

### 3 Equations of State

*“The worth of a State, in the long run,  
is the worth of the individuals composing it.”*

— *John Stuart Mill (1806–1873)*

*“What is Matter?—Never mind.  
What is Mind?—no matter.”*

— *from Punch (1855)*

The equations of state appropriate to the interiors of most stars are simple in one major respect: they may be derived using the assumption that the radiation, gas, fluid, or even solid, is in a state of *local thermodynamic equilibrium*, or LTE. By this we mean that at nearly any position in the star complete thermodynamic equilibrium is as very nearly true as we could wish. It is only near the stellar surface or in highly dynamic events, such as in supernovae, where this assumption may no longer be valid.

The reasons that LTE works so well are straightforward: particle–particle and photon–particle mean free paths are short and collision rates are rapid compared to other stellar length or time scales. (A major exception to this rule involves nuclear reactions, which are usually slow.) Thus two widely separated regions in the star are effectively isolated from one another as far as the thermodynamics are concerned and, for any one region, the Boltzmann populations of ion energy levels are consistent with the local electron kinetic temperature.<sup>1</sup> Note, however, that different regions cannot be *completely* isolated from one another in a real star because, otherwise, energy could not flow between them. Chapter 4 will go into this further.

One typical scale length in a star is the *pressure scale height*,  $\lambda_P$ , given by

$$\lambda_P = - \left( \frac{d \ln P}{dr} \right)^{-1} = \frac{P}{g\rho} \quad (3.1)$$

where the equation of hydrostatic equilibrium (1.6) has been used to eliminate  $dP/dr$ . The constant–density star discussed in the first chapter easily yields an estimate for this quantity of

$$\lambda_P (\rho = \text{constant}) = \frac{\mathcal{R}^2}{2r} \left[ 1 - \left( \frac{r}{\mathcal{R}} \right)^2 \right]$$

using the run of pressure given by (1.41). The central value of  $\lambda_P$  is infinite but through most of the constant–density model it is of order  $\mathcal{R}$ . Near the

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<sup>1</sup> For further discussions of the conditions for LTE see Cox (1968, Chap. 7) and Mihalas (1978, Chap. 5).

surface it decreases rapidly to zero. We compare these lengths to photon mean free paths,  $\lambda_{\text{phot}}$ , which we construct from the opacity by

$$\lambda_{\text{phot}} = (\kappa\rho)^{-1} \text{ cm} . \quad (3.2)$$

This quantity is a measure of how far a photon travels before it is either absorbed or scattered into a new direction (see Chap. 4). Note that opacity has the units of  $\text{cm}^2 \text{ g}^{-1}$ .

For Thomson electron scattering, which is the smallest opacity in most stellar interiors, later work will show that  $\kappa \approx 1 \text{ cm}^2 \text{ g}^{-1}$ . If we consider the sun to be a typical star and set  $\mathcal{R} = \mathcal{R}_{\odot}$  and  $\rho = \langle \rho_{\odot} \rangle \approx 1 \text{ g cm}^{-3}$  in the above, we then find  $\lambda_{\text{phot}}$  is at most a centimeter and  $\lambda_P \sim 10^{11} \text{ cm}$  through the bulk of the interior. Thus  $\lambda_{\text{phot}}$  is smaller than  $\lambda_P$  by many orders of magnitude. We could also have compared  $\lambda_{\text{phot}}$  with a temperature scale height and found the same sort of thing because, for the sun, the temperature decreases by only  $10^{-4} \text{ K cm}^{-1}$  on average from center to surface.

Another simple calculation yields an estimate of how much of a star is *not* in LTE. If the photon mean free path is still of order 1 cm, then the relative radius at which the pressure scale height is equal to the photon mean free path is  $(r/\mathcal{R}) \approx 1 - 10^{-11}$  using the constant-density model. This means, as a crude estimate, that it is within only the last one part in  $10^{11}$  of the radius that the assumption of LTE fails. In realistic models, the assumption of LTE breaks down within the region of the stellar photosphere, which is the only part of a star we can see.

In the following sections we shall quote some results from statistical mechanics, which will eventually be used to derive equations of state for stellar material consisting of gases (including photons) in thermodynamic equilibrium. Because several excellent texts on statistical mechanics are available for reference, many results will be stated without proof. One particular text we recommend is Landau and Lifshitz (1958, or later editions) for its clean style and inclusion of many fundamental physical (and astrophysical) applications. Additional material may be found in Cox (1968), Kippenhahn and Weigert (1990), and Rose (1998, §3.2).

### 3.1 Distribution Functions

The “distribution function” for a species of particle measures the number density of that species in the combined six-dimensional space of coordinates plus momenta. If that function is known for a particular gas composed of a combination of species, then all other thermodynamic variables may be derived given the temperature, density, and composition. For the next few sections we shall assume that the gas, including electrons and photons, is a perfect (sometimes called ideal) gas in that particles comprising the gas interact so weakly that they may be regarded as noninteracting as far as their

thermodynamics is concerned. They may, however, still exchange energy and other conserved properties. Before writing down the distribution function for a perfect gas we first introduce what may be an unfamiliar thermodynamic quantity.

The variables of thermodynamic consequence we have encountered thus far are  $P$ ,  $T$ ,  $\rho$  (or  $V_\rho = 1/\rho$ ),  $S$ ,  $E$ ,  $Q$ , and various number densities,  $n_i$  (see §1.4.1). The latter have been, and will be, given in the units of number  $\text{cm}^{-3}$ . We now introduce  $N_i$ , which is the (specific) number density of an  $i$ th species in the units of number per gram of material with  $N_i = n_i/\rho$ . It is the Lagrangian version of  $n_i$  and it will prove useful because it remains constant even if volume changes.

Another very useful thermodynamic quantity is the *chemical potential*,  $\mu_i$ , defined by<sup>2</sup>

$$\mu_i = \left( \frac{\partial E}{\partial N_i} \right)_{S,V} \quad (3.3)$$

as associated with an  $i$ th species in the material (and is not to be confused with  $\mu_i$ , the ion molecular weight). If there are “chemical” reactions in the stellar mixture involving some subset of species (ions, electrons, photons, molecules, etc.) whose concentrations could, in principle, change by  $dN_i$  as a result of those reactions, then thermodynamic (and chemical) equilibrium requires that

$$\sum_i \mu_i dN_i = 0 \quad (3.4)$$

which we state without proof. Changing  $N_i$  by  $dN_i$  in a real mixture usually means that other components in the mixture must change by an amount related to  $dN_i$  so that not all the  $dN_i$  are independent.

As an example, consider the ionization–recombination reaction



where  $\text{H}^0$  is neutral hydrogen—assumed to have only one bound state in the following discussion— $\text{H}^+$  is the hydrogen ion (a proton), and  $\text{e}^-$  is an electron. We shall neglect the photon that appears on the righthand side of (3.5) in the following because, as we shall show, its chemical potential is zero and will not enter into the application of (3.4). The double-headed arrow is to remind us that the reaction proceeds equally rapidly in both directions in thermodynamic equilibrium. Now write (3.5) in the algebraic form

$$1 \text{H}^+ + 1 \text{e}^- - 1 \text{H}^0 = 0$$

where the coefficients count how many individual constituents are destroyed or created in a single reaction. A more general form for this equation is

<sup>2</sup> A simple example indicating why  $\mu_i$  is a “potential” is given as Ex. 3.6.

$$\sum_i \nu_i C_i = 0. \quad (3.6)$$

The  $C_i$  represent  $H^+$ ,  $H^0$ , and  $e^-$  in the example and the  $\nu_i$ , or *stoichiometric coefficients*, are the numerical coefficients. Obviously the concentrations,  $N_i$ , are constrained in the same way as the  $C_i$ . Thus if  $N_1$  changes by some arbitrary amount  $dN_1$ , then the  $i$ th concentration changes according to

$$\frac{dN_i}{\nu_i} = \frac{dN_1}{\nu_1}.$$

Equation (3.4) then becomes

$$\sum_i \mu_i \frac{dN_1}{\nu_1} \nu_i = \frac{dN_1}{\nu_1} \sum_i \mu_i \nu_i = 0$$

or, since  $dN_1$  is arbitrary,

$$\sum_i \mu_i \nu_i = 0. \quad (3.7)$$

This is the equation for *chemical equilibrium*, which must be part of thermodynamic equilibrium when reactions are taking place.<sup>3</sup>

As another simple, and useful, example consider a classical blackbody cavity filled with radiation in thermodynamic equilibrium with the walls of the cavity. Equilibrium is maintained by the interaction of the photons with material comprising the walls but the number of photons,  $N_\gamma$ , fluctuates about some mean value; that is, photon number is not strictly conserved. Therefore  $dN_\gamma$  need not be zero. Nevertheless, reactions in the cavity must satisfy a symbolic relation of the form  $\sum \mu_i dN_i + \mu_\gamma dN_\gamma = 0$  with  $dN_i = 0$ . The last two statements can only be reconciled if

$$\mu_\gamma = 0 \quad \text{for photons.} \quad (3.8)$$

It is for this reason that photons were not included in the ionization and recombination reaction of (3.5): the vanishing of  $\mu_\gamma$  makes its presence superfluous in the chemical equilibrium equation (3.7).

It is reasonable, and correct, to expect that given  $T$ ,  $\rho$ , and a catalogue of what reactions are possible, we should be able to find all the  $N_i$  for a gas in thermodynamic equilibrium. In other words, information about  $N_i$  is contained in  $\mu_i$  for the given  $T$  and  $\rho$ . In a real gas this connection is difficult to establish because it requires a detailed knowledge of how the particles in the system interact. For a perfect gas things are easier. Any text on statistical mechanics may be consulted for what follows.

<sup>3</sup> We exclude thermonuclear reactions from this discussion for the present because they may proceed very slowly and, usually, only in one direction during stellar nuclear burning.

The relation between the number density of some species of elementary nature (ions, photons, etc.) in coordinate–momentum space and its chemical potential in thermodynamic equilibrium is found from statistical mechanics to be

$$n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\{[-\mu + \mathcal{E}_j + \mathcal{E}(p)]/kT\} \pm 1} . \quad (3.9)$$

We call  $n(p)$  the distribution function for the species (although you will often see this referred to as the “occupation number”). The various quantities are as follows:

- $\mu$  is the chemical potential of the species.
- $j$  refers to the possible energy states of the species (e.g., energy levels of an ion).
- $\mathcal{E}_j$  is the energy of state  $j$  referred to some reference energy level.
- $g_j$  is the degeneracy of state  $j$  (i.e., the number of states having the same energy  $\mathcal{E}_j$ ).
- $\mathcal{E}(p)$  is the kinetic energy as a function of momentum  $p$ .
- a “+” in the denominator is used for Fermi–Dirac particles (fermions of half-integer spin) and a “–” for Bose–Einstein particles (bosons of zero or whole integer spin).
- $h$  is Planck’s constant  $h = 6.6260688 \times 10^{-27}$  erg s.
- $n(p)$  is in the units of number per (cm–unit momentum)<sup>3</sup> where the differential element in coordinate–momentum space is  $d^3\mathbf{r} d^3\mathbf{p}$ .

As we shall demonstrate in the following discussion, (3.9) will lead to all the familiar results from elementary thermodynamics.

To retrieve the physical space number density,  $n$  (cm<sup>−3</sup>), for the species from (3.9) we need only integrate over all momentum space, which, from standard arguments, is assumed to be spherically symmetric; that is,<sup>4</sup>

$$n = \int_p n(p) 4\pi p^2 dp \text{ cm}^{-3} . \quad (3.10)$$

The factor of  $4\pi$  (steradians) comes from the two angular integrations over the surface of a unit sphere.

Because we shall want eventually to consider relativistic particles, the correct form of the kinetic energy,  $\mathcal{E}$ , for a particle of rest mass  $m$  is given by

$$\mathcal{E}(p) = (p^2 c^2 + m^2 c^4)^{1/2} - mc^2 \quad (3.11)$$

<sup>4</sup> We explicitly assume here that the distribution of particles is angularly isotropic in momentum. This is really part of LTE but the assumption will have to be reexamined in Chapter 4 when we put back angular information and partially unravel the integral.

which reduces to  $\mathcal{E}(p) = p^2/2m$  for  $pc \ll mc^2$  in the nonrelativistic limit, and  $\mathcal{E}(p) = pc$  for extremely relativistic particles or those with zero rest mass.

We shall also need an expression for the velocity which, from Hamilton's equations (one of the more elegant and important subjects in the physical sciences), is

$$v = \frac{\partial \mathcal{E}}{\partial p}. \quad (3.12)$$

(As a simple check on this definition of  $v$ , note that  $v \rightarrow p/m$  for  $pc \ll mc^2$  and  $v \rightarrow c$  for the relativistic case, both of which are elementary results.) This is the velocity to use in the following kinetic theory expression for isotropic pressure (as in 1.20)

$$P = \frac{1}{3} \int_p n(p) pv 4\pi p^2 dp. \quad (3.13)$$

Finally, the internal energy is simply

$$E = \int_p n(p) \mathcal{E}(p) 4\pi p^2 dp. \quad (3.14)$$

That completes all that we shall need to construct practical equations of state in the following applications.

## 3.2 Blackbody Radiation

Photons are massless bosons of unit spin. Since they travel at  $c$ , they only have two states (two spin orientations or polarizations) for a given energy and thus the degeneracy factor in (3.9) is  $g = 2$ . From before,  $\mu_\gamma = 0$  and  $\mathcal{E} = pc$ . Because there is only one energy level (no excited states),  $\mathcal{E}_j$  may be taken as zero. Putting this together, we find that the photon number density is given by<sup>5</sup>

$$n_\gamma = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp(pc/kT) - 1} \text{ cm}^{-3}. \quad (3.15)$$

Let  $x = pc/kT$  and use the integral

<sup>5</sup> It may seem contradictory to give one number for the photon density whereas we stated earlier that the photon concentration fluctuates about some mean value—thus giving  $\mu_\gamma = 0$ . But the point is that photons must interact with matter to equilibrate (not with each other unless you delve into quantum electrodynamics) and this is a statistical process. What you get in (3.15) is an average. Fluctuations about that average depend on the particulars of the matter interactions but, as long as there are many interactions, the effect of fluctuations is very small. Much the same can be said about even the ideal gas except there we deal with various conservation rules involving particles, not photons. See, for example, Landau and Lifshitz (1958, Chap. XII).

$$\int_0^{\infty} \frac{x^2 dx}{e^x - 1} = 2 \zeta(3) = 2(1.202 \dots)$$

where  $\zeta(3)$  is a Riemann Zeta function, to find

$$n_{\gamma} = 2\pi \zeta(3) \left( \frac{2kT}{ch} \right)^3 \approx 20.28 T^3 \text{ cm}^{-3}. \quad (3.16)$$

Find, in similar fashion, that the radiation pressure is given by

$$P_{\text{rad}} = \left( \frac{k^4}{c^3 h^3} \frac{8\pi^5}{15} \right) \frac{T^4}{3} = \frac{aT^4}{3} \text{ dyne cm}^{-2} \quad (3.17)$$

and that the energy density is

$$E_{\text{rad}} = aT^4 = 3P_{\text{rad}} \text{ erg cm}^{-3} \quad (3.18)$$

where  $a$  is the radiation constant  $a = 7.56577 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$ . Thus we recover the usual results for blackbody radiation. The nice thing about LTE radiation is that all you have to know is the ambient temperature. Matter density, composition, etc., don't matter, so to speak.

Note that (3.18) is a  $\gamma$ -law equation of state  $P = (\gamma - 1)E$  (as in 1.24 after  $E$  in that equation is converted to energy per unit volume) with  $\gamma = 4/3$ . Thus a star whose equation of state is dominated by radiation is in danger of approaching the  $\gamma = 4/3$  limit discussed in Chapter 1.

