

B. Divisibility

We will denote by $d(n)$ the number of positive divisors of n , by $\sigma(n)$ the sum of those divisors, and by $\sigma_k(n)$ the sum of their k th powers, so that $\sigma_0(n) = d(n)$ and $\sigma_1(n) = \sigma(n)$. We use $s(n)$ for the sum of the **aliquot parts** of n , i.e., the positive divisors of n other than n itself, so that $s(n) = \sigma(n) - n$. The number of distinct prime factors of n will be denoted by $\omega(n)$ and the total number, counting repetitions, by $\Omega(n)$.

Iteration of various arithmetic functions will be denoted, for example, by $s^k(n)$, which is defined by $s^0(n) = n$ and $s^{k+1}(n) = s(s^k(n))$ for $k \geq 0$.

We use the notation $d \mid n$ to mean that d divides n , and $e \nmid n$ to mean that e does not divide n . The notation $p^k \parallel n$ is used to imply that $p^k \mid n$ but $p^{k+1} \nmid n$. By $[m, n]$ we will mean the consecutive integers $m, m+1, \dots, n$.

B1 Perfect numbers.

A **perfect number** is such that $n = s(n)$. Euclid knew that $2^{p-1}(2^p - 1)$ was perfect if $2^p - 1$ is prime. For example, 6, 28, 496, ... ; see the list of Mersenne primes in **A3**. Euler showed that these were the only even perfect numbers.

The existence or otherwise of odd perfect numbers is one of the more notorious unsolved problems of number theory. Euler showed that they have shape $p^\alpha m^2$ where p is prime and $p \equiv \alpha \equiv 1 \pmod{4}$. Touchard showed that they were of shape $12m+1$ or $36m+9$.

The lower bound for an odd perfect number has now been pushed to 10^{300} by Brent, Cohen & te Riele. Brandstein has shown that the largest prime factor is > 500000 and Hagis that the second largest is > 1000 . Cohen has shown that it contains a component (prime power divisor) $> 10^{20}$, and Sayers that there are at least 29 prime factors (not necessarily distinct). Iannucci & Sorli improve this to $\Omega(n) \geq 37$.

Pomerance has shown that an odd perfect number with at most k distinct factors is less than

$$(4k)^{(4k)^{2^{k^2}}}$$

but Heath-Brown has much improved this by showing that if n is an odd

number with $\sigma(n) = an$, then $n < (4d)^{4^k}$, where d is the denominator of a and k is the number of distinct prime factors of n . In particular, if n_k is an odd perfect number with k distinct prime factors, then $n < 4^{4^k}$. Roger Cook further improves this to $n_k < C^{4^k}$ where $C = 3^{512/511} \approx 3.006$ and, for $k = 8$, to $n_8 < D^{4^8}$ with $D = 2^{16/15} \approx 2.094$ if $195 \mid n_8$ or $D = 195^{1/7} \approx 2.123$ otherwise. A recent claim by Nielsen is that $n_k < 2^{4^k}$.

Starni has shown that if $n = p^\alpha M^2$ is an odd perfect number with p prime, $p \perp M$, $p \equiv \alpha \equiv 1 \pmod{4}$, $\alpha + 2$ prime, $(\alpha + 2) \perp (p - 1)$, then $M^2 \equiv 0 \pmod{\alpha + 2}$.

Iannucci has shown that the second largest prime divisor of an odd perfect number exceeds 10^4 , that the third largest exceeds 10^2 , and (with Sorli) that there are at least 37 not necessarily distinct prime factors.

John Leech asks for examples of spoof odd perfect numbers, like Descartes's

$$3^2 7^2 11^2 13^2 22021$$

which is perfect if you pretend that 22021 is prime.

Greg Martin offers the following 'proof' that 4680 is perfect. Write 4680 as $2^3 \cdot 3^2 \cdot (-5) \cdot (-13)$. Then $\sigma(4680) =$

$$(1 + 2 + 2^2 + 2^3)(1 + 3 + 3^2)(1 + (-5))(-13) = 9360 = 2 \cdot 4680.$$

He asks: if you allow $\sigma(-p^n) = \sum_{j=0}^n (-p)^j$, are there others? Dennis Eichhorn and Peter Montgomery found $-84 = 2^2(3)(-7)$ and $-120 = 2^3(3)(-5)$ and noted that $\sigma((-2)^5(3)(7)) = (-2)^5(3)(7)$. Martin also defines $\tilde{\sigma}(p^r) = p^r - p^{r-1} + p^{r-2} - \cdots + (-1)^r$, and asks about $\tilde{\sigma}(k)$ -perfect numbers. For $k = 2$ there are 2, 12, 40, 252, 880, 10880, 75852. For $k = 3$ there are at least 40, including 30240 and $2^{10} 3^4 5^4 11 \cdot 13^2 \cdot 31 \cdot 61 \cdot 157 \cdot 521 \cdot 683$.

Are there $\tilde{\sigma}$ - k -perfect numbers with $k \geq 4$?

Are there infinitely many $\tilde{\sigma}$ - k -perfect numbers?

Are there any odd $\tilde{\sigma}$ -3-perfect numbers? Such a number must be a square.

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B2 Almost perfect, quasi-perfect, pseudoperfect, harmonic, weird, multiperfect and hyperperfect numbers.

Perhaps because they were frustrated by their failure to disprove the existence of odd perfect numbers, numerous authors have defined a number of closely related concepts and produced a raft of problems, many of which seem no more tractable than the original.

For a perfect number, $\sigma(n) = 2n$. If $\sigma(n) < 2n$, n is called **deficient**. A problem in *Abacus* was to prove sum of divisors that every number $n > 3$ is the sum of two deficient numbers, or to find a number that was not. If $\sigma(n) > 2n$, then n is called **abundant**. If $\sigma(n) = 2n - 1$, n has been called **almost perfect**. Powers of 2 are almost perfect; it is not known if any other numbers are. If $\sigma(n) = 2n + 1$, n has been called **quasi-perfect**. Quasi-perfect numbers must be odd squares, but no one knows if there are any. Masao Kishore shows that $n > 10^{30}$ and that $\omega(n) \geq 6$. Hagis & Cohen have improved these results to $n > 10^{35}$ and $\omega(n) \geq 7$. Cattaneo originally claimed to have proved that $3 \nmid n$, but Sierpiński and others have observed that his proof is fallacious. Kravitz, in a letter, makes a more general conjecture, that there are no numbers whose **abundance**, $\sigma(n) - 2n$, is an odd square. In this connexion Graeme Cohen writes that it is interesting that

$$\sigma(2^2 3^2 5^2) = 3(2^2 3^2 5^2) + 11^2$$

and that if $\sigma(n) = 2n + k^2$ with $n \perp k$, then $\omega(n) \geq 4$ and $n > 10^{20}$. He has also shown that if $k < 10^{10}$ then $\omega(n) \geq 6$, and that if $k < 44366047$ then n is primitive abundant (see below). Later, relaxing the condition $n \perp k$, he finds the solution

$$n = 2 \cdot 3^2 \cdot 238897^2, \quad k = 3^2 \cdot 23 \cdot 1999$$

and five solutions $n = 2^2 \cdot 7^2 \cdot p^2$, with

$$\begin{array}{cccccc} p = & 53 & 277 & 541 & 153941 & 358276277 \\ k = & 7 \cdot 29 & 5 \cdot 7 \cdot 23 & 5 \cdot 7 \cdot 43 & 5 \cdot 7 \cdot 103 \cdot 113 & 5 \cdot 7 \cdot 227 \cdot 229 \cdot 521 \end{array}$$

He verifies that the first of these last five is the smallest integer with odd square abundance. Sidney Kravitz has since sent two more solutions,

$$n = 2^3 \cdot 3^2 \cdot 1657^2, \quad k = 3 \cdot 11 \cdot 359,$$

$$n = 2^4 \cdot 31^2 \cdot 7992220179128893^2, \quad k = 44498798693247589.$$

In the latter, 31 divides k . Erdős asks for a characterization of the large numbers for which $|\sigma(n) - 2n| < C$ for some constant C . For example, $n = 2^m$: for other infinite families, see Mąkowski's two papers.

Wall, Crews & Johnson showed that the density of abundant numbers lies between 0.2441 and 0.2909. In an 83-08-17 letter Wall claimed to have narrowed these bounds to 0.24750 and 0.24893. Erdős asks if the density is irrational.

Paul Zimmerman reports that Marc Deléglise has improved the bounds for the density of abundant numbers to 0.2477 ± 0.0003 .

<http://www.mathsoft.com/asolve/constant/abund/abund.html>

Sándor has shown that, for sufficiently large n , there is a deficient number between n and $n + (\ln n)^2$.

Sierpiński called a number **pseudoperfect** if it was the sum of *some* of its divisors; e.g., $20 = 1 + 4 + 5 + 10$. Erdős has shown that their density exists and says that presumably there are integers n which are not pseudoperfect, but for which $n = ab$ with a abundant and b having many prime factors: can b in fact have many factors $< a$?

For $n \geq 3$ Abbott lets $l = l(n)$ be the least integer for which there are n integers $1 \leq a_1 < a_2 < \dots < a_n = l$ such that $a_i | s = \sum a_i$ for each i (so that s is pseudoperfect). He can show that $l(n) > n^{c_1 \ln \ln n}$ for some $c_1 > 0$ and all $n \geq 3$ and that $l(n) < n^{c_2 \ln \ln n}$ for some $c_2 > 0$ and infinitely many n .

Call a number **primitive abundant** if it is abundant, but all its proper divisors are deficient, and **primitive pseudoperfect** if it is pseudoperfect, but none of its proper divisors are. If the harmonic mean of all the divisors of n is an integer, Pomerance called n a **harmonic number**. A. & E. Zachariou call these “Ore numbers” and they call primitive pseudoperfect numbers “irreducible semiperfect”. They note that every multiple of a pseudoperfect number is pseudoperfect and that the pseudoperfect numbers and the harmonic numbers both include the perfect numbers as a proper subset. The last result is due to Ore. All numbers $2^m p$ with $m \geq 1$ and p a prime between 2^m and 2^{m+1} are primitive pseudoperfect, but there are such numbers not of this form, e.g., 770. There are infinitely many primitive pseudoperfect numbers that are not harmonic numbers. The smallest odd primitive pseudoperfect number is 945. Erdős can show that the number of odd primitive pseudoperfect numbers is infinite. He showed that, for sufficiently large n , the number of primitive abundant numbers less than n was bounded above and below by functions of the form $n \exp(-c\sqrt{\ln n \ln \ln n})$. Ivić showed that the constants could be taken to be $12^{-\frac{1}{2}} - \epsilon$ and $6^{\frac{1}{2}} + \epsilon$ and Avidon improved these to $1 - \epsilon$ and $2^{\frac{1}{2}} + \epsilon$.

García extended the list of harmonic numbers to include all 45 which are $< 10^7$, and he found more than 200 larger ones. The least one, apart

from 1 and the perfect numbers, is 140. Are any of them squares, apart from 1? Are there infinitely many of them? If so, find upper and lower bounds on the number of them that are $< x$. Kanold has shown that their density is zero, and Pomerance that a harmonic number of the form $p^a q^b$ (p and q primes) is an even perfect number. If $n = p^a q^b r^c$ is harmonic, is it even?

Which values does the harmonic mean take? Presumably not 4, 12, 16, 18, 20, 22, ...; does it take the value 23? Ore's own conjecture, that every harmonic number is even, implies that there are no odd perfect numbers!

Cohen has listed all 52 harmonic numbers of the form $2^a m$, where m is odd and squarefree and $1 \leq a \leq 11$; 45 of them have $a = 8$. He also shows that there are just 13 harmonic numbers n with $H(n) = nd(n)/\sigma(n) \leq 13$:

$$\begin{array}{cccccccccccccccc} H(n) = & 1 & 2 & 3 & 5 & 6 & 5 & 8 & 9 & 11 & 10 & 7 & 13 & 13 \\ n = & 1 & 6 & 28 & 140 & 270 & 496 & 672 & 1638 & 2970 & 6200 & 8128 & 105664 & 2^{12}8191 \end{array}$$

Bateman, Erdős, Pomerance & Straus show that the set of n for which $\sigma(n)/d(n)$ is an integer has density 1, that the set for which $\sigma(n)/d(n)^2$ is an integer has density $\frac{1}{2}$, and that the number of rationals $r \leq x$ of the form $\sigma(n)/d(n)$ is $o(x)$. They ask for an asymptotic formula for

$$\frac{1}{x} \sum 1$$

where the sum is taken over those $n \leq x$ for which $d(n)$ does *not* divide $\sigma(n)$. They also note that the integers n for which $d(n)$ divides $s(n) = \sigma(n) - n$, have zero density, because for almost all n , $d(n)$ and $\sigma(n)$ are divisible by a high power of 2, while n is divisible only by a low power of 2.

David Wilson, in a 98-08-27 email, conjectured that $\sigma(n) \neq kn+5$. Dan Hoey lists the following values of n for which $\sigma(n) = kn+r$ and suggests that their numbers are finite when r is odd.

r	Values of n for which $\sigma(n) \equiv r \pmod{n}$
2	20, 104, 464, 650, 1952, 130304, 522752
3	4, 18
4	9, 12, 70, 88, 1888, 4030, 5830, 32128, 521728, 1848964
6	25, 180, 8925
7	8, 196
8	10, 49, 56, 368, 836, 11096, 17816, 45356, 77744, 91388, 128768, 254012, 388076, 2087936, 2291936
9	15

Benkoski has called a number **weird** if it is abundant but not pseudo-perfect. For example, 70 is not the sum of any subset of

$$1 + 2 + 5 + 7 + 10 + 14 + 35 = 74$$

There are 24 primitive weird numbers less than a million: 70, 836, 4030, 5830, 7192, Nonprimitive weird numbers include $70p$ with p prime and

$p > \sigma(70) = 144$; $836p$ with $p = 421, 487, 491$, or p prime and ≥ 557 ; also $7192 \cdot 31$. Some large weird numbers were found by Kravitz, and Benkoski & Erdős showed that their density is positive. Here the open questions are: are there infinitely many primitive abundant numbers which are weird? Is every odd abundant number pseudoperfect (i.e., not weird)? Can $\sigma(n)/n$ be arbitrarily large for weird n ? Benkoski & Erdős conjecture “no” in answer to the last question and Erdős offers \$10 and \$25 respectively for solutions to the last two questions.

He also asks if there are extra-weird numbers n for which $\sigma(n) > 3n$, but n is not the sum of distinct divisors of n in two ways without repetitions. For example, 180 does not qualify, because although $\sigma(180) = 546$, $180 = 30 + 60 + 90$ and is the sum of all its other divisors except 6.

One definition of a **practical** number, m , is for every n , $1 \leq n \leq \sigma(m)$ to be expressible as the sum of distinct divisors of m . E.g., 6 is practical, since $4 = 1 + 3$, $5 = 2 + 3$, $7 = 1 + 6$, $8 = 2 + 6$, $9 = 3 + 6$, $10 = 1 + 3 + 6$, $11 = 2 + 3 + 6$, $12 = 1 + 2 + 3 + 6$. Erdős showed in 1950 that the practical numbers have zero asymptotic density. It is known that if $P(x)$ is the number of practical numbers less than x , then

$$x \exp(-\alpha(\ln \ln x)^2) \ll P(x) \ll x/(\ln x)^\beta$$

for some positive constants α and β . The lower bound is due to Margenstern and the upper bound to Hausman & Shapiro. Melfi has shown that every even number is the sum of two practical numbers and that there are infinitely many practical numbers m such that $m \pm 2$ are also practical.

Numbers have been called **multiply perfect**, **multiperfect** or **k -fold perfect** if $\sigma(n) = kn$ with k an integer. For example, ordinary perfect numbers are 2-fold perfect and 120 is 3-fold perfect. Dickson's *History* records a long interest in such numbers. Lehmer has remarked that if n is odd, then n is perfect just if $2n$ is triperfect.

Selfridge and others have observed that there are just six known 3-perfect numbers and they come from $2^h - 1$ for $h = 4, 6, 9, 10, 14, 15$. For example, the third one is illustrated by

$$\sigma(2^8 \cdot 7 \cdot 73 \cdot 37 \cdot 19 \cdot 5) = (2^9 - 1)(2^3)(37 \cdot 2)(19 \cdot 2)(5 \cdot 2^2)(2 \cdot 3).$$

It appears that there may be a similar explanation for the 36 known 4-perfect numbers, the last of which was published by Poulet as long ago as 1929.

For many years the largest known value of k was 8, for which Alan L. Brown gave three examples and Franqui & García two others.

In late 1992 and early 1993, half a dozen examples with $k = 9$ had already been found by Fred Helenius. The smallest is

$$2^{114} \cdot 3^{35} \cdot 5^{17} \cdot 7^{12} \cdot 11^4 \cdot 13^5 \cdot 17^3 \cdot 19^8 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37^4 \cdot 41 \cdot 43 \cdot 47^2 \cdot 53 \cdot 61^2 \cdot 67 \cdot 71 \cdot 73 \cdot 79^2 \cdot 83^2 \cdot 89^2 \cdot 97 \cdot 103 \cdot 109 \cdot 127 \cdot 131^2 \cdot 151 \cdot 157 \cdot 167 \cdot 179^2.$$

$\cdot 197 \cdot 211 \cdot 227 \cdot 331 \cdot 347 \cdot 367 \cdot 379 \cdot 443 \cdot 523 \cdot 599 \cdot 709 \cdot 757 \cdot 829 \cdot 1151 \cdot 1699 \cdot$
 $\cdot 1789 \cdot 2003 \cdot 2179 \cdot 2999 \cdot 3221 \cdot 4271 \cdot 4357 \cdot 4603 \cdot 5167 \cdot 8011 \cdot 8647 \cdot 8713 \cdot$
 $\cdot 14951 \cdot 17293 \cdot 21467 \cdot 29989 \cdot 110563 \cdot 178481 \cdot 530713 \cdot 672827 \cdot 4036961 \cdot$
 $\cdot 218834597 \cdot 16148168401 \cdot 151871210317 \cdot 2646507710984041$

On 97-05-13 Ron Sorli found a 10-fold perfect number, and George Woltman found an 11-fold one on 2001-03-13.

In 1992 we knew of 700 k -perfect numbers with $k \geq 3$. In January, 1993, this number leapt to about 1150 from the discoveries of Fred Helenius which included 114 7-perfect, 327 8-perfect and two 9-perfect numbers. He continued to find dozens of new ones each month, so that it is even less possible to keep this section of the book up-to-date than it is elsewhere; in March 1993 the total neared 1300; a postscript of a 93-09-08 letter from Schroepel gave 1526; by the time he mailed it next day it was 1605. At the end of 2002, there were 8! known multiperfect numbers. These include 1 (for $k = 1$) and 39 Mersenne primes ($k = 2$). The numbers of others are

$k =$	3	4	5	6	7	8	9	10	11	Total
	6	36	65	245	516	1134	2074	923	1	5000

Can k be as large as we wish? Erdős conjectured that $k = o(\ln \ln n)$. It has even been suggested that there may be only finitely many k -perfect numbers for each $k \geq 3$. The first five numbers in the above table may well be complete.

If n is an odd triperfect number, then McDaniel, Cohen, Kishore, Bugulov, Kishore, Cohen & Hagis, Reidlinger, and Kishore have respectively shown that $\omega(n) \geq 9, 9, 10, 11, 11, 11, 12$, and 12. Beck & Najjar, Alexander, and Cohen & Hagis have shown that $n > 10^{50}, 10^{60}, 10^{70}$. Cohen & Hagis have shown that the largest prime factor of n is at least 100129 and that the second largest is at least 1009.

Shigeru Nakamura writes that Bugulov showed, in 1966, that odd k -perfect numbers contain at least ω distinct prime factors, where $(k, \omega) = (3, 11), (4, 21), (5, 54)$ [incorrectly stated in *MR* **37** #5139 & *rNT* A32-96]. Nakamura claims to prove that for an even k -perfect number,

$$\omega > \max\{k^3/81 + \frac{5}{3}, k^5/2500 + 2.9, k^{10}/(14 \cdot 10^8) + 2.9999\}$$

and for an odd k -perfect number,

$$\omega > \max\{k^5/60 + \frac{47}{12}, k^5/50 - 20.8, 737k^{10}/10^9 + 11.5\}.$$

These improve the results of Cohen & Hendy and of Reidlinger; he also gives the improvements $(k, \omega) = (4, 23), (5, 56), (6, 142), (7, 373)$ to those of Bugulov.

Minoli & Bear say that n is **k -hyperperfect** if $n = 1 + k \sum d_i$, where the summation is taken over all proper divisors, $1 < d_i < n$, so that $k\sigma(n) = (k+1)n + k - 1$. For example, 21, 2133 and 19521 are 2-hyperperfect

and 325 is 3-hyperperfect. They conjecture that there are k -hyperperfect numbers for every k .

Cohen & de Riele call numbers (m, k) -**perfect** if $\sigma^m(n) = kn$; e.g., perfect numbers are $(1, 2)$ -perfect, multiperfect numbers are $(1, k)$ -perfect; $(2, 2)$ -perfect numbers have been called superperfect and $(2, k)$ -perfect numbers multiply superperfect. They tabulate all (m, k) -perfect numbers n for $(m, n) = (2, < 10^9)$, $(3, < 2 \cdot 10^8)$, $(4, < 10^8)$ and prove that the equation $\sigma^2(2n) = 2\sigma^2(n)$ has infinitely many solutions. They ask: for any fixed m , are there infinitely many (m, k) -perfect numbers? and: is every n (m, k) -perfect for some m ? For $n \in [1, 400]$ they list the least such m .

