Additional questions for chapter 1

1. *Prove the put-call relationship for American options (current time is $t = 0$ maturity at T):*

$$S - K \leq C_A - P_A \leq S - Ke^{-rT}.$$

Solution:
From the put-call parity for European options we have $S + P_E - C_E = Ke^{-rT}$. We know that $C_A = C_E$, and using the argument of additional flexibility of American type options again, we must have $P_A \geq P_E$. Hence we get $S + P_A - C_A \geq Ke^{-rT}$, or, equivalently $C_A - P_A \leq S - Ke^{-rT}$. We thereby obtain an upper bound for the difference of an American call and the corresponding American put option.

To find the lower bound we construct an arbitrage table assuming that the opposite (strict) inequality is true. We set up the following portfolio: write the put, buy the call, sell the stock short, put K into your bank account. We use T^* to denote either the time of early exercise of the put or expiry, whichever comes earlier. The arbitrage table is:

Portfolio	Current cash flow	$S(T^*) \leq K$	$K < S(T^*)$
Write put	P_A	$-(K - S(T^*))$	0
Buy call	$-C_A$	0	$S(T^*) - K$
Sell stock short	S	$-S(T^*)$	$-S(T^*)$
Lend	$-K$	Ke^{rT^*}	Ke^{rT^*}
Total	$-(C_A - P_A)$ $+(S - K) \;\; > 0$	> 0	$> 0.$

(of course in the case that T^* means early exercise we needn't look at $K < S(T^*)$ since a rational financial agent wouldn't exercise the put under these circumstances.) Since all future cash-flows are positive we have constructed an arbitrage portfolio, contradicting our assumption. We may even have exercised an American call early, which was suboptimal. Hence the inequality $S - K \leq C_A - P_A$ must hold, completing the proof.

2. *Assume a one-period financial market model with three securities on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ *and* $\mathbb{P}(\omega_i) > 0 \quad i = 1, 2, 3$. *The current prices of the securities are* $S(0) = (S_0(0), S_1(0), S_2(0))' = (1, 12, 27)'$. *At time* $t = 1$ *the following outcomes are possible:*

$$S(1) = \begin{pmatrix} S_0(1,\omega_1) & S_0(1,\omega_2) & S_0(1,\omega_3) \\ S_1(1,\omega_1) & S_1(1,\omega_2) & S_1(1,\omega_3) \\ S_2(1,\omega_1) & S_2(1,\omega_2) & S_2(1,\omega_3) \end{pmatrix} = \begin{pmatrix} 1.1 & 1.1 & 1.1 \\ 22 & 11 & 11 \\ 16.5 & 22 & 44 \end{pmatrix}$$

   (a) *Is the model complete? Find the state price vector.*

   (b) *Find the EMM with the first security as numéraire.*

   (c) *Find the EMM with the second security as numéraire.*

   Solution:

   (a) S(1) is a regular matrix. The model is arbitrage-free according to theorem 1.4.1 because the state-price vector is positive. The state-price vector can be found by $\psi = S(T)^{-1}S(0)$. We then get

$$\psi = \begin{pmatrix} -0.909 & 0.909 & 0 \\ 2.954 & -0.1136 & -0.0454 \\ -0.1136 & 0.02272 & 0.0454 \end{pmatrix} \begin{pmatrix} 1 \\ 12 \\ 27 \end{pmatrix} = \begin{pmatrix} 0.1818 \\ 0.3636 \\ 0.3636 \end{pmatrix} = \begin{pmatrix} \frac{2}{11} \\ \frac{4}{11} \\ \frac{4}{11} \end{pmatrix}$$

   As S is regular, the model is complete by theorem 1.4.2.

   (b) When we take the first security as numeraire we get $q = \psi \cdot 1.1 = (0.2, 0.4, 0.4)'$.

   (c) When we take the second security as numeraire, we get

$$\tilde{S}(0) = (1/12, 1, 27/12) = (0.0833, 1, 2.25)$$

   and

$$\tilde{S}(T) = \begin{pmatrix} 0.05 & 0.1 & 0.1 \\ 1 & 1 & 1 \\ 0.75 & 2 & 4 \end{pmatrix}$$

   This gives $q = \tilde{S}(T)^{-1}\tilde{S}(0) = (1/3, 1/3, 1/3)'$.

3. *Assume a single-period market model with three risky assets on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ *and* $\mathbb{P}(\omega_i) > 0 \quad i = 1, 2, 3$. *The prices of the assets at time* $t = 0$ *are* $S(0) = (S_1(0), S_2(0), S_3(0)) = (100, 150, \alpha)$. *At* $t = 1$ *the prices are given by the following matrix:*

$$S(1) = \begin{pmatrix} S_1(1,\omega_1) & S_1(1,\omega_2) & S_1(1,\omega_3) \\ S_2(1,\omega_1) & S_2(1,\omega_2) & S_2(1,\omega_3) \\ S_3(1,\omega_1) & S_3(1,\omega_2) & S_3(1,\omega_3) \end{pmatrix} = \begin{pmatrix} 110 & 110 & 110 \\ 154 & 198 & 143 \\ 176 & 220 & 143 \end{pmatrix}$$

   (a) *Name an equivalent characterization to freedom of arbitrage in single period market models.*

   (b) *What are the possible values for* α, *so that the market remains arbitrage-free?*

   (c) *Assume that* $\alpha = 160$. *Calculate an equivalent martingale measure with the bond as numéraire.*

   (d) *Calculate the price of the asset with payoff-vector* $C(1) = (22, 66, 0)$.