It will be convenient for later purposes to define the energy density per unit frequency ( $\nu$ ) or wavelength ( $\lambda$ ) in the radiation field. These energy densities are usually designated by  $u$  (with an appropriate subscript). Recall that frequency is given by  $\nu = \mathcal{E}/h = pc/h$  and wavelength by  $\lambda = c/\nu$ . If  $u_p$  is the energy density per unit momentum (that is, the integrand of 3.14 with  $E_{\text{rad}} = \int_0^{\infty} u_p dp$ ) and  $u_{\nu}$  and  $u_{\lambda}$  are the corresponding densities per unit frequency and wavelength, then you may easily show

$$u_{\nu} d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu \text{ erg cm}^{-3} \text{ Hz}^{-1} \text{ Hz} \quad (3.19)$$

and

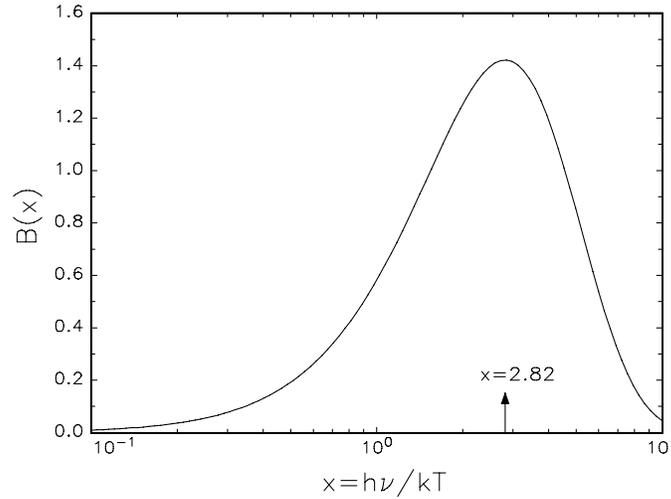
$$u_{\lambda} d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} d\lambda \text{ erg cm}^{-3} \text{ cm}^{-1} \text{ cm}. \quad (3.20)$$

Associated quantities are the *frequency-dependent Planck function*

$$B_{\nu}(T) = \frac{c}{4\pi} u_{\nu} \text{ erg cm}^{-2} \quad (3.21)$$

and the *integrated Planck function*

$$B(T) = \int_0^{\infty} B_{\nu}(T) d\nu = \frac{ca}{4\pi} T^4 = \frac{\sigma}{\pi} T^4 \text{ erg cm}^{-2} \text{ s}^{-1}. \quad (3.22)$$



**Fig. 3.1.** A plot of the function  $B(x) = x^3/[\exp(x) - 1]$  corresponding to the vital part of either  $u_\nu$  or the Planck function  $B_\nu$ . The maximum is at  $x = h\nu/kT = 2.821$ .

The Stefan–Boltzmann constant  $\sigma = 5.6704 \times 10^{-5} \text{ erg cm}^{-2} \text{ K}^{-4} \text{ s}^{-1}$ . We shall make extensive use of these functions when we discuss radiative transfer in the next chapter.

To remind you of what  $u_\nu$  or  $B_\nu$  looks like, we plot the function  $B(x) = x^3/[\exp(x) - 1]$  (as part of 3.19) in Fig. 3.1 where  $x = h\nu/kT$  and multiplicative constants have been ignored. The function is strongly peaked with a maximum at  $x = 2.821 \dots$ . For the center of the sun, with  $T_c \approx 10^7 \text{ K}$ , this peak corresponds to a photon energy of 2.4 keV. (For conversions to eV units see App. B.) Photons of these energies are capable of completely ionizing most of the lighter elements.

### 3.3 Ideal Monatomic Gas

As we shall soon show, the Boltzmann distribution for an ideal gas is characterized by  $(\mu/kT) \ll -1$ . We start off by asserting that this inequality holds for a sample of gas.

To make it simple, assume that the gas particles are nonrelativistic with  $\mathcal{E} = p^2/2m$ ,  $v = p/m$ , and that they have only one energy state  $\mathcal{E} = \mathcal{E}_0$ . These could be, as examples, elementary particles, or a collection of one species of ion in a given state. If  $(\mu/kT) \ll -1$ , then the term  $\pm 1$  in the denominator of (3.9) may be neglected compared to the exponential and the gas becomes purely classical in character with no reference to quantum statistics. The expression for the number density is then

$$n = \frac{4\pi}{h^3} g \int_0^\infty p^2 e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT} dp. \quad (3.23)$$

The integral is elementary and yields  $\mu$  in terms of number density:

$$e^{\mu/kT} = \frac{nh^3}{g(2\pi mkT)^{3/2}} e^{\varepsilon_0/kT}. \quad (3.24)$$

Because we require  $\exp(\mu/kT) \ll 1$  (since  $\mu/kT \ll -1$ ), the righthand side of (3.24) must be small. Thus,  $nT^{-3/2}$  cannot be too large. If this is not true, then other measures must be taken. For example, if  $\mu/kT$  is negative but not terribly less than  $-1$ , it is possible to expand the original integrand for  $n$  (with the  $\pm 1$  statistics term retained) in a power series and then integrate. The additional terms obtained, assuming convergence of the series, represent Fermi–Dirac or Bose–Einstein corrections to the ideal gas. This is done for fermions in Chiu (1968, Chap. 3), and Chandrasekhar (1939, Chapt. X), for example. In any event,  $\mu$  may be computed once  $n$  and  $T$  are given. We assume here that (3.24) is by far the largest contribution to any expansion leading to an expression for  $\mu$  for given  $n$  and  $T$ .

It is easy to take logarithmic differentials of  $n$  that yield the following expressions, and you may easily verify from the literature that they are the distribution functions for a Maxwell–Boltzmann ideal gas:

$$\frac{dn(p)}{n} = \frac{4\pi}{(2\pi mkT)^{3/2}} e^{-p^2/2mkT} p^2 dp \quad (3.25)$$

and, in energy space,

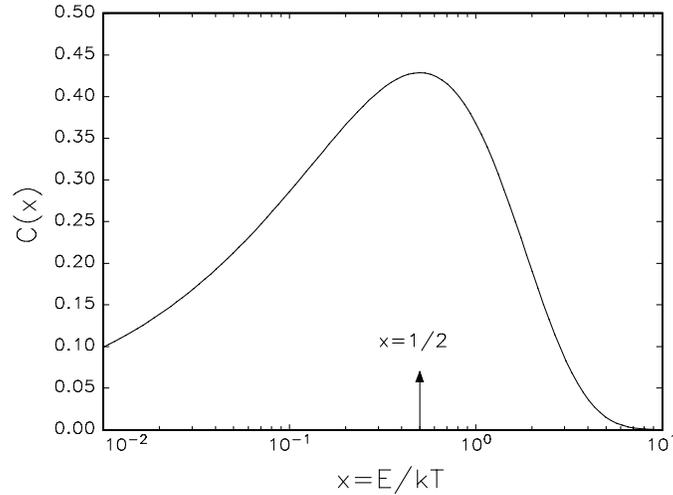
$$\frac{dn(\mathcal{E})}{n} = \frac{2}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} e^{\mathcal{E}/kT} \mathcal{E}^{1/2} d\mathcal{E}. \quad (3.26)$$

The relevant part of (3.26),  $C(x) = x^{1/2} \exp(-x)$ , is shown in Fig. 3.2, where the maximum corresponds to  $\mathcal{E} = kT/2$  (i.e.,  $x = 1/2$ ).

It is easy to show that the average kinetic energy of a particle in this distribution is just  $3kT/2$ , which gives rise to (3.29) below. (To get the average, multiply 3.26 by  $\mathcal{E}$  and integrate from zero energy to infinity.) Thus the “important” particles, in a manner of speaking, of a Maxwell–Boltzmann gas are those with energies near  $kT$ . A major exception to this involves those partaking of fusion reactions. For the solar center  $kT$  is around 1 keV, which is small compared to nuclear energies measured in MeVs. It will turn out (in Chap. 6) that the important fusion reactants are those in the exponential right hand tail of Fig. 3.2, even though their population is small compared to those in the peak of the distribution.

A similar procedure involving the neglect of the  $\pm 1$  statistical factor equivalent to what was done for (3.23) yields the pressure

$$P = g \frac{4\pi}{h^3} \frac{\pi^{1/2}}{8m} (2mkT)^{5/2} e^{\mu/kT} e^{-\varepsilon_0/kT} \quad (3.27)$$



**Fig. 3.2.** A plot of the function  $C(x) = x^{1/2} \exp(-x)$  corresponding to the exciting part of the Maxwell–Boltzmann distribution in energy space. The maximum is at  $x = 1/2$  ( $\mathcal{E} = kT/2$ ).

or, after substituting for  $e^{\mu/kT}$  of (3.24),

$$P = nkT \text{ dyne cm}^{-2} \quad (3.28)$$

which comes as no surprise. This last result is true even if the particles are relativistic (as in Ex. 3.4). The internal energy is

$$E = \frac{3}{2}nkT \text{ erg cm}^{-3} \quad (3.29)$$

using the same procedures. (Note that if reactions are present that change the relative concentrations of particles, then  $E$  must contain information about the energetics of such reactions; see below.) These are all elementary results for the ideal gas so that, given  $n$ ,  $T$ , and composition, then  $P$ ,  $E$ , and  $\mu$  immediately follow.

To tidy up, we return to a statement made at the beginning of this chapter; namely, that “the Boltzmann populations of ion energy levels are consistent with the local electron kinetic temperature” in LTE. We have implicitly assumed here that all species in a mixture have the same temperature, which, in some environments, is not warranted. For the stellar interior the assumption is fine. Thus consider an ion with two energy levels with  $\mathcal{E}_1 > \mathcal{E}_2$ . These levels are populated or depopulated by photon absorption or emission, for example. Because the photon chemical potential is zero, then  $\mu_1 = \mu_2$ . Dividing (3.24) for the two levels yields, after trivial algebra,

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{-(\mathcal{E}_1 - \mathcal{E}_2)/kT} \quad (3.30)$$

which is the Boltzmann population distribution and, if the statistical weights are not strange, means that levels become more sparsely populated as their energy increases.

### 3.4 The Saha Equation

In many situations the number densities of some species cannot be set a priori because “chemical” reactions are taking place. This is the problem referred to in §1.4 where mean molecular weights were computed. If the system is in thermodynamic equilibrium, however, then the chemical potentials of the reacting constituents depend on one another and this additional constraint is sufficient to determine the number densities.

As an example, consider the ionization–recombination reaction brought up earlier:<sup>6</sup>



where  $\chi_{\text{H}} = 13.6$  eV is the ionization potential from the ground state of hydrogen (still assumed to have only one bound level). We assume that no other reactions are taking place that involve the above constituents and, in particular, that the gas is pure hydrogen. Reference to the photon in (3.31) has again been deleted because its chemical potential is zero and does not appear in the equilibrium condition (3.7), which will be invoked shortly.

To obtain the LTE number densities of the electrons and neutral and ionized versions of hydrogen, assume that all gases are ideal so that (3.24) applies. The reference energy levels for all species are established by taking the zero of energy as the just-ionized  $\text{H}^+ + \text{e}^-$  state. (Other choices are possible of course.) Thus  $\mathcal{E}_0$  for electrons and  $\text{H}^+$  is zero, whereas for  $\text{H}^0$  it is  $-\chi_{\text{H}} = -13.6$  eV lower on the energy scale. That is, we need 13.6 eV to convert  $\text{H}^0$  to a free electron and a proton. The ground state of hydrogen has two near-degenerate states corresponding to spin-up or spin-down of the electron relative to the proton spin. For our purposes regard those states as having the same energy (but of course they do not, otherwise 21-cm HI radiation would not exist). Thus the degeneracy factor for  $\text{H}^0$  is  $g^0 = 2$ . The situation for the free electron and  $\text{H}^+$  is a bit more complicated because of the possible problem of double counting. If the spin axis of the proton is taken to be a fixed reference direction, then the free electron may have two spin directions relative to the free proton. Thus,  $g^- = 2$  and  $g^+ = 1$ . The argument could be reversed without having any effect on the following results.

With  $\mu^-$ ,  $\mu^+$ , and  $\mu^0$  denoting the chemical potentials of the components in (3.31), Equation (3.24) then yields

<sup>6</sup> See Ex. 3.1 for a more complicated problem.

$$n_e = \frac{2 [2\pi m_e kT]^{3/2}}{h^3} e^{\mu^- / kT} \quad (3.32)$$

$$n^+ = \frac{[2\pi m_p kT]^{3/2}}{h^3} e^{\mu^+ / kT} \quad (3.33)$$

$$n^0 = \frac{2[2\pi(m_e + m_p)kT]^{3/2}}{h^3} e^{\mu^0 / kT} e^{\chi_H / kT} \quad (3.34)$$

where  $m_e$  and  $n_e$  denote, respectively, the electron mass and number density,  $m_p$  is the proton mass, and the neutral atom mass is set to  $m_e + m_p$ .

Now form the ratio  $n^+ n_e / n^0$  and find

$$\frac{n^+ n_e}{n^0} = \frac{(2\pi kT)^{3/2}}{h^3} \left( \frac{m_e m_p}{m_e + m_p} \right)^{3/2} e^{(\mu^- + \mu^+ - \mu^0) / kT} e^{-\chi_H / kT}.$$

But  $\mu^- + \mu^+ - \mu^0 = 0$  for equilibrium by application of (3.7), so that we obtain the *Saha equation* for the single-level pure hydrogen gas<sup>7</sup>

$$\frac{n^+ n_e}{n^0} = \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_H / kT} \quad (3.35)$$

where the *reduced mass approximation* [ $m_e m / (m_e + m)$ ]  $\approx m_e$  has been used. A numerical version of part of this equation is

$$\left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} = 2.415 \times 10^{15} T^{3/2} \text{ cm}^{-3} \quad (3.36)$$

and note that

$$kT = 8.6173 \times 10^{-5} T \text{ eV} \quad (3.37)$$

where the eV units are handy for energies on the atomic scale.

To find the number densities, and not just ratios, further constraints must be placed on the system. A reasonable one is that of electrical neutrality, which requires that  $n_e = n^+$  for a gas of pure hydrogen. Furthermore, nucleon number must be conserved so that  $n^+ + n^0 = n$ , where  $n$  is a constant if the density ( $\rho$ ) is kept fixed.

We now define the degree of ionization (as in §1.4 and Eq. 1.47)

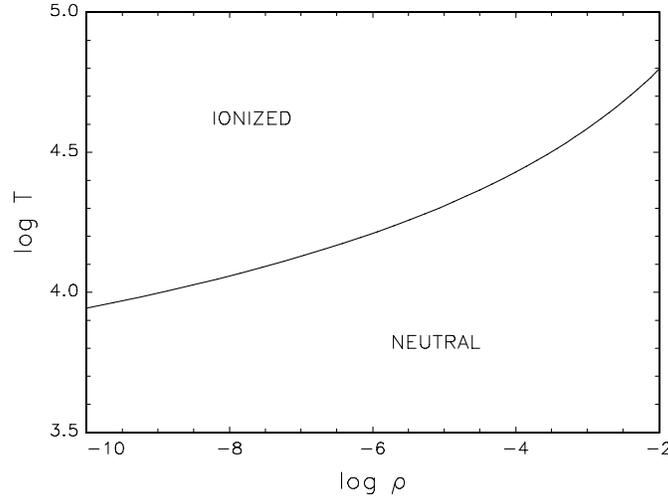
$$y = \frac{n^+}{n} = \frac{n_e}{n} \quad (3.38)$$

so that  $y$  is the fraction of all hydrogen that is ionized. The Saha equation (3.35) is then

<sup>7</sup> Clayton (1968, §1–2) extends this analysis to the case of multiple ionizations in many-electron atoms, which leads to a consideration of partition functions. We shall not need those functions but Clayton's discussion is worth looking into.

$$\frac{y^2}{1-y} = \frac{1}{n} \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_H/kT}. \quad (3.39)$$

For sufficiently high temperatures, with fixed density, we expect the radiation field or collisions effectively to ionize all the hydrogen. This is indeed the case because we see that as  $T \rightarrow \infty$ , then  $y \rightarrow 1$ . Similarly, low temperatures mean less intense radiation fields and recombination wins with  $y \rightarrow 0$ .



**Fig. 3.3.** The half-ionization curve for a mixture of pure hydrogen undergoing the recombination-ionization reaction  $\text{H}^+ + \text{e}^- \rightleftharpoons \text{H}^0 + \chi_{\text{H}}$  (ground state only).