Eswarathasan & Levine define $a(n)/b(n) = \sum_i 1^n 1/i$ with $a(n) \perp b(n)$ and $a(0) = 0$, and for p prime consider the sets $J(p) = \{n \geq 0 : p \mid a(n)\}$ and $I(p) = \{n \geq 0 : p \text{ does not divide } b(n)\}$. Then $J(2) = \{0\}$, $J(3) = \{0, 2, 7, 22\}$ and, for $p \geq 5$, $J(p) \supseteq 0, p-1, p^2-p, p^2-1$; they give necessary and sufficient conditions for equality and show that the primes less than 200 which satisfy these conditions are 5, 13, 17, 23, 41, 67, 73, 79, 107, 113, 139, 149, 157, 179, 191, 193. [$I(5) = \{1, 2, 3, 4, \dots\}$ was the subject of Putnam problem 1997 B3.] It is conjectured that there are infinitely many such primes. In contrast

$$J(7) = \{0, 6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16735, 102728\},$$

but it is also conjectured that $J(p)$ is always finite.

Ron Graham asks if $s(n) = \lfloor n/2 \rfloor$ implies that n is 2 or a power of 3. Luo Shi-Le & Le Mao-Hua have given a partial answer.

Erdős lets $f(n)$ be the smallest integer for which $n = \sum_{i=1}^k d_i$ for some k , where $1 = d_1 < d_2 < \dots$. $d_i = f(n)$ is the increasing sequence of divisors of $f(n)$. Is $f(n) = o(n)$? Or is this true only for almost all n , with $\limsup f(n)/n = \infty$?

n 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28

$f(n)$ 1 - 2 3 - 5 4 7 15 12 21 6 9 13 8 12 30 10 42 19 18 20 57 14 36 46 30 12

Erdős defined n_k to be the smallest integer for which if you partition the proper divisors of n_k into k classes, n_k will always be the sum of distinct divisors from the same class. Clearly $n_1 = 6$, but he was not even able to prove the existence of n_2 .

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B3 Unitary perfect numbers.

If d divides n and $d \perp n/d$, call d a **unitary divisor** of n . A number n which is the sum of its unitary divisors, apart from n itself, is a **unitary perfect number**. There are no odd unitary perfect numbers, and Subbarao conjectures that there are only a finite number of even ones. He, Carlitz & Erdős each offer \$10.00 for settling this question and Subbarao offers 10¢ for each new example. If $n = 2^a m$, where m is odd and has r distinct prime factors, then Subbarao and others have shown that, apart from $2 \cdot 3$, $2^2 \cdot 3 \cdot 5$, $2 \cdot 3^2 \cdot 5$ and $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$, there are no unitary perfect numbers with $a \leq 10$, or with $r \leq 6$. S. W. Graham has shown that the first and third are the only unitary perfect numbers of shape $2^a m$ with m odd and squarefree, and Jennifer DeBoer that the second is the only one of shape $2^a 3^2 m$ with $m \perp 6$ and squarefree.

Wall has found the unitary perfect number

$$2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$$

and shown that it is the fifth such. He can prove that any other unitary perfect number has an odd component greater than 2^{15} . Frey has shown that if $N = 2^m p_1^{a_1} \dots p_r^{a_r}$ is unitary perfect with $N \perp 3$, then $m > 144$, $r > 144$ and $N > 10^{440}$.

Peter Hagis investigates **unitary multiperfect numbers**: there are no odd ones. Write $\sigma^*(n)$ for the sum of the unitary divisors of n . If $\sigma^*(n) = kn$ and n contains t distinct odd prime factors, then $k = 4$ or 6 implies $n > 10^{110}$, $t \geq 51$ and $2^{49} | n$; $k \geq 8$ implies $n > 10^{663}$ and $t \geq 247$; while k odd and $k \geq 5$ imply $n > 10^{461}$, $t \geq 166$ and $2^{166} | n$.

Sitaramaiah & Subbarao call a number **unitary superperfect** if it satisfies the equation $\sigma^*(\sigma^{ast}(n)) = 2n$. They find 22 such numbers below 10^8 .

Cohen calls a divisor d of an integer n a **1-ary divisor** of n if $d \perp n/d$, and he calls d a **k -ary divisor** of n (for $k > 1$), and writes $d|_k n$, if the greatest common $(k-1)$ -ary divisor of d and n/d is 1 (written $(d, n/d)_{k-1} = 1$). In this notation $d|n$ and $d \parallel n$ are written $d|_0 n$ and $d|_1 n$. He also calls p^x an **infinitary divisor** of p^y ($y > 0$) if $p^x |_{y-1} p^y$. This gives rise to infinitary analogs of earlier concepts. Write $\sigma_\infty(n)$ for the sum of the infinitary divisors of n . He found 14 infinitary perfect numbers, i.e., with $\sigma_\infty(n) = kn$ and $k = 2$; 13 numbers with $k = 3$; 7 with $k = 4$; and two with $k = 5$. There are no odd ones, and he conjectures that there are no infinitary multiperfect numbers not divisible by 3.

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B4 Amicable numbers.

Unequal numbers m, n are called **amicable** if each is the sum of the aliquot parts of the other, i.e., $\sigma(m) = \sigma(n) = m + n$. Several thousand (in 2003, ‘million’) such pairs are known. The smaller member, 220, of the smallest pair, occurs in *Genesis*, xxxii, 14, and intrigued the Greeks and Arabs and many others since. For their history see the articles of Lee & Madachy. The *Genesis* reference, from the King James Bible, is achieved by amalgamating 200 females and 20 males. Aviezri Fraenkel writes that in his Pentateuch, they occur at xxxii, 15, and gives the more convincing occurrences of 220 in *Ezra* viii, 20 and in *1 Chronicles* xv, 6; and of 284 in *Nehemiah* xi, 18. He notes that the three places are amicably related: all are connected to the tribe of Levi, whose name derives from the wish of Levi’s mother to be amicably related to his father (*Genesis* xxix, 34).

It is not known if there are infinitely many, but it is believed that there are. In fact Erdős conjectured that the number, $A(x)$, of such pairs with $m < n < x$ is at least $x^{1-\epsilon}$. He improved a result of Kanold to show that $A(x) = o(x)$ and his method can be used to obtain $A(x) \leq cx / \ln \ln \ln x$. Pomerance obtained the further improvement

$$A(x) \leq x \exp\{-c(\ln \ln \ln x \ln \ln \ln x)^{1/2}\}.$$

Erdős conjectured that $A(x) = o(x/(\ln x)^k)$ for every k whereupon Pomerance proved the stronger result

$$A(x) \leq x \exp\{-(\ln x)^{1/3}\}.$$

This implies that the sum of the reciprocals of the amicable numbers is finite, a fact not earlier known. He also notes that his proof can be modified to give the slightly stronger result

$$A(x) \ll x \exp\{-c(\ln x \ln \ln x)^{1/3}\}.$$

Herman te Riele has found all 1427 amicable pairs whose lesser members are less than 10^{10} . He remarks that the quantity $A(x)(\ln x)^3/x^{1/2}$ “remains very close to 174.6”, but I suspect that a much more powerful telescope would require the exponent $1/2$ to be increased much nearer to 1. Moews & Moews have continued the complete search to beyond $2 \cdot 10^{11}$.

Some large amicable pairs, with 32, 40, 81 and 152 decimal digits, discovered by te Riele, are mentioned by Kaplansky under “Mathematics” in the 1975 *Encyclopedia Britannica Yearbook*. The largest previously known had 25 decimal digits. More recently te Riele has constructed, from a “mother” list of 92 known amicable pairs, more than 2000 new pairs of sizes up to 38 decimal digits, and five pairs with from 239 to 282 digits. The largest amicable pair known in mid-1993 has 1041 decimal digits:

$$(2^9 p^{20} q_1 r s t u, 2^9 p^{20} q_2 v)$$

with $p = 5661346302015448219060051$; q_1, q_2 of shape $bc^{20} - 1$ with $b_1 = 5797874220719830725124352$, $b_2 = 5531348900141215019827200$, $c = 5661346302015448219060051$; and $r = 569$, $s = 5039$, $t = 1479911$, $u = 30636732851$; and $v = 136527918704382506064301$. It was found in July 1988 by Holger Wiethaus, a student at Dortmund. On 97-10-04 Mariano Garcia found a pair each of whose members has 4829 digits. This exceeded the pair with 3766 digits found by Frank Zweers the previous August.

Elvin J. Lee has given half a dozen rules for amicable pairs of type $(2^n pq, 2^n rs)$ where p, q, r, s are primes of appropriate shape. E.g.,

$$p = 3 \cdot 2^{n-1} - 1, \quad q = 35 \cdot 2^{n+1} - 29, \quad r = 7 \cdot 2^{n-1} - 1, \quad s = 15 \cdot 2^{n+1} - 13,$$

but the simultaneous discovery of four such primes is a rare event.

Borho, Hoffman & te Riele have made considerable advances, both with proliferation of generalized Thabit rules, and with actual computation. Of the 1427 amicable pairs mentioned above, all but 17 have $m+n \equiv 0 \pmod 9$. The smallest exception is Poulet’s pair

$$2^4 \cdot 331 \cdot \begin{cases} 19 \cdot 6619 \\ 199 \cdot 661 \end{cases}$$

with $m+n \equiv 5 \pmod 9$: te Riele gives the first examples

$$2^4 \cdot \begin{cases} 19^2 \cdot 103 \cdot 1627 \\ 3847 \cdot 16763 \end{cases} \quad \text{and} \quad 2^2 \cdot 19 \cdot \begin{cases} 13^2 \cdot 37 \cdot 43 \cdot 139 \\ 41 \cdot 151 \cdot 6709 \end{cases}$$

with m, n even, $m+n \equiv 3 \pmod 9$.

It is not known if an amicable pair exists with m and n of opposite parity, or with $m \perp n$. Bratley & McKay conjectured that both members of all odd amicable pairs are divisible by 3, but Battiato & Borho produced 15 counterexamples with from 36 to 73 decimal digits. In an 87-05-15 letter te Riele announced a 33-digit specimen (misquoted in UPINT2)

$$5 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19^3 \cdot 23 \cdot 37 \cdot 181 \begin{cases} 101 \cdot 8693 \cdot 19479382229 \\ 365147 \cdot 47307071129 \end{cases}$$

Is this the smallest such pair? Is there an odd amicable pair with one member, but not both, divisible by 3? No pair has been discovered with different smallest prime factors.

Yasutoshi Kohmoto found pairs with each member coprime to 30, for example

$$\begin{aligned}
 &7^2 \cdot 11 \cdot 13^2 \cdot 17^4 \cdot 19^3 \cdot 23 \cdot 29^2 \cdot 31 \cdot 37 \cdot 43 \cdot 59 \cdot 61 \cdot 67 \cdot 83 \cdot 97 \cdot 139 \cdot 173 \times \\
 &181 \cdot 331 \cdot 349 \cdot 577 \times 661 \cdot 1321 \cdot 4349 \cdot 11093 \cdot 41519 \cdot 43973 \cdot 44371 \cdot 88741 \times \\
 &15223567 \cdot 91341401 \cdot 264271333 \cdot 1281651920873 \cdot 47031498888355607 \times \\
 &\quad 41277542598611381429 \cdot 4870750026636143008621 \times \\
 &\text{either} \quad 179 \cdot 1525663971924884048614628301903295821727019 \\
 &\quad \text{or} \quad 274619514946479128750633094342593247910863599
 \end{aligned}$$

An old conjecture of Charles Wall is that odd amicable pairs must be incongruent modulo 4.

On p. 169 of *Mathematical Magic Show*, Vintage Books, 1978, Martin Gardner makes a conjecture about the digital roots of amicable numbers. Lee confirms this in part by showing that if $(2^n pqr, 2^n stu)$ is an amicable pair whose sum is not divisible by 9, then each number is congruent to 7, modulo 9.

Unitary amicable numbers have been studied by Peter Hagis and by Mariano García, who list 82 pairs.

Cohen & de Riele call (a, b) a ϕ -**amicable pair** with multiplier k if $\phi(a) = \phi(b) = (a + b)/k$ for some integer $k \geq 1$, where ϕ is Euler's totient function (see **B36**). They computed all such pairs with larger member $\leq 10^9$ and found 812 pairs whose gcd is squarefree.

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B5 Quasi-amicable or betrothed numbers.

García has called a pair of numbers (m, n) , $m < n$, **quasi-amicable** if

$$\sigma(m) = \sigma(n) = m + n + 1.$$

For example, (48,75), (140,195), (1575,1648), (1050,1925) and (2024,2295). Rufus Isaacs, noting that each of m and n is the sum of the *proper* divisors of the other (i.e., omitting 1 as well as the number itself) has much more appropriately named them **betrothed numbers**.

Mąkowski gave examples of betrothed numbers and also of **amicable triples**

$$\sigma(a) = \sigma(b) = \sigma(c) = a + b + c,$$

e.g., $2^2 3^{25} \cdot 11$, $2^5 3^{27}$, $2^2 3^{271}$. Similarly, in a 92-07-20 letter, Yasutoshi Kohmoto calls the set $\{a, b, c, d\}$ **quadri-amicable** if

$$\sigma(a) = \sigma(b) = \sigma(c) = \sigma(d) = a + b + c + d.$$

As examples which are not multiples of 3 he gives

$$\begin{aligned} a &= x \cdot 173 \cdot 1933058921 \cdot 149 \cdot 103540742849 & b &= x \cdot 173 \cdot 1933058921 \cdot 15531111427499 \\ c &= 336352252427 \cdot 149 \cdot 103540742849 & d &= 336352252427 \cdot 15531111427499 \end{aligned}$$

where x is the product of

$$5^9 \cdot 7^2 \cdot 11^4 \cdot 17^2 \cdot 19 \cdot 29^2 \cdot 67 \cdot 71^2 \cdot 109 \cdot 131 \cdot 139 \cdot 179 \cdot 307 \cdot 431 \cdot 521 \cdot 653 \cdot 1019 \cdot 1279 \cdot 2557 \cdot 3221 \cdot 5113 \cdot 6949$$

with a perfect number $2^{p-1}M_p$, $M_p = 2^p - 1$ being a Mersenne prime (see **A3**) with $p > 3$.

Hagis & Lord have found all 46 pairs of betrothed numbers with $m < 10^7$. All of them are of opposite parity. No pairs are known with m, n having the same parity. If there are such, then $m > 10^{10}$. If $m \perp n$, then mn contains at least four distinct prime factors, and if mn is odd, then mn contains at least 21 distinct prime factors.

Beck & Najar call such pairs *reduced* amicable pairs, and call numbers m, n such that

$$\sigma(m) = \sigma(n) = m + n - 1$$

augmented amicable pairs. They found 11 augmented amicable pairs. They found no reduced or augmented *unitary* amicable or sociable numbers (see **B8**) with $n < 10^5$.

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B6 Aliquot sequences.

Since some numbers are abundant and some deficient, it is natural to ask what happens when you iterate the function $s(n) = \sigma(n) - n$ and produce

an **aliquot sequence**, $\{s^k(n)\}$, $k = 0, 1, 2, \dots$. Catalan and Dickson conjectured that all such sequences were bounded, but we now have heuristic arguments and experimental evidence that some sequences, perhaps almost all of those with n even, go to infinity. The smallest n for which there was ever doubt was 138, but D. H. Lehmer eventually showed that after reaching a maximum

$$s^{117}(138) = 179931\,895322 = 2 \cdot 61 \cdot 929 \cdot 1587569$$

the sequence terminated at $s^{177}(138) = 1$. The next value for which there continues to be real doubt is 276. A good deal of computation by Lehmer, subsequently assisted by Godwin, Selfridge, Wunderlich and others, pushed the calculation as far as $s^{469}(276)$, which was quoted in the first edition.

Thomas Struppeck factored this term and computed two more iterates. Andy Guy wrote a PARI program which started from scratch and overnight verified all the earlier calculations and reached $s^{487}(276)$.

The first few sequences whose fate was unknown are the “Lehmer six” starting from 276, 552, 564, 660, 840 and 966. Our program found that the 840 sequence hit the prime $s^{746}(840) = 601$ and established a new record

$$s^{287}(840) = 3\,463982\,260143\,725017\,429794\,136098\,072146\,586526\,240388 \\ = 2^2 \cdot 64970467217 \cdot 6237379309797547 \cdot 2136965558478112990003$$

for the maximum of a terminating sequence. This has since been beaten by Mitchell Dickerman who found that the 1248 sequence has length 1075 after reaching a maximum $s^{583}(1248) =$

$$1231\,636691\,923602\,991963\,829388\,638861\,714770\,651073\,275257\,065104 = 2^4 p$$

of 58 digits, and by Paul Zimmerman who found that the 446580 sequence terminates at step 4736 with the prime 601.

Godwin investigated the fourteen main sequences starting between 1000 and 2000 whose outcome was unknown and discovered that the sequence 1848 terminated. We have found that those for 2580, 2850, 4488, 4830, 6792, 7752, 8862 and 9540 also terminate.

Wieb Bosma showed that the 3556 sequence terminated. Benito & Verona have shown that the 4170, 7080 and 8262 sequences each terminate, the first of which had a (then) record maximum of 84 decimal digits: $s^{289}(4170) =$

$$329561080342477212747203692863366213833838703158858822327064032192093690321488891836 =$$

$$2^2 \cdot 41 \cdot 97 \cdot 20374357 \cdot 1559593537 \cdot 651966073954976081342107597832287652091395156174990523498331163$$

Wolfgang Creyaufmüller has made extensive calculations, the results of which may be seen at

<http://home.t-online.de/home/wolfgang.creyaufmueller/aliquot>.

Comparatively few have terminated, and many open-ended sequences have appeared. His limits of computation generally exceed 10^{80} . The

current numbers of open-ended sequences in the interval $((k-1)10^5, k \cdot 10^5)$ is

$k =$	1	2	3	4	5	6	7	8	9	10
	922	975	938	877	917	971	958	971	982	985

which seem to support the Guy-Selfridge conjecture rather than the Catalan-Dickson one. His graphs are convincing even though 564 almost bit the dust on one occasion. In October 2003 Paul Zimmermann's statistics for the Lehmer five were:

Sequence	276	552	564	660	966
Length so far	1332	829	3075	531	550
# of digits	123	121	121	118	117

Creyaufmüller stopped calculation of the sequence 389508 when it reached 101 digits at step 7135.

H. W. Lenstra has proved that it is possible to construct arbitrarily long monotonic increasing aliquot sequences. See the quadruple paper cited under **B41**. The last of the following references has a bibliography of 60 items concerning the iteration of number-theoretic functions.

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B7 Aliquot cycles. Sociable numbers.

Aliquot cycles or **sociable numbers**. Poulet discovered two cycles of numbers, showing that $s^k(n)$ can have the periods 5 and 28, in addition to 1 and 2. For $k \equiv 0, 1, 2, 3, 4 \pmod{5}$, $s^k(12496)$ takes the values

$$12496 = 2^4 \cdot 11 \cdot 71, \quad 14288 = 2^4 \cdot 19 \cdot 47, \quad 15472 = 2^4 \cdot 967,$$

$$14536 = 2^3 \cdot 23 \cdot 79, \quad 14264 = 2^3 \cdot 1783.$$

For $k \equiv 0, 1, \dots, 27 \pmod{28}$, $s^k(14316)$ takes the values

14316	19116	31704	47616	83328	177792	295488
629072	589786	294896	358336	418904	366556	274924
275444	243760	376736	381028	285778	152990	122410
97946	48976	45946	22976	22744	19916	17716

After a gap of over 50 years, and the advent of high-speed computing, Henri Cohen discovered nine cycles of period 4, and Borho, David and Root also discovered some. Recently Moews & Moews have made an exhaustive search for such cycles with greatest member less than 10^{10} . There are twenty-four: their smallest members are

1264460 7169104 46722700 330003580 2387776550 4424606020
 2115324 18048976 81128632 498215416 2717495235 4823923384
 2784580 18656380 174277820 1236402232 2879697304 5373457070
 4938136 28158165 209524210 1799281330 3705771825 8653956136

Moews & Moews give five larger 4-cycles, and, in a 90-09-01 letter, another whose least member is:

$$2^6 \cdot 79 \cdot 1913 \cdot 226691 \cdot 207722852483$$

They also found an 8-cycle:

1095447416 1259477224 1156962296 1330251784
 1221976136 1127671864 1245926216 1213138984

Ren Yuanhua had already found three of the 4-cycles and Achim Flammenkamp had also found many of them, as well as a second 8-cycle:

1276254780 2299401444 3071310364 2303482780
 2629903076 2209210588 2223459332 1697298124

and a 9-cycle:

805984760 1268997640 1803863720 2308845400 3059220620
 3367978564 2525983930 2301481286 1611969514

Moews & Moews have continued their exhaustive search to uncover all cycles, of any length, whose member preceding the largest member is less than $3.6 \cdot 10^{10}$. There are three more 4-cycles, with least members

15837081520, 17616303220, 21669628904,

and a 6-cycle, all of whose members are odd:

$$\begin{aligned} 21548919483 &= 3^5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19 \cdot 431, & 23625285957 &= 3^5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 29 \cdot 277, \\ 24825443643 &= 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 20719, & 26762383557 &= 3^4 \cdot 7^2 \cdot 13 \cdot 19 \cdot 27299, \\ 25958284443 &= 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 167 \cdot 1427, & 23816997477 &= 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 218651. \end{aligned}$$

Their continued search to 1.03×10^{11} produced four more 4-cycles. A recent count is

length	4	5	6	8	9	28	Total
number	110	1	2	2	1	1	127

50 four-cycles having been discovered by Blankenagel, Borho & vom Stein. See J. O. M. Pedersen's pages at

<http://amicable.adsl.dk/aliquot/sociable.txt>

It has been conjectured that there are no 3-cycles. On the other hand it has been conjectured that for each k there are infinitely many k -cycles.

Karsten Blankenagel, Walter Borho & Axel vom Stein, New amicable four-cycles, *Math. Comput.*, **72**(2003) 2071–2076.

Walter Borho, Über die Fixpunkte der k -fach iterierten Teilersummenfunktion, *Mitt. Math. Gesellsch. Hamburg*, **9**(1969) 34–48; *MR* **40** #7189.

Achim Flammenkamp, New sociable numbers, *Math. Comput.*, **56**(1991) 871–873.

