Financial Mathematics

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Financial Mathematics

Lecture 1

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Aims and Objectives

- Review basic concepts of probability theory;
- Discuss random variables, their distribution and the notion of independence;
- Calculate functionals and transforms;
- Review basic limit theorems.

Probability theory

To describe a random experiment we use *sample space* Ω, the set of all possible outcomes.

Each point ω of Ω, or *sample point*, represents a possible random outcome of performing the random experiment.

For a set $A \subseteq \Omega$ we want to know the probability $\mathbb{P}(A)$.

The class $\mathcal{F}$ of subsets of Ω whose probabilities $\mathbb{P}(A)$ are defined (call such A *events*) should be be a σ-algebra , i.e. closed under countable, disjoint unions and complements, and contain the empty set $\emptyset$ and the whole space Ω.

Examples. Flip coins, Roll two dice.

Probability theory

We want

(i) $I\!\!P(\emptyset) = 0, I\!\!P(\Omega) = 1,$

(ii) $I\!\!P(A) \geq 0$ for all A,

(iii) If $A_1, A_2, \ldots,$ are disjoint,
$I\!\!P(\bigcup_i A_i) = \sum_i I\!\!P(A_i)$ countable additivity.

(iv) If $B \subseteq A$ and $I\!\!P(A) = 0$,
then $I\!\!P(B) = 0$ (completeness).

A probability space, or Kolmogorov triple, is a triple $(\Omega, \mathcal{F}, I\!\!P)$ satisfying Kolmogorov axioms (i),(ii),(iii), (iv) above.

A probability space is a mathematical model of a random experiment.

Probability theory

Let $(\Omega, \mathcal{F}, I\!P)$ be a probability space. A *random variable* (vector) X is a function $X : \Omega \to I\!R(I\!R^k)$ such that $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$ for all Borel sets $B \in \mathcal{B}(\mathcal{B}(I\!R^k))$.

For a random variable X

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$$

for all $x \in I\!R$.

So define the *distribution function* F_X of X by

$$F_X(x) := I\!P(\{\omega : X(\omega) \leq x\}).$$

Recall: $\sigma(X)$, the σ-algebra *generated* by X.

Probability theory

- Binomial distribution: Number of successes

$$\mathbb{P}(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

- Geometric distribtion: Waiting time

$$\mathbb{P}(N = n) = p(1-p)^{n-1}.$$

- Poisson distribution:

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda}{k!}.$$

- Uniform distribution:

$$f(x) = \frac{1}{b-a} \mathbf{1}_{\{(a,b)\}}.$$

Probability theory

- Exponential distribution:

$$f(x) = \lambda e^{-\lambda x} \mathbf{1}_{\{[0,\infty)\}}.$$

The *expectation* $\mathbb{E}$ of a random variable X on $(\Omega, \mathcal{F}, I\!P)$ is defined by

$$\mathbb{E}\, X := \int_\Omega X dI\!P, \quad \text{or} \quad \int_\Omega X(\omega) dI\!P(\omega).$$

The variance of a random variable is defined as

$$\mathbb{V}\,(X) := \mathbb{E}\,\left[(X - \mathbb{E}\,(X))^2\right] = \mathbb{E}\,\left(X^2\right) - (\mathbb{E}\,X)^2.$$

Probability theory

If X is real-valued with density f,

$$\mathbb{E}\, X := \int x f(x) dx$$

or if X is discrete, taking values $x_n (n = 1, 2, \ldots)$ with probability function $f(x_n)(\geq 0)$,

$$\mathbb{E}\, X := \sum x_n f(x_n).$$

Examples. Moments for some of the above distributions.

Random variables $X_1, \ldots, X_n$ are independent if whenever $A_i \in \mathcal{B}$ for $i = 1, \ldots n$ we have

$$I\!P\left(\bigcap_{i=1}^{n} \{X_i \in A_i\}\right) = \prod_{i=1}^{n} I\!P(\{X_i \in A_i\}).$$

Probability theory

In order for $X_1, \ldots, X_n$ to be independent it is necessary and sufficient that for all $x_1, \ldots x_n \in (-\infty, \infty]$,

$$I\!\!P\left(\bigcap_{i=1}^{n}\{X_i \le x_i\}\right) = \prod_{i=1}^{n} I\!\!P(\{X_i \le x_i\}).$$

Multiplication Theorem If $X_1, \ldots, X_n$ are independent and $\mathbb{E}\,|X_i| < \infty,\ i = 1, \ldots, n$, then

$$\mathbb{E}\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} \mathbb{E}\,(X_i).$$

Probability theory

If X, Y are independent, with distribution functions F, G

$$Z := X + Y,$$

let Z have distribution function H.

Call H the *convolution* of F and G, written $H = F * G$.

Suppose X, Y have densities f, g. Then

$$H(z) = I\!P(X + Y \leq z) = \int_{\{(x,y):x+y\leq z\}} f(x)g(y)dxdy,$$

Thus

$$H(z) = \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{z-x} g(y)dy \right\} dx = \int_{-\infty}^{\infty} f(x)G(z-x)dx.$$

Example. Gamma distribution.

Probability theory

If X is a random variable with distribution function F, its *moment generating function* ϕ_X is

$$\phi(t) := \mathbb{E}\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} dF(x).$$

The mgf *takes convolution into multiplication*: if X, Y are independent,

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t).$$

Observe $\phi^{(k)}(t) = \mathbb{E}\left(X^k e^{tX}\right)$ and $\phi(0) = \mathbb{E}\left(X^k\right)$.

For X on nonnegative integers use the *generating function*

$$\gamma_X(z) = \mathbb{E}\left(z^X\right) = \sum_{k=0}^{\infty} z^k I\!P(Z = k).$$

Probability theory

Conditional expectation For **events**:

$$\mathbb{P}(A|B) := \mathbb{P}(A \cap B)/\mathbb{P}(B) \quad \text{if} \quad \mathbb{P}(B) > 0.$$

Implies the *multiplication rule*:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$$

Leads to the *Bayes rule*

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i)\mathbb{P}(B|A_i)}{\sum_j \mathbb{P}(A_j)\mathbb{P}(B|A_j)}.$$

Probability theory

For *discrete* random variables:

If X takes values $x_1, \ldots, x_m$ with probabilities $f_1(x_i) > 0$,
Y takes values $y_1, \ldots, y_n$ with probabilities $f_2(y_j) > 0$,
(X, Y) takes values (x_i, y_j) with probabilities $f(x_i, y_j) > 0$,
then the **marginal distributions** are

$$f_1(x_i) = \sum_{j=1}^{n} f(x_i, y_j).$$

$$f_2(y_j) = \sum_{i=1}^{m} f(x_i, y_j).$$

Probability theory

$$\mathbb{P}(Y = y_j | X = x_i) = \frac{\mathbb{P}(X = x_i, Y = y_j)}{\mathbb{P}(X = x_i)}$$

$$= \quad \frac{f(x_i, y_j)}{f_1(x_i)} = \frac{f(x_i, y_j)}{\sum_{j=1}^{n} f(x_i, y_j)}.$$

So the conditional distribution

of Y given $X = x_i$

$$f_{Y|X}(y_j|x_i) = \frac{f(x_i, y_j)}{f_1(x_i)} = \frac{f(x_i, y_j)}{\sum_{j=1}^{n} f(x_i, y_j)}.$$

Probability theory

Its expectation is

$$\begin{aligned}\mathbb{E}\left(Y|X=x_i\right) &= \sum_j y_j f_{Y|X}(y_j|x_i) \\ &= \frac{\sum_j y_j f(x_i,y_j)}{\sum_j f(x_i,y_j)}.\end{aligned}$$

Probability theory

Density case. If (X, Y) has density $f(x,y)$,
X has density $f_1(x) := \int_{-\infty}^{\infty} f(x,y)dy$,
Y has density $f_2(y) := \int_{-\infty}^{\infty} f(x,y)dx$.

The conditional density of Y given $X = x$ is:

$$f_{Y|X}(y|x) := \frac{f(x,y)}{f_1(x)} = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)dy}.$$

Its expectation is

$$\begin{aligned}\mathbb{E}\,(Y|X=x) &= \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy \\ &= \frac{\int_{-\infty}^{\infty} y f(x,y)dy}{\int_{-\infty}^{\infty} f(x,y)dy}.\end{aligned}$$

Probability theory

General case.