Solution:

(a) $S(1) \in \mathbb{R}^{m,n}$ and $S(0) \in \mathbb{R}^n$. The financial market is arbitrage-free iff

$$\exists \pi \in \mathbb{R}^n \text{ with } S(T)\pi = S(0) \text{ and } \pi_i > 0 \; \forall i$$

iff

$$\exists \text{ equivalent martingale measure } \mathbb{Q} \text{ with } \mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)}{S_1(T)}\right] = \frac{S(0)}{S_1(0)}.$$

(b) We discount using the first security as numéraire. In order to find the equivalent martingale measure, we have to solve the following equations:

$$\begin{pmatrix} \frac{S_1(1,\omega_1)}{S_1(1,\omega_1)} & \frac{S_1(1,\omega_2)}{S_1(1,\omega_2)} & \frac{S_1(1,\omega_3)}{S_1(1,\omega_3)} \\ \frac{S_2(1,\omega_1)}{S_1(1,\omega_1)} & \frac{S_2(1,\omega_2)}{S_1(1,\omega_2)} & \frac{S_2(1,\omega_3)}{S_1(1,\omega_3)} \\ \frac{S_3(1,\omega_1)}{S_1(1,\omega_1)} & \frac{S_3(1,\omega_2)}{S_1(1,\omega_2)} & \frac{S_3(1,\omega_3)}{S_1(1,\omega_3)} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \frac{S_1(0)}{S_1(0)} \\ \frac{S_2(0)}{S_1(0)} \\ \frac{S_3(0)}{S_1(0)} \end{pmatrix}.$$

Using the values given above yields

$$\begin{pmatrix} 1 & 1 & 1 \\ 1.4 & 1.8 & 1.3 \\ 1.6 & 2 & 1.3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \\ \frac{\alpha}{100} \end{pmatrix}.$$

This can be transformed into

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \frac{17}{2} - \frac{\alpha}{20} \end{pmatrix}.$$

We have $p_2 = \frac{1+p_3}{4}$ and therefore $p_1 = 1 - \frac{1}{4} - \frac{5}{4}p_3$. As $0 < p_3 < 1$ and $0 < p_1 < 1$, we get $p_3 < \frac{3}{5}$. Then $0 < \frac{170-\alpha}{20} < \frac{3}{5}$ and thus $158 < \alpha < 170$.

(c) For $\alpha = 160$, the equivalent martingale measure is $p = (\frac{1}{8}, \frac{3}{8}, \frac{1}{2})$.

(d) The price of the call is $C(0) = p_1 \cdot 20 + p_2 \cdot 60 = 25$.

4. *Consider the following single period market model on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ *and* $\mathbb{P}(\omega_i) > 0 \quad i = 1, 2$. *We have a riskless bond which costs* $B(0) = 1$ *at time* $t = 0$ *and pays* $B(T) = 1.1$ *at time* $t = T$. *We also have a stock with* $S_1(0) = 100$ *and* $S(T, \omega_1) = 150$, $S(T, \omega_2) = 90$.

   (a) *Is the market free of arbitrage? Is it complete?*

   (b) *With the bond as numéraire, what is the EMM?*

   (c) *With the stock as numéraire, what is the EMM?*

   (d) *Determine the arbitrage-free price of a call with strike* $K = 100$ *using*

   - *a replicating portfolio;*
   - *risk-neutral valuation with the bond as numéraire;*
   - *risk-neutral valuation with the stock as numéraire.*

Solution:

(a) We have

$$S(T) = \begin{pmatrix} 1.1 & 1.1 \\ 150 & 90 \end{pmatrix} \quad \text{and} \quad S(0) = \begin{pmatrix} 1 \\ 100 \end{pmatrix}$$

According to theorem 1.4.1 we have no arbitrage iff

$$\exists \psi \in \mathbb{R}^n, \psi_i > 0 \; \forall \; 1 \leq i \leq n \text{ such that } S(T)\psi = S(0).$$

Here we get

$$\psi = S(T)^{-1}S(0) = -\frac{1}{66}\begin{pmatrix} 90 & -1.1 \\ -150 & 1.1 \end{pmatrix}\begin{pmatrix} 1 \\ 100 \end{pmatrix} = \begin{pmatrix} \frac{20}{66} \\ \frac{40}{66} \end{pmatrix}.$$

Thus, the model is arbitrage-free. The model is complete by theorem 1.4.2 because $S(T)$ is invertible.

(b) Using the bond as numéraire, we get

$$q_1 = \frac{\psi_1}{\psi_1 + \psi_2} = \frac{1}{3} \text{ and } q_2 = \frac{2}{3}.$$

We call this measure Q.

