

## *Preface to the second edition*

After five years and six printings it seems only fair to our readers that we should respond to their comments and also correct errors and imperfections to which we have been alerted in addition to those we have discovered ourselves in reviewing the text. This second edition also introduces additional material which earlier constraints of time and space had precluded, and which has, in our view, become more essential as the make-up of our potential readership has become clearer. We hope that we manage to do this in a spirit which preserves the essential features of the text, namely providing the material rigorously and in a form suitable for directed self-study. Thus the focus remains on accessibility, explicitness and emphasis on concrete examples, in a style that seeks to encourage readers to become directly involved with the material and challenges them to prove many of the results themselves (knowing that solutions are also given in the text!).

Apart from further examples and exercises, the new material presented here is of two contrasting kinds. The new Chapter 7 adds a discussion of the comparison of general measures, with the Radon–Nikodym theorem as its focus. The proof given here, while not new, is in our view more constructive and elementary than the usual ones, and we utilise the result consistently to examine the structure of Lebesgue–Stieltjes measures on the line and to deduce the Hahn–Jordan decomposition of signed measures. The common origin of the concepts of variation and absolute continuity of functions and measures is clarified. The main probabilistic application is to conditional expectations, for which an alternative construction via orthogonal projections is also provided in Chapter 5. This is applied in turn in Chapter 7 to derive elementary properties of martingales in discrete time.

The other addition occurs at the end of each chapter (with the exception of Chapters 1 and 5). Since it is clear that a significant proportion of our current

readership is among students of the burgeoning field of mathematical finance, each relevant chapter ends with a brief discussion of ideas from that subject. In these sections we depart from our aim of keeping the book self-contained, since we can hardly develop this whole discipline afresh. Thus we neither define nor explain the origin of the finance concepts we address, but instead seek to locate them mathematically within the conceptual framework of measure and probability. This leads to conclusions with a mathematical precision that sometimes eludes authors writing from a finance perspective.

To avoid misunderstanding we repeat that the purpose of this book remains the development of the ideas of measure and integral, especially with a view to their applications in probability and (briefly) in finance. This is therefore neither a textbook in probability theory nor one in mathematical finance. Both of these disciplines have a large specialist literature of their own, and our comments on these areas of application are intended to assist the student in understanding the mathematical framework which underpins them.

We are grateful to those of our readers and to colleagues who have pointed out many of the errors, both typographical and conceptual, of the first edition. The errors that inevitably remain are our sole responsibility. To facilitate their speedy correction a webpage has been created for the notification of errors, inaccuracies and queries, at <http://www.springeronline.com/1-85233-781-8> and we encourage our readers to use it mercilessly. Our thanks also go to Stephanie Harding and Karen Borthwick at Springer-Verlag, London, for their continuing care and helpfulness in producing this edition.

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The central concepts in this book are Lebesgue measure and the Lebesgue integral. Their role as standard fare in UK undergraduate mathematics courses is not wholly secure; yet they provide the principal model for the development of the abstract measure spaces which underpin modern probability theory, while the Lebesgue function spaces remain the main source of examples on which to test the methods of functional analysis and its many applications, such as Fourier analysis and the theory of partial differential equations.

It follows that not only budding analysts have need of a clear understanding of the construction and properties of measures and integrals, but also that those who wish to contribute seriously to the applications of analytical methods in a wide variety of areas of mathematics, physics, electronics, engineering and, most recently, finance, need to study the underlying theory with some care.

We have found remarkably few texts in the current literature which aim explicitly to provide for these needs, at a level accessible to current undergraduates. There are many good books on modern probability theory, and increasingly they recognise the need for a strong grounding in the tools we develop in this book, but all too often the treatment is either too advanced for an undergraduate audience or else somewhat perfunctory. We hope therefore that the current text will not be regarded as one which fills a much-needed gap in the literature!

One fundamental decision in developing a treatment of integration is whether to begin with measures or integrals, i.e. whether to start with sets or with functions. Functional analysts have tended to favour the latter approach, while the former is clearly necessary for the development of probability. We have decided to side with the probabilists in this argument, and to use the (reasonably) systematic development of basic concepts and results in probability theory as the principal field of application – the order of topics and the

terminology we use reflect this choice, and each chapter concludes with further development of the relevant probabilistic concepts. At times this approach may seem less ‘efficient’ than the alternative, but we have opted for direct proofs and explicit constructions, sometimes at the cost of elegance. We hope that it will increase understanding.

The treatment of measure and integration is as self-contained as we could make it within the space and time constraints: some sections may seem too pedestrian for final-year undergraduates, but experience in testing much of the material over a number of years at Hull University teaches us that familiarity and confidence with basic concepts in analysis can frequently seem somewhat shaky among these audiences. Hence the preliminaries include a review of Riemann integration, as well as a reminder of some fundamental concepts of elementary real analysis.

While probability theory is chosen here as the principal area of application of measure and integral, this is not a text on elementary probability, of which many can be found in the literature.

Though this is not an advanced text, it is intended to be studied (not skimmed lightly) and it has been designed to be useful for directed self-study as well as for a lecture course. Thus a significant proportion of results, labelled ‘Proposition’, are not proved immediately, but left for the reader to attempt before proceeding further (often with a hint on how to begin), and there is a generous helping of exercises. To aid self-study, proofs of the propositions are given at the end of each chapter, and outline solutions of the exercises are given at the end of the book. Thus few mysteries should remain for the diligent.

After an introductory chapter, motivating and preparing for the principal definitions of measure and integral, Chapter 2 provides a detailed construction of Lebesgue measure and its properties, and proceeds to abstract the axioms appropriate for probability spaces. This sets a pattern for the remaining chapters, where the concept of independence is pursued in ever more general contexts, as a distinguishing feature of probability theory.

Chapter 3 develops the integral for non-negative measurable functions, and introduces random variables and their induced probability distributions, while Chapter 4 develops the main limit theorems for the Lebesgue integral and compares this with Riemann integration. The applications in probability lead to a discussion of expectations, with a focus on densities and the role of characteristic functions.

In Chapter 5 the motivation is more functional-analytic: the focus is on the Lebesgue function spaces, including a discussion of the special role of the space  $L^2$  of square-integrable functions. Chapter 6 sees a return to measure theory, with the detailed development of product measure and Fubini’s theorem, now leading to the role of joint distributions and conditioning in probability. Finally,

following a discussion of the principal modes of convergence for sequences of integrable functions, Chapter 7 adopts an unashamedly probabilistic bias, with a treatment of the principal limit theorems, culminating in the Lindeberg–Feller version of the central limit theorem.

The treatment is by no means exhaustive, as this is a textbook, not a treatise. Nonetheless the range of topics is probably slightly too extensive for a one-semester course at third-year level: the first five chapters might provide a useful course for such students, with the last two left for self-study or as part of a reading course for students wishing to continue in probability theory. Alternatively, students with a stronger preparation in analysis might use the first two chapters as background material and complete the remainder of the book in a one-semester course.

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