Suppose that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, $\mathcal{G} \subset \mathcal{F}$

If Y is a non-negative random variable with $\mathbb{E}Y < \infty$, then

$$\mathbb{Q}(B) := \int_B Y d\mathbb{P} \qquad (B \in \mathcal{G})$$

is non-negative, σ-additive – because

$$\int_B Y d\mathbb{P} = \sum_n \int_{B_n} Y d\mathbb{P}$$

if $B = \cup_n B_n$, B_n disjoint – and defined on the σ-algebra $\mathcal{G}$, so is a measure on $\mathcal{G}$.

If $\mathbb{P}(B) = 0$, then $\mathbb{Q}(B) = 0$ also (the integral of anything over a null set is zero), so $\mathbb{Q} << \mathbb{P}$.

Probability theory

By the Radon-Nikodým theorem, there exists a Radon-Nikodým derivative of $\mathbb{Q}$ with respect to $I\!P$ on $\mathcal{G}$, which is $\mathcal{G}$-measurable.

Following Kolmogorov, we call this Radon-Nikodým derivative the conditional expectation of Y given (or conditional on) $\mathcal{G}, \mathbb{E}\,(Y|\mathcal{G})$: this is $\mathcal{G}$-measurable, integrable, and satisfies

$$\int_B Y \, dI\!P = \int_B \mathbb{E}\,(Y|\mathcal{G}) dI\!P \qquad \forall B \in \mathcal{G}.$$

Probability theory

Suppose $\mathcal{G} = \sigma(X)$.

Then $\mathbb{E}(Y|\mathcal{G}) = \mathbb{E}(Y|\sigma(X)) =: \mathbb{E}(Y|X)$.

Its defining property is

$$\int_B Y d\mathbb{P} = \int_B \mathbb{E}(Y|X) d\mathbb{P} \qquad \forall B \in \sigma(X).$$

If $\mathcal{G} = \sigma(X_1, \ldots, X_n)$ write

$$\mathbb{E}\,(Y|\sigma(X_1, \ldots, X_n)) =: \mathbb{E}(Y|X_1, \ldots, X_n)$$

then

$$\int_B Y d\mathbb{P} = \int_B \mathbb{E}(Y|X_1, \ldots, X_n) d\mathbb{P}.$$

Probability theory

Weak Law of Large Numbers If $X_1, X_2, \ldots$ are independent and identically distributed with mean μ, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu \quad \text{in probability.}$$

Central Limit Theorem If $X_1, X_2, \ldots$ are independent and identically distributed with mean μ and variance σ^2, then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} (X_i - \mu)/\sigma \to N(0,1) \quad \text{in distribution.}$$

Financial Mathematics

Lecture 2

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Aims and Objectives

- Derivative Background §1.1;
- Arbitrage §1.2, §1.3;
- Fundamental Pricing Example §1.4;
- Single-period Model §1.4.

Derivative Background

A derivative security, or contingent claim, is a financial contract whose value at expiration date T (more briefly, expiry) is determined exactly by the price (or prices within a prespecified time-interval) of the underlying financial assets (or instruments) at time T (within the time interval $[0, T]$).

Derivative securities can be grouped under three general headings: *Options, Forwards and Futures* and *Swaps*. During this text we will mainly deal with options although our pricing techniques may be readily applied to forwards, futures and swaps as well.

Options

An option is a financial instrument giving one the *right but not the obligation* to make a specified transaction at (or by) a specified date at a specified price. *Call* options give one the right to buy. *Put* options give one the right to sell. *European* options give one the right to buy/sell on the specified date, the expiry date, on which the option expires or matures.

American options give one the right to buy/sell at any time prior to or at expiry.

Options

The simplest call and put options are now so standard they are called *vanilla* options.

Many kinds of options now exist, including so-called *exotic* options. Types include: *Asian* options, which depend on the *average* price over a period, *lookback* options, which depend on the *maximum* or *minimum* price over a period and *barrier* options, which depend on some price level being attained or not.

Terminology

The asset to which the option refers is called the *underlying asset* or the *underlying*. The price at which the transaction to buy/sell the underlying, on/by the expiry date (if exercised), is made, is called the *exercise price* or *strike price*. We shall usually use K for the strike price, time $t = 0$ for the initial time (when the contract between the buyer and the seller of the option is struck), time $t = T$ for the expiry or final time.

Consider, say, a European call option, with strike price K; write $S(t)$ for the value (or price) of the underlying at time t. If $S(t) > K$, the option is *in the money*, if $S(t) = K$, the option is said to be *at the money* and if $S(t) < K$, the option is *out of the money*.

Payoff

The payoff from the option, which is

$$S(T) - K \text{ if } S(T) > K \quad \text{and} \quad 0 \text{ otherwise}$$

(more briefly written as $(S(T) - K)^+$).

Taking into account the initial payment of an investor one obtains the profit diagram below.

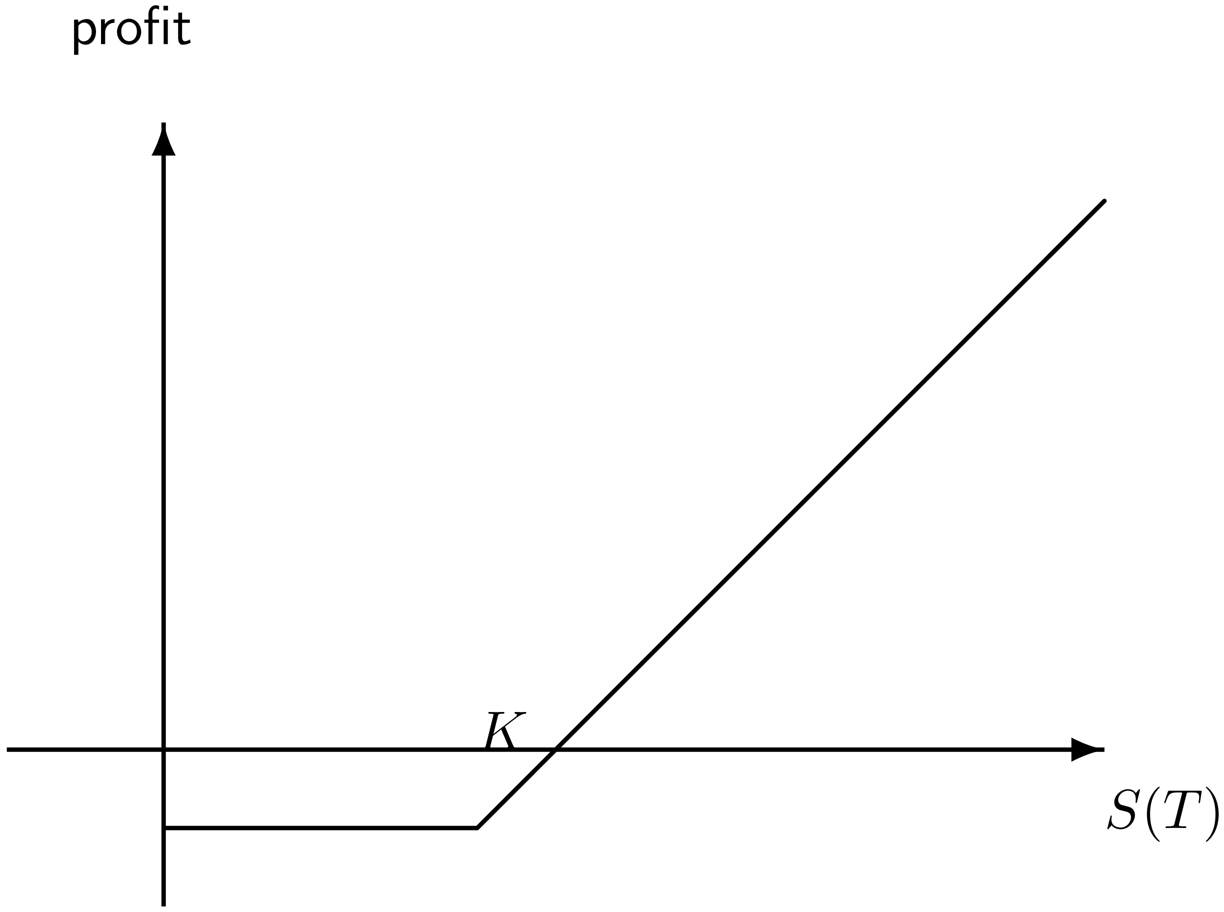

Figure 1: Profit diagram for a European call

Forwards

A *forward contract* is an agreement to buy or sell an asset S at a certain future date T for a certain price K. The agent who agrees to buy the underlying asset is said to have a *long* position, the other agent assumes a *short* position. The settlement date is called *delivery date* and the specified price is referred to as *delivery price*. The *forward price* $f(t,T)$ is the delivery price which would make the contract have zero value at time t. At the time the contract is set up, $t = 0$, the forward price therefore equals the delivery price, hence $f(0,T) = K$. The forward prices $f(t,T)$ need not (and will not) necessarily be equal to the delivery price K during the life-time of the contract.

