Additional questions for chapter 3

1. *Let* $\xi_1, \xi_2, \ldots$ *be independent and identically distributed with* $\phi(\theta) = \mathbb{E}(exp\{\theta\xi_1\}) < \infty$*. Let* $S_n = S_0 + \xi_1 + \ldots + \xi_n$*. Show that*

$$M_n = \frac{\exp\{\theta S_n\}}{\phi(\theta)^n}$$

is a martingale with respect to $\sigma(S_0, \ldots, S_n)$*. Apply the result to the special case*

$$\mathbb{P}(\xi_1 = 1) = p, \quad \textit{and} \quad \mathbb{P}(\xi_1 = -1) = 1 - p.$$

Solution:
Use subsequently measurability and independence

$$\begin{aligned}
\mathbb{E}(M_{n+1}|\mathcal{F}_n) &= \mathbb{E}\left[\exp\{\theta(\xi_{n+1} + S_n)\}/\varphi(\theta)^{n+1}|\mathcal{F}_n\right] \\
&= \frac{\exp\{\theta S_n)\}}{\varphi(\theta)^{n+1}}\mathbb{E}\left[\exp\{\theta\xi_{n+1}\}|\mathcal{F}_n\right] \\
&= \frac{\exp\{\theta S_n)\}}{\varphi(\theta)^{n+1}}\mathbb{E}\left[\exp\{\theta\xi_{n+1}\}\right] = \frac{\exp\{\theta S_n)\}}{\varphi(\theta)^n} = M_n.
\end{aligned}$$

The second part is a straightforward application.

2. *(i) Let $\xi_1, \xi_2, \ldots$ be independent with $\mathbb{E}(\xi_i) = 0$ and $\mathbb{E}(\xi_i^2) = \sigma_i^2$. Let $S_n = S_0 + \xi_1 + \ldots + \xi_n$, where S_0 is a constant, and let $v_n = \sum_{i=1}^n \sigma_i^2$ be the variance of S_n. Show that*

$$M_n = S_n^2 - v_n$$

is a martingale.

(ii) Suppose we are testing the hypothesis that observations $\xi_1, \xi_2, \ldots$ are independent and have density function f but the truth is that $\xi_1, \xi_2, \ldots$ are independent and have density function g where $\{x : f(x) > 0\} = \{x : g(x) > 0\}$. Let

$$h(x) = \begin{cases} f(x)/g(x) & \text{when } g(x) > 0 \\ 0 & \text{when } g(x) = 0 \end{cases}$$

Show that $M_n = h(\xi_1) \cdots h(\xi_n)$ is a martingale.

Solution:
(i) Using the definition of M_n, the independence of (ξ_i) and the property taking out what is known of conditional expectation we get:

$$\begin{aligned}
\mathbb{E}(M_n - M_{n-1}|\mathcal{F}_{n-1}) &= \mathbb{E}((S_{n-1} + \xi_n)^2 - S_{n-1}^2 - \sigma_n^2|\mathcal{F}_{n-1}) \\
&= \mathbb{E}(2S_{n-1}\xi_n + \xi_n^2 - \sigma_n^2|\mathcal{F}_{n-1}) \\
&= 2(S_{n-1})\mathbb{E}(\xi_n|\mathcal{F}_{n-1}) + \mathbb{E}(\xi_n^2|\mathcal{F}_{n-1}) - \mathbb{E}(\sigma_n^2|\mathcal{F}_{n-1}) = 0.
\end{aligned}$$

Thus $\mathbb{E}(M_n|\mathcal{F}_{n-1}) = M_{n-1}$ showing that (M_n) is a martingale.
(ii) This is a special case of a 'product martingale'. Set $\zeta_i = h(\xi_i) = f(\xi_i)/g(\xi_i)$. Then

$$\mathbb{E}(\zeta_i) = \int \frac{f(x)}{g(x)} g(x)dx = \int f(x)dx = 1$$

and the claim follows from the second example in §3.3 in the book.

