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Glossary of Symbols

A.P.	arithmetic progression, $a, a + d, \dots a + kd, \dots$	A5, A6, E10, E33
$a_1 \equiv a_2 \pmod{b}$	a_1 congruent to a_2 , modulo b ; $a_1 - a_2$ divisible by b .	A3, A4, A12, A15, B2, B4, B7, ...
$A(x)$	number of members of a sequence not exceeding x ; e.g. number of amicable numbers not exceeding x	B4, E1, E2, E4
c	a positive constant (not always the same!)	A1, A3, A8, A12, B4, B11, ...
d_n	difference between consecutive primes; $p_{n+1} - p_n$	A8, A10, A11
$d(n)$	the number of (positive) divisors of n ; $\sigma_0(n)$	B, B2, B8, B12, B18, ...
$d n$	d divides n ; n is a multiple of d ; there is an integer q such that $dq = n$	B, B17, B32, B37, B44, C20, D2, E16
$d \nmid n$	d does not divide n	B, B2, B25, D2, E14, E16, ...
e	base of natural logarithms; 2.718281828459045...	A8, B22, B39, D12, ...
E_n	Euler numbers; coefficients in series for $\sec x$	B45
$\exp\{..\}$	exponential function	A12, A19, B4, B36, B39, ...
F_n	Fermat numbers; $2^{2^n} + 1$	A3, A12



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