Options

The payoff from a long position in a forward contract on one unit of an asset with price $S(T)$ at the maturity of the contract is

$$S(T) - K.$$

Compared with a call option with the same maturity and strike price K we see that the investor now faces a downside risk, too. He has the obligation to buy the asset for price K.

Options

A *swap* is an agreement whereby two parties undertake to exchange, at known dates in the future, various financial assets (or cash flows) according to a prearranged formula that depends on the value of one or more underlying assets. Examples are currency swaps (exchange currencies) and interest-rate swaps (exchange of fixed for floating set of interest payments).

Underlying securities

Stocks. Shares

- provide partial ownership of the company, pro rata with investment,
- have value, reflecting both the value of the company's (real) assets and the earning power of the company's dividends.

With publicly quoted companies, shares are quoted and traded on the Stock Exchange. Stock is the generic term for assets held in the form of shares.

Interest Rates

The value of some financial assets depends solely on the level of interest rates (or yields), e.g. Treasury (T-) notes, T-bills, T-bonds, municipal and corporate bonds. These are fixed-income securities by which national, state and local governments and large companies partially finance their economic activity. Fixed-income securities require the payment of interest in the form of a fixed amount of money at predetermined points in time, as well as repayment of the principal at maturity of the security. Interest rates themselves are notional assets, which cannot be delivered. A whole term structure is necessary for a full description of the level of interest rates.

Currencies

A currency is the denomination of the national units of payment (money) and as such is a financial asset. The end of fixed exchange rates and the adoption of floating exchange rates resulted in a sharp increase in exchange rate volatility. International trade, and economic activity involving it, such as most manufacturing industry, involves dealing with more than one currency. A company may wish to hedge adverse movements of foreign currencies and in doing so use derivative instruments.

Indexes

An index tracks the value of a (hypothetical) basket of stocks (FT-SE100, S&P-500, DAX), bonds (REX), and so on. Again, these are not assets themselves. Derivative instruments on indexes may be used for hedging if no derivative instruments on a particular asset (a stock, a bond, a commodity) in question are available and if the correlation in movement between the index and the asset is significant. Furthermore, institutional funds (such as pension funds, mutual funds etc.), which manage large diversified stock portfolios, try to mimic particular stock indexes and use derivatives on stock indexes as a portfolio management tool. On the other hand, a speculator may wish to bet on a certain overall development in a market without exposing him/herself to a particular asset.

Markets

Financial derivatives are basically traded in two ways: on organized exchanges and over-the-counter (OTC). Organised exchanges are subject to regulatory rules, require a certain degree of standardisation of the traded instruments (strike price, maturity dates, size of contract etc.) and have a physical location at which trade takes place.

OTC trading takes place via computers and phones between various commercial and investment banks (leading players include institutions such as Bankers Trust, Goldman Sachs – where Fischer Black worked, Citibank, Chase Manhattan and Deutsche Bank).

Types of Traders

We can classify the traders of derivative securities in three different classes:

Hedgers. Successful companies concentrate on economic activities in which they do best. They use the market to insure themselves against adverse movements of prices, currencies, interest rates etc. Hedging is an attempt to reduce exposure to risk a company already faces.

Types of Traders

Speculators. Speculators want to take a position in the market – they take the opposite position to hedgers. Indeed, speculation is needed to make hedging possible, in that a hedger, wishing to lay off risk, cannot do so unless someone is willing to take it on.

Arbitrageurs. Arbitrageurs try to lock in riskless profit by simultaneously entering into transactions in two or more markets. The very existence of arbitrageurs means that there can only be very small arbitrage opportunities in the prices quoted in most financial markets.

Modelling Assumptions

We impose the following set of assumptions on the financial markets:

- *No market frictions:* No transaction costs, no bid/ask spread, no taxes, no margin requirements, no restrictions on short sales.
- *No default risk:* Implying same interest for borrowing and lending
- *Competitive markets:* Market participants act as price takers
- *Rational agents* Market participants prefer more to less

Arbitrage

We now turn in detail to the concept of arbitrage, which lies at the centre of the relative pricing theory. This approach works under very weak assumptions. All we assume is that they prefer more to less, or more precisely, an increase in consumption without any costs will always be accepted.

The essence of the technical sense of arbitrage is that it should not be possible to guarantee a profit without exposure to risk. Were it possible to do so, arbitrageurs (we use the French spelling, as is customary) would do so, in unlimited quantity, using the market as a 'money-pump' to extract arbitrarily large quantities of riskless profit.

We assume that arbitrage opportunities do not exist!

Arbitrage Relationships

We now use the principle of no-arbitrage to obtain bounds for option prices. We focus on European options (puts and calls) with identical underlying (say a stock S), strike K and expiry date T. Furthermore we assume the existence of a risk-free bank account (bond) with constant interest rate r (continuously compounded) during the time interval $[0, T]$. We start with a fundamental relationship:

We have the following put-call parity between the prices of the underlying asset S and European call and put options on stocks that pay no dividends:

$$S + P - C = Ke^{-r(T-t)}. \tag{1}$$

Arbitrage Relationships

Proof. Consider a portfolio consisting of one stock, one put and a short position in one call (the holder of the portfolio has written the call); write $V(t)$ for the value of this portfolio. Then

$$V(t) = S(t) + P(t) - C(t)$$

for all $t \in [0, T]$. At expiry we have

$$\begin{aligned} V(T) &= S(T) + (S(T) - K)^{-} - (S(T) - K)^{+} \\ &= S(T) + K - S(T) = K. \end{aligned}$$

This portfolio thus guarantees a payoff K at time T. Using the principle of no-arbitrage, the value of the portfolio must at any time t correspond to the value of a sure payoff K at T, that is $V(t) = Ke^{-r(T-t)}$. ■

Having established (1), we concentrate on European calls.

Arbitrage Relationships

The following bounds hold for European call options:

$$\max\left\{S(t) - e^{-r(T-t)}K, 0\right\}$$

$$= \quad \left(S(t) - e^{-r(T-t)}K\right)^{+} \leq C(t) \leq S(t).$$

Arbitrage Relationships

Proof. That $C \geq 0$ is obvious, otherwise 'buying' the call would give a riskless profit now and no obligation later.

Similarly the upper bound $C \leq S$ must hold, since violation would mean that the right to buy the stock has a higher value than owning the stock. This must be false, since a stock offers additional benefits.

Now from put-call parity (1) and the fact that $P \geq 0$ (use the same argument as above), we have

$$S(t) - Ke^{-r(T-t)} = C(t) - P(t) \leq C(t),$$

which proves the last assertion. ∎

Arbitrage Relationships

It is immediately clear that an American call option can never be worth less than the corresponding European call option, for the American option has the added feature of being able to be exercised at any time until the maturity date. Hence (with the obvious notation): $C_A(t) \geq C_E(t)$. The striking result we are going to show (due to R.C. Merton in 1973 is:

For a non-dividend paying stock we have

$$C_A(t) = C_E(t). \tag{2}$$

Arbitrage Relationships

Proof. Exercising the American call at time $t < T$ generates the cash-flow $S(t) - K.$ From the bounds on calls we know that the value of the call must be greater or equal to $S(t) - Ke^{-r(T-t)}$, which is greater than $S(t) - K.$ Hence selling the call would have realised a higher cash-flow and the early exercise of the call was suboptimal. ∎

Arbitrage Relationships

Qualitatively, there are two reasons why an American call should not be exercised early:

(i) Insurance. An investor who holds a call option instead of the underlying stock is 'insured against a fall in stock price below K, and if he exercises early, he loses this insurance.

