}$ are nonzero ($\cos \pi/2 = 0$, $\mathbf{a}$ perpendicular to $\mathbf{b}$); thus

$$\mathbf{a} \cdot \mathbf{b} = 0 \qquad \text{if} \qquad \mathbf{a} \perp \mathbf{b}.$$

Rules of calculation: The following are true:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} && \text{(commutativity);} \\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} && \text{(distributivity);} \\ p(\mathbf{b} \cdot \mathbf{c}) &= (p\,\mathbf{b}) \cdot \mathbf{c} && \text{(associativity).} \end{aligned}$$

The first and last rules are immediately intelligible; the second rule is illustrated in the figure below.

If $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}$ are not coplanar, the rule of distributivity may easily be visualized by a triangle located in space. The vector $\mathbf{a}$ may easily be visualized by a pencil or a pointing rod (compare the figures!).

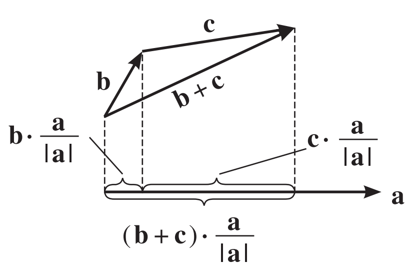


Illustration of the distributivity law.

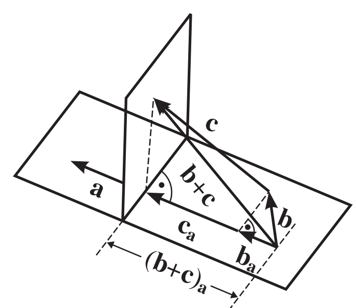


Illustration of the distributivity law in space.

Unit vectors: Unit vectors are understood as vectors of magnitude 1. If $\mathbf{a} \neq \mathbf{0}$, then

$$\mathbf{e} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

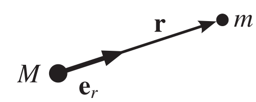
is a unit vector pointing along $\mathbf{a}$. Actually, the magnitude of $\mathbf{e}$ equals 1 since $|\mathbf{e}| = \big|\mathbf{a}/|\mathbf{a}|\big| = |\mathbf{a}|/|\mathbf{a}| = 1$. A possibility frequently used in physics is to assign a direction to a scalarly formulated equation by the unit vector. For example, the gravitational force has the magnitude

$$F = \gamma \frac{mM}{r^2}.$$

It is acting along the connecting line between the two masses M and m, hence

$$\mathbf{F} = -\gamma \frac{mM}{r^2} \frac{\mathbf{r}}{|\mathbf{r}|}.$$

$\mathbf{F}$ is the force applied by the mass M to the mass m. Its direction is given by $-\mathbf{e}_r = -\mathbf{r}/|\mathbf{r}|$. Hence it is acting toward the mass M.



The unit vector pointing from the big mass to the small mass is $\mathbf{e}_r = \mathbf{r}/|\mathbf{r}|$.

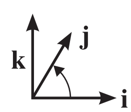
Cartesian unit vectors: The unit vectors pointing along the positive x-, y-, and z-axes of a Cartesian coordinate frame are defined as follows:

$\mathbf{e}_1$ (in x-direction) or also $\mathbf{i}$;

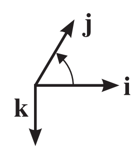
$\mathbf{e}_2$ (in y-direction) or also $\mathbf{j}$;

$\mathbf{e}_3$ (in z-direction) or also $\mathbf{k}$.

There exist two kinds of Cartesian coordinate frames, namely right-handed frames and left-handed frames (compare the figures below).



right-handed system: $\mathbf{k}$ points into the direction of a right-handed screw when $\mathbf{i} \mapsto \mathbf{j}$ is rotated along the shortest possible way.



left-handed system: $\mathbf{k}$ points into the direction of a left-handed screw when $\mathbf{i} \mapsto \mathbf{j}$ is rotated along the shortest possible way.

We shall always use only right-handed frames in these lectures!

Orthonormality relations: $\mathbf{i}, \mathbf{j}, \mathbf{k}$ or $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ will be used in the following always concurrently, depending on convenience.

We now consider the properties of the Cartesian unit vectors with respect to formation of scalar products: Since the enclosed angle is each a right one, the following relations hold:

$$\begin{aligned} &\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \quad \text{(because of } \varphi = 0\text{, hence } \cos 0 = 1\text{);} \\ &\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0 \quad \text{(because of } \varphi = \pi/2\text{, hence } \cos \pi/2 = 0\text{).} \end{aligned} \tag{2.1}$$

These relations are combined by defining

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu}, \qquad \text{where} \quad \delta_{\mu\nu} = \begin{cases} 0 & \text{for } \nu \neq \mu, \\ 1 & \text{for } \nu = \mu, \end{cases}$$

and is called the *Kronecker symbol*.[1] For the three-dimensional space, μ and ν are running from 1 to 3, $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, $\mathbf{e}_3 = \mathbf{k}$.

[1] *Leopold Kronecker*, b. Dec. 7, 1823, Liegnitz (Legnica)—d. Dec. 29, 1891, Berlin. Kronecker was a rich private person who moved to Berlin in 1855. He taught for many years at the university there, without having a chair. Only in 1883, after retirement of his teacher and friend *Kummer*, he took a professorship. His most important publications concern arithmetics, theory of ideals, number theory, and elliptic functions. Kronecker was the leading representative of the *Berlin School*, which claimed the necessity of arithmetization of the entire mathematics.

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