For the pure hydrogen mixture  $n = \rho N_A$  and (3.39) becomes

$$\frac{y^2}{1-y} = \frac{4.01 \times 10^{-9}}{\rho} T^{3/2} e^{-1.578 \times 10^5/T}. \quad (3.40)$$

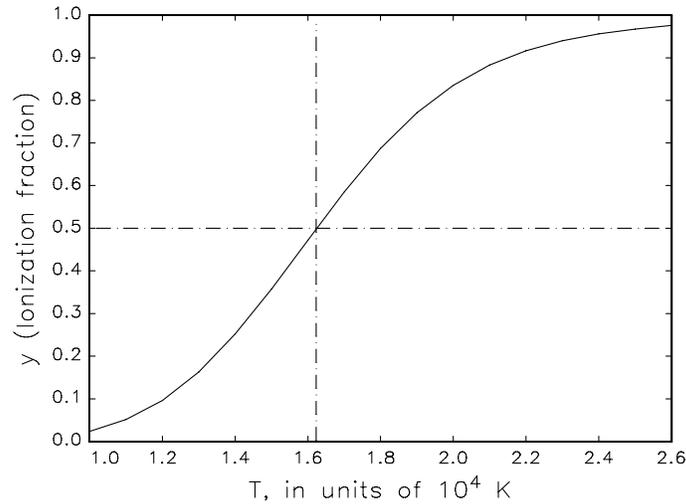
The half-ionized ( $y = 1/2$ ) path in the  $\rho$ - $T$  plane for this mixture is then

$$\rho = 8.02 \times 10^{-9} T^{3/2} e^{-1.578 \times 10^5/T} \text{ g cm}^{-3} \quad (3.41)$$

and this is shown in Fig. 3.3 as a very shallow curve for a range of what are interesting densities.

The dominant factor in (3.40) and (3.41) is the exponential and this is what causes the half-ionization point to depend only weakly on density. For hydrogen ionization from the ground state, the characteristic temperature for ionization-recombination is around  $10^4$  K and you may readily check that the transition from  $y = 0$  to  $y = 1$  takes place very rapidly as the temperature scans across that value (or, more precisely, at the temperature corresponding to  $y = 1/2$  at a particular density). This is shown in Fig. 3.4 for pure hydrogen

at a density of  $10^{-6} \text{ g cm}^{-3}$ . A *rough* rule of thumb is that the transition temperature (where  $y \approx 1/2$ ) is such that  $\chi/kT \sim 10$  to within a factor of three or so depending on density. Thus, for example, the ionization potentials for removing the first and second electrons of helium are 24.6 eV and 54.4 eV, which correspond to transition temperatures of about  $3 \times 10^4 \text{ K}$  and  $6 \times 10^4 \text{ K}$ . (See Ex. 3.1.)



**Fig. 3.4.** Note how the ionization fraction  $y$  changes rapidly for pure hydrogen as temperature is varied through  $1.62 \times 10^4 \text{ K}$  at which  $y = 1/2$ —as indicated by the dashed lines. The density is fixed at  $\rho = 10^{-6} \text{ g cm}^{-3}$ .

As we shall see, the presence of these zones of ionization have profound consequences for the structure of a star. You may wish to consider at this point a mixture of single-level hydrogen and helium (with two stages of ionization) and go through an analysis corresponding to the above to see how the various ions compete for electrons and to find out what the transition temperatures are for the three ionization stages involved. Even for this very practical, but simple, problem, you will find that a computer is essential for your sanity.

If the temperature and density of the hydrogen mixture are fixed, then (3.40) yields the ionization fraction  $y$ . The total hydrogen number density is clearly  $n = \rho N_A$  and thus  $n^+ = n_e = yn$  from (3.38). Chemical potentials, if required, follow from (3.32–3.34). The partial pressures and internal energies, which are additive, yield the total pressure

$$P = n(1 + y)kT \quad (3.42)$$

and total internal energy

$$E = \frac{3}{2}n(1+y)kT + yn\chi_{\text{H}} \quad \text{erg cm}^{-3}. \quad (3.43)$$

The last term in  $E$  appears because we have to take account of the ionization energy. If we wish to ionize the gas ( $y \rightarrow 1$ ) completely, then  $(3nkT/2 + n\chi_{\text{H}})$  erg cm<sup>-3</sup> must be added to the system. Of this amount,  $n\chi_{\text{H}}$  strips off the electrons, and the remainder brings the system up to the common temperature  $T$ .

The real calculation of ionization equilibria is as difficult as that for real equations of state (and the two are intimately connected). In principle, all species, energy levels, and reactions must be considered. In addition, the effects of real interactions must be included (and these depend on composition, temperature, and density), which change the relations between concentration and chemical potential. For textbook examples see Cox (1968, §15.3), and Kippenhahn and Weigert (1990, Chap. 14), with the warning that, in practice, accurate analytic or semianalytic solutions are seldom possible: you are usually faced with computer-generated tables of pressure and the like and the task is to use them intelligently.

### 3.5 Fermi–Dirac Equations of State

The most commonly encountered Fermi–Dirac elementary particles of stellar astrophysics are electrons, protons, and neutrons; all have spin one-half. (Neutrinos also appear but in contexts not usually connected with equations of state.) The emphasis here will be on electrons, but (almost) all that follows may apply to the other fermions as well. The prime motivation for this discussion is that the equation of state in the inner regions of many highly evolved stars, including white dwarfs, is dominated by degenerate electrons and, to a great extent, this determines the structure of such stars.

The number density of Fermi–Dirac particles is given by (3.9) and (3.10) with the choice of +1 in (3.9) and an energy reference level of  $\mathcal{E}_0 = mc^2$ , where  $m$  is the mass of the fermion. (Other choices are indeed possible for  $\mathcal{E}_0$ . They lead to an additive constant in the definition of the chemical potential and you have to watch out for this in the literature.) For these spin 1/2 particles, the statistical weight  $g = 2$ . Transcribing these statements then means that the number density is

$$n = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp\{[-\mu + mc^2 + \mathcal{E}(p)]/kT\} + 1} \quad (3.44)$$

where, in general, from (3.11) and (3.12),

$$\mathcal{E}(p) = mc^2 \left[ \sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1 \right] \quad (3.45)$$

and

$$v(p) = \frac{\partial \mathcal{E}}{\partial p} = \frac{p}{m} \left[ 1 + \left( \frac{p}{mc} \right)^2 \right]^{-1/2}. \quad (3.46)$$

We now explore some consequences of the above.

### 3.5.1 The Completely Degenerate Gas

The “completely degenerate” part of the title of this subsection refers to the unrealistic assumption that the temperature of the gas is absolute zero.<sup>8</sup> In practice this does not happen but, under some circumstances, the gas effectively behaves as if it were at zero temperature and, for fermions in stars, these unusual circumstances are very important. So, in (3.44), note the peculiar behavior of the integrand as  $T \rightarrow 0$ . The exponential tends either to zero or infinity depending on, respectively, whether  $-\mu + mc^2 + \mathcal{E}$  is  $<0$  or  $>0$ . Therefore consider the interesting part of (3.9),

$$F(\mathcal{E}) = \frac{1}{\exp \{ [\mathcal{E} - (\mu - mc^2)] / kT \} + 1} \quad (3.47)$$

where, as  $T \rightarrow 0$ ,  $F(\mathcal{E})$  approaches either zero or unity depending on whether  $\mathcal{E}$  is greater or less than  $\mu - mc^2$ .

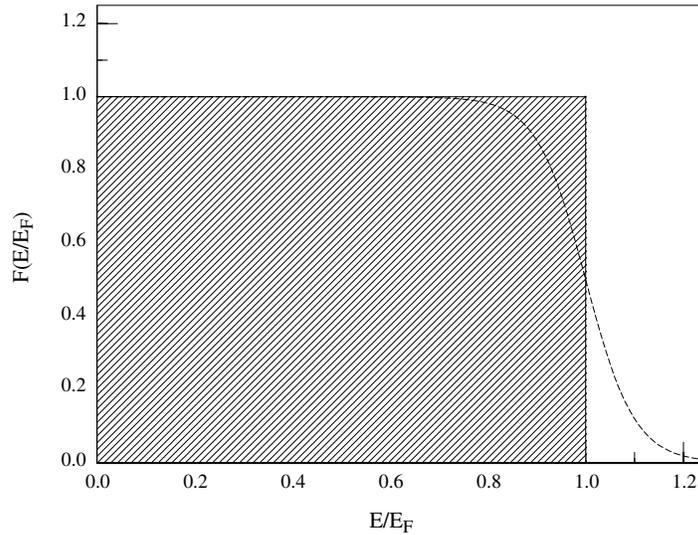
The critical kinetic energy at which  $F(\mathcal{E})$  is discontinuous (for  $T \rightarrow 0$ ) is called the “Fermi energy” and we denote it by  $\mathcal{E}_F$ ; that is, where  $\mathcal{E}_F = \mu - mc^2$ . (But note that we have not yet described how  $\mu$  is found.) The situation is depicted in Fig. 3.5 where, in the unit square corresponding to particle energies  $0 \leq \mathcal{E} \leq \mathcal{E}_F$ ,  $F(\mathcal{E})$  is unity. Fermions are contained only in that energy range and not at energies greater than  $\mathcal{E}_F$  where the distribution function is zero. In this situation we refer to a “filled Fermi sea” of fermions because all the fermions present are swimming in that sea and nowhere else. (Ignore the dashed line for the moment. It shows what happens if the temperature is raised slightly above zero. See §3.5.3.)

The momentum corresponding to the Fermi energy is the Fermi momentum  $p_F$ . It is usually reduced to dimensionless form by setting  $x = p/mc$  and defining  $x_F = p_F/mc$ . Then, from (3.45), we have

$$\mathcal{E}_F = mc^2 \left[ (1 + x_F^2)^{1/2} - 1 \right]. \quad (3.48)$$

In this language, the chemical potential of the system is  $\mu_F = \mathcal{E}_F + mc^2$  and it is the total energy, including rest mass energy, of the most energetic particle

<sup>8</sup> This has almost been achieved in the laboratory by the elegant experiments of DeMarco and Jin (1999), who, using atoms of <sup>40</sup>K at temperatures less than 300 nanoKelvin (!), have made a soup of fermions in their lowest energy states. Similar experiments by Anderson et al. (1995) have done the same for bosons by making a “Bose–Einstein Condensate” (BEC), a form of matter long thought possible but only now demonstrated actually to exist (and the work gained the two senior investigators a Nobel Prize). The two groups, not so incidentally, are in the same institute (JILA) at the University of Colorado.



**Fig. 3.5.** The function  $F(\mathcal{E}/\mathcal{E}_F)$  of (3.47) versus particle kinetic energy in units of  $\mathcal{E}_F$  for zero temperature. Fermions are restricted to the shaded area of unit height and width and do not have energies greater than the Fermi energy  $\mathcal{E}_F$ . The dashed line shows how  $F(\mathcal{E})$  is changed by raising the temperature slightly. (In this case  $\mathcal{E}_F/kT = 20$ .)

(or particles) in the system. If the spin is  $1/2$  ( $g = 2$ ), then all the rest of the particles are locked in pair-wise with spin-up and spin-down paired at each lower energy level by the Pauli exclusion principle.<sup>9</sup> The Fermi sea is then capped by the “Fermi surface” at  $\mathcal{E}_F$ .

The relation between particle number density and the Fermi energy, and thus  $\mu_F$ , is found as follows. Because  $F(\mathcal{E})$  is in the form of a unit step, (3.44) need only be integrated up to  $p_F$ . Hence

$$n = \frac{8\pi}{h^3} \int_0^{p_F} p^2 dp = 8\pi \left( \frac{h}{mc} \right)^{-3} \int_0^{x_F} x^2 dx = \frac{8\pi}{3} \left( \frac{h}{mc} \right)^{-3} x_F^3. \quad (3.49)$$

To deal with astrophysically interesting numbers we shall, from this point on, deal exclusively with electrons unless otherwise noted.

It is traditional, but admittedly confusing, to delete the  $F$  subscript on  $x_F$  so that (3.49) is written

$$n_e = \frac{8\pi}{3} \left( \frac{h}{m_e c} \right)^{-3} x^3 = 5.865 \times 10^{29} x^3 \text{ cm}^{-3} \quad (3.50)$$

<sup>9</sup> The most obvious application of the Pauli exclusion principle is for atoms. Were it not for this curious way nature works, electrons would all cascade down to the lowest energy level of atoms and we all would become very small entities indeed.

for electrons where  $(h/m_e c)$  is the electron Compton wavelength equal to  $2.426 \times 10^{-10}$  cm. The transcription to other spin 1/2 fermions is accomplished merely by changing the mass in (3.50).

To convert this to density units we reintroduce the electron mean molecular weight,  $\mu_e$ , of (1.48–1.49) with  $n_e = \rho N_A / \mu_e$ . Thus

$$\frac{\rho}{\mu_e} = B x^3 \quad (3.51)$$

with

$$B = \frac{8\pi}{3N_A} \left( \frac{h}{m_e c} \right)^{-3} = 9.739 \times 10^5 \text{ g cm}^{-3} \quad (3.52)$$

for electrons. This may be looked upon as a relation that yields  $x$  (i.e.,  $x_F$ ), and, hence,  $\mathcal{E}_F$  and  $p_F$ , once  $\rho/\mu_e$  is given.

Note that the demarcation between nonrelativistic and relativistic mechanics occurs when  $p_F \approx m_e c$  or  $x = x_F \approx 1$ . The corresponding density is  $\rho/\mu_e \approx 10^6 \text{ g cm}^{-3}$ , which, incidentally, is a typical central density for white dwarfs and is near the density at which the “helium flash” takes place (see §2.5). It remains to be shown, however, that temperatures in these contexts are sufficiently low to be effectively zero as far as electrons are concerned.

Looking ahead to neutron star matter, the numerical constant  $B$  in (3.51–3.52) is  $B(\text{neutrons}) = 6.05 \times 10^{15} \text{ g cm}^{-3}$  and  $\mu_e$  in that expression is set to unity; that is, we must replace  $\mu_e$  by the amu weight of the neutron (essentially unity). For typical densities in a neutron star (comparable to nuclear densities of  $\rho \approx 2.7 \times 10^{14} \text{ g cm}^{-3}$ ),  $x \approx 0.35$  and  $\mathcal{E}_F \approx 57 \text{ MeV}$ . This implies that the neutrons are nonrelativistic because the neutron rest mass energy is 939.57 MeV.

The pressure of a completely degenerate electron gas is treated in the same way as that for the number density. It is the integral in (3.13) truncated at the Fermi momentum with  $F(\mathcal{E})$  of (3.47) set to unity. A little work on (3.13) yields

$$P_e = \frac{8\pi}{3} \frac{m_e^4 c^5}{h^3} \int_0^{x_F} \frac{x^4 dx}{(1+x^2)^{1/2}} = A f(x) \quad (3.53)$$

where

$$A = \frac{\pi}{3} \left( \frac{h}{m_e c} \right)^{-3} m_e c^2 = 6.002 \times 10^{22} \text{ dyne cm}^{-2} \quad (3.54)$$

for electrons and

$$f(x) = x(2x^2 - 3)(1+x^2)^{1/2} + 3 \sinh^{-1} x. \quad (3.55)$$

Similarly, the internal energy, from (3.14), is given by the integral

$$E_e = 8\pi \left( \frac{h}{m_e c} \right)^{-3} m_e c^2 \int_0^{x_F} x^2 \left[ (1+x^2)^{1/2} - 1 \right] dx = A g(x) \quad (3.56)$$

with

$$g(x) = 8x^3 \left[ (1 + x^2)^{1/2} - 1 \right] - f(x). \quad (3.57)$$

The units for  $E_e$  are  $\text{erg cm}^{-3}$  as is that for  $A$  when  $\text{dyne cm}^{-2}$  is expressed in those units in (3.54).

It will often prove useful to have limiting forms for  $f(x)$  and  $g(x)$  that correspond to the limits of relativistic or nonrelativistic electrons. These are

$$f(x) \rightarrow \begin{cases} \frac{8}{5}x^5 - \frac{4}{7}x^7 + \dots, & x \ll 1 \\ 2x^4 - 2x^2 + \dots, & x \gg 1 \end{cases} \quad (3.58)$$

and

$$g(x) \rightarrow \begin{cases} \frac{12}{5}x^5 - \frac{3}{7}x^7 + \dots, & x \ll 1 \\ 6x^4 - 8x^3 + \dots, & x \gg 1. \end{cases} \quad (3.59)$$

Note that  $x \ll 1$  implies nonrelativistic particles, and  $x \gg 1$  is the extreme relativistic limit. Also observe that

$$P_e \propto E_e \propto \begin{cases} (\rho/\mu_e)^{5/3}, & x \ll 1 \\ (\rho/\mu_e)^{4/3}, & x \gg 1 \end{cases} \quad (3.60)$$

and the limiting ratios of  $E_e$  to  $P_e$  are

$$\frac{E_e}{P_e} = \frac{g(x)}{f(x)} = \begin{cases} 3/2 \quad (\gamma = 5/3), & x \ll 1 \\ 3 \quad (\gamma = 4/3), & x \gg 1. \end{cases} \quad (3.61)$$

The values for  $\gamma$  are included as a reminder that for a  $\gamma$ -law equation of state the completely degenerate nonrelativistic electron gas acts like a monatomic ideal gas whereas, in the extreme relativistic limit, it behaves like a photon gas.