(ii) Interest on the strike price. When the holder exercises the option, he buys the stock and pays the strike price, K. Early exercise at $t < T$ deprives the holder of the interest on K between times t and T: the later he pays out K, the better.

A fundamental example

We consider a one-period model, i.e. we allow trading only at $t = 0$ and $t = T = 1$(say). Our aim is to value at $t = 0$ a European derivative on a stock S with maturity T.

First idea. Model S_T as a random variable on a probability space $(\Omega, \mathcal{F}, I\!P)$. The derivative is given by $H = f(S_T)$, i.e. it is a random variable (for a suitable function $f(.)$). We could then price the derivative using some discount factor β by using the expected value of the discounted future payoff:

$$H_0 = \mathbb{E}\,(\beta H). \tag{3}$$

Problem. How should we pick the probability measure $I\!P$? According to their preferences investors will have different opinions about the distribution of the price S_T.

A fundamental example

Black-Scholes-Merton approach. Use the no-arbitrage principle and construct a hedging portfolio using only known (and already priced) securities to duplicate the payoff H. We assume

1. Investors are non-satiable, i.e. they always prefer more to less.
2. Markets do not allow arbitrage , i.e. the possibility of risk-free profits.

A fundamental example

From the no-arbitrage principle we see:

If it is possible to duplicate the payoff H *of a derivative using a portfolio* V *of underlying (basic) securities, i.e.* $H(\omega) = V(\omega)$, $\forall \omega$, *the price of the portfolio at* $t = 0$ *must equal the price of the derivative at* $t = 0$.

A fundamental example

Let us assume there are two tradeable assets

- a riskfree bond (bank account) with $B(0) = 1$ and $B(T) = 1$, that is the interest rate $r = 0$ and the discount factor $\beta(t) = 1$. (In this context we use $\beta(t) = 1/B(t)$ as the discount factor).
- a risky stock S with $S(0) = 10$ and two possible values at $t = T$

$$S(T) = \begin{cases} 20 & \text{with probability } p \\ 7.5 & \text{with probability } 1 - p. \end{cases}$$

A fundamental example

We call this setting a $(B, S)-$ market. The problem is to price a European call at $t = 0$ with strike $K = 15$ and maturity T, i.e. the random payoff $H = (S(T) - K)^+$. We can evaluate the call in every possible state at $t = T$ and see $H = 5$ (if $S(T) = 20$) with probability p and $H = 0$ (if $S(T) = 7.5$) with probability $1 - p$.

A fundamental example

The key idea now is to try to find a portfolio combining bond and stock, which synthesizes the cash flow of the option. If such a portfolio exists, holding this portfolio today would be equivalent to holding the option – they would produce the same cash flow in the future. Therefore the price of the option should be the same as the price of constructing the portfolio, otherwise investors could just restructure their holdings in the assets and obtain a riskfree profit today.

A fundamental example

We briefly present the constructing of the portfolio $\boldsymbol{\theta} = (\theta_0, \theta_1)$, which in the current setting is just a simple exercise in linear algebra. If we buy θ_1 stocks and invest θ_0 £ in the bank account, then today's value of the portfolio is

$$V(0) = \theta_0 + \theta_1 \cdot S(0).$$

In state 1 the stock price is 20 £ and the value of the option 5 £, so

$$\theta_0 + \theta_1 \cdot 20 = 5.$$

In state 2 the stock price is 7.5 £ and the value of the option 0 £, so

$$\theta_0 + \theta_1 \cdot 7.5 = 0.$$

A fundamental example

We solve this and get $\theta_0 = -3$ and $\theta_1 = 0.4$. So the value of our portfolio at time 0 in £ is

$$V(0) = -3B(0) + 0.4S(0) = 1$$

$V(0)$ is called the no-arbitrage price. Every other price allows a riskless profit, since if the option is too cheap, buy it and finance yourself by selling short the above portfolio (i.e. sell the portfolio without possessing it and promise to deliver it at time $T = 1$ – this is riskfree because you own the option). If on the other hand the option is too dear, write it (i.e. sell it in the market) and cover yourself by setting up the above portfolio.

A fundamental example

We see that the no-arbitrage price is independent of the individual preferences of the investor (given by certain probability assumptions about the future, i.e. a probability measure $I\!P$). But one can identify a special, so called risk-neutral, probability measure $I\!P^*$, such that

$$
\begin{aligned}
H_0 &= \mathbb{E}^*(\beta H) \\
&= (p^* \cdot \beta(S_1 - K) + (1-p^*) \cdot 0) \\
&= 1.
\end{aligned}
$$

A fundamental example

In the above example we get from $1 = p^*5 + (1 - p^*)0$ that $p^* = 0.2$

This probability measure $I\!P^*$ is equivalent to $I\!P$, and the discounted stock price process, i.e. $\beta_t S_t,\ t = 0, 1$ follows a $I\!P^*$-martingale. In the above example this corresponds to $S(0) = p^* S(T)^{up} + (1 - p^*) S(T)^{down}$, that is $S(0) = \mathbb{E}^*\left(\beta S(T)\right)$.

A fundamental example

We will show that the above generalizes. Indeed, we will find that *the no-arbitrage condition is equivalent to the existence of an equivalent martingale measure* (first fundamental theorem of asset pricing) and that the property that we can *price assets using the expectation operator is equivalent to the uniqueness of the equivalent martingale measure*.

A single-period model

We proceed to formalise and extend the above example and present in detail a simple model of a financial market. Despite its simplicity it already has all the key features needed in the sequel (and the reader should not hesitate to come back here from more advanced chapters to see the bare concepts again).

We consider a single period model, i.e. we have two time-indices, say $t = 0$, which is the current time (date), and $t = T$, which is the terminal date for all economic activities considered.

A single-period model

The financial market contains $d+1$ traded financial assets, whose prices at time $t=0$ are denoted by the vector $S(0) \in \mathbb{R}^{d+1}$,

$$S(0) = (S_0(0), S_1(0), \ldots, S_d(0))'$$

(where $'$ denotes the transpose of a vector or matrix). At time T, the owner of financial asset number i receives a random payment depending on the state of the world. We model this randomness by introducing a *finite* probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a finite number $|\Omega| = N$ of points (each corresponding to a certain state of the world) $\omega_1, \ldots, \omega_j, \ldots, \omega_N$, each with positive probability: $\mathbb{P}(\{\omega\}) > 0$, which means that every state of the world is possible.

A single-period model

$\mathcal{F}$ is the set of subsets of Ω (events that can happen in the world) on which $I\!\!P(.)$ is defined (we can quantify how probable these events are), here $\mathcal{F} = \mathcal{P}(\Omega)$ the set of all subsets of Ω.

We can now write the random payment arising from financial asset i as

$$
\begin{aligned}
& S_i(T) \\
= \quad & (S_i(T, \omega_1), \ldots, S_i(T, \omega_j), \ldots, S_i(T, \omega_N))'.
\end{aligned}
$$

A single-period model

At time $t = 0$ the agents can buy and sell financial assets. The portfolio position of an individual agent is given by a *trading strategy* φ, which is an $\mathbb{R}^{d+1}$ vector,

$$\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_d)'.$$

Here φ_i denotes the quantity of the ith asset bought at time $t = 0$, which may be negative as well as positive (recall we allow short positions).

A single-period model

The dynamics of our model using the trading strategy φ are as follows: at time $t = 0$ we invest the amount

$$S(0)'\varphi = \sum_{i=0}^{d} \varphi_i S_i(0)$$

and at time $t = T$ we receive the random payment $S(T,\omega)'\varphi = \sum_{i=0}^{d} \varphi_i S_i(T,\omega)$ depending on the realised state ω of the world. Using the $(d+1) \times N$-matrix $\vec{S}$, whose columns are the vectors $S(T,\omega)$, we can write the possible payments more compactly as $\vec{S}'\varphi$.