(c) First, we transform the prices using the stock as numéraire:

$$\tilde{S}(0) = \begin{pmatrix} 1/100 \\ 1 \end{pmatrix}; \; \tilde{S}(T) = \begin{pmatrix} \frac{1.1}{150} & \frac{1.1}{90} \\ 1 & 1 \end{pmatrix}$$

We get the risk-neutral probabilities by

$$\begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} = \tilde{S}^{-1}(T)\tilde{S}(0) = \begin{pmatrix} \frac{5}{11} \\ \frac{6}{11} \end{pmatrix}.$$

We call this measure $\tilde{Q}$.

(d) Now lets calculate the risk-neutral price of a European call option using 3 different ways

- The payoff of the call at time $t = T$ is $C^u(T) = 50$ and $C^d(T) = 0$. We set up the replicating portfolio $\varphi = (\varphi_o, \varphi_1)'$ using the equalities

$$\varphi_0 B(T, \omega_1) + \varphi_1 S_1(T, \omega_1) = C(T, \omega_1)$$
$$\varphi_0 B(T, \omega_2) + \varphi_1 S_1(T, \omega_2) = C(T, \omega_2)$$

  This gives $\varphi = (-\frac{750}{11}, \frac{5}{6})'$. The costs for setting up this portfolio are $C(0) = \varphi_0 B(0) + \varphi_1 S_1(0) = \frac{500}{33}$. This is the arbitrage-free price of the call option.
- Now we use the bond as numéraire and the risk-neutral probabilities we already calculated. We get

$$C(0)/B(0) = \mathbb{E}_Q[C(T)/B(T)] = \frac{50}{1.1} \cdot \frac{1}{3} = \frac{500}{33}.$$

  As $B(0) = 1$, we get $C(0) = \frac{500}{33}$.
- With the stock as numéraire and the measure $\tilde{Q}$ we obtain:

$$C(0)/S_1(0) = \mathbb{E}_{\tilde{Q}}[C(T)/S_1(T)] = \frac{50}{150} \cdot \frac{5}{11} = \frac{5}{33}.$$

  As $S_1(0) = 100$, $C(0) = \frac{500}{33}$.

Thus, the results for all 3 ways are the same.

5. *Consider the following single period market model on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ *and* $\mathbb{P}(\omega_i) > 0 \quad i = 1, 2, 3$. *There exists one bond and one stock with possible future outcomes at time* $t = T$:

$$
\begin{aligned}
B(T) &= (B(T, \omega_1), B(T, \omega_2), B(T, \omega_3)) = (1.1, 1.1, 1.1) \\
S(T) &= (S(T, \omega_1), S(T, \omega_2), S(T, \omega_3)) = (143, 121, 88)
\end{aligned}
$$

and $B(0) = 1$, $S(0) = 100$. *What are the possible values for a call on the stock with strike* $K = 100$, *maturing at* T, *so that the resulting model is still arbitrage free?*

Solution:
We have to solve the equations:

$$\begin{pmatrix} 1 & 1 & 1 \\ 130 & 110 & 80 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 100 \end{pmatrix}$$

This gives

$$p_1 + p_2 + p_3 = 1 \quad \text{and} \quad 2p_2 + 5p_3 = 3$$

As we also have $0 < p_1, p_2, p_3 < 1$, we get

$$1 > p_2 = 1.5 - 2.5p_3 > 0 \quad \text{and} \quad 1 > p_1 = -0.5 + 1.5p_3 > 0.$$

Thus we have

$$0.2 < p_3 < 0.6 \quad \text{and} \quad \frac{1}{3} < p_3 < 1.$$

All in all, the set of possible equivalent martingale measures is characterized by

$$p_1 = -0.5 + 1.5p_3 \quad \text{and} \quad p_2 = 1.5 - 2.5p_3 \quad \text{and} \quad p_3 \in \left(\frac{1}{3}, \frac{3}{5}\right).$$

For the call, we have the payoff $C(T) = (43, 21, 0)$. The price is

$$C(0) = \frac{1}{1.1}(43 \cdot (-0.5 + 1.5p_3) + 21 \cdot (1.5 - 2.5p_3)) = \frac{1}{1.1}(10 + 12p_3).$$

The set of possible call prices, so that the resulting market is still arbitrage-free is

$$\frac{14}{1.1} < C(0) < \frac{17.2}{1.1}.$$

Alternatively, it is possible to solve

$$\begin{pmatrix} 1.1 & 1.1 & 1.1 \\ 143 & 121 & 88 \\ 43 & 21 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 100 \\ x \end{pmatrix}$$

for ψ_1, ψ_2 and ψ_3 and determine the values for x so that $\psi_1, \psi_2, \psi_3 \geq 0$.

Springer
http://www.springer.com/978-1-85233-458-1

Risk-Neutral Valuation
Pricing and Hedging of Financial Derivatives
Bingham, N.H.; Kiesel, R.
2004, XVIII, 438 p., Hardcover
ISBN: 978-1-85233-458-1