3. *Let* $X = \{X_n\}_{n\in\mathbb{N}_0}$ *be an integrable stochastic process which is adapted to the filtration* $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$. *Show that X has a decomposition*

$$X_n = X_0 + M_n + A_n$$

where $\{M_n\}_{n\in\mathbb{N}_0}$ *is a martingale with* $M_0 = 0$ *and* $\{A_n\}_{n\in\mathbb{N}_0}$ *is a predictable process with* $A_0 = 0$. *Show that the decomposition is unique. Also show that* $\{A_n\}_{n\in\mathbb{N}_0}$ *is monotonously increasing iff X is a submartingale.*
(Hint:$\mathbb{E}[X_n - X_{n-1}|\mathcal{F}_{n-1}] = A_n - A_{n-1}$*)*

Solution:
First we define $\{A_n\}_{n\in\mathbb{N}_0}$ recursively by $A_0 = 0$ and the hint

$$A_n = A_{n-1} + \mathbb{E}[X_n - X_{n-1}|\mathcal{F}_{n-1}]$$

Then A_n is $\mathcal{F}_{n-1}$ measurable by induction and the measurability of conditional expectation. Thus, $\{A_n\}_{n\in\mathbb{N}_0}$ is predictable.
Now define $M_n = X_n - X_0 - A_n$. Then we see that $\{M_n\}_{n\in\mathbb{N}_0}$ is clearly integrable and adapted. We only have to show that it is a martingale.

$$\mathbb{E}[M_n|\mathcal{F}_{n-1}] = \mathbb{E}[X_n|\mathcal{F}_{n-1}] - X_0 - A_n =$$
$$\mathbb{E}[X_n|\mathcal{F}_{n-1}] - X_0 - A_{n-1} - \mathbb{E}[X_n - X_{n-1}|\mathcal{F}_{n-1}] = X_{n-1} - X_0 - A_{n-1} = M_{n-1}$$

It remains to show uniqueness.
Assume that X has a second decomposition $X_n = X_0 + M'_n + A'_n$. Then we get $M'_n - M_n = A'_n - A_n$. We see that $M'_n - M_n$ is predictable. Thus

$$M'_n - M_n = \mathbb{E}[M'_n - M_n|\mathcal{F}_{n-1}] = M'_{n-1} - M_{n-1}$$

Therefore $M'_1 - M_1 = M'_0 - M_0 = 0$ and then $M'_1 = M_1$. By induction we see that $M'_n = M_n$. Then we immediately have $A'_n = A_n$ and the uniqueness has been shown.

It remains to show that $\{A_n\}_{n\in\mathbb{N}_0}$ is monotonously incresing iff X is a submartingale. This follows immediately by the definition of A_n.

4. *Assume that* $(\xi_n)_{n\in\mathbb{N}}$ *is a sequence of independent random variables with* $\mathbb{E}[\xi_n] = 0\ \forall n$ *and* $\mathbb{E}[\exp(\xi_n)] < \infty\ \forall n$. *Furthermore*

$$\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$$

and

$$S_0 = 0 \quad S_n = \sum_{k=1}^{n} \xi_k.$$

   (a) Show that S_n *is a martingale with respect to* $(\mathcal{F}_n)_{n\geq 0}$.

   (b) Show that $P_n = \exp(S_n)$ *is a submartingale with respect to* $(\mathcal{F}_n)_{n\geq 0}$.

   (c) Now assume $\xi_n \sim N(0, \sigma_n^2)$. *Determine the Doob-decomposition of* P_n.

Solution:

(a) We have $\mathbb{E}[|S_n|] \leq \sum_{k=1}^{n} \mathbb{E}[|\xi_k|] < \infty$ and S_n is $\mathcal{F}_n$-measurable by definition of $\mathcal{F}_n$.

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}\left[\sum_{k=1}^{n+1} \xi_k \middle| \mathcal{F}_n\right] = \sum_{k=1}^{n} \xi_k + \mathbb{E}[\xi_{n+1}|\mathcal{F}_n] =$$
$$= \sum_{k=1}^{n} \xi_k + \mathbb{E}[\xi_{n+1}] = \sum_{k=1}^{n} \xi_k = S_n$$

(b) We define $f(x) = \exp(x)$. As f is convex, by applying the conditional Jensen formula we get

$$\mathbb{E}[f(S_{n+1})|\mathcal{F}_n] \geq f(\mathbb{E}[S_{n+1}|\mathcal{F}_n]) = f(S_n)$$

In addition to this, P_n is adapted, as S_n is adapted and $f(x) = \exp(x)$ is Borel-measurable. Apart from this, $\mathbb{E}[|P_n|] = \prod_{k=1}^{n} \mathbb{E}[\exp(\xi_k)] < \infty$. Thus, $P_n = \exp(S_n)$ is a submartingale with respect to $(\mathcal{F}_n)_{n\geq 0}$.