A single-period model

What does an *arbitrage opportunity* mean in our model? As arbitrage is 'making something out of nothing'; an arbitrage strategy is a vector $\varphi \in I\!R^{d+1}$ such that $S(0)'\varphi = 0$, our net investment at time $t = 0$ is zero, and

$$S(T,\omega)'\varphi \geq 0,\ \forall \omega \in \Omega \text{ and there exists a } \omega \in \Omega \text{ such that } S(T,\omega)'\varphi > 0.$$

We can equivalently formulate this as: $S(0)'\varphi < 0$, we borrow money for consumption at time $t = 0$, and

$$S(T,\omega)'\varphi \geq 0,\ \forall \omega \in \Omega,$$

i.e we don't have to repay anything at $t = T$. Now this means we had a 'free lunch' at $t = 0$ at the market's expense.

A single-period model

We agreed that we should not have arbitrage opportunities in our model. The consequences of this assumption are surprisingly far-reaching.

So assume that there are no arbitrage opportunities. If we analyse the structure of our model above, we see that every statement can be formulated in terms of Euclidean geometry or linear algebra. For instance, absence of arbitrage means that the space

$$\begin{aligned} \Gamma \\ = \quad & \left\{ \begin{pmatrix} x \\ y \end{pmatrix}, x \in I\!R, y \in I\!R^N : \right. \\ & \left. x = -S(0)'\varphi, y = \vec{S}'\varphi, \varphi \in I\!R^{d+1} \right\} \end{aligned}$$

A single-period model

and the space

$$
\begin{aligned}
& \mathbb{R}_+^{N+1} \\
= \quad & \{z \in \mathbb{R}^{N+1} : z_i \geq 0 \quad \forall \ 0 \leq i \leq N \\
& \exists\, i \ \text{ such that } z_i > 0\}
\end{aligned}
$$

have no common points. A statement like that naturally points to the use of a separation theorem for convex subsets, the separating hyperplane theorem Using such a theorem we come to the following characterisation of no arbitrage.

A single-period model

There is no arbitrage if and only if there exists a vector

$$\psi \in I\!R^N, \quad \psi_i > 0, \quad \forall \; 1 \le i \le N$$

such that

$$\vec{S}\psi = S(0). \tag{4}$$

Proof. The implication '$\Leftarrow$' follows straightforwardly: assume that $S(T,\omega)'\varphi \ge 0, \; \omega \in \Omega$ for a vector $\varphi \in I\!R^{d+1}$.Then

$$S(0)'\varphi = (\vec{S}\psi)'\varphi = \psi'\vec{S}'\varphi \ge 0,$$

since $\psi_i > 0, \; \forall 1 \le i \le N$. So no arbitrage opportunities exist.

A single-period model

To show the implication '$\Rightarrow$' we use a variant of the separating hyperplane theorem. Absence of arbitrage means the Γ and $I\!R_+^{N+1}$ have no common points. This means that $K \subset I\!R_+^{N+1}$ defined by

$$K = \left\{ z \in I\!R_+^{N+1} : \sum_{i=0}^{N} z_i = 1 \right\}$$

and Γ do not meet.

A single-period model

But K is a compact and convex set, and by the separating hyperplane theorem, there is a vector $\lambda \in I\!R^{N+1}$ such that for all $z \in K$

$$\lambda' z > 0$$

but for all $(x, y)' \in \Gamma$

$$\lambda_0 x + \lambda_1 y_1 + \ldots + \lambda_N y_N = 0.$$

Now choosing $z_i = 1$ successively we see that $\lambda_i > 0,\ i = 0, \ldots N$, and hence by normalising we get $\psi = \lambda / \lambda_0$ with $\psi_0 = 1.$ Now set $x = -S(0)'\varphi$ and $y = \vec{S}'\varphi$ and the claim follows. ■

A single-period model

The vector ψ is called a *state-price vector*. We can think of ψ_j as the marginal cost of obtaining an additional unit of account in state ω_j. We can now reformulate the above statement to:

There is no arbitrage if and only if there exists a state-price vector.

A single-period model

Using a further normalisation, we can clarify the link to our probabilistic setting. Given a state-price vector $\psi = (\psi_1, \ldots, \psi_N)$, we set $\psi_0 = \psi_1 + \ldots + \psi_N$ and for any state ω_j write $q_j = \psi_j/\psi_0$. We can now view $(q_1, \ldots, q_N)$ as probabilities and define a new probability measure on Ω by $\mathbb{Q}(\{\omega_j\}) = q_j,\ j = 1, \ldots, N$. Using this probability measure, we see that for each asset i we have the relation

$$\frac{S_i(0)}{\psi_0} = \sum_{j=1}^{N} q_j S_i(T, \omega_j) = \mathbb{E}_{\mathbb{Q}}(S_i(T)).$$

Hence the normalized price of the financial security i is just its expected payoff under some specially chosen 'risk-neutral' probabilities.

A single-period model

So far we have not specified anything about the denomination of prices. From a technical point of view we could choose any asset i as long as its price vector $(S_i(0), S_i(T,\omega_1), \ldots, S_i(T,\omega_N))'$ only contains positive entries, and express all other prices in units of this asset. We say that we use this asset as *numéraire*. Let us emphasise again that arbitrage opportunities do not depend on the chosen numéraire. It turns out that appropriate choice of the numéraire facilitates the probability-theoretic analysis in complex settings, and we will discuss the choice of the numéraire in detail later on.

A single-period model

For simplicity, let us assume that asset 0 is a riskless bond paying one unit in all states $\omega \in \Omega$ at time T. This means that $S_0(T,\omega) = 1$ in all states of the world $\omega \in \Omega$. By the above analysis we must have

$$\frac{S_0(0)}{\psi_0} = \sum_{j=1}^{N} q_j S_0(T,\omega_j) = \sum_{j=1}^{N} q_j 1 = 1,$$

and ψ_0 is the discount on riskless borrowing. Introducing an interest rate r, we must have $S_0(0) = \psi_0 = (1+r)^{-T}$.

A single-period model

We can now express the price of asset i at time $t = 0$ as

$$S_i(0) = \sum_{j=1}^{N} q_j \frac{S_i(T, \omega_j)}{(1+r)^T} = \mathbb{E}_{\mathbb{Q}} \left(\frac{S_i(T)}{(1+r)^T} \right).$$

We rewrite this as

$$\frac{S_i(T)}{(1+r)^0} = \mathbb{E}_{\mathbb{Q}} \left(\frac{S_i(T)}{(1+r)^T} \right).$$

A single-period model

In the language of probability theory we just have shown that the processes $S_i(t)/(1+r)^t, \ t = 0, T$ are $\mathbb{Q}$-martingales. (Martingales are the probabilists' way of describing fair games.) It is important to notice that under the given probability measure $I\!P$ (which reflects an individual agent's belief or the markets' belief) the processes $S_i(t)/(1+r)^t, \ t = 0, T$ generally do not form $I\!P$-martingales.

A single-period model

We use this to shed light on the relationship of the probability measures $I\!P$ and $\mathbb{Q}$. Since $\mathbb{Q}(\{\omega\}) > 0$ for all $\omega \in \Omega$ the probability measures $I\!P$ and $\mathbb{Q}$ are equivalent and because of the argument above we call $\mathbb{Q}$ an *equivalent martingale measure*. So we arrived at yet another characterisation of arbitrage:

There is no arbitrage if and only if there exists an equivalent martingale measure.

A single-period model

We also see that risk-neutral pricing corresponds to using the expectation operator with respect to an equivalent martingale measure. This concept lies at the heart of stochastic (mathematical) finance and will be the *golden thread* (or *roter Faden*) throughout this lecture.

A single-period model

We now know how the given prices of our $(d+1)$ financial assets should be related in order to exclude arbitrage opportunities, but how should we price a newly introduced financial instrument? We can represent this financial instrument by its random payments

$$\delta(T) = (\delta(T,\omega_1), \ldots, \delta(T,\omega_j), \ldots, \delta(T,\omega_N))'$$

(observe that $\delta(T)$ is a vector in $\mathbb{R}^N$) at time $t = T$ and ask for its price $\delta(0)$ at time $t = 0$.