### 3.5.2 Application to White Dwarfs

As a simple, but important, application of completely degenerate fermion statistics, consider zero temperature stars in hydrostatic equilibrium whose internal pressures are due solely to electron degenerate material and whose densities and composition are constant throughout.

The easiest way to look at this is to apply the virial theorem in the hydrostatic form  $3(\gamma - 1)U = -\Omega$  from (1.25). Because the star is assumed to have constant density,  $\Omega = -(3/5)(GM^2/\mathcal{R})$ . If  $E_e$  is the volumetric energy density (with no contribution from the zero temperature ions), then  $U = VE_e$  where  $V$  is the total stellar volume  $V = (4\pi/3)\mathcal{R}^3$ . In the nonrelativistic limit  $E_e = 12Ax^5/5$  from (3.56) and (3.59),  $x$  may be expressed in terms of  $\rho/\mu_e$  via (3.51) and  $\rho$ , in turn, may be eliminated in favor of  $\mathcal{M}$  and  $\mathcal{R}$  by  $\rho = \mathcal{M}/(4\pi\mathcal{R}^3/3)$ . If the entire virial theorem is also cast in a form containing only  $\mathcal{M}$  and  $\mathcal{R}$ , and if the constants  $B$  and  $A$  of (3.52) and (3.54) are given in terms of fundamental constants, then a little algebra yields the nonrelativistic mass–radius relation

$$\mathcal{M} = \frac{1}{4} \left( \frac{3}{4\pi} \right)^4 \left( \frac{h^2 N_A}{m_e G} \right)^3 \frac{N_A^2}{\mu_e^5} \frac{1}{\mathcal{R}^3} \text{ for constant density.} \quad (3.62)$$

This relation has the remarkable property that *as mass increases, radius decreases* and is quite unlike the homology result for main sequence stars discussed in the first chapter. And this result is what we promised you several times in Chapter 2.

For electrons, this yields the numeric expression

$$\frac{\mathcal{M}}{\mathcal{M}_\odot} \approx 10^{-6} \left( \frac{\mathcal{R}}{\mathcal{R}_\odot} \right)^{-3} \left( \frac{2}{\mu_e} \right)^5. \quad (3.63)$$

We state, without proof for now, that the interiors of white dwarf stars are almost entirely supported by electron degeneracy pressure, and that they typically have masses around  $0.6 \mathcal{M}_\odot$ . If the electrons are nonrelativistic, then (3.63) yields a typical radius of  $\mathcal{R} \approx 0.01 \mathcal{R}_\odot$  for  $\mu_e = 2$  (completely ionized  ${}^4\text{He}$ ,  ${}^{12}\text{C}$ ,  ${}^{16}\text{O}$ , etc.). This radius is very close to that of the earth's with  $\mathcal{R}_\oplus = 6.38 \times 10^8$  cm. An exact analysis involving integration of the hydrostatic equation using the nonrelativistic equation of state shows that (3.63) gives the correct result provided that the numerical coefficient is increased by (only!) a factor of two.

If  $\mu_e$  in (3.62) is replaced by unity and the particle mass is taken to be that of the neutron, then the neutron star equivalent of (3.63) becomes

$$\frac{\mathcal{M}}{\mathcal{M}_\odot} \approx 5 \times 10^{-15} \left( \frac{\mathcal{R}}{\mathcal{R}_\odot} \right)^{-3} \quad (\text{neutron stars}) \quad (3.64)$$

in the nonrelativistic limit. For  $\mathcal{M} = \mathcal{M}_\odot$ ,  $\mathcal{R} \approx 11$  km, which is in the right ballpark. Note that general relativistic effects have been completely ignored, but this is the least of our sins because the nuclear force makes our noninteracting equation of state inaccurate.

You will have realized by now that the simple arguments outlined above for mass-radius relations contain a serious flaw. The nonrelativistic degenerate electron pressure depends solely on density and composition (through  $\mu_e$ ); that is, in numeric form and using (3.51), (3.53), and (3.58)

$$P_e = 1.004 \times 10^{13} \left( \frac{\rho}{\mu_e} \right)^{5/3} \text{ dyne cm}^{-2} \quad (3.65)$$

and, as may easily be verified, the corresponding extreme relativistic expression is

$$P_e = 1.243 \times 10^{15} \left( \frac{\rho}{\mu_e} \right)^{4/3} \text{ dyne cm}^{-2}. \quad (3.66)$$

Thus if  $\rho$  and  $\mu_e$  are constant, then so is  $P_e$  by virtue of the equation of state. But a constant pressure is inconsistent with hydrostatic equilibrium

and, in fact, (1.41) is the correct solution for the pressure through a constant-density star. Thus  $P_e$  is not a constant and neither is  $E_e$  as assumed above. The trouble is that we have overconstrained the problem by insisting on the constancy of  $\rho$  combined with the degenerate equation of state.

The correct way to construct equilibrium degenerate models is to use the general expression for the pressure given by (3.53) along with the relation between  $\rho/\mu_e$  and dimensionless Fermi momentum of (3.51). This yields a pressure–density relation, which is then put into the equation of hydrostatic equilibrium. The resulting equation is then combined with the equation of mass conservation yielding a second-order differential equation that must be integrated numerically. We shall not go into the tedious details here because more than adequate discussions are given in Chandrasekhar (1939, Chap. 11) and Cox (1968, §25.1), and, in any case, such solutions are easy to come by using modern numerical techniques. (See, for example, Chap. 7.) Important results are summarized below.

In the limit of extreme relativistic degeneracy, where (3.66) is appropriate, you may easily convince yourself by using dimensional analysis that the total stellar mass depends only on  $\mu_e$  and not on radius. An exact analysis yields

$$\frac{M}{M_\odot} = \frac{M_\infty}{M_\odot} = 1.456 \left( \frac{2}{\mu_e} \right)^2 \quad (3.67)$$

where  $M_\infty$  is the *Chandrasekhar limiting mass*.<sup>10</sup> A virial analysis similar to that used to find (3.62), but done in the relativistic limit, yields a result differing from the above by only a change in the constant (a 1.75 instead of 1.456). We assume you will try to verify this and, if you do, you should also find that the full virial expression (1.25) implies  $d^2I/dt^2$  becomes negative if the total mass exceeds  $M_\infty$ . The interpretation is that electron degenerate objects (of fixed  $\mu_e$ ) cannot have masses exceeding the Chandrasekhar limit without collapsing the object. Increased densities and pressure cannot halt the collapse because the relativistic limit has already been reached. In the nonrelativistic limit, on the other hand, a new configuration may be reached by decreasing the radius as indicated by (3.63). Extreme relativistic equations of state, including that for photons, are too “soft” compared to the effects of self-gravity. (You can’t make the particles exceed the speed of light to try to increase pressures!) This conclusion might have been anticipated because extreme relativistic effects imply  $\gamma \rightarrow 4/3$ .

<sup>10</sup> The exact value of this limiting mass depends on physics we have not included in our analysis. Hamada and Salpeter (1961), for example, consider the effects of electrostatic interactions and electron captures on various nuclei. For single white dwarfs with normal masses and compositions, these effects are not that significant. However, we can imagine massive objects formed by various means in binary systems where such effects could well give a stable maximum mass less than the Chandrasekhar limiting mass, as discussed earlier in §2.13.

The astrophysical significance of the Chandrasekhar limiting mass is just as we discussed in Chapter 2. If electron degenerate configurations are good representations of white dwarfs, and if those objects are the final end product of evolution for most stars, then the late stages of evolution are severely constrained. That is, if a star does not finally rid itself of enough mass to eventually leave a white dwarf with  $\mathcal{M} \lesssim 1.46\mathcal{M}_\odot$  (assuming a reasonable value of  $\mu_e$  near 2), then something catastrophic will happen at some time in its life. Since there are so many white dwarfs in the sky, a large fraction of stars either start off with sufficiently low masses, or they manage to rid themselves of the excess mass.

The regime intermediate between nonrelativistic and full relativistic degeneracy is intractable using simple means, and full-scale models must be calculated (and you may try this by using the code `WD.FOR` on the CD-ROM). The following useful and quite accurate mass–radius relation bridging the two regimes (fit to actual calculations) is based on one given by Eggleton (1982) for electrons:

$$\frac{\mathcal{R}}{\mathcal{R}_1} = 2.02 \left[ 1 - \left( \frac{\mathcal{M}}{\mathcal{M}_\infty} \right)^{4/3} \right]^{1/2} \left( \frac{\mathcal{M}}{\mathcal{M}_\infty} \right)^{-1/3}. \quad (3.68)$$

Here,  $\mathcal{M}/\mathcal{M}_\infty$  is given by (3.67), and  $\mathcal{R}_1$  is defined by

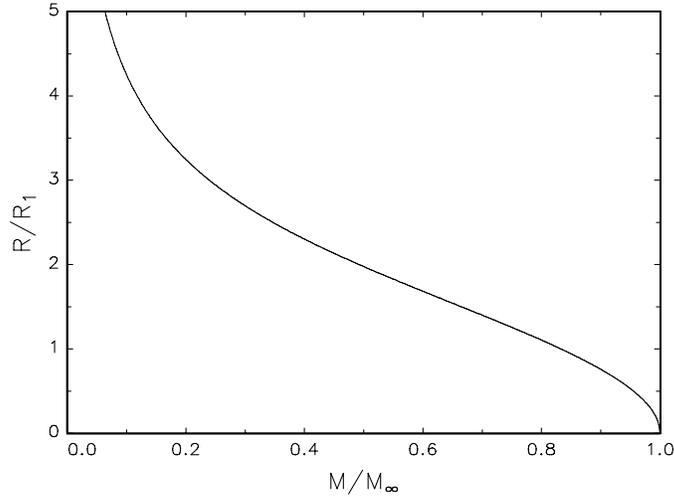
$$\frac{\mathcal{R}_1}{\mathcal{R}_\odot} = 5.585 \times 10^{-3} \left( \frac{2}{\mu_e} \right). \quad (3.69)$$

This radius is a typical scale length for electron degenerate objects. The relativistic and nonrelativistic limits of (3.68) go to the correct values as  $\mathcal{R} \rightarrow 0$  (relativistic) or  $\mathcal{M}$  becomes small (nonrelativistic). It is shown plotted in Fig. 3.6.

We shall have more to say about white dwarfs in Chapter 10. One crucial item that has not been addressed here, and that pertains to these objects, is the effect of temperature on degeneracy. After all, if white dwarfs were really at zero temperature we wouldn't see them.

### 3.5.3 Effects of Temperature on Degeneracy

The crucial step in deriving some of the thermodynamics of the completely degenerate zero temperature fermion gas was the realization that the distribution function becomes a unit step function at a kinetic energy equal to  $\mu - mc^2$ . If the zero temperature condition is relaxed, the distribution function follows suit. Suppose the temperature is low—on some scale yet to be determined—but not zero. Fermions deep in the Fermi sea, at energies much less than  $\mathcal{E}_F$ , need roughly an additional  $\mathcal{E}_F$  energy units to move around in energy. That is, if the energy input to the system, as measured by  $kT$ , is much smaller than  $\mathcal{E}_F$ , then low-energy particles are excluded from promotion to already occupied upper energy levels by the Pauli exclusion principle.



**Fig. 3.6.** The mass–radius relation for zero temperature white dwarfs with constant  $\mu_e$ . (See Eqs. 3.68–3.69.)

Fermions near the top of the Fermi sea don’t have that difficulty and they may find themselves elevated into states with energies greater than  $\mathcal{E}_F$ . Thus as temperature is raised from zero, the stepped end of the distribution function smooths out to higher energies. This is the effect shown in Fig. 3.5 by the dashed line. If temperatures rise high enough, we expect the effects of Fermi–Dirac statistics to be washed out completely and the gas should merge into a Maxwell–Boltzmann distribution. With this discussion as a guide, it should be apparent that a rough criterion for the transition from degeneracy to near- or nondegeneracy is  $\mathcal{E}_F \approx kT$ . The dashed line in Fig. 3.5 shows the effect of a rise in temperature corresponding to  $\mathcal{E} = 20kT$ . The effect on the distribution function is rather small, as would be expected, but the gas is no longer completely degenerate. A better description is that the gas is *partially degenerate*. As an example of the transition to nondegeneracy we apply the criterion  $\mathcal{E} \approx kT$  to nonrelativistic electrons.

The Fermi energy of a nonrelativistic electron gas is  $\mathcal{E}_F = mc^2 x_F^2/2$ , which is easily obtained by expanding the radical in (3.48) for small  $x_F$ . The dimensionless Fermi momentum  $x_F$  is then converted to  $\rho/\mu_e$  using (3.51). After this is applied to  $\mathcal{E}_F \approx kT$ , and numbers put in, the criterion becomes

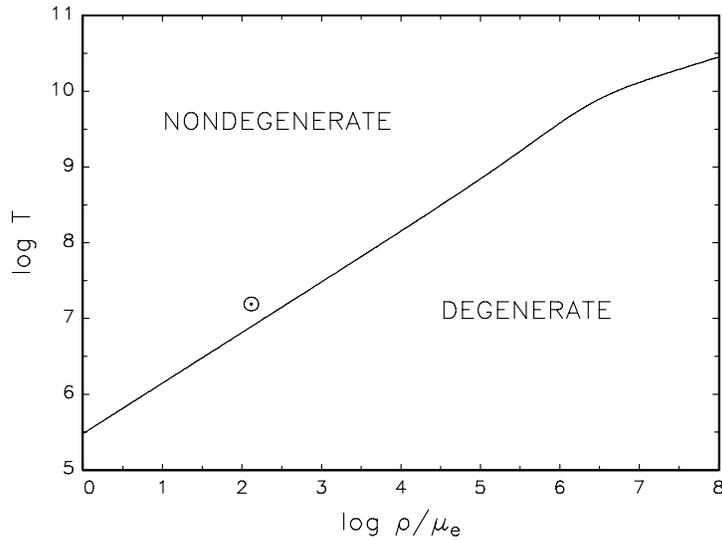
$$\frac{\rho}{\mu_e} \approx 6.0 \times 10^{-9} T^{3/2} \text{ g cm}^{-3}. \quad (3.70)$$

If  $\rho/\mu_e$  exceeds the value implied by the righthand side of (3.70) for a given temperature, then the gas is considered degenerate. Realize though that this is a rough statement: there is no clean demarcation line on the  $T$ – $\rho/\mu_e$  plane that distinguishes degenerate from nondegenerate electrons.

The extreme relativistic equivalent to (3.70) is

$$\frac{\rho}{\mu_e} \approx 4.6 \times 10^{-24} T^3 \text{ g cm}^{-3}. \quad (3.71)$$

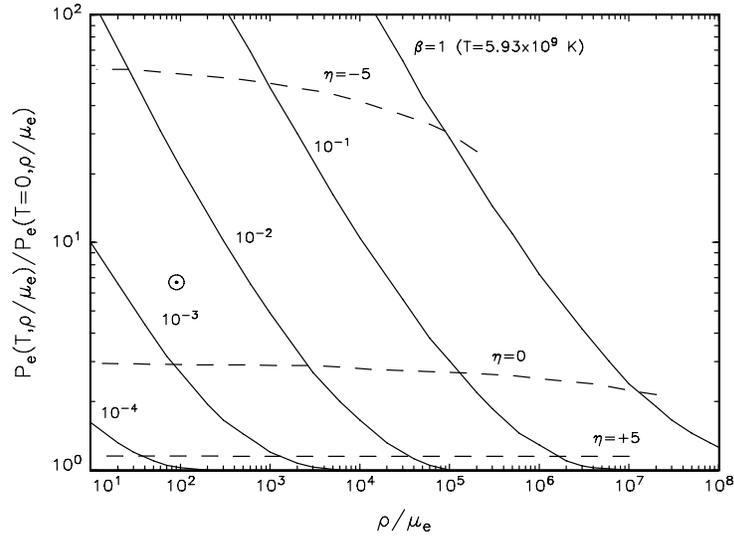
The density near which special relativistic effects become important was estimated earlier as  $\rho/\mu_e \approx 10^6 \text{ g cm}^{-3}$ . Equations (3.70) and (3.71) are illustrated in Fig. 3.7 where the transition near  $10^6 \text{ g cm}^{-3}$  has been smoothed. Note that the center of the present-day sun, as indicated in the figure, is nondegenerate but close enough to the transition line that good solar models include the effects of Fermi–Dirac statistics.