David Moews & Paul C. Moews, A search for aliquot cycles below 10^{10} , *Math. Comput.*, **57**(1991) 849–855; *MR* **92e**:11151.

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OEIS: A003416.

B8 Unitary aliquot sequences.

The ideas of aliquot sequence and aliquot cycle can be adapted to the case where only the *unitary* divisors are summed, leading to **unitary aliquot sequences** and . We use $\sigma^*(n)$ and $s^*(n)$ for the analogs of $\sigma(n)$ and $s(n)$ when just the unitary divisors are summed (compare **B3**).

Are there unbounded unitary aliquot sequences? Here the balance is more delicate than in the ordinary aliquot sequence case. The only sequences which deserve serious consideration are those involving odd multiples of 6, which is a unitary perfect number as well as an ordinary one. Now the sequences tend to increase if $3 \parallel n$, but decrease when a higher power of 3 is present, and it is a moot point as to which situation will dominate. Once a term of a sequence is $6m$, with m odd, then $\sigma^*(6m)$ is an even multiple of 6, making $s^*(6m)$ an odd multiple of 6 again, except in the extremely rare case that m is 4 raised to an odd power.

te Riele pursued all unitary aliquot sequences for $n < 10^5$. The only one which did not terminate or become periodic was 89610. Later calculations showed that this reached a maximum,

$$645\,856\,907\,610\,421\,353\,834 = 2 \cdot 3^2 \cdot 13 \cdot 19 \cdot 73 \cdot 653 \cdot 3047409443791$$

at its 568th term, and terminated at its 1129th.

One can hardly expect typical behavior until the expected number of prime factors is large. Since this number is $\ln \ln n$, such sequences are well beyond computer range. Of 80 sequences examined near 10^{12} , all have terminated or become periodic. One sequence exceeded 10^{23} .

Unitary amicable pairs and unitary sociable numbers may occur rather more frequently than their ordinary counterparts. Lal, Tiller & Summers found cycles of periods 1, 2, 3, 4, 5, 6, 14, 25, 39 and 65. Examples of unitary amicable pairs are (56430, 64530) and (1080150, 1291050), while (30, 42, 54) is a 3-cycle and

$$(1482, 1878, 1890, 2142, 2178)$$

is a 5-cycle.

Cohen (see **B3** for definitions and a reference) finds 62 infinitary amicable pairs with smaller member less than a million, eight infinitary aliquot cycles of order 4 and three of order 6. The only other such cycle of order less than 17 and least member less than a million is of order 11:

448800, 696864, 1124448, 1651584, 3636096, 6608784,
5729136, 3736464, 2187696, 1572432, 895152.

A type of aliquot sequence which can be unbounded has been suggested by David Penney & Carl Pomerance and is based on Dedekind's function: see **B41**. It was in fact the subject of Chapter 7 of de Riele's thesis.

Erdős, looking for a number-theoretic function whose iterates might be bounded, suggested defining $w(n) = n \sum 1/p_i^{\alpha_i}$ where $n = \prod p_i^{\alpha_i}$, and $w^k(n) = w(w^{k-1}(n))$. Note that $w(n) \perp n$. Can it be proved that $w^k(n)$, $k = 1, 2, \dots$, is bounded? Is $|\{w(n) : 1 \leq n \leq x\}| = o(x)$?

Erdős & Selfridge called n a **barrier** for a number-theoretic function $f(m)$ if, for all $m < n$, $m + f(m) \leq n$. Euler's ϕ -function (see **B36**) and $\sigma(m)$ increase too fast to have barriers, but does $\omega(m)$ have infinitely many barriers? The numbers 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 17, 18, 20, 24, 26, 28, 30, \dots , are barriers for $\omega(m)$. Does $\Omega(m)$ have infinitely many barriers? Selfridge observes that 99840 is the largest barrier for $\Omega(m)$ that is $< 10^5$. Mąkowski observes that $n = 1$ is a barrier for every function, and that 2 is a barrier for every function $f(n)$ with $f(1) = 1$; in particular for $d(m)$, the number of divisors of m . The inequality

$$\max\{d(n-1) + n - 1, d(n-2) + n - 2\} \geq n + 2$$

holds for $n \geq 7$, but not for $n = 6$. But $d(n-1) + n - 1 \geq n + 1$ for $n \geq 3$, so $d(m)$ has no barriers ≥ 3 . Does

$$\max_{m < n} (m + d(m)) = n + 2$$

have infinitely many solutions? It is very doubtful. The first few are $n = 5, 8, 10, 12, 24$; Jud McCranie found no others below 10^{10} .

Paul Erdős, A mélange of simply posed conjectures with frustratingly elusive solutions, *Math. Mag.*, **52**(1979) 67–70.

P. Erdős, Problems and results in number theory and graph theory, *Congressus Numerantium* **27**, Proc. 9th Manitoba Conf. Numerical Math. Comput., 1979, 3–21.

Richard K. Guy & Marvin C. Wunderlich, Computing unitary aliquot sequences – a preliminary report, *Congressus Numerantium* **27**, Proc. 9th Manitoba Conf. Numerical Math. Comput., 1979, 257–270.

P. Hagis, Unitary amicable numbers, *Math. Comput.*, **25**(1971) 915–918; *MR* **45** #8599.

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M. Lal, G. Tiller & T. Summers, Unitary sociable numbers, *Congressus Numerantium* **7**, Proc. 2nd Manitoba Conf. Numerical Math., 1972, 211–216; *MR* **50** #4471.

Rudolph M. Najar, The unitary amicable pairs up to 10^8 , *Internat. J. Math. Math. Sci.*, **18**(1995) 405–410; *MR* **96c**:11011.

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H. J. J. te Riele, *Further Results on Unitary Aliquot Sequences*, NW12/73, Mathematisch Centrum, Amsterdam, 1973; reviewed *Math. Comput.*, **32**(1978) 945.

H. J. J. te Riele, *A Theoretical and Computational Study of Generalized Aliquot Sequences*, MCT72, Mathematisch Centrum, Amsterdam, 1976; reviewed *Math. Comput.*, **32**(1978) 945–946; *MR* **58** #27716.

C. R. Wall, Topics related to the sum of unitary divisors of an integer, PhD thesis, Univ. of Tennessee, 1970.

OEIS: A005236, A068597, A087281.

B9 Superperfect numbers.

Suryanarayana defines **superperfect numbers** n by $\sigma^2(n) = 2n$, i.e., $\sigma(\sigma(n)) = 2n$. He and Kanold show that the even ones are just the numbers 2^{p-1} where $2^p - 1$ is a . Are there any odd superperfect numbers? If so, Kanold shows that they are perfect squares, and Dandapat and others that n or $\sigma(n)$ is divisible by at least three distinct primes.

More generally, Bode defines **m -superperfect numbers** as numbers n for which $\sigma^m(n) = 2n$, and shows that for $m \geq 3$ there are no even m -superperfect numbers. He also shows that for $m = 2$ there is no superperfect number $< 10^{10}$. Hunsucker & Pomerance have raised this bound to 7×10^{24} and have shown that if n is an odd super perfect number, then $n\sigma(n)$ has at least 5 distinct prime factors, and that the number of distinct prime factors in n , together with the number of distinct prime factors in $\sigma(n)$ is at least 7. These results are announced in the paper with Dandapat.

If $\sigma^2(n) = 2n + 1$, it would be consistent with earlier terminology to call n quasi-superperfect. The Mersenne primes are such. Are there others? Are there “almost superperfect numbers” for which $\sigma^2(n) = 2n - 1$?

Erdős asks if $(\sigma^k(n))^{1/k}$ has a limit as $k \rightarrow \infty$. He conjectures that it is infinite for each $n > 1$.

Schinzel asks if $\liminf \sigma^k(n)/n < \infty$ for each k , as $n \rightarrow \infty$, and observes that it follows for $k = 2$ from a deep theorem of Rényi. Mąkowski & Schinzel give an elementary proof for $k = 2$ that the limit is 1. Helmut Maier has used sieve methods to prove the result for $k = 3$.

Sitaramaiah & Subbarao call a number **unitary superperfect** if it satisfies the equation $\sigma^*(\sigma^*(n)) = 2n$. They note that the equation $\sigma^*(\sigma^*(n)) = 2n + 1$ has no solutions and that $\sigma^*(\sigma^*(n)) = 2n - 1$ has only $n = 1$ and 3. They list the unitary superperfect numbers less than 10^8 : 2, 9, 165, 238, 1640, 4320, 10250, 10824, 13500, 23760, 58500, 66912, 425880, 520128, 873180, 931392, 1899744, 2129400, 2253888, 3276000, 4580064, 4668300, and they conjecture that there are infinitely many unitary superperfect numbers, of which only a finite number are odd. They also list the solutions $n = 10, 30, 288, 660, 720, 2146560$ of $\sigma^*(\sigma^*(n)) = kn$ for $k = 3$ and the solution $n = 18$ for $k = 4$.

Dieter Bode, Über eine Verallgemeinerung der vollkommenen Zahlen, Dissertation, Braunschweig, 1971.

G. G. Dandapat, J. L. Hunsucker & C. Pomerance, Some new results on odd perfect numbers, *Pacific J. Math.*, **57**(1975) 359–364; **52** #5554.

P. Erdős, Some remarks on the iterates of the ϕ and σ functions, *Colloq. Math.*, **17**(1967) 195–202.

J. L. Hunsucker & C. Pomerance, There are no odd super perfect numbers less than $7 \cdot 10^{24}$, *Indian J. Math.*, **17**(1975) 107–120; *MR* **82b**:10010.

H.-J. Kanold, Über “Super perfect numbers,” *Elem. Math.*, **24**(1969) 61–62; *MR* **39** #5463.

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Andrzej Mąkowski, On two conjectures of Schinzel, *Elem. Math.*, **31**(1976) 140–141.

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V. Sitaramaiah & M. V. Subbarao, On the equation $\sigma^*(\sigma^*(n)) = 2n$, *Utilitas Math.*, **53**(1998) 101–124; *MR* **99a**:11009.

D. Suryanarayana, Super perfect numbers, *Elem. Math.*, **24**(1969) 16–17; *MR* **39** #5706.

D. Suryanarayana, There is no superperfect number of the form $p^{2\alpha}$, *Elem. Math.*, **28**(1973) 148–150; *MR* **48** #8374.

B10 Untouchable numbers.

Erdős has proved that there are infinitely many n such that $s(x) = n$ has no solution. Alane calls such n **untouchable**. In fact Erdős showed that the untouchable numbers have positive lower density. Here are the untouchable numbers less than 1000:

2 5 52 88 96 120 124 146 162 178 188 206 210 216 238 246
 248 262 268 276 288 290 292 304 306 322 324 326 336 342 372 406
 408 426 430 448 472 474 498 516 518 520 530 540 552 556 562 576
 584 612 624 626 628 658 668 670 714 718 726 732 738 748 750 756
 766 768 782 784 792 802 804 818 836 848 852 872 892 894 896 898
 902 916 926 936 964 966 976 982 996

In view of the plausibility of the Goldbach conjecture (**C1**), it seems likely that 5 is the only odd untouchable number since if $2n + 1 = p + q + 1$ with p and q prime, then $s(pq) = 2n + 1$. Can this be proved independently? Are there arbitrarily long sequences of consecutive even numbers which are untouchable? How large can the gaps between untouchable numbers be?

Wouter Meeussen, in a 98-08-27 email, calls an integer $m < n$ **unrelated** to n if it is neither a divisor of n nor relatively prime to it, and defines the function $W(n) = n - d(n) - \phi(n) + 1$, the number of integers unrelated to n . For $n < 14$ the only examples are: 4 unrelated to 6; 6 unrelated to 8 and 9; 4, 6, 8 unrelated to 10; and 8, 9, 10 unrelated to 12, so that $W(6) = W(8) = W(9) = 1$ and $W(10) = W(12) = 3$. Meeussen asks if any of the numbers

2, 13, 67, 93, 123, 133, 141, 173, 187, 193, 205, 217, 229, 245, 253, 257, 283, 285, 293, 303, 317, 319, 325, 333, 341, 389, 393, 397, 405, 415, 427, 445, 453, 467, 473, 483, 491, 493, 509, 525, 527, 533, 537, 549, 557, 571, 573, 581, 587, 589, 595, 609, 621, 635, 643, 653, 655, 667, 669, 673, 679, 685, 701, 709, 723, 765, 777, 779, 789, 797, 811, 813, 833, 843, 845, 869, 877, 893, 899, 901, 907, 915, 921, 941, 957, 973, 997, ...

can occur as a value of $W(n)$.

Felice Russo found the following **unitary untouchable numbers**: 2, 3, 4, 5, 7, 374, 702, 758, 998, i.e., numbers n for which $s^*(x) = n$ has no solution, $s^*(x) = \sigma^*(x) - x$ being the sum of the unitary divisors of x other than x itself. They are the only ones < 1000 . David Wilson found 862 unitary untouchable numbers $\leq 10^5$. What is a good estimate for their number $\leq x$?

P. Erdős, Über die Zahlen der Form $\sigma(n) - n$ und $n - \phi(n)$, *Elem. Math.*, **28**(1973) 83–86; *MR* **49** #2502.

Paul Erdős, Some unconventional problems in number theory, *Astérisque*, **61**(1979) 73–82; *MR* **81h**:10001.

OEIS: A005114, A063948.

B11 Solutions of $m\sigma(m) = n\sigma(n)$.

Leo Moser observed that while $n\phi(n)$ determines n uniquely, $n\sigma(n)$ does not. [$\phi(n)$ is Euler's totient function; see **B36**.] For example, $m\sigma(m) = n\sigma(n)$ for $m = 12$, $n = 14$. The multiplicativity of $\sigma(n)$ now ensures an infinity of solutions, $m = 12q$, $n = 14q$, where $q \perp 42$. So Moser asked if

there is an infinity of *primitive* solutions, in the sense that (m^*, n^*) is *not* a solution for any $m^* = m/d$, $n^* = n/d$, $d > 1$. The example we've given is the least of the set $m = 2^{p-1}(2^q - 1)$, $n = 2^{q-1}(2^p - 1)$, where $2^p - 1$, $2^q - 1$ are distinct Mersenne primes, so that only a finite number of such solutions is known. Another set of solutions is $m = 2^7 \cdot 3^2 \cdot 5^2 \cdot (2^p - 1)$, $n = 2^{p-1} \cdot 5^3 \cdot 17 \cdot 31$, where $2^p - 1$ is a Mersenne prime other than 3 or 31; also $p = 5$ gives a primitive solution on deletion of the common factor 31. There are other solutions, such as $m = 2^4 \cdot 3 \cdot 5^3 \cdot 7$, $n = 2^{11} \cdot 5^2$ and $m = 2^9 \cdot 5$, $n = 2^3 \cdot 11 \cdot 31$. An example with $m \perp n$ is $m = 2^5 \cdot 5$, $n = 3^3 \cdot 7$. If $m\sigma(m) = n\sigma(n)$, is m/n bounded?

Erdős observed that if n is squarefree, then integers of the form $n\sigma(n)$ are distinct. He also proved that the number of solutions of $m\sigma(m) = n\sigma(n)$ with $m < n < x$ is $cx + o(x)$. In answer to the question, are there three distinct numbers l , m , n such that $l\sigma(l) = m\sigma(m) = n\sigma(n)$, Mąkowski observes that for distinct Mersenne primes M_{p_i} , $1 \leq i \leq s$, we have $n_i\sigma(n_i)$ is constant for $n_i = A/M_{p_i}$, where $A = \prod_{j=1}^s M_{p_j}$. Is there an infinity of primitive solutions of the equation $\sigma(a)/a = \sigma(b)/b$? Without restricting the solutions to being primitive, Erdős showed that their number with $a < b < x$ is at least $cx + o(x)$; with the restriction $a \perp b$ no solution is known at all.

Erdős believes that the number of solutions of $x\sigma(x) = n$ is less than $n^{\epsilon/\ln \ln n}$ for every $\epsilon > 0$, and says that the number may be less than $(\ln n)^c$.

Jean-Marie De Koninck asks if $n = 1782$ is the only non-trivial solution of $\sigma(n) = (\text{rad } n)^2$, where $\text{rad } n$, the radical of n , is its greatest squarefree divisor:

$$\sigma(1782) = \sigma(2 \cdot 3^4 \cdot 11) = (2+1) \cdot \frac{3^5-1}{3-1} \cdot (11+1) = (2 \cdot 3 \cdot 11)^2$$

At the conclusion of the article by Huard, Ou, Spearman & Williams on convolution sums of divisor functions is mentioned the possibility of finding further identities and of connexions with Ramanujan's τ -function.

Nicolae Ciprian Bonciocat, Congruences for the convolution of divisor sum function, *Bull. Greek Math. Soc.*, **46**(2002) 161–170; *MR 2003e:11111*.

P. Erdős, Remarks on number theory II: some problems on the σ function, *Acta Arith.*, **5**(1959) 171–177; *MR 21 #6348*.

James G. Huard, Zhiming M. Ou, Blair K. Spearman & Kenneth S. Williams, Elementary evaluation of certain convolution sums involving divisor functions, *Number Theory for the Millenium II (Urbana IL, 2000)* 229–274, AKPeters, Natick MA; 2002.

B12 Analogs with $d(n)$, $\sigma_k(n)$.

Analogous questions may be asked with $\sigma_k(n)$ in place of $\sigma(n)$, where $\sigma_k(n)$ is the sum of the k -th powers of the divisors of n . For example, are there distinct numbers m and n such that $m\sigma_2(m) = n\sigma_2(n)$? For $k = 0$ we have

$md(m) = nd(n)$ for $(m, n) = (18, 27), (24, 32), (56, 64)$ and $(192, 224)$. The last pair can be supplemented by 168 to give three distinct numbers such that $ld(l) = md(m) = nd(n)$. There are primitive solutions (m, n) of shape

$$m = 2^{qt-1}p, \quad n = 2^{pt \cdot 2^{tu}-1}q$$

where p and $q = u + p \cdot 2^{tu}$ are primes, but it does not immediately follow that these are infinitely numerous. Many other solutions can be constructed; for example $(2^{70}, 2^{63} \cdot 71)$, $(3^{19}, 3^{17} \cdot 5)$ and $(5^{51}, 5^{49} \cdot 13)$.

Bencze proves the inequalities

$$\frac{n^k + 1}{2} \geq \frac{\sigma_k(n)}{\sigma_{k-l}(n)} \geq \sqrt{n^l}$$

for $0 \leq l \leq k$ and gives no fewer than 60 applications.

Mihály Bencze, A contest problem and its application (Hungarian), *Mat. Lapok Ifjúsági Folyóirat (Románia)*, **91**(1986) 179–186.

OEIS: A000005, A033950, A036762-036763, A039819, A051278-051280.

B13 Solutions of $\sigma(n) = \sigma(n+1)$.

Sierpiński has asked if $\sigma(n) = \sigma(n+1)$ infinitely often. Hunsucker, Nebb & Stearns extended the tabulations of Mąkowski and of Mientka & Vogt and have found just 113 solutions

$$14, 206, 957, 1334, 1364, 1634, 2685, 2974, 4364, \dots$$

less than 10^7 . They also obtain statistics concerning the equation $\sigma(n) = \sigma(n+l)$, of which Mientka & Vogt had asked: for what l (if any) is there an infinity of solutions? They found many solutions if l is a factorial, but only two solutions for $l = 15$ and $l = 69$. They also ask whether, for each l and m , there is an n such that $\sigma(n) + m = \sigma(n+l)$.

Jud McCranie found 832 solutions of $\sigma(n) = \sigma(n+1)$ for $n < 4.25 \times 10^9$ and 2189 solutions of $\sigma(n) = \sigma(n+2)$ in the same range. An example is

$$\begin{aligned} 4236745811 &= 64399 \times 65789 \\ 4236745813 &= 64499 \times 65687 \end{aligned}$$

for which $\sigma(n) = \sigma(n+2) = 4236876000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23 \cdot 43$. He found no solutions for $\sigma(n) = \sigma(n+1) = \sigma(n+2)$ in that range.

Hunsucker, Nebb & Stearns conjectured that if $\sigma(n) = \sigma(n+1)$ and neither n nor $n+1$ is squarefree, then $n \equiv 0$ or $-1 \pmod{4}$ but Haukkanen gave the counterexamples $n = 52586505 = 3^2 \cdot 5 \cdot 71 \cdot 109 \cdot 151$, $n+1 = 2 \cdot 7^2 \cdot 43 \cdot 12479$ and $n = 164233250 = 2 \cdot 5^3 \cdot 353 \cdot 1861$, $n+1 = 3^5 \cdot 7^2 \cdot 13 \cdot 1061$.

He also observed that for no $n \leq 2 \cdot 10^8$ is $\sigma(n) = \sigma(n+1) = \sigma(n+2)$.

If n and $n+2$ are twin primes, then $\sigma(n+2) = \sigma(n) + 2$. Mąkowski found the composite solutions $n = 434, 8575, 8825$ and Haukkanen showed that these were the only ones $\leq 2 \cdot 10^8$.

One can ask corresponding questions for $\sigma_k(n)$, the sum of the k -th powers of the divisors of n [For $k = 0$, see **B15**.] The only solution of $\sigma_2(n) = \sigma_2(n+1)$ is $n = 6$, since $\sigma_2(2n) > \sigma_2(2n+1)$ for $n > 7$ and $\sigma_2(2n) > 5n^2 > (\pi^2/8)(2n-1)^2 > \sigma_2(2n-1)$. Note that $\sigma_2(24) = \sigma_2(26)$; Erdős doubts that $\sigma_2(n) = \sigma_2(n+2)$ has infinitely many solutions, and thinks that $\sigma_3(n) = \sigma_3(n+2)$ has no solutions at all. De Koninck shows that $\sigma_2(n) = \sigma_2(n+l)$ has only finitely many solutions for l odd, whereas Schinzel's Hypothesis H (see **A**) implies that there are infinitely many solutions for l even.