A single-period model

The natural idea is to use an equivalent probability measure $\mathbb{Q}$ and set

$$\delta(0) = \mathbb{E}_{\mathbb{Q}}(\delta(T)/(1+r)^T)$$

(recall that all time $t = 0$ and time $t = T$ prices are related in this way). Unfortunately, as we don't have a unique martingale measure in general, we cannot guarantee the uniqueness of the $t = 0$ price. Put another way, we know every equivalent martingale measure leads to a reasonable relative price for our newly created financial instrument, but which measure should one choose?

A single-period model

The easiest way out would be if there were only one equivalent martingale measure at our disposal – and surprisingly enough the classical economic pricing theory puts us exactly in this situation! Given a set of financial assets on a market the underlying question is whether we are able to price any new financial asset which might be introduced in the market, or equivalently whether we can replicate the cash-flow of the new asset by means of a portfolio of our original assets. If this is the case and we can replicate every new asset, the market is called *complete*.

A single-period model

In our financial market situation the question can be restated mathematically in terms of Euclidean geometry: do the vectors $S_i(T)$ span the whole $I\!R^N$? This leads to:

Suppose there are no arbitrage opportunities. Then the model is complete if and only if the matrix equation

$$\vec{S}'\varphi = \delta$$

has a solution $\varphi \in I\!R^{d+1}$ for any vector $\delta \in I\!R^N$.

A single-period model

Linear algebra immediately tells us that the above theorem means that the number of independent vectors in $\vec{S}'$ must equal the number of states in Ω. In an informal way we can say that if the financial market model contains 2 (N) states of the world at time T it allows for 1 $(N-1)$ sources of randomness (if there is only one state we know the outcome). Likewise we can view the numéraire asset as risk-free and all other assets as risky. We can now restate the above characterisation of completeness in an informal (but intuitive) way as:

A financial market model is complete if it contains at least as many independent risky assets as sources of randomness.

A single-period model

The question of completeness can be expressed equivalently in probabilistic language, as a question of representability of the relevant random variables or whether the σ-algebra they generate is the full σ-algebra.

If a financial market model is complete, traditional economic theory shows that there exists a unique system of prices. If there exists only one system of prices, and every equivalent martingale measure gives rise to a price system, we can only have a unique equivalent martingale measure.

The (arbitrage-free) market is complete if and only if there exists a unique equivalent martingale measure.

Financial Mathematics

Lecture 3

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Aims and Objectives

- Review basic facts of conditional expectation §2.5;
- Introduce discrete-parameter martingales §3.3;
- Discuss main properties of martingales §3.4;
- Discussion of the optional stopping theorem §3.5;
- Discussion of the Snell envelope §3.6;

Conditional Expectation

Recall the defining property: For X and Y random variables $\mathbb{E}\,(Y|\sigma(X)) = \mathbb{E}\,(Y|X)$ is defined as the $\sigma(X)$-measurable random variable such that

$$\int_B Y d\mathbb{P} = \int_B \mathbb{E}\,(Y|X) d\mathbb{P} \qquad \forall B \in \sigma(X) \tag{5}$$

To define $\mathbb{E}\,(Y|\mathcal{G})$ for a general σ-algebra $\mathcal{G}$, replace $\sigma(X)$ with $\mathcal{G}$ in (5).

Conditional Expectation

From the definition linearity of conditional expectation follows from the linearity of the integral. Further properties

1. $\mathcal{G} = \{\emptyset, \varnothing\}, \quad \mathbb{E}\,(Y|\{\emptyset, \varnothing\}) = \mathbb{E}\,Y.$
2. $\mathcal{G} = \mathcal{F}, \quad \mathbb{E}\,(Y|\mathcal{F}) = Y \quad I\!\!P - a.s..$
3. If Y is $\mathcal{G}$-measurable, $\mathbb{E}\,(Y|\mathcal{G}) = Y \quad I\!\!P - a.s..$
4. If Y is $\mathcal{G}$-measurable, $\mathbb{E}\,(YZ|\mathcal{G}) = Y\mathbb{E}\,(Z|\mathcal{G}) \quad I\!\!P - a.s.$ (we call this 'taking out what is known' in view of the above).

Conditional Expectation

5. If $\mathcal{G}_0 \subset \mathcal{G}$, $\mathbb{E}\left[\mathbb{E}\left(Y|\mathcal{G}\right)|\mathcal{G}_0\right] = \mathbb{E}\left[Y|\mathcal{G}_0\right] \quad a.s.$ This is the so-called *tower property*).
6. *Conditional mean formula.* $\mathbb{E}\left[\mathbb{E}\left(Y|\mathcal{G}\right)\right] = \mathbb{E}\, Y \quad I\!\!P - a.s.$
7. *Role of independence.* If Y is independent of $\mathcal{G}$,

$$\mathbb{E}\left(Y|\mathcal{G}\right) = \mathbb{E}\, Y \quad a.s.$$

Discrete-Parameter Martingales

A process $X = (X_n)$ is called a *martingale* relative to $((\mathcal{F}_n), I\!P)$ if

(i) X is adapted (to $(\mathcal{F}_n)$);
(ii) $\mathbb{E}\,|X_n| < \infty$ for all n;
(iii) $\mathbb{E}\,[X_n|\mathcal{F}_{n-1}] = X_{n-1} \quad I\!P - a.s. \quad (n \geq 1)$.

X is a *supermartingale* if in place of (iii)

$$\mathbb{E}\,[X_n|\mathcal{F}_{n-1}] \leq X_{n-1} \quad I\!P - a.s. \quad (n \geq 1);$$

X is a *submartingale* if in place of (iii)

$$\mathbb{E}\,[X_n|\mathcal{F}_{n-1}] \geq X_{n-1} \quad I\!P - a.s. \quad (n \geq 1).$$

Discrete-Parameter Martingales

Using (iii) we see that the best forecast of unobserved future values of (X_k) based on information at time $\mathcal{F}_n$ is X_n; in more mathematical terms, the $\mathcal{F}_n$ measurable random variable Y which minimises $\mathbb{E}\left((X_{n+1} - Y)^2|\mathcal{F}_n\right)$ is X_n.

Martingales also have a useful interpretation in terms of dynamic games: a martingale is 'constant on average', and models a fair game; a supermartingale is 'decreasing on average', and models an unfavourable game; a submartingale is 'increasing on average', and models a favourable game.

Discrete-Parameter Martingales

X is a submartingale (supermartingale) if and only if $-X$ is a supermartingale (submartingale); X is a martingale if and only if it is both a submartingale and a supermartingale.

(X_n) is a martingale if and only if $(X_n - X_0)$ is a martingale. So we may without loss of generality take $X_0 = 0$ when convenient.

Discrete-Parameter Martingales

If X is a martingale, then for $m < n$ using the iterated conditional expectation and the martingale property repeatedly

$$\begin{aligned}
\mathbb{E}\left[X_n|\mathcal{F}_m\right] &= \mathbb{E}\left[\mathbb{E}\left(X_n|\mathcal{F}_{n-1}\right)|\mathcal{F}_m\right] \\
&= \mathbb{E}\left[X_{n-1}|\mathcal{F}_m\right] \\
&= \ldots = \mathbb{E}\left[X_m|\mathcal{F}_m\right] = X_m,
\end{aligned}$$

and similarly for submartingales, supermartingales.

Discrete-Parameter Martingales

Examples of a martingale include: sums of independent, integrable zero-mean random variables (submartingales: positive mean; supermartingale: negative mean). Also

Example. Accumulating data about a random variable: If $\xi \in L^1(\emptyset, \mathcal{F}, I\!\!P)$, $M_n := \mathbb{E}\left(\xi|\mathcal{F}_n\right)$ (so M_n represents our best estimate of ξ based on knowledge at time n), then using iterated conditional expectations

$$\begin{aligned} \mathbb{E}\left[M_n|\mathcal{F}_{n-1}\right] &= \mathbb{E}\left[\mathbb{E}\left(\xi|\mathcal{F}_n\right)|\mathcal{F}_{n-1}\right] \\ &= \mathbb{E}\left[\xi|\mathcal{F}_{n-1}\right] = M_{n-1}, \end{aligned}$$

so (M_n) is a martingale. One has the convergence

$$M_n \to M_\infty := \mathbb{E}\left[\xi|\mathcal{F}_\infty\right] \quad a.s. \quad \text{and in } L^1.$$

Martingale Convergence

We turn now to the theorems that make martingales so powerful a tool.