**Fig. 3.7.** The domains of nondegenerate and degenerate electrons in the  $T$ – $\rho/\mu_e$  plane. The location of the center of the present-day sun in these coordinates is indicated by the  $\odot$  sign.

A better idea of how the transition from degeneracy to nondegeneracy takes place with respect to temperature and  $\rho/\mu_e$  requires explicit evaluation of the Fermi–Dirac integrals. In general, this involves numeric integration, although there are some useful series expansions and we shall discuss one of these in a bit. The reader is referred to Cox (1968) and other references at the end of this chapter for a full discussion but the results are summarized in Fig. 3.8, which is derived from the numeric tabulations in App. A2 of Cox and his §24.4. Cloutman (1989) discusses some techniques for computing the Fermi–Dirac integrals and includes a FORTRAN program listing (see also Eggleton et al., 1973, and Antia, 1993).

Plotted versus  $\rho/\mu_e$  in Fig. 3.8 is the ratio of electron pressure at nonzero temperature,  $P_e(T, \rho/\mu_e)$ , to the electron pressure for complete degeneracy



**Fig. 3.8.** The domains of nondegenerate and degenerate electrons in temperature and density as expressed by the ratio  $P_e(T, \rho/\mu_e)/P_e(T=0, \rho/\mu_e)$ . Temperatures are given in units of  $\beta = kT/m_e c^2$ , where  $\beta = 1$  corresponds to  $5.93 \times 10^9$  K. The dashed lines are lines of constant  $\eta$ , which is sometimes called the “degeneracy parameter” and is related to the chemical potential (see text). The position of the solar center is indicated by  $\odot$ .

at zero temperature,  $P_e(T=0, \rho/\mu_e)$ . Values near unity for this ratio imply strong degeneracy for  $P_e(T, \rho/\mu_e)$ , whereas large values mean that the gas is nondegenerate and, if large enough, the Maxwell–Boltzmann expression may be used. The solar center is indicated in the figure, and its position implies that degeneracy accounts for some 15% of the total pressure at that location.

Note that the effects of electron–positron pairs created by the radiation field are not included here. These become important if temperatures approach or exceed  $kT \approx m_e c^2$  (i.e.,  $T \gtrsim 6 \times 10^9$  K). We shall discuss pair-created electrons briefly in Chapter 6, where they play a role in creating neutrinos.

A parameter called  $\eta$  is plotted as dashed lines on the figure and an  $\eta$  of five, for example, corresponds to the situation where the true pressure is only about 15% greater than if the gas were completely degenerate. Along the dashed line labeled “ $\eta = 0$ ,” a degenerate estimate for the pressure would be too low by about a factor of three. Transferring this line to the temperature versus density plane results in a plot that is very similar to that of Fig. 3.7. Finally, the parameter  $\eta$ , which is commonly used in the literature (but not by everyone), is related to the electron chemical potential defined here by  $\eta = (\mu - m_e c^2)/kT$ .

For strongly, but not completely, degenerate gases, there are useful expansions for number density, pressure, and internal energy that are often quoted

in the literature. We shall not derive complete versions of those expansions here (see the references) but they all depend on the mild relaxation of the shape of the distribution function near  $\mathcal{E}_F$ . One of them is the following.

Following Landau and Lifshitz (1958, §57) we write any of the Fermi–Dirac integrals (for number density, etc.) in the kinetic energy-dependent form

$$I(\mu, T) = \int_0^\infty \frac{G(\mathcal{E}) d\mathcal{E}}{\exp [(-\mu + mc^2 + \mathcal{E}) / kT] + 1} . \quad (3.72)$$

The integral  $I$  may be expressed as an asymptotic (but not necessarily convergent) series whose leading terms are

$$I(\mu, T) = \int_0^{\mu'} G(\mathcal{E}) d\mathcal{E} + \frac{\pi^2}{6} \frac{\partial G}{\partial \mathcal{E}} (kT)^2 + \frac{7\pi^4}{360} \frac{\partial^3 G}{\partial \mathcal{E}^3} (kT)^4 + \dots \quad (3.73)$$

where  $\mu' = \mu - mc^2$  and all the partials are evaluated at  $\mu'$ . It is assumed that  $\mu' / kT$  is much larger than unity.

It is a simple, but tedious, exercise to transform the integrals for  $n$ ,  $P$ , and  $E$  of, respectively, (3.10), (3.13), and (3.14), into their energy space counterparts and then to find  $G(\mathcal{E})$ . Another way, however, is to transform all of the elements in the expansion (3.73) into  $x = p/mc$ -space using (3.45); that is,  $\mathcal{E} = mc^2 [(1 + x^2)^{1/2} - 1]$ . A big part of this was done when the expressions for the completely degenerate electron gas were written down in the equations for  $n_e$  (3.49),  $P_e$  (3.53), and  $E_e$  (3.56). Thus, for example, the leading term in the expansion of (3.73) for  $n_e$  is simply (neglecting constants)

$$n_e \text{ (first term)} \propto \int_0^{x_f} x^2 dx .$$

Here  $x_f$  takes the place of  $\mu' = \mu - mc^2$  and, since we have converted from energy to  $x$ -space, it should be obvious that the relation between  $x_f$  and  $\mu'$  is

$$\mu' = \mu - m_e c^2 = m_e c^2 \left[ (1 + x_f^2)^{1/2} - 1 \right] . \quad (3.74)$$

This relation is given in the same spirit as was done for the completely degenerate case where the Fermi energy was related to the chemical potential by  $\mathcal{E}_F = \mu - m_e c^2$  and  $\mathcal{E}_F$  was given in terms of  $x_F$  through (3.48). In that instance,  $x_F$  and, hence,  $\mu$  were found by fixing the number density  $n_e$  and using (3.49). The same sort of thing can be done here except there is an additional complication because temperature also appears in the thermodynamics; that is,  $n_e$  must be a function of both  $x_f$  (or  $\mu$ ) and  $T$ . This all can be accomplished by performing the indicated operations in the expansion (3.73). Carrying out this enterprise is left to you as an exercise in elementary calculus, but the result, to second-order in temperature, is

$$n_e = \frac{8\pi}{3} \left( \frac{h}{m_e c} \right)^{-3} x_f^3 \left[ 1 + \pi^2 \frac{1 + 2x_f^2}{2x_f^4} \left( \frac{kT}{m_e c^2} \right)^2 + \dots \right] \text{ cm}^{-3} . \quad (3.75)$$

This expansion is useful only if the second term in the brackets is small compared to unity. A useful rule of thumb is to be wary if it exceeds 0.1 to 0.2. In any case, given any two of  $n_e$  (or  $\rho/\mu_e$ ),  $T$ , or  $x_f$  (or  $\mu$ ), the third follows. Looked at another way (and we shall use this shortly), (3.75) may be used to find out how the chemical potential changes with respect to temperature for fixed  $n_e$  or  $\rho/\mu_e$ . Note that as  $T \rightarrow 0$ , the number density approaches the completely degenerate expression (3.49) with  $x_f \rightarrow x_F$ , and  $\mu' \rightarrow \mathcal{E}_F$ .

The corresponding expansions truncated to second order in  $kT$  for pressure and internal energy are

$$P_e = Af(x_f) \left[ 1 + 4\pi^2 \frac{x_f(1+x_f^2)^{1/2}}{f(x_f)} \left( \frac{kT}{m_e c^2} \right)^2 \right] \quad (3.76)$$

$$E_e = Ag(x_f) \left[ 1 + 4\pi^2 \frac{(1+3x_f^2)(1+x_f^2)^{1/2} - (1+2x_f^2)}{x_f g(x_f)} \left( \frac{kT}{m_e c^2} \right)^2 \right] \quad (3.77)$$

where  $f(x_f)$  and  $g(x_f)$  are given, respectively, by (3.55) and (3.57). Note that  $P_e$  is in dyne  $\text{cm}^{-2}$  and  $E_e$  is the volumetric energy density in  $\text{erg cm}^{-3}$  (and *not* specific energy density in  $\text{erg g}^{-1}$ ).

These equations will be used to find such things as specific heats and temperature exponents for the almost completely degenerate electron gas.

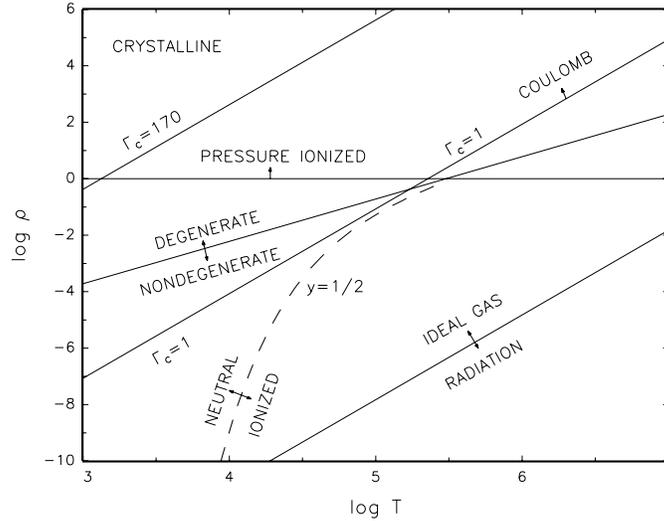
**Note:** As a matter of practicality,  $x_f$  is often computed as if the gas were completely degenerate. Thus if the correction term for temperature is very small, then  $x$  (or  $x_F$ ) of (3.50) is used instead of  $x_f$  as a good approximation for direct calculation of  $n_e$ ,  $P_e$ , and  $E_e$  in (3.75–3.77). This is what we shall usually do here.

### 3.6 “Almost Perfect” Equations of State

In real gases, interactions have to be taken into account that modify the “perfect” results given above. In addition, a stellar equation of state might consist of many components with radiation, Maxwell–Boltzmann, and degenerate gases competing in importance. This short section will not attempt to show how imperfections are treated in detail but will indicate where some are important in practical situations. The results of this discussion are summarized in Fig. 3.9 for a hypothetical gas composed of pure hydrogen.

In an almost-ideal gas, a measure of the interaction energy between ions is the Coulomb potential between two ions. If the ionic charge is  $Z$ , then the potential is  $Z^2 e^2/a$ , where  $a$  is some typical separation between the ions. Coulomb effects are expected to become important when this energy is comparable to  $kT$ . Thus form the ratio

$$\Gamma_C \equiv \frac{Z^2 e^2}{akT} \quad (3.78)$$



**Fig. 3.9.** A composite showing how the  $\rho$ - $T$  plane is broken up into regions dominated by pressure ionization, degeneracy, radiation, ideal gas, crystallization, and ionization-recombination. The gas is assumed to be pure hydrogen.

where  $\Gamma_C = 1$  is the rough demarcation between where Coulomb effects might be important or not, and a  $\Gamma_C > 1$  implies they probably are important. The distance  $a$  is usually taken as the radius of a Wigner-Seitz sphere whereby  $(4\pi a^3/3) = (1/n_i)$  and  $n_i$  is the ion number density. If the gas consists of pure ionized hydrogen and  $\Gamma_C = 1$ , then (3.78) becomes

$$\rho = 8.49 \times 10^{-17} T^3 \text{ g cm}^{-3}. \quad (3.79)$$

If the density is greater than that implied by (3.79) for a given temperature, then you can be reasonably certain that a perfect gas is not as perfect as could be desired. This line is shown in Fig. 3.9. You may check, from the material given previously, that the centers of very low mass ZAMS stars are encroaching upon both this line and the one for degeneracy effects. Carefully done stellar models of these stars contain corrections for these effects.

If  $\Gamma_C$  becomes large enough, then Coulomb effects overwhelm those of thermal agitation and the gas settles down into a crystal. The best estimates as to how this takes place yield a  $\Gamma_C$  of around 170 for the transition. With this value of  $\Gamma_C$  in a hydrogen gas (which is kind of silly for a crystallizing composition but fine for talking purposes), (3.79) becomes

$$\rho = 4.2 \times 10^{-10} T^3 \text{ g cm}^{-3}. \quad (3.80)$$

This is not an academic issue because some portions of very cool white dwarfs are thought to turn crystalline, but with carbon and/or oxygen rather than hydrogen.

We have already discussed the Saha equation for pure hydrogen, and the density–temperature relation for half-ionization was given by (3.41). That relation is also shown in Fig. 3.9 as the dashed line. In deriving the Saha equation it was implicitly assumed that the energy levels of the hydrogen atom (had we included all of them) were known and that their energies were independent of conditions in the ambient environment. This cannot be true in general. If the gas is dense, then the electrostatic field of one atom should influence a neighboring atom and hence disturb atomic levels. In the extreme, we can imagine this continuing until electron clouds practically rub and electrons are ionized off the parent atoms. This is a crude description of *pressure ionization*. To estimate under what conditions this occurs, take the rubbing picture seriously and find at what density the Wigner–Seitz radius equals the radius of the first Bohr orbit of hydrogen ( $0.53 \times 10^{-8}$  cm). A very easy calculation says that this takes place when

$$\rho \approx 1 \text{ g cm}^{-3}. \quad (3.81)$$

This density is shown in Fig. 3.9 as the line that terminates ordinary Saha ionization. Such densities are commonplace in stellar interiors and lead to the statement that the larger bulk of those interiors are ionized as far as the lighter elements are concerned independent of the effects of the radiation field.<sup>11</sup>

We finally ask under what conditions radiation pressure dominates over ideal gas pressure or the other way round. That is, where does  $aT^4/3 = \rho N_A kT/\mu$ ? With the assumption of complete ionization in hydrogen this becomes

$$\rho = 1.5 \times 10^{-23} T^3 \text{ g cm}^{-3} \quad (3.82)$$

as shown in the figure. This ends the discussion of the major factors determining pressures and internal energies in simple environments.<sup>12</sup>

### 3.7 Adiabatic Exponents and Other Derivatives

For the most part, all we need in the way of thermodynamic variables to construct a simplified stellar model is the internal energy and pressure as a function of density, temperature, and composition (as was done in Chap. 1). To construct realistic models, and to evolve them in time, however, we need several thermodynamic derivatives. We shall assume, at first, that the detailed

<sup>11</sup> As a side comment, note that several lines in the figure cross at  $T \approx 3 \times 10^5$  K and  $\rho \approx 1 \text{ g cm}^{-3}$ . You can be assured that computing accurate equations of state in that region of the  $T$ – $\rho$  plane is a nightmare.

<sup>12</sup> We have purposely ignored equations of state at ultrahigh densities such as are found in neutron stars and the collapsing cores of supernovae. This is a difficult subject itself worthy of a monograph. For further reading we suggest chapters 2 and 8 of Shapiro and Teukolsky (1983) and Bethe (1990, §§3-4).

composition, including concentrations of ions, etc., has been determined and that chemical reactions are not taking place. We also assume that you have some facility in transforming thermodynamic functions under reversible conditions and that you are familiar with their properties.

### 3.7.1 Keeping the Composition Fixed

If changes in temperature and density (or volume) do not cause corresponding changes in the relative concentrations of various species of atoms or ions in the stellar mixture, then the calculation of thermodynamic derivatives is not particularly difficult. We now examine this situation and ignore until later those complications arising from chemical reactions.