J.-M. De Koninck, On the solutions of $\sigma_2(n) = \sigma_2(n+l)$, *Ann. Univ. Sci. Budapest. Sect. Comput.*, **21**(2002) 127–133; *MR* **2003h**:11007.

Richard K. Guy & Daniel Shanks, A constructed solution of $\sigma(n) = \sigma(n+1)$, *Fibonacci Quart.*, **12**(1974) 299; *MR* **50** #219.

Pentti Haukkanen, Some computational results concerning the divisor functions $d(n)$ and $\sigma(n)$, *Math. Student*, **62**(1993) 166–168; *MR* **90j**:11006.

John L. Hunsucker, Jack Nebb & Robert E. Stearns, Computational results concerning some equations involving $\sigma(n)$, *Math. Student*, **41**(1973) 285–289.

W. E. Mientka & R. L. Vogt, Computational results relating to problems concerning $\sigma(n)$, *Mat. Vesnik*, **7**(1970) 35–36.

OEIS: A002961.

B14 Some irrational series.

Is $\sum_{n=1}^{\infty} (\sigma_k(n)/n!)$ irrational? It is for $k = 1$ and 2 .

Erdős established the irrationality of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1} = \sum_{n=1}^{\infty} \frac{d(n)}{2^n}$$

and Peter Borwein showed that

$$\sum_{n=1}^{\infty} \frac{1}{q^n + r} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n + r}$$

are irrational if q is an integer other than $0, \pm 1$ and r is a rational other than 0 or $-q^n$.

Peter B. Borwein, On the irrationality of $\sum 1/(q^n + r)$, *J. Number Theory*, **37**(1991) 253–259.

Peter B. Borwein, On the irrationality of certain series, *Math. Proc. Cambridge Philos. Soc.*, **112**(1992) 141–146; *MR* **93g**:11074.

P. Erdős, On arithmetical properties of Lambert series, *J. Indian Math. Soc.(N.S.)* **12**(1948) 63–66.

P. Erdős, On the irrationality of certain series: problems and results, in *New Advances in Transcendence Theory*, Cambridge Univ. Press, 1988, pp. 102–109.

P. Erdős & M. Kac, Problem 4518, *Amer. Math. Monthly* **60**(1953) 47. Solution R. Breusch, **61**(1954) 264–265.

B15 Solutions of $\sigma(q) + \sigma(r) = \sigma(q + r)$.

Max Rumney (*Eureka*, **26**(1963) 12) asked if the equation $\sigma(q) + \sigma(r) = \sigma(q + r)$ has infinitely many solutions which are primitive in a sense similar to that used in **B11**. If $q + r$ is prime, the only solution is $(q, r) = (1, 2)$. If $q + r = p^2$ where p is prime, then one of q and r , say q , is prime, and $r = 2^n k^2$ where $n \geq 1$ and k is odd. If $k = 1$, there is a solution if $p = 2^n - 1$ is a and $q = p^2 - 2^n$ is prime; this is so for $n = 2, 3, 5, 7, 13$ and 19 . For $k = 3$ there are no solutions, and none for $k = 5$ with $n < 189$. For $k = 7$, $n = 1$ and 3 give $(q, r, q + r) = (5231, 2 \cdot 7^2, 73^2)$ and $(213977, 2^3 \cdot 7^2, 463^2)$. Other solutions are $(k, n) = (11, 1), (11, 3), (19, 5), (25, 1), (25, 9), (49, 9), (53, 1), (97, 5), (107, 5), (131, 5), (137, 1), (149, 5), (257, 5), (277, 1), (313, 3)$ and $(421, 3)$. Solutions with $q + r = p^3$ and p prime are $\sigma(2) + \sigma(6) = \sigma(8)$ and

$$\sigma(11638687) + \sigma(2^2 \cdot 13 \cdot 1123) = \sigma(227^3)$$

Erdős asks how many solutions (not necessarily primitive) are there with $q + r < x$; is it $cx + o(x)$ or is it of higher order? If $s_1 < s_2 < \dots$ are the numbers for which $\sigma(s_i) = \sigma(q) + \sigma(s_i - q)$ has a solution with $q < s_i$, what is the density of the sequence $\{s_i\}$?

M. Sugunamma, PhD thesis, Sri Venkataswara Univ., 1969.

B16 Powerful numbers. Squarefree numbers.

Erdős & Szekeres studied numbers n such that if a prime p divides n , then p^i divides n where i is a given number greater than one. Golomb named these numbers **powerful** and exhibited infinitely many pairs of consecutive ones. In answer to his conjecture that 6 was not representable as the difference of two powerful numbers, Władysław Narkiewicz noted that $6 = 5^4 7^3 - 463^2$, and that there were infinitely many such representations. In fact in 1971 Richard P. Stanley (unpublished, since a simultaneous discovery was made by Peter Montgomery) used the theory of the Bhaskara (Pell) equation to show that every non-zero integer is the difference between two powerful numbers and that 1 is the difference between two non-square powerful numbers, each in infinitely many ways. A typical result of Stanley is that

if $a_1 = 39$, $b_1 = 1$, $a_n = 24335a_{n-1} + 7176b_{n-1}$ and $b_n = 82524a_{n-1} + 24335b_{n-1}$, then $2^3(a_n)^2 - 23^3(b_n)^2 = 1$.

Many have investigated which numbers are the difference of two powers, $m^p - n^q$, with $m, n \geq 1$, $p, q \geq 2$. Can any of the following numbers be so expressed?

6, 14, 34, 42, 50, 58, 62, 66, 70, 78, 82, 86, 90, 102, 110, 114, 130, 134, 158, 178, 182, 202, 206, 210, 226, 230, 238, 246, 254, 258, 266, 274, 278, 302, 306, 310, 314, 322, ...

Erdős denotes by $u_1^{(k)} < u_2^{(k)} < \dots$ the integers all of whose prime factors have exponents $\geq k$; sometimes called **k -full numbers**. He asks if the equation $u_{i+1}^{(2)} - u_i^{(2)} = 1$ has infinitely many solutions which do not come from Pell equations $x^2 - dy^2 = \pm 1$. Is there a constant c , such that the number of solutions with $u_i < x$ is less than $(\ln x)^c$? Does $u_{i+1}^{(3)} - u_i^{(3)} = 1$ have no solutions? Do the equations $u_{i+2}^{(2)} - u_{i+1}^{(2)} = 1$, $u_{i+1}^{(2)} - u_i^{(2)} = 1$ have no simultaneous solutions? And several other questions, some of which have been answered by Mąkowski.

For example, Mąkowski notes that $7^3x^2 - 3^3y^2 = 1$ has infinitely many solutions, and that this is not usually counted as a Bhaskara (Pell) equation. He also notes that

$$(2^{k+1} - 1)^k, \quad 2^k(2k + 1 - 1)^k \quad \text{and} \quad (2^{k+1} - 1)^{k+1}$$

are k -full numbers in A.P., and that if a_1, a_2, \dots, a_s are k -full and in A.P. with common difference d then

$$a_1(a_s + d)^k, \quad a_2(a_s + d)^k, \quad \dots, \quad a_s(a_s + d)^k, \quad (a_s + d)^{k+1}$$

are $s + 1$ such numbers. As

$$a^k(a^l + \dots + 1)^k + a^{k+1}(a^l + \dots + 1)^k + \dots + a^{k+l}(a^l + \dots + 1)^k = a^k(a^l + \dots + 1)^{k+1},$$

the sum of $l + 1$ k -ful numbers can be k -full. He says that these last two questions become difficult when we require that the numbers be relatively prime. Heath-Brown has shown that every sufficiently large number is the sum of three powerful numbers; his proof would be much shortened if his conjecture could be proved that the quadratic form $x^2 + y^2 + 125z^2$ represents every sufficiently large $n \equiv 7 \pmod{8}$. Erdős suggested that this may follow from work of Duke and Iwaniec: in fact see the paper by Moroz.

Are there only finitely many powerful numbers n such that $n^2 - 1$ is also powerful? (See the Granville reference at **D2**.)

If $A(x)$ is the number of squareful integers $\leq x$ and $\Delta(x) = A(x) - a_1x^{1/2} - a_2x^{1/3}$ with a_1, a_2 known constants, then, assuming the Riemann Hypothesis, Cao showed that $\Delta(x) = O(x^{5/33+\epsilon})$, and Cai replaced $5/33$ by $4/27$.

Liu Hong-Quan shows that the number of 3-full numbers in the interval $(x, x + x^{\frac{2}{3}+\mu})$ is asymptotic to Cx^μ if $\frac{11}{92} < \mu < \frac{1}{3}$.

Nitaj proves the conjecture of Erdős that $x + y = z$ has solutions in relatively prime 3-full numbers, with $|z| \rightarrow \infty$. Cohn further shows that this can be done infinitely often with none of x, y, z being a perfect cube.

Gang Yu improves Menzer's estimate for the number of 4-full numbers $\leq x$ from $x^{35/316}(\ln x)^3$ to $x^{3626/35461+\epsilon}$.

Huxley & Trifonov show that the number of square-full numbers among $N + 1, \dots, N + h$ is, for N sufficiently large in terms of ϵ and $h \geq \frac{1}{\epsilon} N^{\frac{5}{8}} (\ln N)^{\frac{5}{16}}$ is

$$\frac{\zeta(\frac{3}{2})}{2\zeta(3)} \frac{h}{\sqrt{N}} + O\left(\frac{\epsilon h}{\sqrt{N}}\right)$$

Earlier results, with exponents $\frac{2}{3}$, 0.6318, 0.6308, and $\frac{49}{78}$ in place of $\frac{5}{8}$, were obtained by Bateman & Grosswald, Heath-Brown, Liu, and Filaseta & Trifonov.

If p is a prime, $p \equiv 1 \pmod{4}$, and $\frac{1}{2}(t + u\sqrt{p})$ is the fundamental unit of $Q(\sqrt{p})$ (i.e., (t, u) are the least positive integers satisfying the Bhaskara equation $t^2 - pu^2 = 1$), then the Ankeny-Artin-Chowla conjecture asserts that $p \nmid u$ for any p . It was proved for all $p < 10^{11}$ by van der Poorten, te Riele & Williams. The conjecture is false if p is not prime; Gerry Myerson believes that 46 and 430 are the two smallest counterexamples.

At the other end of the spectrum from the powerful numbers are the **squarefree** numbers, with no repeated prime divisors. If we denote the sequence of squarefree numbers by $\{f_n\} = \{1, 2, 3, 5, 6, 7, 10, \dots\}$, then it is well known that $f_{n+1} - f_n = 1$ for infinitely many n and $\limsup(f_{n+1} - f_n) = \infty$. Panaitopol further shows that

$$\limsup(\min\{f_{n+1} - f_n, f_n - f_{n-1}\}) = \infty$$

and that if $e_n = f_{n+1} - 2f_n + f_{n-1}$ and $g_n = f_{n+1}^2 - 2f_n^2 + f_{n-1}^2$, then each of $e_n < 0$, $e_n = 0$, $e_n > 0$, $g_n < 0$, $g_n > 0$ holds for infinitely many n , and $g_n \neq 0$ for all $n > 1$. He asks if there exist, for each positive integer k , an index n such that $f_{n+1} - f_n = k$?

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B17 Exponential-perfect numbers

If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then Straus & Subbarao call d an **exponential divisor** (e-divisor) of n if $d|n$ and $d = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$ where $b_j|a_j$ ($1 \leq j \leq r$), and they call n **e-perfect** if $\sigma_e(n) = 2n$, where $\sigma_e(n)$ is the sum of the e-divisors of n . Some examples of e-perfect numbers are

$$\begin{aligned} &2^2 \cdot 3^2, \quad 2^2 \cdot 3^3 \cdot 5^2, \quad 2^3 \cdot 3^2 \cdot 5^2, \quad 2^4 \cdot 3^2 \cdot 11^2, \quad 2^4 \cdot 3^3 \cdot 5^2 \cdot 11^2, \\ &2^6 \cdot 3^2 \cdot 7^2 \cdot 13^2, \quad 2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 13^2, \quad 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 13^2, \quad 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 139^2 \\ &\text{and} \quad 2^{19} \cdot 3^2 \cdot 5^5 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 19^2 \cdot 37^2 \cdot 79^2 \cdot 109^2 \cdot 157^2 \cdot 313^2. \end{aligned}$$

If m is squarefree, $\sigma_e(m) = m$, so if n is e-perfect and m is squarefree with $m \perp n$, then mn is e-perfect. So it suffices to consider only powerful (**B16**) e-perfect numbers.

Straus & Subbarao show that there are no odd e-perfect numbers, in fact no odd n which satisfy $\sigma_e(n) = kn$ for any integer $k > 1$. They also show that for each r the number of (powerful) e-perfect numbers with r prime factors is finite, and that the same holds for **e-multiperfect numbers** ($k > 2$).

Is there an e-perfect number which is *not* divisible by 3 (equivalently, not divisible by 36)?

Straus & Subbarao conjecture that there is only a finite number of numbers *not* divisible by any given prime p .

Are there any e-multiperfect numbers?

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OEIS A051377, A054979-054980.

B18 Solutions of $d(n) = d(n + 1)$.

Claudia Spiro has proved that $d(n) = d(n + 5040)$ has infinitely many solutions and Heath-Brown used her ideas to show that there are infinitely many numbers n such that $d(n) = d(n + 1)$, and Pinner has extended this to $d(n) = d(n + a)$ for any integer a . Many examples arise from pairs of consecutive numbers which are products of just two distinct primes, and it has been conjectured that there is an infinity of *triples* of consecutive products of two primes, n , $n + 1$, $n + 2$. For example, $n = 33, 85, 93, 141, 201, 213, 217, 301, 393, 445, 633, 697, 921, \dots$. It is clearly not possible to have *four* such numbers, but it *is* possible to have longer sequences of consecutive numbers with the same number of divisors. For example,

$$d(242) = d(243) = d(244) = d(245) = 6 \quad \text{and}$$

$$d(40311) = d(40312) = d(40313) = d(40314) = d(40315) = 8.$$

How long can such sequences be?

For 5 and 6 consecutive numbers, Haukkanen (see ref. at **B13**) showed that the least n is respectively 11605 and 28374. In an 87-07-16 letter Stephane Vandemergel sent the sequence of seven numbers: $171893 = 19 \cdot 83 \cdot 109$, $171894 = 2 \cdot 3 \cdot 28649$, $171895 = 5 \cdot 31 \cdot 1109$, $171896 = 2^3 \cdot 21487$, $171897 = 3 \cdot 11 \cdot 5209$, $171898 = 2 \cdot 61 \cdot 1409$, $171899 = 7 \cdot 13 \cdot 1889$, each with 8 divisors. In 1990, Ivo Düntsch & Roger Eggleton discovered several such sequences of 7 numbers, two of 8 and one of 9, each with 48 divisors; the last example starts at 17796126877482329126044, presumably not the smallest of its kind. At the beginning of 2002 Jud McCranie gave 1043710445721 as the smallest first member of eight consecutive numbers with the same number of divisors.

Erdős believes that there are sequences of length k for every k , but does not see how to give an upper bound for k in terms of n .

Erdős, Pomerance & Sárközy showed that the number of $n \leq x$ with $d(n) = d(n+1)$ is $\ll x/(\ln \ln x)^{1/2}$, and Hildebrand showed that this number is $\gg x/(\ln \ln x)^3$. The former authors also showed that the number of $n \leq x$ with the ratio $d(n)/d(n+1)$ in the set $\{2^{-3}, 2^{-2}, 2^{-1}, 1, 2, 2^2, 2^3\}$ is $\asymp x/(\ln \ln x)^{1/2}$.

Erdős showed that the density of numbers n with $d(n+1) > d(n)$ is $\frac{1}{2}$. This, with the above results, settles a conjecture of S. Chowla. Fabrykowski & Subbarao extend this to the case with $n+h$ in place of $n+1$.

Erdős also lets

$$1 = d_1 < d_2 < \cdots < d_\tau = n$$

be the set of all divisors of n , listed in order, defines

$$f(n) = \sum_{i=1}^{\tau-1} d_i/d_{i+1}$$

and asks us to prove that $\sum_{n=1}^x f(n) = (1 + o(1))x \ln x$.

Erdős & Mirsky ask for the largest k so that the numbers $d(n), d(n+1), \dots, d(n+k)$ are all distinct. They only have trivial bounds; probably $k = (\ln n)^c$.

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OEIS: A000005, A005237-005238, A006558, A006601, A019273, A039665, A049051.

B19 $(m, n + 1)$ and $(m + 1, n)$ with same set of prime factors. The *abc*-conjecture.

Motzkin & Straus asked for all pairs of numbers m, n such that m and $n + 1$ have the same set of distinct prime factors, and similarly for n and $m + 1$. It was thought that such pairs were necessarily of the form $m = 2^k + 1$, $n = m^2 - 1$ ($k = 0, 1, 2, \dots$) until Conway observed that if $m = 5 \cdot 7$, $n + 1 = 5^4 \cdot 7$, then $n = 2 \cdot 3^7$, $m + 1 = 2^2 \cdot 3^2$. Are there others?

Similarly, Erdős asks if there are numbers m, n ($m < n$) other than $m = 2^k - 2$, $n = 2^k(2^k - 2)$ such that m and n have the same prime factors and similarly for $m + 1, n + 1$. Mąkowski found the pair $m = 3 \cdot 5^2$, $n = 3^5 \cdot 5$ for which $m + 1 = 2^2 \cdot 19$, $n + 1 = 2^6 \cdot 19$. Compare problem **B29**.

Pomerance has asked if there are any odd numbers $n > 1$ such that n and $\sigma(n)$ have the same prime factors. He conjectures that there are not.

The example $1 + 2 \cdot 3^7 = 5^4 7$ in the first paragraph is of interest in connexion with the **abc-conjecture**:

Many of the classical problems of number theory (Goldbach conjecture, twin primes, the Fermat problem, Waring's problem, the Catalan conjecture) owe their difficulty to a clash between multiplication and addition. Roughly, if there's an additive relation between three numbers, their prime factors can't all be small.

Suppose that $A + B = C$ with $\gcd(A, B, C) = 1$. Define the radical R to be the maximum squarefree integer dividing abc and the power P by

$$P = \frac{\ln \max(|A|, |B|, |C|)}{\ln R}$$

then for a given $\eta > 1$ are there only finitely many triples $\{A, B, C\}$ with $P \geq \eta$? Another form of this conjecture is that $\limsup P = 1$; both forms of the conjecture seem to be hopelessly beyond reach.

Joe Kanapka, a student of Noam Elkies, has produced a list of all examples with $C < 2^{32}$ and $P > 1.2$. There are nearly 1000 of them. The “top ten” according to

[`http://www.math.unicaen.fr/~nitaj/abc.html#Ten`](http://www.math.unicaen.fr/~nitaj/abc.html#Ten)*abc*

(which has an extensive bibliography) are

P	A	B	C	author
1.62991	2	$3^{10} \cdot 109$	23^5	Reyssat
1.62599	11^2	$3^2 \cdot 5^6 \cdot 7^3$	$2^{21} \cdot 23$	de Weger (D10)
1.62349	$19 \cdot 1307$	$7 \cdot 29^2 \cdot 31^8$	$2^8 \cdot 3^{22} \cdot 5^4$	Browkin-Brzeziński
1.58076	283	$5^{11} \cdot 13^2$	$2^8 \cdot 3^8 \cdot 17^3$	Br-Br, Nitaj
1.56789	1	$2 \cdot 3^7$	$5^4 \cdot 7$	Lehmer (B29)
1.54708	7^3	3^{10}	$2^{11} \cdot 29$	de Weger
1.54443	$7^2 \cdot 41^2 \cdot 311^3$	$11^{16} \cdot 13^2 \cdot 79$	$2 \cdot 3^3 \cdot 5^{23} \cdot 953$	Nitaj
1.53671	5^3	$2^9 \cdot 3^{17} \cdot 13^2$	$11^5 \cdot 17 \cdot 31^3 \cdot 137$	Montgomery-teRiele
1.52700	$13 \cdot 19^6$	$2^{30} \cdot 5$	$3^{13} \cdot 11^2 \cdot 31$	Nitaj
1.52216	$3^{18} \cdot 23 \cdot 2269$	$17^3 \cdot 29 \cdot 31^8$	$2^{10} \cdot 5^2 \cdot 7^{15}$	Nitaj

Browkin & Brzeziński generalize the *abc*-conjecture (which is their case $n = 3$) to an “ n -conjecture” on $a_1 + \dots + a_n = 0$ in coprime integers with non-vanishing subsums. With R and P defined analogously, they conjecture that $\limsup P = 2n - 5$. They prove that $\limsup P \geq 2n - 5$. They give a lot of examples for the *abc*-conjecture with $P > 1.4$. Their method is to look for rational numbers approximating roots of integers (note that the best example above is connected to the good approximation $23/9$ for $109^{1/5}$). Abderrahmane Nitaj used a similar method. Some of these were found independently by Robert Styer (**D10**). The Catalan relation $1 + 2^3 = 3^2$ gives a comparatively poor $P \approx 1.22629$.

For connexions between the *abc*-conjecture and the Fermat problem, see the Granville references at **D2**. Indeed, if $A = a^p$, $B = b^p$, $C = c^p$ and the Fermat equation $A + B = C$ is satisfied, then Gerhard Frey’s elliptic curve

$$y^2 = x(x - A)(x + B)$$

has discriminant $16(abc)^{2p}$.

This area has had several stimuli: two being the proof of Fermat’s Last Theorem and the announcement of the Beal prize. I thank Andrew Granville for the following remarks.