(c) The Doob-decomposition is

$$A_n = \sum_{k=1}^{n} \mathbb{E}[P_k - P_{k-1}|\mathcal{F}_{k-1}] = \sum_{k=1}^{n} P_{k-1}(e^{\frac{1}{2}\sigma_k^2} - 1).$$

The martingale part M_n of the decomposition is

$$M_n = P_n - A_n - P_0 = P_n - \sum_{k=1}^{n} P_{k-1}(e^{\frac{1}{2}\sigma_k^2} - 1) - 1$$

Then $P_n = M_n + A_n + P_0$, where A_n is predictable, M_n is a martingale and P_0 a constant.

5. *We assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a standard filtered probability space with $\mathbb{F} = (\mathcal{F}_n)_{n=0}^{\infty}$ a filtration. Let $U = (U_n)_{n=0}^{\infty}$ be an adapted sequence and consider the discrete stochastic exponential*

$$\mathcal{E}_n(U) = \prod_{k=1}^{n}(1 + \Delta U_k), \quad \mathcal{E}_0(U) = 1,$$

where $\Delta U_n = U_n - U_{n-1}$. Consider the difference equation

$$\Delta X_n = X_{n-1}\Delta U_n, \quad X_0 = 1. \tag{DE}$$

(i) *Verify that $\mathcal{E}_n(U)$ is a solution of (DE).*

(ii) *Assume $\mathcal{E}_n(U) \neq 0$. Show that $\mathcal{E}_n(U)$ is a martingale if (U_n) is a martingale.*

(iii) *Let $(\alpha_n)_{n=0}^{\infty}$ be a deterministic series of positive numbers and $V = (V_n)_{n=0}^{\infty}$ be an adapted sequence. Set*

$$A_n = \sum_{k=1}^{n} \mathbb{E}\left(e^{\alpha_k \Delta V_k} - 1 \middle| \mathcal{F}_{k-1}\right).$$

Prove that

$$Z_n = \exp\left\{\sum_{k=1}^{n} \alpha_k \Delta V_k\right\} \mathcal{E}_n^{-1}(A), \quad Z_0 = 1$$

is a martingale. (Hint: You may use $\mathcal{E}_n(U)^{-1} = \mathcal{E}_n(-Y)$, where $\Delta Y_n = \Delta U_n - \frac{(\Delta U_n)^2}{(1+\Delta U_n)}$).

Solution:

(i) Define $X_n := \mathcal{E}_n(U)$. Then

$$\begin{aligned} \Delta X_n &= \mathcal{E}_n(U) - \mathcal{E}_{n-1}(U) \\ &= (1 + \Delta U_n - 1) \cdot \prod_{k=1}^{n-1}(1 + \Delta U_k) \\ &= \Delta U_n \cdot X_{n-1}. \end{aligned}$$

(ii) Let (U_n) be a martingale. Adaptedness of $(\mathcal{E}_n(U))$ can be seen. Integrability of $(\mathcal{E}_n(U))$ is provided by assumption. Define $X_n := \mathcal{E}_n(U)$. Then, according to (i), (X_n) solves (DE). Thus, by (DE) and the martingale-property of (U_n)

$$\begin{aligned} E(\Delta X_n|\mathcal{F}_{n-1}) &= E(X_{n-1} \cdot \Delta U_n|\mathcal{F}_{n-1}) \\ &= X_{n-1} \cdot E(\Delta U_n|\mathcal{F}_{n-1}) \\ &= X_{n-1} \cdot 0 = 0, \end{aligned}$$

which shows that (X_n) is a martingale.

(iii) Adaptedness of (Z_n) follows from adaptedness of (A_n), integrability of (Z_n) is provided by assumption. Applying the hint to (A_n) and simplifying yields

$$\mathcal{E}_n(A)^{-1} = \prod_{k=1}^{n}\left(1 - \frac{\Delta A_k}{1 + \Delta A_k}\right).$$

With this result, we get

$$
\begin{aligned}
Z_n &= \exp\left\{\sum_{k=1}^{n} \alpha_k \cdot \Delta V_k\right\} \cdot \mathcal{E}_n(A)^{-1} \\
&= e^{\alpha_n \cdot \Delta V_n} \cdot \left(1 - \frac{\Delta A_n}{1+\Delta A_n}\right) \cdot \prod_{k=1}^{n-1} e^{\alpha_k \cdot \Delta V_k} \cdot \left(1 - \frac{\Delta A_k}{1+\Delta A_k}\right) \\
&= e^{\alpha_n \cdot \Delta V_n} \cdot \left(1 - \frac{\Delta A_n}{1+\Delta A_n}\right) \cdot Z_{n-1}.
\end{aligned}
$$