#### Specific Heats

The first derivatives encountered in elementary thermodynamics are specific heats. In general form these are defined by

$$c_\alpha = \left( \frac{dQ}{dT} \right)_\alpha \quad (3.83)$$

where  $\alpha$  is kept fixed as  $T$  changes. In the following,  $Q$  will have the units of erg  $g^{-1}$  and thus the specific heats will have units of erg  $g^{-1} K^{-1}$ . The most useful variables for  $\alpha$  for us are  $P$ ,  $\rho$ , or the specific volume  $V_\rho = 1/\rho$ . (We shall also have occasion to use the ordinary volume,  $V$ .) From the first law for a reversible process (and see 1.11)

$$dQ = dE + P dV_\rho = dE + P d\left(\frac{1}{\rho}\right) = dE - \frac{P}{\rho^2} d\rho \quad (3.84)$$

so that

$$c_{V_\rho} = \left( \frac{dQ}{dT} \right)_\rho = \left( \frac{\partial E}{\partial T} \right)_\rho \quad \text{erg } g^{-1} K^{-1}. \quad (3.85)$$

For an ideal monatomic gas  $E = 3N_A kT/2\mu$  erg  $g^{-1}$  (from 3.29) so that  $c_{V_\rho} = 3N_A k/2\mu$  and  $E = c_{V_\rho} T$ . Note that the composition has not been mentioned here except in the mean molecular weight  $\mu$ : it is kept fixed by assumption.

To find  $c_P$ , recall (from any of many thermodynamic texts) that  $c_P$  and  $c_{V_\rho}$  (or  $c_V$ ) are related by

$$c_P - c_{V_\rho} = -T \left( \frac{\partial P}{\partial T} \right)_{(\rho \text{ or } V_\rho)}^2 \left( \frac{\partial P}{\partial V_\rho} \right)_T^{-1}. \quad (3.86)$$

To cast this in a form that will prove more suitable for later purposes we reintroduce the power law expression for the equation of state given in Chapter 1 by (1.67):

$$P = P_0 \rho^{\chi_\rho} T^{\chi_T} \quad (3.87)$$

where  $P_0$ ,  $\chi_\rho$ , and  $\chi_T$  are constants. This means that the last two are also defined by

$$\chi_T = \left( \frac{\partial \ln P}{\partial \ln T} \right)_{(\rho \text{ or } V_\rho)} = \frac{T}{P} \left( \frac{\partial P}{\partial T} \right)_{(\rho \text{ or } V_\rho)} \quad (3.88)$$

and

$$\chi_\rho = \left( \frac{\partial \ln P}{\partial \ln \rho} \right)_T = - \left( \frac{\partial \ln P}{\partial \ln V_\rho} \right)_T = \frac{\rho}{P} \left( \frac{\partial P}{\partial \rho} \right)_T = - \frac{1}{\rho P} \left( \frac{\partial P}{\partial V_\rho} \right)_T. \quad (3.89)$$

Thus

$$c_P - c_{V_\rho} = \frac{P}{\rho T} \frac{\chi_T^2}{\chi_\rho} \quad \text{erg g}^{-1} \text{ K}^{-1}. \quad (3.90)$$

For an ideal monatomic gas  $\chi_\rho = \chi_T = 1$  and

$$c_P - c_{V_\rho} = \frac{N_A k}{\mu} \quad \text{erg g}^{-1} \text{ K}^{-1} \quad (\text{ideal gas}), \quad (3.91)$$

which gives the elementary result  $c_P = 5N_A k/2\mu$ .

We also define  $\gamma$  (yes, another  $\gamma$ ), the ratio of specific heats, to be

$$\gamma = \frac{c_P}{c_{V_\rho}} = 1 + \frac{P}{\rho T c_{V_\rho}} \frac{\chi_T^2}{\chi_\rho} \quad (3.92)$$

which will be discussed shortly. This  $\gamma$  need not be the  $\gamma$  of the  $\gamma$ -law equation of state, but sometimes it is—see later.

### Adiabatic Exponents

The dimensionless adiabatic exponents, the “ $\Gamma$ s,” measure the thermodynamic response of the system to adiabatic changes and will be used extensively. (Two of them,  $\Gamma_1$  and  $\Gamma_2$ , were already introduced in Chap. 1.) They are defined as follows:

$$\Gamma_1 = \left( \frac{\partial \ln P}{\partial \ln \rho} \right)_{\text{ad}} = - \left( \frac{\partial \ln P}{\partial \ln V_\rho} \right)_{\text{ad}} \quad (3.93)$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \left( \frac{\partial \ln P}{\partial \ln T} \right)_{\text{ad}} = \frac{1}{\nabla_{\text{ad}}} \quad (3.94)$$

which also defines  $\nabla_{\text{ad}}$ , and

$$\Gamma_3 - 1 = \left( \frac{\partial \ln T}{\partial \ln \rho} \right)_{\text{ad}} = - \left( \frac{\partial \ln T}{\partial \ln V_\rho} \right)_{\text{ad}}. \quad (3.95)$$

As in Chapter 1, the subscript “ad” means that the indicated partials are to be evaluated at constant entropy. (We shall not need it directly, but extensive

use will be made of  $\nabla_{\text{ad}}$  in later chapters.) It will shortly become clear why the  $\Gamma_i$  appear in such curious combinations in the definitions, but first note that not all the  $\Gamma_i$  are independent. You may easily show that

$$\frac{\Gamma_3 - 1}{\Gamma_1} = \frac{\Gamma_2 - 1}{\Gamma_2} = \nabla_{\text{ad}}. \quad (3.96)$$

Computation of the  $\Gamma_i$  is tedious and not particularly enlightening. Complete and clear derivations may be found in Cox (1968), but we suggest you try to derive the expressions that follow using the more compact methods given in Landau and Lifshitz (1958), for example. They start from fundamentals and then use powerful yet simple Jacobian transformations to derive what is needed. All you need watch out for is the distinction between  $V$  and  $V_\rho$ . When you get done, realize that there are many variations in the ways that the  $\Gamma_i$  may be expressed and the following may not always be the most efficient to use; that is, you may wish to rearrange things. The adiabatic exponents are

$$\Gamma_3 - 1 = \frac{P}{\rho T} \frac{\chi_T}{c_{V_\rho}} = \frac{1}{\rho} \left( \frac{\partial P}{\partial E} \right)_\rho \quad (3.97)$$

$$\Gamma_1 = \chi_T (\Gamma_3 - 1) + \chi_\rho = \frac{\chi_\rho}{1 - \chi_T \nabla_{\text{ad}}} \quad (3.98)$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \nabla_{\text{ad}}^{-1} = c_P \frac{\rho T}{P} \frac{\chi_\rho}{\chi_T} = \frac{\chi_\rho}{\Gamma_3 - 1} + \chi_T. \quad (3.99)$$

The last exponent,  $\gamma$ , is given by

$$\gamma = \frac{c_P}{c_{V_\rho}} = \frac{\Gamma_1}{\chi_\rho} = 1 + \frac{\chi_T}{\chi_\rho} (\Gamma_3 - 1) = \frac{\Gamma_3 - 1}{\chi_\rho} \frac{1}{\nabla_{\text{ad}}}. \quad (3.100)$$

Note that the righthand side result for  $\Gamma_3$  implies that  $P = (\Gamma_3 - 1)\rho E$  so that the  $\gamma$  in the  $\gamma$ -law equation of state of (1.24) is  $\Gamma_3$  and, generally, not one of the other gammas. Lay the blame for any possible confusion here on the quirks of historical nomenclature.

Explicit values for all the exponents and specific heats, etc., for interesting gases follow below. Remember, however, that there are still no chemical reactions going on so that the relative concentrations of ions and electrons are fixed despite changes in temperature and density.

### Mixtures of Ideal Gases and Radiation

For a monatomic ideal gas  $\chi_\rho$  and  $\chi_T$  are equal to unity and  $\Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma = 5/3$ . A pure radiation “gas” has  $\chi_\rho = 0$ ,  $\chi_T = 4$ , and  $\Gamma_1 = \Gamma_2 = \Gamma_3 = 4/3$ . Note that  $\gamma = \Gamma_1/\chi_\rho \rightarrow \infty$  in this case.

If  $\gamma = \Gamma_1 = \Gamma_2 = \Gamma_3$  of the same constant value, as can be satisfied by an ideal gas, then

$$P \propto \rho^\gamma \quad (3.101)$$

$$P \propto T^{\gamma/(\gamma-1)} \quad (3.102)$$

$$T \propto \rho^{(\gamma-1)} \quad (3.103)$$

along *adiabats*. This is the result usually quoted in elementary physics texts for adiabatic behavior: it is collectively true only if the exponents satisfy the above equality.

In modeling simple stars, it often turns out that an equation of state consisting of a mixture of ideal gas and radiation suffices:

$$P = \frac{\rho N_A k T}{\mu} + \frac{a T^4}{3} = P_g + P_{\text{rad}} \quad \text{dyne cm}^{-2} \quad (3.104)$$

and

$$E = \frac{3 N_A k T}{2 \mu} + \frac{a T^4}{\rho} \quad \text{erg g}^{-1}. \quad (3.105)$$

We can find the density and temperature exponents almost by inspection so that

$$\chi_\rho = \frac{P_g}{P} \equiv \beta \quad (3.106)$$

which also defines  $\beta$ , the ratio of gas ( $P_g$ ) to total pressure, and

$$\chi_T = 4 - 3\beta. \quad (3.107)$$

(This  $\beta$  is not to be confused with  $\beta = kT/m_e c^2$  introduced earlier.) Further analysis, using the general expressions given previously, yields

$$c_{V\rho} = \frac{3 N_A k}{2 \mu} \left( \frac{8 - 7\beta}{\beta} \right) \quad \text{erg g}^{-1} \text{ K}^{-1} \quad (3.108)$$

$$\Gamma_3 - 1 = \frac{2}{3} \left( \frac{4 - 3\beta}{8 - 7\beta} \right) \quad (3.109)$$

$$\Gamma_1 = \beta + (4 - 3\beta)(\Gamma_3 - 1) \quad (3.110)$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{32 - 24\beta - 3\beta^2}{2(4 - 3\beta)} \quad (3.111)$$

and, finally,

$$\gamma = \frac{\Gamma_1}{\beta}. \quad (3.112)$$

It is easy to confirm that all quantities go to their proper limits as  $\beta \rightarrow 1$  (ideal gas) or  $\beta \rightarrow 0$  (pure radiation) and that all quantities are intermediate between their pure gas and radiation values for intermediate  $\beta$ .

### Mixtures of Degenerate and Ideal Gases

The first thing we shall find is the specific heat at constant volume for an almost completely degenerate electron gas. Recall our earlier discussion of the temperature corrections to such a gas where the number density,  $n_e$ , was given as a function of  $T$  and  $x_f$  in (3.75). If the volume or density of the gas is fixed while temperature is varied, then  $n_e$  does not change but  $x_f$  must. Thus  $(\partial n_e / \partial T)_\rho = 0$ . If this operation is performed on (3.75), then the righthand side of the resulting equation contains  $(\partial x_f / \partial T)_\rho$ , which may be solved to first order in  $T$  as

$$\left(\frac{\partial x_f}{\partial T}\right)_\rho = -\frac{\pi^2 k^2}{m_e^2 c^4} \frac{1 + 2x_f^2}{3x_f^3} T. \quad (3.113)$$

When you derive this you will find that it is missing a denominator of the form  $[1 + \mathcal{O}(T^2)]$ , where  $\mathcal{O}(T^2)$  contains terms that are of order  $T^2$ . Those terms must be ignored because they are of the same order as other correction terms that would have appeared if the equation for  $n_e$  had been carried out to higher order in temperature. Thus (3.113) is correct to first order in  $T$ .

To find the specific heat we have to differentiate  $E_e$  of (3.77) with respect to  $T$  while keeping density fixed. This operation yields, through the chain rule, nasty terms such as  $[dg(x_f)/dx_f](\partial x_f / \partial T)_\rho$ . When these are all straightened out (see Chandrasekhar 1939, Chap. 10, §6), we find

$$c_{V_\rho(e)} = \frac{8\pi^3 m_e^4 c^5}{3h^3 T \rho} \left(\frac{kT}{m_e c^2}\right)^2 x_f (1 + x_f^2)^{1/2} \quad (3.114)$$

for electrons or

$$c_{V_\rho(e)} = \frac{1.35 \times 10^5}{\rho} T x_f (1 + x_f^2)^{1/2} \text{ erg g}^{-1} \text{ K}^{-1}. \quad (3.115)$$

Note the presence of  $\rho$  in the (3.115). It is required because this specific heat is a specific specific heat (from the units). As before, it is reasonable to replace  $x_f$  with  $x_F$  or  $x$  using (3.49–3.50) provided that temperature correction terms are small in all of  $n_e$ ,  $P_e$ , and  $E_e$ . In any case, note the important result that the electron specific heat for the nearly degenerate gas is proportional to temperature.

From here on, we have to make some reasonable physical assumptions about the nature of the stellar gas. Because of pressure ionization, we expect all or most of the nuclear species to be completely ionized so that all electrons are free to swim in the Fermi sea. Thus pressure and energy, as additive quantities, are determined by bare ions and the free electrons. Radiation should play no significant role because, if it did, the temperatures would be so high that electrons would no longer be nearly degenerate—which we assumed at the onset. (See Fig. 3.9.) Thus the total pressure consists of  $P = P_e + P_I$ , where “I” means “ions.” Internal energies and specific heats

are also additive. The reason we bring this up is because the rest of the thermodynamic derivatives are, for the most part, logarithmic (like the  $\Gamma$ s) and we cannot simply add them together. It is best to give an example.

The temperature exponent of pressure,  $\chi_T$ , is  $(T/P) (\partial P/\partial T)_\rho$  from (3.88) where  $P$  is the *total* pressure. We cannot separate  $\chi_T$  into components describing just the electrons or just the ions. We had the same problem when treating the gas and radiation mixture of the previous section but the calculations there were fairly straightforward. Here, however, the complexity of the electron gas equation of state makes things computationally more difficult. Nevertheless, we can compute all the derivatives fairly easily if we assume that temperatures are very low. If this is the case, then electron degeneracy pressure greatly exceeds that of the ions and  $P_e \gg P_i$ . The same is not true for the partials of pressure with respect to temperature. By following the same course of analysis as was outlined above for the specific heat, you should verify that  $(\partial P_e/\partial T)_\rho \propto T$ . (See 3.76.) On the other hand,  $(\partial P_i/\partial T)_\rho = N_A k \rho / \mu_i$  where  $\mu_i$  is the ion mean molecular weight. (The ions are still assumed to be ideal.) Thus for low enough temperatures the temperature derivative of electron pressure may be neglected compared to that of the ions. The net result is that for low temperatures

$$\chi_T \rightarrow \frac{N_A k \rho T}{\mu_i P_e} \quad (3.116)$$

and, as  $T \rightarrow 0$ , so does  $\chi_T$ . The electrons have nothing to say in the matter.

The density exponent  $\chi_\rho = (\rho/P) (\partial P/\partial \rho)_T$  of (3.89) is easier. The electron pressure dominates both terms for low temperatures so that

$$\chi_\rho \rightarrow \frac{\rho}{P_e} \left( \frac{\partial P_e}{\partial \rho} \right)_T \rightarrow \begin{cases} 5/3 & \text{nonrelativistic} \\ 4/3 & \text{relativistic.} \end{cases} \quad (3.117)$$

The limiting forms come directly from the pressure-density relations (3.60) for the degenerate gas.

The rest of the derivatives require that the specific heats be found. We already have  $c_{V_\rho(e)}$  (from 3.114) and we know that the ion specific heat is  $3N_A k/2\mu_i$  (from, e.g., 3.85) and it is a constant. Therefore, for sufficiently low temperatures

$$c_{V_\rho} \rightarrow c_{V_\rho(i)} = \frac{3N_A k}{2\mu_i} = \frac{1.247 \times 10^8}{\mu_i} \text{ erg g}^{-1} \text{ K}^{-1} \quad (3.118)$$

and the electrons do not matter. (But always check that the temperatures are “sufficiently low.”) It may seem strange at first that the electrons, which may have a lot of total kinetic energy tied up in their Fermi sea, have a low specific heat. But most of that energy is locked in, so to speak, because of the exclusion principle and the vast majority of electrons have nowhere to go in energy space. Thus increasing or lowering the temperature of the electrons

does little to change their total kinetic energy. The ideal gas ions do not have that constraint.

The combination of pressure dominance by electrons, low sensitivity of pressure to temperature (small  $\chi_T$ ), and low specific heats (only the ions matter), all add up to a potentially explosive situation when very reactant nuclear fuels are present, as in the helium flash.