The problem has been much studied recently by several authors. Darmon & Granville showed that if we fix integers x , y , z with $1/x + 1/y + 1/z < 1$ then there are only finitely many triples of coprime integers a , b , c satisfying $a^x + b^y = c^z$. This is proved independently of any assumption, and fits well with the conjecture that x , y , $z > 2$ imply that a , b , c have a common factor since in this case $1/x + 1/y + 1/z < 1$ unless $x = y = z = 3$, but of course Euler, and possibly Fermat, knew that there are no solutions in that case. Following this result there has been extensive computer searching and exactly ten solutions have been found with $1/x + 1/y + 1/z < 1$ and a , b , c coprime:

$$\begin{array}{ll}
1 + 2^3 = 3^2 & 17^7 + 76271^3 = 21063928^2 \\
2^5 + 7^2 = 3^4 & 1414^3 + 2213459^2 = 65^7 \\
7^3 + 13^2 = 2^9 & 9262^3 + 15312283^2 = 113^7 \\
2^7 + 17^3 = 71^2 & 43^8 + 96222^3 = 30042907^2 \\
3^5 + 11^4 = 122^2 & 33^8 + 1549034^2 = 15613^3
\end{array}$$

The five big solutions were found by clever computations by Beukers & Zagier. From this, several people have conjectured that the only solutions do have some exponent equal to 2, and wonder if this is always the case. Granville is loth to believe or disbelieve such a statement. For a year he and Cohen believed that the five small solutions above were the only five, and then were totally shocked by these computations — and he sees no good reason to suppose that we’ve seen the last of the solutions!

The technique used by Darmon & Granville was to reduce the problem to applications of Faltings’s Theorem. This is why they always say ‘at most finitely many solutions’. Recently Darmon & Merel, and also Poonen, have revisited these problems, and tried to reduce several examples of x, y, z to applications of Wiles’s Theorem (Darmon and Granville had done a couple of examples of this in their paper, but it is done much more skillfully in the recent papers). Darmon & Merel, and Poonen, prove that there are no coprime solutions with exponents $(x, x, 3)$ with $x \geq 3$.

As Mauldin pointed out, the *abc*-conjecture is relevant to this. Gerald Tenenbaum has long suggested an explicit and plausible version of the *abc*-conjecture: If $a + b = c$ in coprime positive integers then $c \leq (\text{product of } p|abc)^2$. Assuming then that $a^x + b^y = c^z$ with $a, b, c > 0$ we’d have $c^{(z/2)} \leq abc < c^{z(1/x+1/y+1/z)}$ and thus $1/x + 1/y + 1/z > 1/2$. This leaves us with a list of cases to consider if we insist that $x, y, z > 2$: $(3, 3, z > 3)$, $(3, 4, z > 3)$, $(3, 5, z > 4)$, $(3, 6, z > 6)$, $(4, 4, z > 4)$, and a finite list.

I am also indebted to Andrew Bremner and the AMS for permission to reproduce the review of Mauldin’s paper:

This note announces the award of a substantial monetary prize (since this article was written, fixed at \$50,000) to any person who provides a solution to the “Beal Conjecture”, stated as the following: *Let A, B, C, x, y, z be positive integers with $x, y, z > 2$. If $A^x + B^y = C^z$ (1), then A, B, C have a non-trivial common factor.*

The story of this conjecture is an interesting one, and told at slightly greater length in the author’s follow-up letter to the

Notices in March, 1998. Andrew Beal is a successful Texas businessman, with enthusiasm for number theory. He has a particular interest in Fermat and his methods, and evidently formulated this conjecture after several years of study following the announcement in 1993 of Andrew Wiles's work on Fermat's Last Theorem. So often, the amateur number-theorist turns out to be a well-intentioned crank; what is remarkable here is how close the stated problem is to current research activity by leaders in the field. In fact, the problem is essentially many decades old, and Brun [1914] asks many similar questions. The formulation in the 1980s by Masser, Oesterle & Szpiro of the *abc*-conjecture has had great influence on the discipline, and in fact a corollary of the *abc*-conjecture is that there are no solutions to the Beal Prize problem when the exponents are sufficiently large. The prize problem itself was implicitly posed by Andrew Granville in the Unsolved Problems section of the West Coast Number Theory Meeting, Asilomar, 1993 ("Find examples of $x^p + y^q = z^r$ with $1/p + 1/q + 1/r < 1$ other than $2^3 + 1^7 = 3^2$ and $7^3 + 13^2 = 2^9$ "), and is stated and discussed in van der Poorten's book "Notes on Fermat's Last Theorem" (1996). The resolution by Wiles of Fermat's Last Theorem disposes of a special case of the prize problem; and Darmon and Granville prove the deep result that if $1/x + 1/y + 1/z < 1$ then there can only be finitely many triples of coprime integers A, B, C satisfying $A^x + B^y = C^z$ (ten solutions are known). Recently, Darmon & Merel have shown there can exist no coprime solutions to (1) with the exponents $(x, x, 3)$, $x \geq 3$.

There has been some feeling expressed that the Conjecture should best be referred to as the "Beal Prize problem", though there is no doubt that Andrew Beal after much computation independently arrived at and formulated the conjecture without knowledge of the current literature. With a nod to T. S. Eliot, the matter of naming Conjectures can be as difficult as the naming of Cats.

See also **D2**.

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B20 Cullen and Woodall numbers.

Some interest has been shown in the **Cullen numbers**, $n \cdot 2^n + 1$, which are all composite for $2 \leq n \leq 1000$, except for $n = 141$. This is probably a good example of the Law of Small Numbers, because for small n , where

the density of primes is large, the Cullen numbers are very likely to be composite because Fermat's (little) theorem tells us that $(p-1)2^{p-1}+1$ and $(p-2)2^{p-2}+1$ are both divisible by p . Moreover, as John Conway observes, the Cullen numbers are divisible by $2n-1$ if that is a prime of shape $8k \pm 3$. He asks if p and $p \cdot 2^p + 1$ can both be prime. Wilfrid Keller notes that Conway's remark can be generalized as follows. Write $C_n = n \cdot 2^n + 1$, $W_n = n \cdot 2^n - 1$: then a prime p divides $C_{(p+1)/2}$ and $W_{(3p-1)/2}$ or it divides $C_{(3p-1)/2}$ and $W_{(p+1)/2}$ according as the Legendre symbol (see **F5**) $\left(\frac{2}{p}\right)$ is -1 or $+1$. Known Cullen numbers include $n = 1, 141, 4713, 5795, 6611, 18496, 32292, 32469, 59656, 90825, 262419$ and 361275 .

The corresponding numbers (which have been called Woodall primes) $n \cdot 2^n - 1$ are prime for $n = 2, 3, 6, 30, 75, 81, 115, 123, 249, 362, 384, 462, 512$ (i.e. M_{521}), $751, 822, 5312, 7755, 9531, 12379, 15822, 18885, 22971, 23005, 98726, 143018, 151023, 667071$. In parallel with Conway's question above, Keller notes that here $3, 751$ and 12379 are primes.

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B21 $k \cdot 2^n + 1$ composite for all n .

Let $N(x)$ be the number of odd positive integers k , not exceeding x , such that $k \cdot 2^n + 1$ is prime for no positive integer n . Sierpiński used covering congruences (see **F13**) to show that $N(x)$ tends to infinity with x . For example, if

$$k \equiv 1 \pmod{641 \cdot (2^{32} - 1)} \quad \text{and} \quad k \equiv -1 \pmod{6700417},$$

then every member of the sequence $k \cdot 2^n + 1$ ($n = 0, 1, 2, \dots$) is divisible by just one of the primes $3, 5, 17, 257, 641, 65537$ or 6700417 . He also noted that at least one of $3, 5, 7, 13, 17, 241$ will always divide $k \cdot 2^n + 1$ for certain other values of k .

Erdős & Odlyzko have shown that

$$\left(\frac{1}{2} - c_1\right)x \geq N(x) \geq c_2x.$$

What is the least value of k such that $k \cdot 2^n + 1$ is composite for all values of n ? Selfridge discovered that one of $3, 5, 7, 13, 19, 37, 73$ always divides

$78557 \cdot 2^n + 1$. He also noted that there is a prime of the form $k \cdot 2^n + 1$ for each $k < 383$ and Hugh Williams discovered the prime $383 \cdot 2^{6393} + 1$.

In the first edition we wrote that the determination of the least k may now be within computer reach, though Keller has expressed his doubts about this. Extensive calculations have been made by Baillie, Cormack & Williams, by Keller, and by Buell & Young. Continuing activities by these and many others, including seventeenorbust.com have reduced the 35 possibilities of the second edition to twelve. The answer seems almost certain to be $k = 78557$, but there remain the possibilities

4847	5359	10223	19249	21181	22699
24737	27653	28433	33661	55459	67607

Riesel (see references at **B20**) investigated the corresponding question for $k \cdot 2^n - 1$. For $k = 509203, 762701, 992077$, the covering set of divisors is $\{3, 5, 7, 13, 17, 241\}$; for $k = 777149, 790841$, the covering set of divisors is $\{3, 5, 7, 13, 19, 37, 73\}$. There are no other values of $k < 10^6$ covered by the following six covering sets given by Stanton:

$\{3, 5, 7, 13, 19, 37, 73\}$, $\{3, 5, 7, 13, 19, 37, 109\}$, $\{3, 5, 7, 11, 13, 31, 41, 61, 151\}$,
 $\{3, 5, 7, 11, 13, 19, 31, 37, 41, 61, 181\}$, $\{3, 5, 7, 13, 17, 241\}$, $\{3, 5, 7, 13, 17, 97, 257\}$.
 It seems that $k = 509203$ is the smallest number such that $k \cdot 2^n - 1$ are all composite for every integer $n > 0$. An enormous amount of prime finding is needed to establish this – when Meng wrote there were more than 1000 values of k still to be investigated.

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OEIS: A076336, A076337.

B22 Factorial n as the product of n large factors.

Straus, Erdős & Selfridge have asked that $n!$ be expressed as the product of n factors, with the least one, l , as large as possible. For example, for $n = 56$, $l = 15$,

$$56! = 15 \cdot 16^3 \cdot 17^3 \cdot 18^8 \cdot 19^2 \cdot 20^{12} \cdot 21^9 \cdot 22^5 \cdot 23^2 \cdot 26^4 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53$$

Selfridge has two conjectures: (a) that, except for $n = 56$, $l \geq \lfloor 2n/7 \rfloor$; (b) that for $n \geq 300000$, $l \geq n/3$. If the latter is true, by how much can 300000 be reduced?

Straus was reputed to have shown that for $n > n_0 = n_0(\epsilon)$, $l > n/(e+\epsilon)$, but a proof was not found in his Nachlaß. It is clear from Stirling's formula that this is best possible. It is also clear that l is a monotonic, though not strictly monotonic, increasing function of n . On the other hand it does not take all integer values: for $n = 124$, 125 , l is respectively 35 and 37. Erdős

asks how large the gaps in the values of l can be, and can l be constant for arbitrarily long stretches?

Alladi & Grinstead write $n!$ as a product of prime powers, each as large as $n^{\delta(n)}$ and let $\alpha(n) = \max \delta(n)$ and show that $\lim_{n \rightarrow \infty} \alpha(n) = e^{c-1} = \alpha$, say, where

$$c = \sum_2^{\infty} \frac{1}{k} \ln \frac{k}{k-1} \quad \text{so that} \quad \alpha = 0.809394020534 \dots$$

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B23 Equal products of factorials.

Suppose that $n! = a_1! a_2! \dots a_r!$, $r \geq 2$, $a_1 \geq a_2 \geq \dots \geq a_r \geq 2$. A trivial example is $a_1 = a_2! \dots a_r! - 1$, $n = a_2! \dots a_r!$ Dean Hickerson notes that the only nontrivial examples with $n \leq 410$ are $9! = 7!3!3!2!$, $10! = 7!6! = 7!5!3!$ and $16! = 14!5!2!$ and asks if there are any others. Jeffrey Shallit & Michael Easter have extended the search to $n = 18160$ and Chris Caldwell has shown that any other n is greater than 10^6 .

Erdős observes that if $P(n)$ is the largest prime factor of n and if it were known that $P(n(n+1))/\ln n$ tends to infinity with n , then it would follow that there are only finitely many nontrivial examples.

He & Graham have studied the equation $y^2 = a_1! a_2! \dots a_r!$. They define the set F_k to be those m for which there is a set of integers $m = a_1 > a_2 > \dots > a_r$ with $r \leq k$ which satisfies this equation for some y , and write D_k for $F_k - F_{k-1}$. They have various results, for example: for almost all primes p , $13p$ does not belong to F_5 ; and the least element of D_6 is 527. If $D_4(n)$ is the number of elements of D_4 which are $\leq n$, they do not know the order of growth of $D_4(n)$. They conjecture that $D_6(n) > cn$ but cannot prove this.

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B24 The largest set with no member dividing two others.

Let $f(n)$ be the size of the largest subset of $[1, n]$ no member of which divides two others. Erdős asks how large can $f(n)$ be? By taking $[m + 1, 3m + 2]$ it is clear that one can have $\lceil 2n/3 \rceil$. D.J. Kleitman shows that $f(29) = 21$ by taking $[11, 30]$ and omitting 18, 24 and 30, which then allows the inclusion of 6, 8, 9 and 10. However, this example does not seem to generalize. In fact Lebensold has shown that if n is large, then

$$0.6725n \leq f(n) \leq 0.6736n.$$

Erdős also asks if $\lim f(n)/n$ is irrational.

Dually, one can ask for the largest number of numbers $\leq n$, with no number a multiple of any two others. Kleitman's example serves this purpose also. More generally, Erdős asks for the largest number of numbers with no one divisible by k others, for $k > 2$. For $k = 1$, the answer is $\lceil n/2 \rceil$.

For some related problems, see **E2**.

Driss Abouabdillah & Jean M. Turgeon, On a 1937 problem of Paul Erdős concerning certain finite sequences of integers none divisible by another, Proc. 15th S.E. Conf. Combin. Graph Theory Comput., Baton Rouge, 1984, *Congr. Numer.*, **43**(1984) 19–22; *MR* **86h**:11020.

Neil J. Calkin & Andrew Granville, On the number of co-prime-free sets, *Number Theory (New York, 1991–1995)*, Springer, New York, 1996, 9–18; *MR* **97j**:11006.

P. J. Cameron & P. Erdős, On the number of sets of integers with various properties, *Number Theory (Banff, 1988)*, de Gruyter, Berlin, 1990, 61–79; *MR* **92g**:11010.

P. Erdős, On a problem in elementary number theory and a combinatorial problem, *Math. Comput.*, (1964) 644–646; *MR* **30** #1087.

Kenneth Lebensold, A divisibility problem, *Studies in Appl. Math.*, **56**(1976–77) 291–294; *MR* **58** #21639.

Emma Lehmer, Solution to Problem 3820, *Amer. Math. Monthly*, **46**(1939) 240–241.

B25 Equal sums of geometric progressions with prime ratios.

Bateman asks if $31 = (2^5 - 1)/(2 - 1) = (5^3 - 1)/(5 - 1)$ is the only prime which is expressible in more than one way in the form $(p^r - 1)/(p - 1)$ where p is prime and $r \geq 3$ and $d \geq 1$ are integers. Trivially one has $7 = (2^3 - 1)/(2 - 1) = ((-3)^3 - 1)/(-3 - 1)$, but there are no others $< 10^{10}$. If the condition that p be prime is relaxed, the problem goes back to Goormaghtigh and we have the solution

$$8191 = (2^{13} - 1)/(2 - 1) = (90^3 - 1)/(90 - 1)$$

E. T. Parker observed that the very long proof by Feit & Thompson that every group of odd order is solvable would be shortened if it could be proved that $(p^q - 1)/(p - 1)$ never divides $(q^p - 1)/(q - 1)$ where p, q are distinct odd primes. In fact it has been conjectured that these two expressions are relatively prime, but Nelson Stephens noticed that when $p = 17$, $q = 3313$ they have a common factor $2pq + 1 = 112643$. McKay has established that $p^2 + p + 1 \nmid 3^p - 1$ for $p < 53 \cdot 10^6$.

Karl Dilcher quotes Nelson Stephens to the effect that if $p^q - 1$ and $q^p - 1$ have a common factor r , then r is of shape $2\lambda pq + 1$ and has searched with all such $r < 8 \cdot 10^{10}$. Also with $1 \leq \lambda \leq 10$ and $p < q < 10^7$, and with $p = 3$ and $q < 10^{14}$.

P. T. Bateman & R. M. Stemmler, Waring's problem for algebraic number fields and primes of the form $(p^r - 1)/(p^d - 1)$, *Illinois J. Math.*, **6**(1962) 142–156; *MR* **25** #2059.

Ted Chinburg & Melvin Henriksen, Sums of k th powers in the ring of polynomials with integer coefficients, *Bull. Amer. Math. Soc.*, **81**(1975) 107–110; *MR* **51** #421; *Acta Arith.*, **29**(1976) 227–250; *MR* **53** #7942.

Karl Dilcher & Josh Knauer, On a conjecture of Feit and Thompson, (preprint, Williams60, Banff, May 2003).

A. Mąkowski & A. Schinzel, Sur l'équation indéterminée de R. Goormaghtigh, *Mathesis*, **68**(1959) 128–142; *MR* **22** # 9472; **70**(1965) 94–96.

N. M. Stephens, On the Feit-Thompson conjecture, *Math. Comput.*, **25**(1971) 625; *MR* **45** #6738.

B26 Densest set with no l pairwise coprime.

Erdős asks what is the maximum k so that the integers a_i , $1 \leq a_1 < a_2 < \dots < a_k \leq n$ have no l among them which are pairwise relatively prime.

He conjectures that this is the number of integers $\leq n$ which have one of the first $l - 1$ primes as a divisor. He says that this is easy to prove for $l = 2$ and not difficult for $l = 3$; he offers \$10.00 for a general solution.

Dually one can ask for the largest subset of $[1, n]$ whose members have pairwise least common multiples not exceeding n . If $g(n)$ is the cardinality of such a maximal subset, then Erdős showed that

$$\frac{3}{2\sqrt{2}}n^{1/2} - 2 < g(n) \leq 2n^{1/2}$$

where the first inequality follows by taking the integers from 1 to $(n/2)^{1/2}$ together with the even integers from $(n/2)^{1/2}$ to $(2n)^{1/2}$. Choi improved the upper bound to $1.638n^{1/2}$.

Rudolf F. Ahlswede & L. G. Khachatrian, Maximal sets of numbers not containing $k + 1$ pairwise coprime integers, *Acta Arith.*, **72**(1995) 77–100; *MR* **96k**:11020.

Neil J. Calkin & Andrew Granville, On the number of coprime-free sets, *Number Theory (New York, 1991–1995)* 9–18, Springer, New York, 1996; *MR* **97j**:11006.

S. L. G. Choi, The largest subset in $[1, n]$ whose integers have pairwise l.c.m. not exceeding n , *Mathematika*, **19**(1972) 221–230; **47** #8461.

S. L. G. Choi, On sequences containing at most three pairwise coprime integers, *Trans. Amer. Math. Soc.*, **183**(1973) 437–440; **48** #6052.

P. Erdős, Extremal problems in number theory, *Proc. Sympos. Pure Math. Amer. Math. Soc.*, **8**(1965) 181–189; *MR* **30** #4740.

B27 The number of prime factors of $n+k$ which don't divide $n+i$, $0 \leq i < k$.

Erdős & Selfridge define $v(n; k)$ as the number of prime factors of $n + k$ which do not divide $n + i$ for $0 \leq i < k$, and $v_0(n)$ as the maximum of $v(n; k)$ taken over all $k \geq 0$. Does $v_0(n) \rightarrow \infty$ with n ? They show that $v_0(n) > 1$ for all n except 1, 2, 3, 4, 7, 8 and 16. More generally, define $v_l(n)$ as the maximum of $v(n; k)$ taken over $k \geq l$. Does $v_l(n) \rightarrow \infty$ with n ? They are unable to prove even that $v_1(n) = 1$ has only a finite number of solutions. Probably the greatest n for which $v_1(n) = 1$ is 330.

They also denote by $V(n; k)$ the number of primes p for which p^α is the highest power of p dividing $n + k$, but p^α does not divide $n + i$ for $0 \leq i < k$, and by $V_l(n)$ the maximum of $V(n; k)$ taken over $k \geq l$. Does $V_1(n) = 1$ have only a finite number of solutions? Perhaps $n = 80$ is the largest solution. What is the largest n such that $V_0(n) = 2$?

Some further problems are given in their paper.

P. Erdős & J. L. Selfridge, Some problems on the prime factors of consecutive integers, *Illinois J. Math.*, **11**(1967) 428–430.

A. Schinzel, Unsolved problem 31, *Elem. Math.*, **14**(1959) 82–83.

OEIS: A059756-059757.

B28 Consecutive numbers with distinct prime factors.

Selfridge asked: do there exist n consecutive integers, each having either two distinct prime factors less than n or a repeated prime factor less than n ? He gives two examples:

1. the numbers $a + 11 + i$ ($1 \leq i \leq n = 115$) where $a \equiv 0 \pmod{2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2}$ and $a + p \equiv 0 \pmod{p^2}$ for each prime p , $13 \leq p \leq 113$;
2. the numbers $a + 31 + i$ ($1 \leq i \leq n = 1329$) where $a + p \equiv 0 \pmod{p^2}$ for each prime p , $37 \leq p \leq 1327$ and $a \equiv 0 \pmod{2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31^2}$.

It is harder to find examples of n consecutive numbers, each one divisible by two distinct primes less than n or by the square of a prime $< n/2$, though he believes that they could be found by computer.

This is related to the problem: find n consecutive integers, each having a composite common factor with the product of the other $n - 1$. If the composite condition is relaxed, and one asks merely for a common factor greater than 1, then $2184 + i$ ($1 \leq i \leq n = 17$) is a famous example.

Alfred Brauer, On a property of k consecutive integers, *Bull. Amer. Math. Soc.*, **47**(1941) 328–331; *MR* **2**, 248.

Ronald J. Evans, On blocks of N consecutive integers, *Amer. Math. Monthly* **76**(1969) 48–49.

Ronald J. Evans, On N consecutive integers in an arithmetic progression, *Acta Sci. Math. Univ. Szeged*, **33**(1972) 295–296; *MR* **47** #8408.