A supermartingale is 'decreasing on average'. Recall that a decreasing sequence (of real numbers) that is bounded below converges (decreases to its greatest lower bound or infimum). This suggests that a supermartingale which is bounded below converges a.s.. More is true. Call X L^1-bounded if

$$\sup_n \mathbb{E}\,|X_n| < \infty.$$

An L^1-bounded supermartingale is a.s. convergent: there exists X_∞ finite such that

$$X_n \to X_\infty \quad (n \to \infty) \quad a.s.$$

Martingale Convergence

Doob's Martingale Convergence Theorem
An L^1-bounded martingale converges a.s..

We say that

$$X_n \to X_\infty \quad \text{in } L^1$$

if

$$\mathbb{E}\,|X_n - X_\infty| \to 0 \quad (n \to \infty).$$

For a class of martingales, one gets convergence in L^1 as well as almost surely.

Martingale Convergence

The following are equivalent for martingales $X = (X_n)$:

(i) X_n converges in L^1;
(ii) X_n is L^1-bounded, and its a.s. limit X_∞ (which exists, by above) satisfies

$$X_n = \mathbb{E}\,[X_\infty|\mathcal{F}_n];$$

(iii) There exists an integrable random variable X with

$$X_n = \mathbb{E}\,[X|\mathcal{F}_n].$$

Such martingales are called regular or *uniformly integrable*.

Doob Decomposition

Let $X = (X_n)$ be an adapted process with each $X_n \in L^1$. Then X has an (essentially unique) Doob decomposition

$$X = X_0 + M + A : \quad X_n = X_0 + M_n + A_n \quad \forall n \tag{6}$$

with M a martingale null at zero, A a predictable process null at zero. If also X is a submartingale ('increasing on average'), A is increasing: $A_n \leq A_{n+1}$ for all n, a.s.

Doob decomposition

Proof. If X has a Doob decomposition (6),

$$
\begin{aligned}
& \mathbb{E}[X_n - X_{n-1} | \mathcal{F}_{n-1}] \\
= \quad & \mathbb{E}[M_n - M_{n-1} | \mathcal{F}_{n-1}] + \mathbb{E}[A_n - A_{n-1} | \mathcal{F}_{n-1}].
\end{aligned}
$$

The first term on the right is zero, as M is a martingale. The second is $A_n - A_{n-1}$, since A_n (and A_{n-1}) is $\mathcal{F}_{n-1}$-measurable by previsibility. So

$$
\mathbb{E}[X_n - X_{n-1} | \mathcal{F}_{n-1}] = A_n - A_{n-1}, \tag{7}
$$

and summation gives

$$
A_n = \sum_{k=1}^{n} \mathbb{E}[X_k - X_{k-1} | \mathcal{F}_{k-1}], \quad a.s.
$$

Doob decomposition

We use this formula to *define* (A_n), clearly previsible. We then use (6) to *define* (M_n), then a martingale, giving the Doob decomposition (6).

If X is a submartingale, the LHS of (7) is ≥ 0, so the RHS of (7) is ≥ 0, i.e. (A_n) is increasing. ∎

Martingale Transforms

Now think of a gambling game, or series of speculative investments, in discrete time. There is no play at time 0; there are plays at times $n = 1, 2, \ldots$, and

$$\Delta X_n := X_n - X_{n-1}$$

represents our net winnings per unit stake at play n. Thus if X_n is a martingale, the game is 'fair on average'.

Call a process $C = (C_n)_{n=1}^{\infty}$ *predictable* (or *previsible*) if

$$C_n \text{ is } \mathcal{F}_{n-1} - \text{measurable for all } n \geq 1.$$

Think of C_n as your stake on play n (C_0 is not defined, as there is no play at time 0).

Martingale Transforms

Previsibility says that you have to decide how much to stake on play n based on the history *before* time n (i.e., up to and including play $n-1$). Your winnings on game n are $C_n \Delta X_n = C_n(X_n - X_{n-1})$. Your total (net) winnings up to time n are

$$Y_n = \sum_{k=1}^{n} C_k \Delta X_k = \sum_{k=1}^{n} C_k (X_k - X_{k-1}).$$

We write

$$Y = C \bullet X, \quad Y_n = (C \bullet X)_n, \quad \Delta Y_n = C_n \Delta X_n$$

($(C \bullet X)_0 = 0$ as $\sum_{k=1}^{0}$ is empty), and call $C \bullet X$ the *martingale transform* of X by C.

Martingale Transforms

(i) If C is a bounded non-negative predictable process and X is a supermartingale, $C \bullet X$ is a supermartingale null at zero.
(ii) If C is bounded and predictable and X is a martingale, $C \bullet X$ is a martingale null at zero.

Martingale Transforms

Proof. With $Y = C \bullet X$ as above,

$$\begin{aligned}\mathbb{E}\left[Y_n - Y_{n-1}|\mathcal{F}_{n-1}\right] &= \mathbb{E}\left[C_n(X_n - X_{n-1})|\mathcal{F}_{n-1}\right] \\ &= C_n\mathbb{E}\left[(X_n - X_{n-1})|\mathcal{F}_{n-1}\right]\end{aligned}$$

(as C_n is bounded, so integrable, and $\mathcal{F}_{n-1}$-measurable, so can be taken out)

$$\leq 0$$

in case (i), as $C \geq 0$ and X is a supermartingale,

$$= 0$$

in case (ii), as X is a martingale. ∎

Interpretation. You can't beat the system! In the martingale case, previsibility of C means we can't foresee the future (which is realistic and fair). So we expect to gain nothing – as we should.

Martingale Transforms

Martingale Transform Lemma: An adapted sequence of real integrable random variables (M_n) is a martingale iff for any bounded previsible sequence (H_n),

$$\mathbb{E}\left(\sum_{k=1}^{n} H_k \Delta M_k\right) = 0 \quad (n = 1, 2, \ldots).$$

Proof. If (M_n) is a martingale, X defined by $X_0 = 0$,

$$X_n = \sum_{k=1}^{n} H_r \Delta M_r \quad (n \geq 1)$$

is the martingale transform $H \bullet M$, so is a martingale.

Martingale Transforms

Conversely, if the condition of the proposition holds, choose j, and for any $\mathcal{F}_j$-measurable set A write $H_n = 0$ for $n \neq j+1$, $H_{j+1} = I_A$. Then (H_n) is previsible, so the condition of the proposition, $\mathbb{E}\left(\sum_1^n H_r \Delta M_r\right) = 0$, becomes

$$\mathbb{E}\left[I_A(M_{j+1} - M_j)\right] = 0.$$

Since this holds for every set $A \in \mathcal{F}_j$, the definition of conditional expectation gives

$$\mathbb{E}\left(M_{j+1}|\mathcal{F}_j\right) = M_j.$$

Since this holds for every j, (M_n) is a martingale. ∎

Stopping Times and Optional Stopping

A random variable T taking values in $\{0, 1, 2, \ldots; +\infty\}$ is called a *stopping time* (or optional time) if

$$\{T \le n\} = \{\omega : T(\omega) \le n\} \in \mathcal{F}_n \quad \forall n \le \infty.$$

Equivalently,

$$\{T = n\} \in \mathcal{F}_n \quad n \le \infty,$$

or

$$\{T \ge n\} \in \mathcal{F}_n, \quad n \le \infty.$$

Stopping Times and Optional Stopping

Think of T as a time at which you decide to quit a gambling game: whether or not you quit at time n depends only on the history up to and including time n – NOT the future. Thus stopping times model gambling and other situations where there is no foreknowledge, or prescience of the future; in particular, in the financial context, where there is no insider trading.