Having found the above, it should be a simple matter for you to verify the following:  $c_P = c_{V_\rho(i)}$ ,  $\Gamma_3 - 1 = 2/3$ ,  $\Gamma_1 = \chi_\rho$ , and  $\nabla_{\text{ad}} = 2/3\chi_\rho$ .

### 3.7.2 Allowing for Chemical Reactions

We now give an example of how the thermodynamic derivatives are found when chemical reactions are taking place. For simplicity, the ideal gas, one-state hydrogen atom will again be used, and radiation in the equation of state will be ignored. As usual, real calculations are very difficult and you are referred to Cox (1968, §9.18) for a fuller discussion. As you will see, even in the simple example given here, the analysis is made difficult because relative concentrations of particles vary as temperatures and densities change.

Because we assume that all changes in the system take place along paths in thermodynamic equilibrium, which implies chemical equilibrium, the Saha equation of (3.35) holds and

$$\frac{n^+ n_e}{n^0} = \mathcal{B} T^{3/2} e^{-\chi_{\text{H}}/kT} \quad (3.119)$$

where  $\mathcal{B}$  is

$$\mathcal{B} = \left( \frac{2\pi m_e k}{h^2} \right)^{3/2} = 2.415 \times 10^{15} \text{ cm}^{-3} \text{ K}^{-3/2} \quad (3.120)$$

and the other symbols are the same as those in §3.4. Define  $N$  (as in §3.1) so that  $N\rho \equiv n = n^+ + n^0$ . Thus  $N$  is the total ion plus neutral atom number density per unit mass and it is independent of density and will not change as the system is compressed or expanded. With the usual definition of  $y = n^+/n = n_e/n$ , the pressure may be written

$$P = (n_e + n^+ + n^0)kT = (1 + y)N\rho kT \quad \text{dyne cm}^{-2} \quad (3.121)$$

and the specific internal energy is (see 3.43)

$$E = (1 + y) \frac{n}{\rho} \frac{3kT}{2} + y \frac{n}{\rho} \chi_{\text{H}} \quad \text{erg g}^{-1} \quad (3.122)$$

or

$$E = (1 + y)N \frac{3kT}{2} + yN \chi_{\text{H}} \quad \text{erg g}^{-1} \quad (3.123)$$

where the energetics of the reaction are accounted for.

Having the pressure and internal energy now allows us to compute the thermodynamic derivatives. First note that the analysis leading to the determination of those derivatives in the previous discussion involved only taking partials with respect to either temperature or density with the other kept fixed: concentrations were never mentioned in that analysis. But this implies that partials with respect to concentrations (i.e., the  $N_i$ ) were never needed. Thus the general expressions derived for the specific heats, the  $\Gamma$ s, etc., are formally correct and all we need do is put in the correct pressures and internal energies that contain the information about chemical equilibrium. To carry this out in detail, however, still requires some effort. We start with easier quantities,  $\chi_T$  and  $\chi_\rho$ , and leave most of the rest of the work to you.

The ionization fraction  $y$  is given by a slightly rewritten version of (3.39):

$$\frac{y^2}{1-y} = \frac{\mathcal{B}}{N\rho} T^{3/2} e^{-\chi_H/kT}. \quad (3.124)$$

We now have the three relations  $P = P(\rho, T, y)$ ,  $E = E(\rho, T, y)$ , and the Saha equation. Take total differentials of the first two to find

$$dP = P \left[ \frac{dT}{T} + \frac{d\rho}{\rho} + \frac{dy}{1+y} \right]$$

and

$$dE = \frac{3}{2} NkT(1+y) \left[ \frac{dT}{T} + \frac{2}{3} \left( \frac{3}{2} + \frac{\chi_H}{kT} \right) \frac{dy}{1+y} \right].$$

Recall that  $N$  remains fixed because it is the number of hydrogen nuclei per gram and cannot change with temperature, density, or volume.

Also take the differential of the Saha equation (3.124) and divide the result by the Saha equation itself to find

$$\frac{dy}{1+y} = \mathcal{D}(y) \left[ \left( \frac{3}{2} + \frac{\chi_H}{kT} \right) \frac{dT}{T} - \frac{d\rho}{\rho} \right]$$

where

$$\mathcal{D}(y) = \frac{y(1-y)}{(2-y)(1+y)}. \quad (3.125)$$

Note that  $\mathcal{D}(1) = \mathcal{D}(0) = 0$  and, for general  $0 \leq y \leq 1$ ,  $\mathcal{D}(y) \geq 0$ . It reaches a maximum at the half-ionization point  $y = 1/2$  where  $\mathcal{D}(1/2) = 1/9$ .

The lefthand side of the differentiated Saha equation appears explicitly in the expressions for  $dP$  and  $dE$ . Therefore, use that equation to eliminate any reference to  $dy$  in  $dP$  and  $dE$  and find, for  $dE$ ,

$$dE = \frac{3}{2} NkT(1+y) \left\{ \left[ 1 + \mathcal{D} \frac{2}{3} \left( \frac{3}{2} + \frac{\chi_H}{kT} \right)^2 \right] \frac{dT}{T} - \mathcal{D} \frac{2}{3} \left( \frac{3}{2} + \frac{\chi_H}{kT} \right) \frac{d\rho}{\rho} \right\}.$$

From this find directly

$$c_{V\rho} = \frac{3}{2}Nk(1+y) \left[ 1 + \mathcal{D}(y) \frac{2}{3} \left( \frac{3}{2} + \frac{\chi_{\text{H}}}{kT} \right)^2 \right] \text{ erg g}^{-1} \text{ K}^{-1}. \quad (3.126)$$

Note that  $Nk = N_{\text{A}}k/\mu_{\text{i}}$  and  $Nky = N_{\text{A}}k/\mu_{\text{e}}$  from which may be found  $\mu_{\text{i}}$  of (1.45) and  $\mu_{\text{e}}$  of (1.48).

Treating the pressure differential in like fashion we find

$$\frac{dP}{P} = \left[ 1 + \mathcal{D} \left( \frac{3}{2} + \frac{\chi_{\text{H}}}{kT} \right) \right] \frac{dT}{T} + (1 - \mathcal{D}) \frac{d\rho}{\rho}$$

so that

$$\chi_{\rho} = 1 - \mathcal{D}(y) \quad (3.127)$$

and

$$\chi_{\text{T}} = 1 + \mathcal{D}(y) \left( \frac{3}{2} + \frac{\chi_{\text{H}}}{kT} \right). \quad (3.128)$$

Because  $\mathcal{D} \geq 0$ , we have  $\chi_{\rho} \leq 1$  and  $\chi_{\text{T}} \geq 1$ . The interpretation here is that if temperature rises, keeping density fixed, we get more free electrons liberated and the pressure rises more so than the rise due to temperature alone. Hence  $\chi_{\text{T}}$  increases above its nominal value of unity without ionization or recombination; that is,  $\chi_{\text{T}}$  must be greater than or equal to unity. If density increases, keeping temperature constant, then recombination decreases the number of free electrons per gram and thus  $\chi_{\rho}$  can fall below unity.

The  $\Gamma_i$  may now be calculated using equations (3.97) through (3.99) in the forms that contain  $\chi_{\rho}$ ,  $\chi_{\text{T}}$ , and  $c_{V\rho}$ . After a bit of algebra the results are

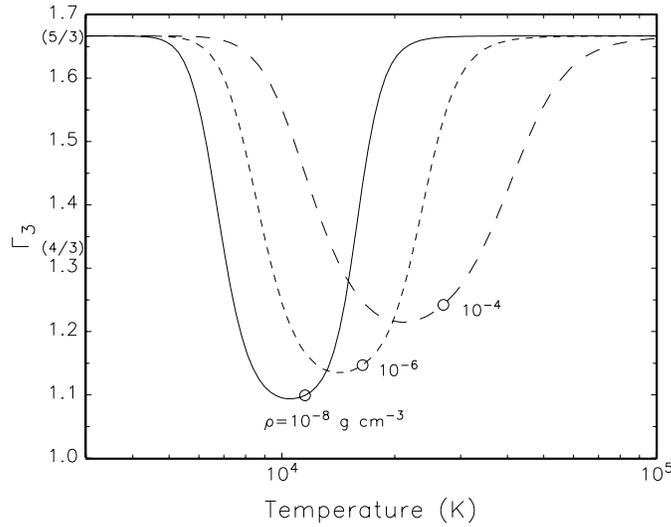
$$\Gamma_3 - 1 = \frac{2 + 2\mathcal{D}(y) (3/2 + \chi_{\text{H}}/kT)}{3 + 2\mathcal{D}(y) (3/2 + \chi_{\text{H}}/kT)^2} \quad (3.129)$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{5 + 2\mathcal{D}(y) \{ \chi_{\text{H}}/kT + (3/2 + \chi_{\text{H}}/kT) (5/2 + \chi_{\text{H}}/kT) \}}{2 + 2\mathcal{D}(y) (3/2 + \chi_{\text{H}}/kT)} \quad (3.130)$$

and  $\Gamma_1$  follows from (3.96).

Note that as  $y$  approaches zero or unity (so that  $\mathcal{D} \rightarrow 0$ ) all the  $\Gamma_i$  approach their ideal gas values of  $5/3$ . This is as it should be. If the gas is completely neutral or totally ionized, then the equation of state is of its usual ideal gas form since  $y$  is not changing. It is the intermediate case that is interesting.

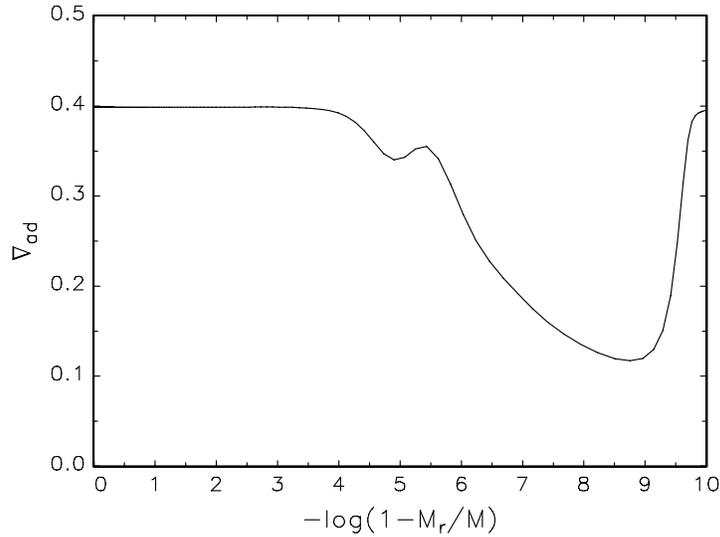
To compute the  $\Gamma_i$  the scheme is, choose  $\rho$  (or  $n$ ) and  $T$ , find  $y$  from the Saha equation (3.124) (and  $\mathcal{D}$  by means of 3.125), and then apply the above expressions. A typical result is shown in Fig. 3.10, where  $\Gamma_3$  is plotted as a function of temperature for three densities. The half-ionization point,  $y = 1/2$ , is indicated. Note that if  $T$  is near the typical hydrogen ionization temperature of  $10^4$  K,  $\Gamma_3$  drops rapidly from its value of  $5/3$  to much lower values. Even the dangerous  $4/3$  may be passed by in the process. A word to the wise: always watch out for temperatures near  $10^4$  K in a hydrogen-rich mixture.



**Fig. 3.10.** The adiabatic exponent  $\Gamma_3$  for an ionizing pure one-state hydrogen gas is plotted as a function of temperature. Results are shown for three densities. The  $\circ$  indicates the half-ionization point. The fiducial points  $5/3$  and  $4/3$  are also shown.

The reason why  $\Gamma_3$  (and the other  $\Gamma$ s) behaves the way it does when ionization is taking place is quite simple. First suppose no ionization or recombination processes can operate in an almost completely neutral gas so that concentrations remain constant as the system is compressed adiabatically. In that case  $\Gamma_3 = 5/3$ ,  $T \sim \rho^{2/3}$ , (as in 3.103) and the gas heats up. If, however, we allow ionization to take place, then compression may still heat up the gas, but ionization is much more sensitive to temperature changes than to changes in density. Hence, ionization is accelerated. But this takes energy and that energy is paid for at the expense of the thermal motion in the gas. Thus the temperature tends not to rise as rapidly as  $\rho^{2/3}$  and  $\Gamma_3$  is smaller than its value with no ionization.

As we shall see in chapters to come, all the  $\Gamma_i$  are important in some respect or another:  $\Gamma_3$  says something about how the heat content of the gas responds to compression;  $\Gamma_1$  is intimately tied up with dynamics (partially through the sound speed); the behavior of  $\Gamma_2$  and  $\nabla_{\text{ad}}$  may be a deciding factor in whether convection may take place. As an example, Fig. 3.11 shows the run of  $\nabla_{\text{ad}}$  through a ZAMS model sun. The abscissa is  $-\log(1 - \mathcal{M}_r/\mathcal{M})$ , the stellar center is at the left, and the surface is to the right. Such an axis emphasizes the outer layers of the model. Thus, a value of “9” on this axis corresponds to  $1 - \mathcal{M}_r/\mathcal{M} = 10^{-9}$  or a mass point that is within  $10^{-9}$  of the total mass. The dips in  $\nabla_{\text{ad}}$  signal ionization. The one at “5” takes place at a temperature of about  $10^5$  K and is the first  $^4\text{He}$  ionization zone. The broad trough around 8-9 is at about  $10^4$  K and corresponds to a combination of



**Fig. 3.11.** Plotted is  $\nabla_{\text{ad}}$  (of 3.96) versus  $-\log(1 - \mathcal{M}_r/\mathcal{M})$  for a ZAMS model sun. The region from roughly “3” on this scale to almost the surface is convective due, in part, to the depression of  $\nabla_{\text{ad}}$ .

second  ${}^4\text{He}$  and  ${}^1\text{H}$  ionization. Suffice it to say that the whole region with depressed  $\nabla_{\text{ad}}$  is convective, for reasons that will be explained in Chapter 5.

The effects of radiation or other ionizing species and energy levels are included in more complete analyses than what we have done here (see Cox, 1968). In addition, the effects of pressure ionization (among other things) have to be included in many situations. Even though the results we have obtained are useful for many calculations in stellar structure, you should be aware that real models are usually constructed using tabular equations of state with  $P$ ,  $E$ , and, sometimes, various derivatives given as functions of temperature and density for a fixed nuclear composition. Very often these are included in tabulations of opacities—which we discuss in the next chapter.

### 3.8 Exercises

**Exercise 3.1.** We have already explored the Saha equation using a pure hydrogen gas as an example. Now consider the more complicated  ${}^4\text{He}$  atom with its two electrons. Assume, as in the hydrogenic example, that the neutral atom and first ionized ion are in their respective ground states. The ionization potential to remove the first (second) electron is  $\chi_1 = 24.587$  eV ( $\chi_2 = 54.416$  eV). To agree on a common nomenclature, let  $n_e$ ,  $n_0$ ,  $n_1$ , and  $n_2$  be the number densities of, respectively, electrons, neutral atoms, and first and second ionized ions. The total number density of atoms plus ions of the

pure helium gas is denoted by  $n$ . Furthermore, define  $z_e$  as the ratio  $n_e/n$ ; and, in like manner, let  $z_i$  be  $n_i/n$ , where  $i = 0, 1, 2$ . The gas is assumed to be electrically neutral. For the following you will also need the degeneracy factors for the atoms and ions and these are to be found in Allen (1973), Lang (1991), or Cox (1999) as the data-type references given at the end of Chapter 1.