Heiko Harborth, Eine Eigenschaft aufeinanderfolgender Zahlen, *Arch. Math. (Basel)* **21**(1970) 50–51; *MR* **41** #6771.

Heiko Harborth, Sequenzen ganzer Zahlen, *Zahlentheorie (Tagung, Math. Forschungsinst. Oberwolfach, 1970)* 59–66; *MR* **51** #12775.

S. S. Pillai, On m consecutive integers I, *Proc. Indian Acad. Sci. Sect. A*, **11**(1940) 6–12; *MR* **1**, 199; II **11**(1940) 73–80; *MR* **1**, 291; III **13**(1941) 530–533; *MR* **3**, 66; IV *Bull. Calcutta Math. Soc.*, **36**(1944) 99–101; *MR* **6**, 170.

OEIS: A059756-059757.

B29 Is x determined by the prime divisors of $x + 1, x + 2, \dots, x + k$?

Alan R. Woods asks if there is a positive integer k such that every x is uniquely determined by the (sets of) prime divisors of $x + 1, x + 2, \dots$,

$x + k$. Perhaps $k = 3$?

For primes less than 23 there are four ambiguous cases for $k = 2$: $(x+1, x+2) = (2,3)$ or $(8,9)$; $(6,7)$ or $(48,49)$; $(14,15)$ or $(224,225)$; $(75,76)$ or $(1215,1216)$. The first three of these are members of the infinite family $(2^n - 2, 2^n - 1)$, $(2^n(2^n - 2), (2^n - 1)^2)$. Compare **B19**.

D. H. Lehmer, On a problem of Størmer, *Illinois J. Math.*, **8**(1964) 57–79; *MR* **28** #2072.

OEIS: A059756-059757.

B30 A small set whose product is square.

Erdős, Graham & Selfridge want us to find the least value of t_n so that the integers $n + 1, n + 2, \dots, n + t_n$ contain a subset the product of whose members with n is a square. The Thue–Siegel theorem implies that $t_n \rightarrow \infty$ with n , faster than a power of $\ln n$.

I was asked for a justification or reference for this last sentence. Andrew Granville kindly supplied the following comments:

The point is that if you do have such a subset then there is an integer point (n, m) on some hyperelliptic curve

$$y^2 = x(x + i_1)(x + i_2) \cdots (x + i_k)$$

where $0 < i_1 < i_2 < \dots < i_k \leq t_n$. If t_n were to be $< T$ for infinitely many n then some such curve would have infinitely many rational points (or even integer points), contradicting Faltings's Theorem if $k \geq 3$, and Thue's Theorem for $k \geq 0$. Thus $t_n \rightarrow \infty$.

More difficult would be to estimate quite how fast we can prove $t_n \rightarrow \infty$. To do this one needs some effective version of Faltings's or Thue's Theorem. There is probably a pretty good effective version of Thue's Theorem, especially for hyperelliptic curves.

It is amusing to note that the *abc*-conjecture is certainly applicable to this question, via Elkies's paper (see ref. at **B19**) or Langevin, though this would take some working out. Presumably $t_n > n^c$ for some $c > 0$ (assuming *abc*) though I have not proved this! Perhaps Silverman is interested in this question.

Joseph Silverman responded:

Granville's argument that $t_n \rightarrow \infty$ is fine, but it depends on the fact that a hyperelliptic curve has only finitely many integer

points (due to Siegel, I believe, not Thue). It seems to me that Theorem 1 of my paper with Evertse might be helpful. Let

$$f(X) = X(X + i_1) \cdots (X + i_k) \quad \text{with} \quad 0 < i_1 < \cdots < i_k$$

and assume that $k \geq 3$, since the case $k = 2$ can be dealt with separately. Then Theorem 1(b) can be applied with $K = \mathbb{Q}$, $m = 1$, S is the infinite place of \mathbb{Q} together with the primes dividing $D(f)$, the discriminant of f , $s = |S| = 1 + \nu(D(f))$, R_S is the ring of S -integers in \mathbb{Q} , $L = K = \mathbb{Q}$, $M = 1$, $n = 2$, $\kappa_n(L) = 0$. This appears to give that the number of integer solutions to $Y^2 = f(X)$ is $\leq 7^{4+9s} = 7^{13+9\nu(D(f))}$.

Selfridge (W. No. Theory problem 97:22) says that it is conjectured that 6 and 392 are the only numbers of shape $n = rs^2$ with $r > 1$ and squarefree for which there do not exist a, b with $n < a < b < r(s+1)^2$ and nab a square.

To revert to the opening paragraph, Selfridge has shown that $t_n \leq \max(P(n), 3\sqrt{n})$, where $P(n)$ is the largest prime factor of n .

Alternatively, is it true that for every c there is an n_0 so that for every $n > n_0$ the products $\prod a_i$, taken over $n < a_1 < \cdots < a_k < n + (\ln n)^c$ ($k = 1, 2, \dots$) are all distinct? Erdős, Graham & Selfridge proved this for $c < 2$.

Selfridge conjectures that if n is not a square, and t is the next larger number than n such that nt is a square, then, unless $n = 8$ or 392, it is always possible to find r and s , $n < r < s < t$ such that nrs is a square. E.g., if $n = 240 = 2^4 \cdot 3 \cdot 5$ then $t = 375 = 3 \cdot 5^3$ and we can find $r = 243 = 3^5$ and $s = 245 = 5 \cdot 7^2$. Selfridge and Meyerowitz have confirmed the conjecture for $n < 10^{30000}$.

Several of the papers referred to at **D10** are relevant here.

P. Erdős & Jan Turk, Products of integers in short intervals, *Acta Arith.*, **44**(1984) 147–174; *MR* **86d**:11073.

Paul Erdős, Janice Malouf, John Selfridge & Esther Szekeres, Subsets of an interval whose product is a power, Paul Erds memorial collection. *Discrete Math.*, **200**(1999) 137–147; *MR* **2000e**:11017.

Jan-Hendrik Evertse & J. H. Silverman, Uniform bounds for the number of solutions to $Y^n = f(X)$ *Math. Proc. Cambridge Philos. Soc.*, **100**(1986) 237–248; *MR* **87k**:11034.

L. Hajdu & Ákos Pintér, Square product of three integers in short intervals, *Math. Comput.*, **68**(1999) 1299–1301; **99j**:11027.

Michel Langevin, Cas d'égalité pour le théorème de Mason et applications de la conjecture (abc) , *C.R. Acad. Sci. Paris Sér. I Math.*, **317**(1993) 441–444; *MR* **94j**:11027.

Shorey, T. N.(6-TIFR) Perfect powers in products of integers from a block of consecutive integers. *Acta Arith.* 49 (1987), no. 1, 71–79.88m:11002

T. N. Shorey & Yu. V. Nesterenko, Perfect powers in products of integers from a block of consecutive integers, II *Acta Arith.*, **76**(1996) 191–198; *MR* **97d**:11005.

OEIS: A068568.

B31 Binomial coefficients.

Earl Ecklund, Roger Eggleton, Erdős & Selfridge (see **B23**) write the **binomial coefficient** $\binom{n}{k} = n!/k!(n-k)!$ as a product UV in which every prime factor of U is at most k and every prime factor of V is greater than k . There are only finitely many cases with $n \geq 2k$ for which $U > V$. They determine all such cases except when $k = 3, 5$ or 7 .

S. P. Khare lists all cases with $n \leq 551$: $k = 3$, $n = 8, 9, 10, 18, 82, 162$; $k = 5$, $n = 10, 12, 28$; and $k = 7$, $n = 21, 30, 54$.

Most binomial coefficients $\binom{n}{k}$ with $n \geq 2k$ have a prime factor $p \leq n/k$. After some computing with Lacampagne & Erdős, Selfridge conjectured that this inequality is true whenever $n > 17.125k$. A slightly stronger conjecture is that any such binomial coefficient has least prime factor $p \leq n/k$ or $p \leq 17$ with just 4 exceptions: $\binom{62}{6}$, $\binom{959}{56}$, $\binom{474}{66}$, $\binom{284}{28}$ for which $p = 19, 19, 23$ and 29 respectively.

These authors define the **deficiency** of the binomial coefficient $\binom{n+k}{k}$, $k \leq n$, as the number of i for which $b_i = 1$, where $n+i = a_i b_i$, $1 \leq i \leq k$, the prime factors of b_i are greater than k , and $\prod a_i = k!$. Then

$\binom{44}{8}$, $\binom{74}{10}$, $\binom{174}{12}$, $\binom{239}{14}$, $\binom{5179}{27}$, $\binom{8413}{28}$, $\binom{8414}{28}$ and $\binom{96622}{42}$ each have deficiency 2; $\binom{46}{10}$, $\binom{47}{10}$, $\binom{241}{16}$, $\binom{2105}{25}$, $\binom{1119}{27}$ and $\binom{6459}{33}$ have deficiency 3; $\binom{47}{11}$ has deficiency 4; and $\binom{284}{28}$ has deficiency 9; and they conjecture that there are no others with deficiency greater than 1. Are there only finitely many binomial coefficients with deficiency 1?

Erdős & Selfridge noted that if $n \geq 2k \geq 4$, then there is at least one value of i , $0 \leq i \leq k-1$, such that $n-i$ does not divide $\binom{n}{k}$, and asked for the least n_k for which there was only one such i . For example, $n_2 = 4$, $n_3 = 6$, $n_4 = 9$, $n_5 = 12$. $n_k \leq k!$ for $k \geq 3$.

For a positive integer k , the Erdős-Selfridge function is the least integer $g(k) > k+1$ such that all prime factors of $\binom{g(k)}{k}$ exceed k . The first few values are

$k =$	2	3	4	5	6	7	8	9	10	11	12	13
$g(k) =$	6	7	7	23	62	143	44	159	46	47	174	2239

They are far from being monotonic. Ecklund, Erdős & Selfridge conjecture that \limsup & \liminf of $g(k+1)/g(k)$ are ∞ & 0 . They also conjecture that, for each n and for $k > k_0(n)$, $g(k) > k^n$, that $\lim g(k)^{1/k} = 1$ and that $g(k) < e^{c\pi(k)}$. For example, they conjecture that $g(k) > k^3$ for $k > 35$ and that $g(k) > k^5$ for $k > 100$. They show that $g(k) < k^2 L_k P_l$ with

$l = \lfloor 6k/\ln k \rfloor$, where L_k is the l.c.m. of $1, 2, \dots, k$ and P_l is the product of the primes not exceeding l and Erdős, Lacampagne & Selfridge show that $g(k) > ck^2/\ln k$. The tables of $g(k)$ have been extended by Scheidler & Williams (ref. at **B33**) to $k \leq 140$ and, with the help of Lukes, to $k \leq 200$.

$$g(200) = 520\,8783889271\,0191382732$$

Granville shows that the abc-conjecture implies that there are only finitely many powerful binomial coefficients $\binom{n}{k}$ with $3 \leq k \leq n/2$.

Harry Ruderman asks for a proof or disproof that for every pair (p, q) of nonnegative integers there is a positive integer n such that

$$\frac{(2n-p)!}{n!(n+q)!}$$

is an integer.

A problem which has briefly baffled good mathematicians is: is $\binom{n}{r}$ ever prime to $\binom{n}{s}$, $0 < r < s \leq n/2$? The negative answer follows from the identity

$$\binom{n}{s} \binom{s}{r} = \binom{n}{r} \binom{n-r}{s-r}.$$

Erdős & Szekeres ask if the greatest prime factor of the g.c.d. is always greater than r ; the only counterexample with $r > 3$ that they noticed is

$$\gcd\left(\binom{28}{5}, \binom{28}{14}\right) = 2^3 \cdot 3^3 \cdot 5$$

On 98-01-12 David Gale reported that a Berkeley student proposed a natural generalization, of the noncoprimality of two binomial coefficients, to trinomial coefficients, with the further generalization to k -nomial coefficients. If $n = a + b + c$, let $T(n; a, b, c) = n!/(a!b!c!)$. The conjecture, supported by fairly extensive computer evidence, is that for $0 < a, b, c < n$, every *three* $T(n; a, b, c)$ have a common factor. George Bergman and Hendrik Lenstra and others have some relevant partial results.

Wolstenholme's theorem states that if n is a prime > 3 , then

$$\binom{2n-1}{n} \equiv 1 \pmod{n^3}.$$

There are no composite solutions for $n < 10^9$ and it is conjectured that there are none. Call a prime p satisfying $\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}$ a **Wolstenholme prime**. McIntosh shows that p is Wolstenholme just if it divides the numerator of the Bernoulli number B_{p-3} (see **A17**). The two known such are 16843 and 2124679. He conjectures that there are infinitely many and that none satisfies $\binom{2p-1}{p-1} \equiv 1 \pmod{p^5}$.

For other problems and results on the divisors of binomial coefficients, see **B33**.

Emre Alkan, Variations on Wolstenholme's theorem, *Amer. Math. Monthly*, **101**(1994) 1001–1004.

D. F. Bailey, Two p^3 variations of Lucas's theorem, *J. Number Theory*, **35**(1990) 208–215; *MR 90f*:11008.

M. Bayat, A generalization of Wolstenholme's theorem, *Amer. Math. Monthly*, **104**(1997) 557–560 (but see Gessel reference).

Daniel Berend & Jørgen E. Harmse, On some arithmetical properties of middle binomial coefficients, *Acta Arith.*, **84**(1998) 31–41.

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P. Erdős & J. L. Selfridge, Problem 6447, *Amer. Math. Monthly* **90**(1983) 710; **92**(1985) 435–436.

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Ira M. Gessel, Wolstenholme revisited, *Amer. Math. Monthly*, **105**(1998) 657–658; *MR 99e*:11009.

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Andrew Granville, On the scarcity of powerful binomial coefficients, *Mathematika* **46**(1999) 397–410; *MR 2002b*:11029.

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Hong Shao-Fang, A generalization of Wolstenholme's theorem, *J. South China Normal Univ. Natur. Sci. Ed.*, **1995** 24–28; *MR 99f*:11004.

Gerhard Larcher, On the number of odd binomial coefficients, *Acta Math. Hungar.*, **71**(1996) 183–203.

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Harry D. Ruderman, Problem 714, *Cruz Math.*, **8**(1982) 48; **9**(1983) 58.

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J. W. Sander, On the order of prime powers dividing $\binom{2n}{n}$, *Acta Math.*, **174**(1995) 85–118; *MR 96b*:11018.

J. W. Sander, A story of binomial coefficients and primes, *Amer. Math. Monthly*, **102**(1995) 802–807; *MR 96m*:11015.

Renate Scheidler & Hugh C. Williams, A method of tabulating the number-theoretic function $g(k)$, *Math. Comput.*, **59**(1992) 199, 251–257; *MR 92k*:11146.

David Segal, Problem E435, partial solution by H.W. Brinkman, *Amer. Math. Monthly*, **48**(1941) 269–271.

OEIS: A034602.

B32 Grimm's conjecture.

Grimm has conjectured that if $n + 1, n + 2, \dots, n + k$ are all composite, then there are distinct primes p_{i_j} such that $p_{i_j} | (n + j)$ for $1 \leq j \leq k$. For example

1802 1803 1804 1805 1806 1807 1808 1809 1810

are respectively divisible by

53 601 41 19 43 139 113 67 181

and

114 115 116 117 118 119 120 121 122 123 124 125 126

by

19 23 29 13 59 17 2 11 61 41 31 5 7

Ramachandra, Shorey & Tijdeman proved, under the hypothesis of Schinzel mentioned in **A2**, that there are only finitely many exceptions to Grimm's conjecture.

Erdős & Selfridge asked for an estimate of $f(n)$, the least number such that for each m there are distinct integers $a_1, a_2, \dots, a_{\pi(n)}$ in the interval $[m + 1, m + f(n)]$ with $p_i | a_i$ where p_i is the i th prime. They and Pomerance show that, for large n ,

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B33 Largest divisor of a binomial coefficient.

What can one say about the largest divisor, less than n , of the binomial coefficient $\binom{n}{k} = n!/k!(n-k)!$? Erdős points out that it is easy to show that it is at least n/k and conjectures that there may be one between cn and n for any $c < 1$ and n sufficiently large. Marilyn Faulkner showed that if p is the least prime $> 2k$ and $n \geq p$, then $\binom{n}{k}$ has a prime divisor $\geq p$, except for $\binom{9}{2}$ and $\binom{10}{3}$. Earl Ecklund showed that if $n \geq 2k > 2$ then $\binom{n}{k}$ has a prime divisor $p \leq n/2$, except for $\binom{7}{3}$.

John Selfridge conjectures that if $n \geq k^2 - 1$, then, apart from the exception $\binom{62}{6}$, there is a prime divisor $\leq n/k$ of $\binom{n}{k}$. Among those binomial coefficients whose least prime factor p is $\geq n/k$ there may be only a finite number with $p \geq 13$, but there could be infinitely many with $p = 7$. That there are infinitely many with $p = 5$ was proved by Erdős, Lacampagne & Selfridge (**B31**).

A classical theorem, discovered independently by Sylvester and Schur, stated that the product of k consecutive integers, each greater than k , has a prime divisor greater than k . Leo Moser conjectured that the Sylvester-Schur theorem holds for primes $\equiv 1 \pmod{4}$, in the sense that for n sufficiently large (and $\geq 2k$), $\binom{n}{k}$ has a prime divisor $\equiv 1 \pmod{4}$ which is greater

than k . However, Erdős does not think that this is true, but it may not be at all easy to settle. In this connexion John Leech notices that the fourteen integers 280213, ..., 280226 have no prime factor of the form $4m + 1 > 13$.

Thanks to Ira Gessel and John Conway, we can say that the generalization of the **Catalan numbers** $\frac{1}{n+1} \binom{2n}{n}$, requested in the first edition by Neil Sloane, is $\frac{(n,r)}{n} \binom{n}{r}$, which is always an integer (multiply by n and by r and Euclid knew that (n, r) is a linear combination of n and r). These are also known as generalized ballot numbers and they occur when enumerating certain lattice paths.

If $f(n)$ is the sum of the reciprocals of those primes $< n$ which do not divide $\binom{2n}{n}$, then Erdős, Graham, Ruzsa & Straus conjectured that there is an absolute constant c so that $f(n) < c$ for all n . Erdős also conjectured that $\binom{2n}{n}$ is never squarefree for $n > 4$. Since $4 \mid \binom{2n}{n}$ unless $n = 2^k$, it suffices to consider

$$\binom{2^{k+1}}{2^k}.$$

Sárközy proved this for n sufficiently large and Sander has shown, in a precise sense, that binomial coefficients near the centre of the Pascal triangle are not squarefree. Granville & Ramaré completed Sárközy's proof by showing that $k > 300000$ was sufficiently large, and checking it computationally for $2 \leq k \leq 300000$. They also improved Sander's result by showing that there is a constant δ , $0 < \delta < 1$, such that if $\binom{n}{k}$ is squarefree then k or $n - k$ must be $< n^\delta$ for sufficiently large n . They conjecture that k or $n - k$ must in fact be $< (\ln n)^{2-\delta}$, and that this is best possible in the sense that there are infinitely many squarefree $\binom{n}{k}$ with $\frac{1}{2}n > k > c(\ln n)^2$ for some $c > 0$. They prove such a result for $\frac{1}{2}n > k > \frac{1}{5} \ln n$. They show that there is a constant $\rho_k > 0$ such that the number of $n \leq N$ with $\binom{n}{k}$ squarefree is $\sim \rho_k N$. Since $\rho_k < c/k^2$ for some $c > 0$, they conjecture that there is a constant $\gamma > 0$ such that the number of squarefree entries in the first N rows of Pascal's triangle is $\sim \gamma N$.

Erdős has also conjectured that for $k > 8$, 2^k is not the sum of distinct powers of 3 [$2^8 = 3^5 + 3^2 + 3 + 1$]. If that's true, then for $k \geq 9$,

$$3 \nmid \binom{2^{k+1}}{2^k}.$$

In answer to the question, is $\binom{342}{171}$ the largest $\binom{2n}{n}$ which is not divisible by the square of an odd prime, Eugene Levine gave the examples $n = 784$ and 786. Erdős feels sure that there are no larger such n .

Denote by $e = e(n)$ the largest exponent such that, for some prime p , p^e divides $\binom{2n}{n}$. It is not known whether $e \rightarrow \infty$ with n . On the other hand Erdős cannot disprove $e > c \ln n$.

Ron Graham offers \$100.00 for deciding if $\binom{2n}{n}, 105 = 1$ infinitely often. Kummer knew that n , when written in base 3 or 5 or 7, would have to have only the digits 0, 1 or 0, 1, 2 or 0, 1, 2, 3 respectively. H. Gupta & S. P. Khare found the 14 values 1, 10, 756, 757, 3160, 3186, 3187, 3250, 7560, 7561, 7651, 20007, 59548377, 59548401 of n less than 7^{10} , while Peter Montgomery, Khare and others found many larger values.

Erdős, Graham, Ruzsa & Straus showed that for any *two* primes p, q there are infinitely many n for which $\binom{2n}{n}, pq = 1$. If $g(n)$ is the smallest odd prime factor of $\binom{2n}{n}$, then $g(3160) = 13$ and $g(n) \leq 11$ for $3160 < n < 10^{10000}$.

More complicated quotients of products of factorials which yield integers have been considered by Picon.

Gould repeated Hermite's 1889 observations that

$$\frac{m}{\binom{m}{n}} \mid \binom{m}{n} \quad \text{and} \quad \frac{m-n+1}{\binom{m}{n+1}} \mid \binom{m}{n}$$

and asked for more general a, b, c, r, s, u, v such that

$$\frac{am + bn + c}{(rm + s, un + v)} \mid \binom{m}{n}$$

John McKay notes that $\binom{72-1}{72/2}$ is squarefree, and states that $\binom{2n-1}{n}$ is square-free only for $n = 1, 2, 3, 4, 6, 9, 10, 12, 36$ with $n \leq 500$.

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B34 If there's an i such that $n - i$ divides $\binom{n}{k}$.