Now, with the definition of A_n, we find

$$
1 - \frac{\Delta A_n}{1+\Delta A_n} = \frac{1}{E(\exp\{\alpha_n \cdot \Delta V_n\}|\mathcal{F}_{n-1})}.
$$

Applying this gives

$$
\begin{aligned}
\Delta Z_n &= Z_{n-1} \cdot \left(e^{\alpha_n \cdot \Delta V_n} \cdot \left(1 - \frac{\Delta A_n}{1+\Delta A_n}\right) - 1\right) \\
&= Z_{n-1} \cdot \left(\frac{e^{\alpha_n \cdot \Delta V_n}}{E(\exp\{\alpha_n \cdot \Delta V_n\}|\mathcal{F}_{n-1})} - 1\right),
\end{aligned}
$$

and thus

$$
\begin{aligned}
E(\Delta Z_n|\mathcal{F}_{n-1}) &= Z_{n-1} \cdot E\left(\left.\frac{e^{\alpha_n \cdot \Delta V_n}}{E(\exp\{\alpha_n \cdot \Delta V_n\}|\mathcal{F}_{n-1})}\right| \mathcal{F}_{n-1}\right) - Z_{n-1} \\
&= \frac{Z_{n-1}}{E(\exp\{\alpha_n \cdot \Delta V_n\}|\mathcal{F}_{n-1})} \cdot E\left(e^{\alpha_n \cdot \Delta V_n}|\mathcal{F}_{n-1}\right) - Z_{n-1} \\
&= 0,
\end{aligned}
$$

completing the proof that (Z_n) is a martingale.

6. *Let σ and τ be two stopping times with respect to the filtration $(\mathcal{F}_n)_{n=0}^{\infty}$. Show that $\mathcal{F}_{\sigma\wedge\tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$. Also show that the events*

$$\{\tau < \sigma\},\ \{\sigma < \tau\},\ \{\tau \le \sigma\},\ \{\sigma \le \tau\},\ \{\sigma = \tau\}$$

belong to $\mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

Solution:
From Proposition 3.5.2 in the book we know that as $\sigma \wedge \tau \le \sigma$ and also $\sigma \wedge \tau \le \tau$ we have $\mathcal{F}_{\sigma\wedge\tau} \subset \mathcal{F}_\sigma$ and $\mathcal{F}_{\sigma\wedge\tau} \subset \mathcal{F}_\tau$. Thus $\mathcal{F}_{\sigma\wedge\tau} \subset \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

For the other direction assume $A \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$. Therefore

$$A \cap \{\sigma \le n\} \in \mathcal{F}_n \quad \forall n$$

and

$$A \cap \{\tau \le n\} \in \mathcal{F}_n \quad \forall n.$$

We have

$$\begin{aligned} A \cap \{\sigma \wedge \tau \le n\} &= A \cap (\{\sigma \le n\} \cup \{\tau \le n\}) \\ &= (A \cap \{\sigma \le n\} \cup (A \cap \{\tau \le n\}) \in \mathcal{F}_n \quad \forall n. \end{aligned}$$

From this it follows that $A \in \mathcal{F}_{\sigma\wedge\tau}$.

We now have to verify that the mentioned sets are in $\mathcal{F}_{\sigma\wedge\tau}$. First show $\{\tau \le \sigma\} \in \mathcal{F}_\sigma$:

$$\{\tau \le \sigma\} \cap \{\sigma \le n\} = \bigcup_{k=0}^{n} \{\sigma = k\} \cap \{\tau \le k\} \in \mathcal{F}_n.$$

Analogously we get $\{\tau \le \sigma\} \in \mathcal{F}_\tau$ by:

$$\{\tau \le \sigma\} \cap \{\tau \le n\} = \bigcup_{k=0}^{n} \{\tau = k\} \cap \{\sigma > k\} \in \mathcal{F}_n.$$

The result for $\{\sigma \le \tau\}$ follows by reversing the roles of σ and τ.

We also have:

$$\{\tau = \sigma\} \cap \{\sigma \le n\} = \bigcup_{k=0}^{n} \{\sigma = k\} \cap \{\tau = k\} \in \mathcal{F}_n.$$

Thus, we have $\{\sigma = \tau\} \in \mathcal{F}_\sigma$. By reversing the role of σ and τ, we get the result.

The result for the last two sets follows by taking complements.

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