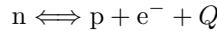
1. Following the hydrogenic case, construct the ratios  $n_e n_1/n_0$  and  $n_e n_2/n_1$ . In doing so you must take care to establish the zero points of energy for the various constituents. One way to do this is to use  $mc^2$  arguments. For example, the first ionization has  $m_e c^2 + m_1 c^2 = m_0 c^2 + \chi_1$  in obvious notation. The reference energies  $\mathcal{E}$  to be used in (3.24) for each constituent are then taken to be the  $mc^2$ s. This establishes the relative order of the  $\mathcal{E}$ s. The final form you obtain should not contain chemical potentials (and you must show why this is true).
2. Apply  $n = n_0 + n_1 + n_2$ , overall charge neutrality, and recast the above Saha equations so that only  $z_1$  and  $z_2$  appear as unknowns. The resulting two equations have temperature and  $n$  or, equivalently,  $\rho = 4n/N_A$  as independent parameters.
3. Simultaneously solve the two Saha equations for  $z_1$  and  $z_2$  for temperatures in the range  $1 \times 10^4 \leq T \leq 2 \times 10^5$  K with a fixed value of density from among the choices  $\rho = 10^{-4}$ ,  $10^{-6}$ , or  $10^{-8}$  g cm $^{-3}$ . Choose a dense grid in temperature because you will soon plot the results. (These cases will prove useful when discussing pure helium opacities in Chap. 4.) Once you have found  $z_1$  and  $z_2$ , also find  $z_e$  and  $z_0$  for the same range of temperature. Note that this is a numerical exercise and use of a computer is strongly advised.
4. Now plot all your  $z_i$  as a function of temperature for your chosen value of  $\rho$ . (Plot  $z_0$ ,  $z_1$ , and  $z_2$  on the same graph.) This is an essential step because it will make clear how the ionization responds to temperature changes.
5. Find the half-ionization points on your plot. The two temperatures you obtain (for fixed density) will correspond to the single half-ionization point for pure hydrogen.

**Exercise 3.2.** We earlier established that photon mean free paths were very short in a star except in the very outermost layers. This means that photons must follow a tortuous path to escape eventually from a star and must take a long time in doing so. To estimate this time, assume that a photon is created at the center of a star and thereafter undergoes a long series of random walk scatterings off electrons until it finally reaches the surface. The mean free path associated with each scattering is  $\lambda_{\text{phot}} = (n_e \sigma_e)^{-1}$ , where  $\sigma_e$  is the Thomson scattering cross section  $\sigma_e \approx 0.7 \times 10^{-24}$  cm $^2$  (see §4.4.1). For simplicity, assume that the star has a constant density so that  $\lambda_{\text{phot}}$  is also constant. This is an order-of-magnitude problem, so don't worry too

much about constants of order unity. (Real diffusion in stars is much more complicated and your estimate for the time will be an underestimate.)

1. Using one-dimensional random walk arguments, show that  $L \approx \mathcal{R}^2/\lambda_{\text{phot}}$  is the *total* distance a photon must travel if it starts its scattering career at stellar center and eventually ends up at the surface at  $\mathcal{R}$ .
2. Since the photons travel at the speed of light,  $c$ , find the time,  $\tau_{\text{phot}}$ , required for the photon to travel from the stellar center to the surface. (Assume that any scattering process takes place instantaneously.)
3. Give an estimate for  $\tau_{\text{phot}}$ , in the units of years, for a star of mass  $\mathcal{M}/\mathcal{M}_{\odot}$  and radius  $\mathcal{R}/\mathcal{R}_{\odot}$ .

**Exercise 3.3.** Neutron stars are assumed to be objects with  $\mathcal{M} \sim \mathcal{M}_{\odot}$ ,  $\mathcal{R} \sim 10$  km ( $\langle \rho \rangle \sim 10^{14}$  g cm $^{-3}$ ) where internal temperatures ( $kT$ ) are small compared to the Fermi energies of electrons, protons, and neutrons (which are assumed to be the only particles present). To demonstrate that the name neutron star is apt, consider the following. Assume that the stellar temperature is zero and that chemical equilibrium exists between electrons, protons, and neutrons. The reaction connecting them is



where  $Q = 0.782$  MeV and we are neglecting the electron anti-neutrino, which should appear on the right-side of the reaction. Further assume that the electrons are completely relativistic but that protons and neutrons are nonrelativistic.

1. Convince yourself that the ‘‘Saha’’ equation is

$$\mathcal{E}_n + Q = \mathcal{E}_p + \mathcal{E}_e$$

where the  $\mathcal{E}$ s are the Fermi energies of the respective particles. Do your ‘‘self-convincing’’ two ways: (a) argue from the chemical potential equation of the reaction; (b) make a physical argument based on the energetics of the reaction and the Pauli exclusion principle.

2. Now find the number densities of the particles as a function of density. Assume charge neutrality, so that  $n_e = n_p$ , and use the Saha equation to find  $n_e$ ,  $n_p$ , and  $n_n$  for densities in the range  $10^{13} \lesssim \rho \lesssim 2 \times 10^{14}$  g cm $^{-3}$ . You may take the density as being  $\rho = (n_p + n_n)m$  where  $m$  is the mass of either proton or neutron.
3. Plot your number density results as a function of density and, if possible, compare to what you might find in the literature.

**Exercise 3.4.** Show for the ideal gas ( $\mu/kT \ll -1$ ) that  $P = nkT$  is a general result independent of whether the particles are relativistic, nonrelativistic, or anything in between. (Hint: integrate 3.13 by parts after inserting 3.12.)

**Exercise 3.5.** Verify (3.29) by computing the average kinetic energy of a Maxwell–Boltzmann distribution.

**Exercise 3.6.** To give an idea why the chemical potential is referred to as a “potential,” consider the following, as discussed in Landau and Lifshitz (1958, §25). They state that a body subject to an external field is in equilibrium if the sum of the local chemical potential at every position in the body—here call it  $\mu_{\text{local}}(\mathbf{r})$ —and the potential of the external field,  $\psi(\mathbf{r})$ , is a constant; that is,

$$\mu_{\text{tot}} \equiv \mu_{\text{local}}(\mathbf{r}) + \psi(\mathbf{r}) = \text{constant}.$$

To make things simple, consider a one-dimensional situation where the external field is gravitational and the local gravity,  $g$ , is everywhere constant so that  $\psi(\mathbf{r}) = -mgz$  where  $z$  is height and  $m$  is the mass of a particle in the body. Further assume that the particles compose an ideal gas.

1. Using the ideal gas results, show that

$$\left( \frac{\partial P}{\partial \mu_{\text{local}}} \right)_T = \frac{\rho}{m}.$$

2. Compute  $d\mu_{\text{tot}}/dz$  and finally show that

$$\frac{dP}{dz} = -g\rho$$

which is the elementary result for the equation of hydrostatic equilibrium in a constant gravity field; that is, you have shown that the chemical potential is part of a potential! For a more complicated situation, see

▷ Aronson, E., & Hansen, C.J. 1972, ApJ, 177, 145,

who give an example of the “gravo–thermo catastrophe.”

**Exercise 3.7.** This problem deals with corrections to Maxwell–Boltzmann thermodynamics due to the effects of weak electron degeneracy. Suppose  $\mu/kT$  is still very much less than  $-1$  as discussed in §3.3, but we wish to include some effects of Fermi–Dirac statistics; i.e., what are the effects due to the  $+1$  in the distribution function (3.9)?

1. If the exponential term in (3.9) is still large then, we can use the expansion  $1/(a+1) \approx (1-1/a)/a$  to first order in the large quantity  $a$ . If you assume, as an approximation, that  $\mu/kT$  of (3.24) is still given by

$$e^{\mu/kT} = \frac{n_0 h^3}{2(2\pi m_e kT)^{3/2}} \equiv K$$

where  $n_0$  is the electron number density in the pure Maxwell–Boltzmann limit, then show that the number density,  $n$ , for weak degeneracy is

$$n = n_0 \left[ 1 - 2^{-3/2} K \right].$$

2. Similarly show that the new pressure is

$$P = n_0 kT \left[ 1 - 2^{-5/2} K \right].$$

**Exercise 3.8.** Section 3.6 discusses “imperfections” in equations of state that make life difficult for the stellar modeler. One of these imperfections arises from electrostatic interactions between ions. These cause modifications in the ideal gas equation of state. The severity of the modifications depends on density and temperature in the sense that low temperatures and/or high densities means you have to work harder. One method of attacking the problem is to use Debye–Hückel theory wherein it is assumed that (for, say, a one-component composition) that the average inter-ion spacing  $r_0$  is large compared to the Debye length

$$r_D = \left( \frac{kT}{4\pi e^2 \rho \zeta N_A} \right)^{1/2}.$$

Here  $\zeta = Z(Z+1)/A$  where  $Z$  is the ion charge and  $A$  is its atomic mass (in rounded off amu’s). This statement is equivalent to

$$n_Z \ll \left( \frac{kT}{Z^2 e^2} \right)^3$$

where  $n_Z$  is the ion number density. If this condition is satisfied then, we find the following expression for the pressure:

$$P = nkT \left[ 1 - \frac{e^3 (\pi N_A \rho)^{1/2}}{3 (kT)^{3/2}} \mu \zeta^{3/2} \right]$$

which becomes, after putting in numbers,

$$P = nkT \left[ 1 - 0.32 \frac{\rho^{1/2}}{T_6^{3/2}} \mu \zeta^{3/2} \right].$$

Here  $n$  is the total number density (ions plus electrons),  $\mu$  is the mean molecular weight, and  $T_6$  is the temperature in units of  $10^6$  K. You may check these expressions by consulting Cox (1968, §15.5) or Clayton (1968, §2.3). In any case, write this as

$$P = nkT(1 - B)$$

where, for this analysis to work at all,  $B$  must be small compared to unity. If it gets moderately large (say 0.1 or larger), then electrostatic effects are considered to be significant. Now do the following.

1. Consult the literature (or the *Supplemental Material* section of Chap. 2) for properties of ZAMS models. Make believe these are composed of pure ionized hydrogen ( $\mu = 1/2$ ) and compute  $B$  at model center for a selection of these models starting with  $60M_\odot$  and ending at  $0.08M_\odot$ .

2. What do you conclude from this exercise? Where, in mass on the ZAMS, do you think electrostatic corrections begin to be important?

**Exercise 3.9.** Having already found the ionization fractions for pure helium in a previous exercise, let's go one step further—but “we” have done most of the work for you here. The FORTRAN program `GETEOS.F90` to be found on the CD-ROM on the endcover of this text was written by W. Dean Ponnell (an old colleague of ours) to compute the pressure and internal energy (among other things) for an ideal gas plus radiation. (This code is also part of `ZAMS.FOR` also found on the CD.) You input the hydrogen (one ionization state) mass fraction  $X$ , the helium (two ionization stages) mass fraction  $Y$ , the temperature  $T$ , and the specific volume  $V_\rho$ . The output from `GETEOS` consists of pressure  $P$ , internal energy  $E$  (in ergs  $\text{g}^{-1}$ ), the electron pressure (PE),  $(\partial P/\partial V_\rho)_T$  (PV),  $(\partial P/\partial T)_{V_\rho}$  (PT),  $(\partial E/\partial V_\rho)_T$  (EV), and  $(\partial E/\partial T)_{V_\rho}$  (ET). The code is sparsely annotated but you should try to see what goes on. The variable `GES` is our  $1/\mu_e$  and it is iterated upon until all the Saha equations are satisfied. One way to unravel the code (in your mind, not when using the code) is to set `XHE` and `XHE2` (the helium ionization potentials) to infinity, thus shutting off the ionization of that element. The metals consist partially of Mg, Si, and Fe, included as a single element, with potential `XM`. Set `XM` to infinity also. The rest of the metals are Na and Al, which are always assumed to be ionized. A driver code at the beginning is just an example and you will have to change it to get all the output quantities from EOS. Note that this is in FORTRAN 90. In using the code be aware that it doesn't always like  $X$  or  $Y$  (or  $Z = 1 - X - Y$ ) to be zero. But you can set them to some very small number.

1. Use this code, with your version of the driver, to compute various pressures, etc., for interesting combinations of the input quantities.
2. Find the  $\Gamma$ s for nearly pure hydrogen and compare to what was shown in Fig. 3.10. The output from `GETEOS` gives you all you need.
3. Do the same for nearly pure helium to show the effects of the two ionization stages. And we are sure your instructor can think of lots more things to keep you busy! **Note:** pressure ionization is not included in this code but, by the time the density reaches that level, the major constituents (H and He) are already ionized.

### 3.9 References and Suggested Readings

#### Introductory Remarks

The place to go for general information on stellar equations of state is

- ▷ Cox, J.P. 1968, *Principles of Stellar Structure*, in two volumes (New York: Gordon & Breach).

In particular, see his Chaps. 9–11, 15, and 24. We also recommend Part III of

- ▷ Kippenhahn, R., & Weigert, A. 1990, *Stellar Structure and Evolution* (Berlin: Springer-Verlag)
- and §3.2 of
- ▷ Rose, W.K. 1998, *Advanced Stellar Astrophysics* (Cambridge: Cambridge University Press).
  - ▷ Clayton, D.D. 1968, *Principles of Stellar Evolution and Nucleosynthesis* (New York: McGraw-Hill)
- also contains useful material.

A favorite text of ours is

- ▷ Landau, L.D., & Lifshitz, E.M. 1958, *Statistical Physics* (London: Pergamon)
- and its later editions. We recommend it for its clarity (but it is not easy) and wealth of practical applications. You will even find material about neutron stars in it.

A complete discussion of what conditions must be met to use the approximation of LTE sensibly may be found in

- ▷ Mihalas, D. 1978, *Stellar Atmospheres*, 2nd ed. (San Francisco: Freeman).
- Anyone thinking seriously about studying stars should try to find a copy. The last we heard, it is out of print, but permission might be granted by the publisher to reproduce it (but check for royalty fees).

### §3.3: Ideal Monatomic Gas

A complete monograph discussion of Fermi–Dirac equations of state for use in stars was first published by

- ▷ Chandrasekhar S. 1939, *An Introduction to the Study of Stellar Structure* (Chicago: University of Chicago Press).
- It should be available in paperback Dover editions and is well worth buying at modest cost. We shall refer to this work quite often. Other versions may be found in §3.5 of
- ▷ Chiu, H.-Y. 1968, *Stellar Physics*, Vol. 1. (Waltham, MA: Blaisdell)
- and Chapter 24 of
- ▷ Cox, J.P. 1968, *Principles of Stellar Structure*, in two volumes (New York: Gordon & Breach).

### §3.4: The Saha Equation

Systematic application of the Saha equation to multicomponent mixtures is not easy. The bookkeeping required to keep track of all the energy levels is a daunting task, to say nothing of getting information on level parameters. See Chapter 15 of

- ▷ Cox, J.P. 1968, *Principles of Stellar Structure*, in two volumes (New York: Gordon & Breach)
- and Chapter 14 of

- ▷ Kippenhahn, R., & Weigert, A. 1990, *Stellar Structure and Evolution* (Berlin: Springer-Verlag).

### §3.5: Fermi–Dirac Equations of State

Chandrasekhar (1939) and Cox (1968) (see above) are standard references. The references to fermionic matter and Bose–Einstein condensates are

- ▷ Anderson, M.H., Ensher, J.R., Matthews, M.R., Wieman, C.E., & Cornell, E.A. 1995, *Science*, 269, 198.
- ▷ DeMarco, B., & Jin, D.S. 1999, *Science*, 285, 1703.

The reference to Peter Eggleton (1982) is given as a private communication in

- ▷ Truran, J.T., & Livio, M. 1986, *ApJ*, 308, 721,

who use it in some work concerning nova systems. We have extended Eggleton’s mass–radius fit for white dwarfs to accommodate general  $\mu_e$ .

- ▷ Cloutman, L.D. 1989, *ApJS*, 71, 677

is a good source of numerical techniques for computing the Fermi–Dirac integrals. See also

- ▷ Eggleton, P., Faulkner, J., & Flannery, B. 1973, *A&A*, 23, 325

for a thermodynamically self-consistent and efficient computation of the equation of state for arbitrarily degenerate and arbitrarily relativistic ionized gases.

- ▷ Antia, H.M. 1993, *ApJS*, 84, 101

gives rational expansions for the Fermi–Dirac integrals. Early work on realistic corrections to the perfect Fermi–Dirac gas includes

- ▷ Hamada, T., & Salpeter, E.E. 1961, *ApJ*, 134, 683.

### §3.6: “Almost Perfect” Equations of State

Our Fig. 3.9 is our version of Fig. 1 of

- ▷ Fontaine, G., Graboske, H.C., & Van Horn, H.M. 1977, *ApJS*, 35, 293.

This paper has an excellent discussion of the problems that arise when ionization (including pressure effects) and electron degeneracy must be accounted for. Their results are in the form of tables.

We have not discussed nuclear equations of state. To get an idea of what may be involved see chapters 2, 8, and 9 of

- ▷ Shapiro, S.L., & Teukolsky, S.A. 1983, *Black Holes, White Dwarfs, and Neutron Stars* (New York: John Wiley & Sons)

and the review article by

- ▷ Bethe, H.A. 1990, *RevModPhys*, 62, 801.



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