If $H_{k,n}$ is the proposition: there is an i , $0 \leq i < k$ such that $n - i$ divides $\binom{n}{k}$, then Erdős asked if $H_{k,n}$ is true for all k when $n \geq 2k$. Schinzel gave the counterexample $n = 99215$, $k = 15$. If H_k is the proposition: $H_{k,n}$ is true for all n , then Schinzel showed that H_k is false for $k = 15, 21, 22, 33, 35$ and thirteen other values of k . He showed that H_k is true for all other $k \leq 32$ and asked if there are infinitely many k , other than prime-powers, for which H_k is true: he conjectures not and later reported that it is true for $k = 34$, but for no other non-prime-powers between 34 and 201.

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B35 Products of consecutive numbers with the same prime factors.

Let $f(n)$ be the least integer such that at least one of the numbers $n, n+1, \dots, n+f(n)$ divides the product of the others. It is easy to see that $f(k!) = k$ and $f(n) > k$ for $n > k!$ Erdős has also shown that

$$f(n) > \exp((\ln n)^{1/2-\epsilon})$$

for an infinity of values of n , but it seems difficult to find a good upper bound for $f(n)$.

Erdős asks if $(m+1)(m+2)\cdots(m+k)$ and $(n+1)(n+2)\cdots(n+l)$ with $k \geq l \geq 3$ can contain the same prime factors infinitely often. For example $(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot 7 \cdot 8 \cdot 9 \cdot 10$ and $14 \cdot 15 \cdot 16$ and $48 \cdot 49 \cdot 50$; also $(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12$ and $98 \cdot 99 \cdot 100$. For $k = l \geq 3$ he conjectures that this happens only finitely many times.

If $L(n; k)$ is the l.c.m. of $n+1, n+2, \dots, n+k$, then Erdős conjectures that for $l > 1$, $n \geq m+k$, $L(m; k) = L(n; l)$ has only a finite number of solutions. Examples are $L(4; 3) = L(13; 2)$ and $L(3; 4) = L(19; 2)$. He asks if there are infinitely many n such that for all k ($1 \leq k < n$) we have $L(n; k) > L(n-k; k)$. What is the largest $k = k(n)$ for which this inequality can be reversed? He notes that it is easy to see that $k(n) = o(n)$, but he believes that much more is true. He expects that for every $\epsilon > 0$ and $n > n_0(\epsilon)$, $k(n) < n^{1/2+\epsilon}$ but cannot prove this.

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B36 Euler's totient function.

Euler's **totient function**, $\phi(n)$, is the number of numbers not greater than n and prime to n . For example $\phi(1) = \phi(2) = 1$, $\phi(3) = \phi(4) = \phi(6) = 2$, $\phi(5) = \phi(8) = \phi(10) = \phi(12) = 4$, $\phi(7) = \phi(9) = 6$. Are there infinitely many pairs of consecutive numbers, $n, n+1$, such that $\phi(n) = \phi(n+1)$? For example, $n = 1, 3, 15, 104, 164, 194, 255, 495, 584, 975$. It is not even known if $|\phi(n+1) - \phi(n)| < n^\epsilon$ has an infinity of solutions for each $\epsilon > 0$. Jud McCranie found 1267 solutions of $\phi(n) = \phi(n+1)$ with $n < 10^{10}$, the largest of which is $n = 9985705185$, $\phi(n) = \phi(n+1) = 2^{11}3^{57} \cdot 11$. He also looked for solutions of $\phi(n+k) = \phi(n)$. Schinzel conjectures that for every even k there are infinitely many solutions, but observes that the corresponding conjecture with k odd is implausible. Indeed, when k is an odd multiple of 3, in a search to $n = 10^{10}$, McCranie found only a few solutions. For $k = 3$ only $n = 3$ and $n = 5$. In fact, the only value of $k \equiv 3 \pmod{6}$ which yielded as many as 13 solutions was $k = 141$ and

the largest solution found was $n = 715$ with $k = 245$, except that $k = 27$ yielded a very atypical $n = 4135966808$. Other odd k were still yielding a steady supply of solutions; the most for $k < 101$ being 1673 solutions for $k = 47$, and the least being 278 for $k = 55$, except that $k = 35$ gave only 29 solutions, 12 of them $> 10^9$, including $n = 9423248800$.

Sierpiński has shown that there is at least one solution for each k and Schinzel & Wakulicz that there are at least two for each $k < 2 \cdot 10^{58}$. For even k Holt raises this to $k < 1.38 \cdot 10^{26595411}$.

Mąkowski has shown that $\phi(n+k) = 2\phi(n)$ has at least one solution for every k . For the equation $\phi(n+k) = 3\phi(n)$ see the solution to Problem E3215 in *Amer. Math. Monthly*, **96**(1989) 63–64.

McCranie found no solutions of $\phi(n+k) = \phi(n) + \phi(k)$ for $k = 3$, nor any of $\phi(n) = \phi(n+1) = \phi(n+2)$, for $n < 10^{10}$, apart from $\phi(5186) = \phi(5187) = \phi(5188) = 2^5 3^4$. Other curiosities are $\phi(25930) = \phi(25935) = \phi(25940) = \phi(25942) = 2^7 3^4$ and $\phi(404471) = \phi(404473) = \phi(404477) = 2^8 3^2 5^{27}$.

Sid Graham, Jeffrey Holt & Carl Pomerance can show that $|\phi(n) - \phi(n+2)| < n^{9/10}$ infinitely often by looking at $n = 2(2p-1)$, where p is a prime, and using a theorem of Chen that says that there are infinitely many primes p such that either $2p-1$ is a prime or $2p-1$ is a product of two prime factors, both of which exceed $p^{1/10}$. With a little work, they could improve the $9/10$ exponent. On the other hand, they don't see any similar argument for $|\phi(n) - \phi(n+1)|$. This seems to be a fundamentally different problem.

Erdős lets a_1, \dots, a_t be the longest sequence for which

$$a_1 < \dots < a_t \leq n \quad \text{and} \quad \phi(a_1) < \dots < \phi(a_t)$$

and suggests that $t = \pi(n)$. Can one even prove $t < (1 + o(1))\pi(n)$ or at least $t = o(n)$? Similar questions can be asked about $\sigma(n)$.

Nontotients are positive *even* values of n for which $\phi(x) = n$ has no solution; for example, $n = 14, 26, 34, 38, 50, 62, 68, 74, 76, 86, 90, 94, 98$. The number, $\#(y)$, of these less than y has been calculated by the Lehmers.

y	10^3	10^4	$2 \cdot 10^4$	$3 \cdot 10^4$	$4 \cdot 10^4$	$5 \cdot 10^4$	$6 \cdot 10^4$	$7 \cdot 10^4$	$8 \cdot 10^4$	$9 \cdot 10^4$
$\#(y)$	210	2627	5515	8458	11438	14439	17486	20536	23606	26663

Zhang Ming-Zhi has shown that for every positive integer m , there is a prime p such that mp is a nontotient.

Browkin & Schinzel have proved the conjecture of Sierpiński and Erdős that there are infinitely many noncototients, by showing that none of the numbers $2^k \cdot 509203$, $k = 1, 2, \dots$ is of the form $x - \phi(x)$.

Erdős & Hall have shown that the number, $\Phi(y) = y - \#(y)$, of n for which $\phi(x) = n$ has a solution is $ye^{f(y)}/\ln y$, where $f(y)$ lies between $c(\ln \ln \ln y)^2$ and $c(\ln y)^{1/2}$. Maier & Pomerance more recently showed that the lower bound was correct, with $c \approx 0.8178$. Erdős conjectures that

$\Phi(cy)/\Phi(y) \rightarrow c$, and that this, if true, may be the best substitute that one can find for an asymptotic formula for $\Phi(y)$.

Noncototients are positive values of n for which $x - \phi(x) = n$ has no solution; for example, $n = 10, 26, 34, 50, 52, 58, 86, 100$. Sierpiński and Erdős conjecture that there are infinitely many noncototients.

Erdős once asked if it was true that for every ϵ there is an n with $\phi(n) = m$, $m < \epsilon n$ and for no $t < n$ is $\phi(t) = m$; perhaps there are many such n .

Michael Ecker has asked for which values of x do each of the series $\sum_{n=1}^{\infty} \phi(n)/n^x$ and $\sum_{n=1}^{\infty} (-1)^{n+1} \phi(n)/n^x$ converge.

Donald Newman has shown that if a, b, c, d are nonnegative integers with $a, c > 0$ and $ad - bc \neq 0$, then there exists a positive integer n such that $\phi(an + b) < \phi(cn + d)$, but there is no such $n < 2 \cdot 10^6$ such that $\phi(30n + 1) < \phi(30n)$. In fact Greg Martin showed that the least such n has 1116 decimal digits. It is

$n = 232909\ 8101754967\ 9381404968\ 4205233780\ 0048598859\ 6605123536\ 3345311075$
 $8883445287\ 2315452798\ 4260176895\ 8541826348\ 0290710927\ 1610432287\ 6529769074$
 $6757436240\ 0134090318\ 3559621214\ 7678571289\ 1544538210\ 9667040369\ 9088529244$
 $6155135679\ 7175658080\ 6376638384\ 6220120606\ 1438265094\ 3354025008\ 5111624970$
 $4645413809\ 3448637568\ 8208918750\ 6406746299\ 4246549936\ 9036578640\ 3317590359$
 $7936930268\ 5371156272\ 2454663962\ 2786562195\ 1101808240\ 6922599602\ 0309133058$
 $9296656888\ 0117910114\ 1606263156\ 5320593772\ 2871189137\ 2860899790\ 1791216356$
 $1086654763\ 0608074012\ 1528236888\ 6801201524\ 7913832745\ 1088404280\ 9290483149$
 $1212278487\ 9758304016\ 8324367515\ 3225518564\ 0249324065\ 4924915110\ 7252158598$
 $0547438748\ 6893071593\ 6348123396\ 5802331725\ 0336638626\ 1895716897\ 4043547448$
 $8796632179\ 7108144561\ 9618789985\ 4720743031\ 0030363607\ 8827273695\ 5511620897$
 $2543511024\ 6701964021\ 0458490818\ 1160442733\ 1227553783\ 5908215100\ 9160756717$
 $8842569576\ 6995480382\ 1767317189\ 5383249326\ 8006674329\ 9353118643\ 7659910632$
 $8654198923\ 7095772215\ 4266351039\ 8085481508\ 2886896882\ 0675198820\ 3811355236$
 $4636120238\ 3915218571\ 0178014630\ 1149110878\ 4343253284\ 3935116502\ 5450659792$
 $3969653616\ 8138977106\ 2175669382\ 7471154701\ 1512223204\ 4334740818\ 0047964860$

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OEIS: A000010, A001274, A001494, A003275, A005277-005278, A007015, A007617, A050472-050473, A051953.

B37 Does $\phi(n)$ properly divide $n - 1$?

D. H. Lehmer has conjectured that there is no composite value of n such that $\phi(n)$ is a divisor of $n - 1$, i.e., that for no value of n is $\phi(n)$ a *proper* divisor of $n - 1$. Such an n must be a Carmichael number (**A13**). He showed that it would have to be the product of at least seven distinct primes, and Lieuwens has shown that if $3|n$, then $n > 5.5 \cdot 10^{571}$ and $\omega(n) \geq 212$; if the smallest prime factor of n is 5, then $\omega(n) \geq 11$; if the smallest prime factor of n is at least 7, then $\omega(n) \geq 13$. This supersedes and corrects the work of Schuh. Masao Kishore has shown that at least 13 primes are needed in any case, and Cohen & Hagi have improved this to 14. Siva Rama Prasad & Subbarao improve Lieuwens's 212 result to $\omega(n) \geq 1850$ and Hagi to $\omega(n) \geq 298848$. Siva Rama Prasad & Rangamma show that if $3|n$, n composite, $M\phi(n) = n - 1$, $M \neq 4$, then $\omega(n) \geq 5334$.

Pomerance has proved that the number of composite n less than x for which $\phi(n) | n - 1$ is

$$O(x^{1/2}(\ln x)^{3/4}(\ln \ln x)^{-1/2})$$

and Shan Zun improved the exponent $\frac{3}{4}$ to $\frac{1}{2}$.

Schinzel notes that if $n = p$ or $2p$, where p is prime, then $\phi(n) + 1$ divides n and asks if the converse is always true. Segal (see paper with Cohen) observes that Schinzel's question reduces to that of Lehmer, that it arises in group theory, and may have been raised by G. Hajós (see Miech's paper, though there it is attributed to Gordon).

Lehmer gives eight solutions to $\phi(n) | n + 1$, namely $n = 2$, $n = 2^{2^k} - 1$ for $1 \leq k \leq 5$, $n = n_1 = 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929$ and $n = n_1 \cdot 83623937$. [Note that $353 = 11 \cdot 2^5 + 1$, $929 = 29 \cdot 2^5 + 1$, $83623937 = 11 \cdot 29 \cdot 2^{18} + 1$ and $(353 - 2^8)(929 - 2^8) = 2^{16} - 2^8 + 1$.] This exhausts the solutions with less than seven factors. Victor Meally notes that $n = n_1 \cdot 83623937 \cdot 699296672132097$ would be a solution were the largest factor a prime, but Peter Borwein notes that this is divisible by 73. The Borweins & Roland Girgensohn conjecture that there are no more solutions.

See Erick Wong's thesis, quoted at **A17**.

If n is prime, it divides $\phi(n)d(n) + 2$. Is this true for any composite n other than $n = 4$? Subbarao also notes that if n is prime, then $n\sigma(n) \equiv 2 \pmod{\phi(n)}$, and also only if $n = 4, 6$ or 22 . Jud McCranie finds no others with $n < 10^{10}$ for either problem.

Subbarao has an analogous conjecture to Lehmer's, based on the function $\phi^*(n) = \prod (p^\alpha - 1)$, where the product is taken over the maximal prime power divisors of n , $p^\alpha || n$. He conjectures that $\phi^*(n) | (n - 1)$ if and only if n is a power of a prime. He also has a 'dual' of Lehmer's conjecture, namely that $\psi(n) \equiv 1 \pmod n$ only when n is a prime, where $\psi(n)$ is Dedekind's function (see **B41**). Again, Jud McCranie finds no counterexamples $< 10^{10}$ to either of these conjectures.

Ron Graham makes the following conjecture

¿ For all k there are infinitely many n such that $\phi(n) | (n - k)$?

He observes that it is true for $k = 0$, $k = 2^a$ ($a \geq 0$) and $k = 2^a 3^b$ ($a, b > 0$) for example. Pomerance (see *Acta Arith.* paper quoted in **B2**) has treated Graham's problem.

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Shan Zun, On composite n for which $\phi(n)|n - 1$, *J. China Univ. Sci. Tech.*, **15**(1985) 109–112; *MR* **87h**:11007.

OEIS: A051948.

B38 Solutions of $\phi(m) = \sigma(n)$.

Are there infinitely many pairs of numbers m, n such that $\phi(m) = \sigma(n)$? Since for p prime $\phi(p) = p - 1$ and $\sigma(p) = p + 1$ this question would be answered affirmatively if there were infinitely many twin primes (**A7**). Also if there were infinitely many Mersenne primes (**A3**) $M_p = 2^p - 1$, since $\sigma(M_p) = 2^p = \phi(2^{p+1})$. However there are many solutions other than these, sometimes displaying little noticeable pattern, e.g., $\phi(780) = 192 = \sigma(105)$.

Erdős remarks that the equation $\phi(x) = n!$ is solvable, and (apart from $n = 2$) $\sigma(y) = n!$ is probably solvable also. Charles R. Wall can show that $\psi(n) = n!$ is solvable for $n \neq 2$, where ψ is Dedekind's function (see **B41**).

Jean-Marie De Koninck asks if R is the radical of n , i.e., the greatest squarefree divisor of n , then are $n = 1$ and 1782 the only solutions of $\sigma(n) = R^2$?

Jean-Marie De Koninck, Problem 10966(b), *Amer. Math. Monthly*, **109**(2002) 759.

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B39 Carmichael's conjecture.

Carmichael's conjecture. For every n it appears to be possible to find an m , not equal to n , such that $\phi(m) = \phi(n)$ and for a few years early in the last century it was thought that Carmichael had proved this. Klee verified the conjecture for $\phi(n) < 10^{400}$, and for all $\phi(n)$ not divisible by $2^{42} \cdot 3^{47}$. Masai & Valette have raised the bound to 10^{100000} , and Schläfly & Wagon to $10^{10900000}$. Pomerance has shown that if n is such that for every prime p for which $p - 1$ divides $\phi(n)$ we have p^2 divides n , then n is a counterexample. He can also show (unpublished) that if the first k primes $p \equiv 1 \pmod{q}$ (where q is prime) are all less than q^{k+1} , then there are no numbers n which satisfy his theorem. This also implies the truth of his

conjecture that $p_k - 1 \mid \prod_{i < k} p_i(p_i - 1)$. The truth of this last conjecture for all k also implies that there are no numbers n which satisfy his theorem.

Define the **multiplicity** of an integer as the number of times it occurs as a value of $\phi(n)$. For example, 6 has multiplicity 4 because $\phi(n) = 6$ for $n = 7, 9, 14, 18$ and no other values of n . The multiplicity may be zero (for any odd $n > 1$, and $n = 14, 26, 34, \dots$), but not, according to the Carmichael conjecture, equal to one. Sierpiński conjectured that all integers greater than 1 occur as multiplicities and Erdős has shown that if a multiplicity occurs once it occurs infinitely often. Kevin Ford has proved Sierpiński's conjecture. He also showed that if there is a counterexample to Carmichael's conjecture, then a positive proportion of totients are counterexamples.

There are examples of *even* numbers n such that there is no *odd* number m such that $\phi(m) = \phi(n)$. Lorraine Foster has given $n = 33817088 = 2^9 \cdot 257^2$ as the least such.

Erdős proved that if $\phi(x) = k$ has exactly s solutions, then there are infinitely many other k for which there are exactly s solutions, and that $s > k^c$ for infinitely many k . If C is the least upper bound of those c for which this is true, then Wooldridge showed that $C \geq 3 - 2\sqrt{2} > 0.17157$. Pomerance used Hooley's improvement on the Brun–Titchmarsh theorem to improve this to $C \geq 1 - 625/512e > 0.55092$ and notes that further improvements by Iwaniec enable him to get $C > 0.55655$ so that $s > k^{5/9}$ for infinitely many k . Erdős conjectures that $C = 1$. In the other direction Pomerance also shows that

$$s < k \exp\{-(1 + o(1)) \ln k \ln \ln k / \ln \ln k\}$$

and gives a heuristic argument to support the belief that this is best possible.

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OEIS: A000010, A000142, A000203, A005100-005101, A014197, A054973, A055486-055489, A055506.

B40 Gaps between totatives.

If $a_1 < a_2 < \dots < a_{\phi(n)}$ are the integers less than n and prime to it, then Erdős conjectured that $\sum (a_{i+1} - a_i)^2 < cn^2/\phi(n)$ and offered \$500.00 for a proof. Hooley showed that, for $1 \leq \alpha < 2$, $\sum (a_{i+1} - a_i)^\alpha \ll n(n/\phi(n))^{\alpha-1}$ and that $\sum (a_{i+1} - a_i)^2 \ll n(\ln \ln n)^2$, Vaughan established the conjecture “on the average” and he & Montgomery finally won the prize.

Jacobsthal asked what bounds can be placed on $J(n) = \max(a_{i+1} - a_i)$. Erdős asks if, for infinitely many x , there are two integers n_1, n_2 , $n_1 < n_2 < x$, $n_1 \perp n_2$, $J(n_1) > \ln x$, $J(n_2) > \ln x$.

In a 95-03-30 letter Bernardo Recamán asks if the set of totatives of every sufficiently large n contains a Pythagorean triple, and whether, for each k , it contains an arithmetic progression of length k .

P. Erdős, On the integers relatively prime to n and on a number-theoretic function considered by Jacobsthal, *Math. Scand.*, **10**(1962) 163–170; *MR* **26** #3651.

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B41 Iterations of ϕ and σ .

Pomerance asks if, for each positive integer n , there is a positive integer k such that $\sigma^k(n)/n$ is an integer. E.g., $(n, k) = (1, 1), (2, 2), (3, 4), (4, 2), (5, 5), (6, 4), (7, 5), \dots$

There is a close relative to the sum of divisors and the sum of the unitary divisors function, which complements Euler's totient function and which is often named for Dedekind. If $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, denote by $\psi(n)$ the product $\prod p_i^{a_i-1}(p_i + 1)$, i.e., $\psi(n) = n \prod (1 + p^{-1})$, where the product is taken over the distinct prime divisors of n . It is easy to see that iteration of the function leads eventually to terms of the form $2^a 3^b$ where b is fixed and a increases by one in successive terms. Given any value of b there are infinitely many values of n which lead to such terms, for example, $\psi^k(2^a 3^b 7^c) = 2^{a+4k} 3^b 7^{c-k}$ ($0 \leq k \leq c$) and $\psi^k(2^a 3^b 7^c) = 2^{a+5k-c} 3^b$ ($k > c$).

That there are values of n for which the iterates of the function $\psi(n) - n$ are unbounded as the number of iterations tends to infinity is the subject of te Riele's thesis (reference at **B8**); the least such n is 318.

If we average ψ with the ϕ -function, $\frac{1}{2}(\phi + \psi)$, and iterate, we produce sequences whose terms become constant whenever they are prime powers; for example $24, \frac{1}{2}(8+48) = 28, \frac{1}{2}(12+48) = 30, \frac{1}{2}(8+72) = 40, \frac{1}{2}(16+72) = 44, \frac{1}{2}(20+72) = 46, \frac{1}{2}(22+72) = 47, \frac{1}{2}(46+48) = 47, \dots$. Charles R. Wall gives examples where iteration leads to an unbounded sequence: start with 45, 48, ... or 50, 55, ... and continue 56, 60, 80, 88, 92, 94, 95, 96, ... ; each term after the 35th is the double of the last but seven!

We can also average the σ - and ϕ -functions, and iterate. Since $\phi(n)$ is always even for $n > 2$ and $\sigma(n)$ is odd when n is a square or twice a square, we will sometimes get a noninteger value. For example, 54, 69, 70, 84, 124, 142, 143, 144, $225\frac{1}{2}$; in this case we say that the sequence **fractures**. It is easy to show that $(\sigma(n) + \phi(n))/2 = n$ just if $n = 1$ or a prime, so sequences can become constant, for example, 60, 92, 106, 107, 107, Are there sequences which increase indefinitely without fracturing?

Of course, if we iterate the ϕ -function, it eventually arrives at 1. Call the least integer k for which $\phi^k(n) = 1$ the **class** of n .

k	n														
1	2														
2	3	4 6													
3	5 7	8 9 10 12 14 18													
4	11 13 15	16 19 20 21 22 24 26 ...													
5	17 23 25 29 31	32 33 34 35 37 39 40 43...													
6	41 47 51 53 55 59 61	64 65 67 68 69 71 73 ...													
7	83 85 89 97 101 103 107 113 115 119 121 122 123 125 128 ...														

The set of least values of the classes is $M = \{2, 3, 5, 11, 17, 41, 83, \dots\}$. Shapiro conjectured that M contained only prime values, but Mills found several composite members. If S is the union, for all k , of the members of class k which are $< 2^k$, then

$$S = \{3; 5, 7; 11, 13, 15; 17, 23, 25, 29, 31; 41, 47, 51, 53, 55, 59, 61; 83, 85, \dots\}$$

and Shapiro showed that the factors of an element of S is also in S . Catlin showed that if m is an odd element of M , then the factors of M are in M , and that there are finitely many primes in M just if there are finitely many odd numbers in M . Does S contain infinitely many odd numbers? Does M contain infinitely many odd numbers?

Pillai showed that the class, $k = k(n)$, of n satisfies

$$\left\lfloor \frac{\ln n}{\ln 3} \right\rfloor \leq k(n) \leq \left\lfloor \frac{\ln n}{\ln 2} \right\rfloor$$

and it's easy to see (look at $2^a 3^b$) that $k(n)/\ln n$ is dense in the interval $[1/\ln 3, 1/\ln 2]$. What is the average and normal behavior of $k(n)$? Erdős, Granville, Pomerance & Spiro conjecture that there is a constant α such that the normal order of $k(n)$ is $\alpha \ln n$ and prove this under the assumption of the Elliott-Halberstam conjecture. They also showed that the normal order of $\phi^h(n)/\phi^{h+1}(n)$ is $he^\gamma \ln \ln \ln n$ for each positive integer h , where γ is Euler's constant. See their paper for many unsolved problems: for example, if $\sigma^k(n)$ is the k th iterate of the sum of divisors function, they are unable to prove or disprove any of the following statements.

- $\dot{?}$ for every $n > 1$, $\sigma^{k+1}(n)/\sigma^k(n) \rightarrow 1$ as $k \rightarrow \infty$?
- $\dot{?}$ for every $n > 1$, $\sigma^{k+1}(n)/\sigma^k(n) \rightarrow \infty$ as $k \rightarrow \infty$?
- $\dot{?}$ for every $n > 1$, $(\sigma^k(n))^{1/k} \rightarrow \infty$ as $k \rightarrow \infty$?
- $\dot{?}$ for every $n > 1$, there is some k with $n|\sigma^k(n)$?
- $\dot{?}$ for every $n, m > 1$, there is some k with $m|\sigma^k(n)$?
- $\dot{?}$ for every $n, m > 1$, there are some k, l with $\sigma^k(m) = \sigma^l(n)$?

Miriam Hausman has characterized those integers n which are solutions of the equation $n = m\phi^k(n)$; they are mainly of the form $2^a 3^b$.

Finucane iterated the function $\phi(n) + 1$ and asked: in how many steps does one reach a prime? Also, given a prime p , what is the distribution of the values of n whose sequences end in p ? Are 5, 8, 10, 12 the only numbers which lead to 5? And 7, 9, 14, 15, 16, 18, 20, 24, 30 the only ones leading to 7?

Erdős similarly asked about the iteration of $\sigma(n) - 1$. Does it always end on a prime, or can it grow indefinitely? In none of the cases of iteration of $\sigma(n) - 1$, of $(\psi(n) + \phi(n))/2$, or of $(\phi(n) + \sigma(n))/2$ is he able to show that the growth is slower than exponential. For several results and conjectures, consult the quadruple paper cited below.

Atanassov defines some additive analogs of ϕ and σ , poses 17 questions and answers only three of them.

Iannucci, Moujie & Chen define **perfect totient numbers** as those n for which $n = \phi(n) + \phi^2(n) + \phi^3(n) + \cdots + 2 + 1$. All powers of 3 are perfect totient numbers. They find 30 others $< 5 \cdot 10^9$ of which the largest is 4764161215.

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OEIS: A000010, A003434, A005239, A007755, A019268, A040176, A049108.

B42 Behavior of $\phi(\sigma(n))$ and $\sigma(\phi(n))$.

Erdős asks us to prove that $\phi(n) > \phi(n - \phi(n))$ for almost all n , but that $\phi(n) < \phi(n - \phi(n))$ for infinitely many n .

Mañkowski & Schinzel prove that $\limsup \phi(\sigma(n))/n = \infty$,

$$\limsup \phi(\phi(n))/n = \frac{1}{2}, \quad \text{and} \quad \liminf \sigma(\phi(n))/n \leq \frac{1}{2} + \frac{1}{2^{34} - 4}$$

and they ask if $\sigma(\phi(n))/n \geq \frac{1}{2}$ for all n . They point out that even $\inf \sigma(\phi(n))/n > 0$ is not proved, but Pomerance has since established this, using Brun's method. Graeme Cohen and later Segal each thought that he had proved the main result, but it remains open.

John Selfridge, Fred Hoffman & Rich Schroeppel found 24 solutions of $\phi(\sigma(n)) = n$, namely

2^k for $k = 0, 1, 3, 7, 15, 31$; $2^2 \cdot 3$; $2^8 \cdot 3^3$; $2^{10} \cdot 3^3 \cdot 11^2$; $2^{12} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$; $2^4 \cdot 3 \cdot 5$; $2^4 \cdot 3^2 \cdot 5$; $2^9 \cdot 3 \cdot 5^2 \cdot 31$; $2^9 \cdot 3^2 \cdot 5^2 \cdot 31$; $2^5 \cdot 3^4 \cdot 5 \cdot 11$; $2^5 \cdot 3^4 \cdot 5^2 \cdot 11$; $2^8 \cdot 3^4 \cdot 5 \cdot 11$; $2^8 \cdot 3^4 \cdot 5^2 \cdot 11$; $2^5 \cdot 3^6 \cdot 7^2 \cdot 13$; $2^6 \cdot 3^6 \cdot 7^2 \cdot 13$; $2^{13} \cdot 3^7 \cdot 5 \cdot 7^2$; $2^{13} \cdot 3^7 \cdot 5^2 \cdot 7^2$; $2^{21} \cdot 3^3 \cdot 5 \cdot 11^3 \cdot 31$; $2^{21} \cdot 3^3 \cdot 5^2 \cdot 11^3 \cdot 31$; and there are, of course, 24 corresponding solutions of $\sigma(\phi(m)) = m$.

Terry Raines, in January 1995, found ten further solutions:

$2^8 \cdot 3^6 \cdot 7^2 \cdot 13$; $2^9 \cdot 3^4 \cdot 5^2 \cdot 11^2 \cdot 31$; $2^{13} \cdot 3^3 \cdot 5^4 \cdot 7^3$; $2^{13} \cdot 3^8 \cdot 5 \cdot 7^3$; $2^{13} \cdot 3^6 \cdot 5 \cdot 7^3 \cdot 13$; $2^{13} \cdot 3^3 \cdot 5^4 \cdot 7^4$; $2^{13} \cdot 3^8 \cdot 5^2 \cdot 7^3$; $2^{13} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 13$; $2^{21} \cdot 3^5 \cdot 5 \cdot 11^3 \cdot 31$; $2^{21} \cdot 3^5 \cdot 5^2 \cdot 11^3 \cdot 31$;

and on Independence Day, 1996, Graeme Cohen added

$$2^{13} \cdot 3^5 \cdot 5^4 \cdot 7^4 \quad \text{and} \quad 2^{24} \cdot 3^9 \cdot 5^7 \cdot 11 \cdot 13$$

Are there others? An infinite number?

Golomb observes that if $q > 3$ and $p = 2q - 1$ are primes and $m \in \{2, 3, 8, 9, 15\}$, then $n = pm$ is a solution of $\phi(\sigma(n)) = \phi(n)$. Undoubtedly there are infinitely many such and undoubtedly no one will prove this in the foreseeable future. There are other solutions, 1, 3, 15, 45, ...; an infinite number? He gives the solutions 1, 87, 362, 1257, 1798, 5002, 9374 to $\sigma(\phi(n)) = \sigma(n)$. He also notes that if p and $(3^p - 1)/2$ are primes (e.g., $p = 3, 7, 13, 71, 103$), then $n = 3^{p-1}$ is a solution of $\sigma(\phi(n)) = \phi(\sigma(n))$; and shows that $\sigma(\phi(n)) - \phi(\sigma(n))$ is both positive and negative infinitely often and asks what is the proportion of each?

There are numerous questions that one may ask about these two roughly dual functions. Zhang Ming-Zhi notes that if n is prime, then it divides $\phi(n) + \sigma(n)$, but if $n = p^\alpha$ with $\alpha > 1$ or $n = p^\alpha q$ with p, q distinct primes, then this is not so. He finds 17 composite n less than 10^7 which divide $\phi(n) + \sigma(n)$: 312, 560, 588, 1400, 23760, 59400, 85632, 147492, 153720, 556160, 569328, 1590816, 2013216 and four others.

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OEIS: A000010, A000203, A001229, A006872, A033631, A051487-051488, A066831, A067382-0670385.

B43 Alternating sums of factorials.

The numbers $3! - 2! + 1! = 5$,
 $4! - 3! + 2! - 1! = 19$,
 $5! - 4! + 3! - 2! + 1! = 101$,
 $6! - 5! + 4! - 3! + 2! - 1! = 619$,
 $7! - 6! + 5! - 4! + 3! - 2! + 1! = 4421$,
 and $8! - 7! + 6! - 5! + 4! - 3! + 2! - 1! = 35899$

are each prime. Are there infinitely many such? Here are the factors of $A_n = n! - (n-1)! + (n-2)! - + \dots - (-1)^n 1!$ for the next few values of n :

n	A_n	n	A_n
9	$79 \cdot 4139$	19	15578717622022981 (prime)
10	3301819 (prime)	20	$8969 \cdot 210101 \cdot 1229743351$
11	$13 \cdot 2816537$	21	$113 \cdot 167 \cdot 4511191 \cdot 572926421$
12	$29 \cdot 15254711$	22	$79 \cdot 239 \cdot 56947572104043899$
13	$47 \cdot 1427 \cdot 86249$	23	$85439 \cdot 289993909455734779$
14	$211 \cdot 1679 \cdot 229751$	24	$12203 \cdot 24281 \cdot 2010359484638233$
15	1226280710981 (prime)	25	$59 \cdot 555307 \cdot 455254005662640637$
16	$53 \cdot 6581 \cdot 56470483$	26	$1657 \cdot 234384986539153832538067$
17	$47 \cdot 7148742955723$	27	$127^2 \cdot 271 \cdot 1163 \cdot 2065633479970130593$
18	$2683 \cdot 2261044646593$	28	$61 \cdot 221171 \cdot 21820357757749410439949$

The example $n = 27$ shows that these numbers are not necessarily square-free. Wilfrid Keller has continued the calculations for $n \leq 335$; A_n is prime for $n = 41, 59, 61, 105$ and 160 .

If there is a value of n such that $n+1$ divides A_n , then $n+1$ will divide A_m for all $m > n$, and there would be only a finite number of prime values. This problem has been answered by Miodrag Živković, who has shown that $p = 3612703$ divides $A(n)$ for all $n \geq p$.

There are questions if $0!$ is included. The numbers are now even, and only $2! - 1! + 0! = 2$ is prime. Kevin Buzzard reported that a teenage friend asked for which n does n divide $A_n + (-1)^n$? If \mathcal{S} is the set of such n , then $a \in \mathcal{S}$, $b \in \mathcal{S}$, $a \perp b$ imply $ab \in \mathcal{S}$ and $c \in \mathcal{S}$, $d \mid c$ imply $d \in \mathcal{S}$. So one needs only search for prime powers. Buzzard found only 2, 4, 5, 13, 37, 463 less than 160000, and asks: is \mathcal{S} just the set of divisors of 4454060?

Miodrag Živković, The number of primes $\sum_{i=1}^n (-1)^{n-i} i!$ is finite, *Math. Comput.*, **68**(1999) 403–409; *MR99c*:11163.

OEIS: A000142, A001272, A002981-002982, A003422, A005165.

B44 Sums of factorials.

Đ. Kurepa defined $!n = 0! + 1! + 2! + \dots + (n-1)!$ and asks if $!n \not\equiv 0 \pmod n$ for all $n > 2$. Slavić established this for $3 \leq n \leq 1000$. The conjecture is that $(!n, n!) = 2$. Wagstaff verified the conjecture for $n < 50000$, and Mijajlović and Gogić independently for $n \leq 10^6$. Mijajlović notes that for $K_n = !(n+1) - 1 = 1! + 2! + \dots + n!$ we have $3|K_n$ for $n \geq 2$, $9|K_n$ for $n \geq 5$ and $99|K_n$ for $n \geq 10$. Wilfrid Keller has since extended this and found no new divisibilities for K_n with $n < 10^6$. In a 91-03-21 letter, and a February 1998 preprint, Reg. Bond offered an as yet unpublished proof of the conjecture.

Miodrag Živković (see ref. at **B43**) has shown that $(54503)^2$ divides $!26540$, so that $!n$ is not always squarefree for $n > 3$.

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OEIS: A000142, A003422, A005165, A007489, A014144, A049782.

B45 Euler numbers.

The coefficients in the expansion of $\sec x = \sum E_n(ix)^n/n!$ are the **Euler numbers**, and arise in several combinatorial contexts. $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1385$, $E_{10} = -50521$, $E_{12} = 2702765$, $E_{14} = -199360981$, $E_{16} = 19391512145$, $E_{18} = -2404879675441$, \dots . Is it true that for any prime $p \equiv 1 \pmod 8$, $E_{(p-1)/2} \not\equiv 0 \pmod p$? Is it true for $p \equiv 5 \pmod 8$?

V. I. Arnol'd, Bernoulli-Euler updown numbers associated with function singularities, their combinatorics and arithmetics, *Duke Math. J.*, **63**(1991) 537–555; *MR* **93b**:58020.

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Barry J. Powell, Advanced problem 6325, *Amer. Math. Monthly* **87**(1980) 826.

B46 The largest prime factor of n .

Erdős denotes by $P(n)$ the largest prime factor of n and asks if there are infinitely many primes p such that $(p-1)/P(p-1) = 2^k$? Or $= 2^k \cdot 3^l$?

If $n > 2$, then $P(n)$, $P(n+1)$, $P(n+2)$ are all distinct. Show that each of the six permutations of {low, medium, high} occurs infinitely often, and that they occur with equal frequency. $2^k - 2$, $2^k - 1$, 2^k show that medium, high, low occurs for infinitely many k because $P(2^k - 1) \rightarrow \infty$ as $k \rightarrow \infty$ by a theorem of Bang (or Mahler). To see that low, medium high occurs infinitely often, ask if $p-1$, p , $p+1$ works for p prime. No! Try $p^2 - 1$, p^2 , $p^2 + 1$. Maybe. If $P(p^2 + 1) < p$, try $p^4 - 1$, p^4 , $p^4 + 1$. Eventually, for each prime p , there will be a value of k such that $P(p^{2^k} + 1) > p$.

Selfridge settled the low, high, medium case with 2^k , $2^k + 1$, $2^k + 2$ and Tijdeman gave the following argument for medium, low, high: consider the possibilities $2^k - 1$, 2^k , $2^k + 1$; $2^{2k} - 1$, 2^{2k} , $2^{2k} + 1$; $2^{4k} - 1$, 2^{4k} , $2^{4k} + 1$;

...

Mabkhout showed that $P(n^4 + 1) \geq 137$ for all $n > 3$; he used a classical result of Størmer, quoted at **D10**.

P. Erdős & Carl Pomerance, On the largest prime factors of n and $n+1$, *Aequationes Math.*, **17**(1978) 311–321; *MR* **58** #476.

M. Mabkhout, Minoration de $P(x^4 + 1)$, *Rend. Sem. Fac. Sci. Univ. Cagliari*, **63**(1993) 135–148; *MR* **96e**:11039.

B47 When does $2^a - 2^b$ divide $n^a - n^b$?

Selfridge notices that $2^2 - 2$ divides $n^2 - n$ for all n , that $2^{2^2} - 2^2$ divides $n^{2^2} - n^2$ and $2^{2^{2^2}} - 2^{2^2}$ divides $n^{2^{2^2}} - n^{2^2}$ and asks for what a and b does $2^a - 2^b$ divide $n^a - n^b$ for all n . The case $n = 3$ was proposed as E2468*, *Amer. Math. Monthly*, **81**(1974) 405 by Harry Ruderman. In his solution (**83**(1976) 288–289) Bill Vélez omits $(b, a-b) = (0, 1)$ as trivial and gives 13 other solutions, (1,1), (1,2), (2,2), (3,2), (1,4), (2,4), (3,4), (4,4), (2,6), (3,6), (2,12), (3,12), (4,12). Remarks by Pomerance (**84**(1977) 59–60) show

that results of Schinzel complete Vélez's solution. The problem was also solved by Sun Qi & Zhang Ming Zhi.

A. Schinzel, On primitive prime factors of $a^n - b^n$, *Proc. Cambridge Philos. Soc.*, **58**(1962) 555–562.

Sun Qi & Zhang Ming-Zhi, Pairs where $2^a - 2^b$ divides $n^a - n^b$ for all n , *Proc. Amer. Math. Soc.*, **93**(1985) 218–220; *MR* **86c**:11004.

Marian Văjăitu & Alexandru Zaharescu, A finiteness theorem for a class of exponential congruences, *Proc. Amer. Math. Soc.*, **127**(1999) 2225–2232; *MR* **99j**:11003.

B48 Products taken over primes.

David Silverman noticed that if p_n is the n -th prime, then

$$\prod_{n=1}^m \frac{p_n + 1}{p_n - 1}$$

is an integer for $m = 1, 2, 3, 4$ and 8 and asked is it ever again an integer? Equivalently, as Mąkowski observes (reference at **B16**), for what $n = \prod_{r=1}^m p_r$ does $\phi(n)$ divide $\sigma(n)$? For example, if $\sigma(n) = 4\phi(n)$ then $2n$ is either perfect or abundant, $\sigma(2n) \geq 4n$. Jud McCranie checked that the product is not an integer for $8 < m \leq 98222287$, i.e for primes p , $23 \leq p < 2 \cdot 10^9$. He notes the connexion with Sophie Germain primes (primes p such that $2p + 1$ is also prime) and with Cunningham chains (**A7**).

Wagstaff asked for an elementary proof (e.g., without using properties of the Riemann ζ -function) that

$$\prod \frac{p^2 + 1}{p^2 - 1} = \frac{5}{2}$$

where the product is taken over all primes. It seems very unlikely that there is a proof which doesn't involve analytical methods. At first glance it might appear that the fractions might cancel, but none of the numerators are divisible by 3. Euler's proof is

$$\prod \frac{p^2 + 1}{p^2 - 1} = \prod \frac{p^4 - 1}{(p^2 - 1)^2} = \prod \frac{1 - p^{-4}}{(1 - p^{-2})^2} = \frac{\zeta^2(2)}{\zeta(4)} = \frac{(\pi^2/6)^2}{\pi^4/90} = \frac{5}{2}.$$

This uses $\sum n^{-k} = \prod (1 - p^{-k})^{-1}$ and $\sum n^{-2} = \pi^2/6$ and $\sum n^{-4} = \pi^4/90$. Wagstaff regards the first as elementary, but not the latter two. He would like to see a direct proof of $2(\sum n^{-2})^2 = 5 \sum n^{-4}$ or of

$$4 \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=n+1}^{\infty} \frac{1}{m^2} = 3 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

David Borwein & Jonathan M. Borwein, On an intriguing integral and some series related to $\zeta(4)$, *Proc. Amer. Math. Soc.*, **123**(1995) 1191–1198; *MR* **95e**:11137.

B49 Smith numbers.

Albert Wilansky named **Smith numbers** from his brother-in-law's telephone number

$$4937775 = 3 \cdot 5 \cdot 5 \cdot 65837,$$

the sum of whose digits is equal to the sum of the digits of its prime factors, and they soon caught the public fancy. Trivially, any prime is a Smith number: so are 4, 22, 27, 58, 85, 94, 121, Oltikar & Wayland gave the examples $3304(10^{317} - 1)/9$ and $2 \cdot 10^{45}(10^{317} - 1)/9$ and the race to find larger and larger Smith numbers was on. Yates has given

$$10^{3913210}(10^{1031} - 1)(10^{4594} + 3 \cdot 10^{2297} + 1)^{1476}$$

with 10694985 decimal digits, but has since beaten his own record with a 13614513-digit Smith number.

Patrick Costello, A new largest Smith number, *Fibonacci Quart.*, **40**(2002) 369–371.

Patrick Costello & Kathy Lewis, Lots of Smiths, *Math. Mag.*, **75**(2002) 223–226.

Stephen K. Doig, Math Whiz makes digital discovery, *The Miami Herald*, 1986-08-22; *Coll. Math. J.*, **18**(1987) 80.

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Samuel Yates, How odd the Smiths are, *J. Recreational Math.*, **19**(1987) 168–174.

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OEIS: A006753, A019506, A059754.



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