Stopping Times and Optional Stopping

Doob's Optional Stopping Theorem Let T be a stopping time, $X = (X_n)$be a supermartingale, and assume that one of the following holds:

(i) T is bounded ($T(\omega) \leq K$ for some constant K and all $\omega \in \Omega$);
(ii) $X = (X_n)$ is bounded ($|X_n(\omega)| \leq K$ for some K and all n, ω);
(iii) $\mathbb{E}\, T < \infty$ and $(X_n - X_{n-1})$ is bounded.
Then X_T is integrable, and

$$\mathbb{E}\, X_T \leq \mathbb{E}\, X_0.$$

If X is a martingale, then

$$\mathbb{E}\, X_T = \mathbb{E}\, X_0.$$

Stopping Times and Optional Stopping

Write $X_n^T := X_{n \wedge T}$ for the sequence (X_n) stopped at time T.

(i) If (X_n) is adapted and T is a stopping time, the stopped sequence $(X_{n \wedge T})$ is adapted.
(ii) If (X_n) is a martingale (supermartingale) and T is a stopping time, (X_n^T) is a martingale (supermartingale).

Stopping Times and Optional Stopping

Proof. If $\phi_j := \mathbf{1}_{\{j \le T\}}$,

$$X_{T\wedge n} = X_0 + \sum_{j=1}^{n} \phi_j (X_j - X_{j-1})$$

(as the right is $X_0 + \sum_{j=1}^{T\wedge n}(X_j - X_{j-1})$, which telescopes to $X_{T\wedge n}$). Since $\{j \le T\}$ is the complement of $\{T < j\} = \{T \le j-1\} \in \mathcal{F}_{j-1}$, (ϕ_n) is predictable. So (X_n^T) is adapted.

If (X_n) is a martingale, so is (X_n^T) as it is the martingale transform of (X_n) by (ϕ_n).

Stopping Times and Optional Stopping

Since by predictability of (ϕ_n)

$$\begin{aligned}
\mathbb{E}\left(X_{T\wedge n}|\mathcal{F}_{n-1}\right) &= X_0 + \sum_{j=1}^{n-1} \phi_j (X_j - X_{j-1}) \\
&\quad +\phi_n(\mathbb{E}\left[X_n|\mathcal{F}_{n-1}\right] - X_{n-1}) \\
&= X_{T\wedge(n-1)} \\
&\quad +\phi_n(\mathbb{E}\left[X_n|\mathcal{F}_{n-1}\right] - X_{n-1}),
\end{aligned}$$

$\phi_n \geq 0$ shows that if (X_n) is a supermartingale (submartingale), so is $(X_{T\wedge n})$. ■

Examples

1. Simple Random Walk Recall the simple random walk: $S_n := \sum_1^n X_k$, where the X_n are independent tosses of a fair coin, taking values ± 1 with equal probability $1/2$. Suppose we decide to bet until our net gain is first $+1$, then quit. Let T be the time we quit; T is a stopping time. The stopping time T has been analysed in detail; see e.g.(**?**), §5.3.

Examples

From this, note:

(i) $T < \infty$ a.s.: the gambler will certainly achieve a net gain of $+1$ eventually;
(ii) $\mathbb{E}\,T = +\infty$: the mean waiting-time until this happens is infinity.
Hence also:
(iii) No bound can be imposed on the gambler's maximum net loss before his net gain first becomes $+1$.

Examples

At first sight, this looks like a foolproof way to make money out of nothing: just bet until you get ahead (which happens eventually, by (i)), then quit. However, as a gambling strategy, this is hopelessly impractical: because of (ii), you need unlimited time, and because of (iii), you need unlimited capital – neither of which is realistic.

Notice that the optional stopping theorem fails here: we start at zero, so $S_0 = 0$, $ES_0 = 0$; but $S_T = 1$, so $ES_T = 1$.

Examples

This example shows two things:

(a) The Optional Stopping Theorem does indeed need conditions, as the conclusion may fail otherwise (none of the conditions (i) – (iii) in the OST are satisfied in the example above).
(b) Any practical gambling (or trading) strategy needs to have some integrability or boundedness restrictions to eliminate such theoretically possible but practically ridiculous cases.

Examples

2. The Doubling Strategy The strategy of doubling when losing - *the martingale*, according to the *Oxford English Dictionary* – has similar properties. We play until the time T of our first win. Then T is a stopping time, and is geometrically distributed with parameter $p = 1/2$. If $T = n$, our winnings on the nth play are 2^{n-1} (our previous stake of 1 doubled on each of the previous $n - 1$ losses). Our cumulative losses to date are $1 + 2 + \ldots + 2^{n-2} = 2^{n-1} - 1$ (summing the geometric series), giving us a net gain of 1.

Examples

The mean time of play is $\mathbb{E}(T) = 2$ (so doubling strategies accelerate our eventually certain win to give a finite expected waiting time for it). But no bound can be put on the losses one may need to sustain before we win, so again we would need unlimited capital to implement this strategy – which would be suicidal in practice as a result.

Examples

3. The Saint Petersburg Game A single play of the Saint Petersburg game consists of a sequence of coin tosses stopped at the first head; if this is the rth toss, the player receives a prize of \$ 2^r. (Thus the expected gain is $\sum_{r=1}^{\infty} 2^{-r}.2^r = +\infty$, so the random variable is not integrable, and martingale theory does not apply.) Let S_n denote the player's cumulative gain after n plays of the game. The question arises as to what the 'fair price' of a ticket to play the game is. It turns out that fair prices exist (in a suitable sense), but the fair price of the nth play varies with n – surprising, as all the plays are replicas of each other.

The Snell Envelope

If $Z = (Z_n)_{n=0}^N$ is a sequence adapted to a filtration $(\mathcal{F}_n)$, the sequence $U = (U_n)_{n=0}^N$ defined by

$$\begin{cases} U_N := Z_N, \\ U_n := \max(Z_n, E(U_{n+1}|\mathcal{F}_n)) \quad (n \le N-1) \end{cases}$$

is called the *Snell envelope* of Z.

The Snell Envelope

The Snell envelope (U_n) of (Z_n) is a supermartingale, and is the smallest supermartingale dominating (Z_n) (that is, with $U_n \geq Z_n$ for all n).

Proof. First, $U_n \geq E(U_{n+1}|\mathcal{F}_n)$, so U is a supermartingale, and $U_n \geq Z_n$, so U dominates Z.

Next, let $T = (T_n)$ be any other supermartingale dominating Z; we must show T dominates U also. First, since $U_N = Z_N$ and T dominates Z, $T_N \geq U_N$.

The Snell Envelope

Assume inductively that $T_n \geq U_n$. Then

$$T_{n-1} \geq \mathbb{E}\,(T_n|\mathcal{F}_{n-1}) \geq \mathbb{E}\,(U_n|\mathcal{F}_{n-1}),$$

and as T dominates Z

$$T_{n-1} \geq Z_{n-1}.$$

Combining,

$$T_{n-1} \geq \max(Z_{n-1}, \mathbb{E}\,(U_n|\mathcal{F}_{n-1})) = U_{n-1}.$$

By repeating this argument (or more formally, by backward induction), $T_n \geq U_n$ for all n, as required. ■

The Snell Envelope

$T_0 := \inf\{n \geq 0 : U_n = Z_n\}$ is a stopping time, and the stopped sequence $(U_n^{T_0})$ is a martingale.

Proof. Since $U_N = Z_N$, $T_0 \in \{0, 1, \ldots, N\}$ is well-defined. For $k = 0$, $\{T_0 = 0\} = \{U_0 = Z_0\} \in \mathcal{F}_0$; for $k \geq 1$,

$$
\begin{aligned}
& \{T_0 = k\} \\
= \quad & \{U_0 > Z_0\} \cap \cdots \cap \{U_{k-1} > Z_{k-1}\} \cap \{U_k = Z_k\} \\
& \in \mathcal{F}_k.
\end{aligned}
$$

So T_0 is a stopping time.

The Snell Envelope

$$U_n^{T_0} = U_{n\wedge T_0} = U_0 + \sum_{j=1}^{n} \phi_j \Delta U_j,$$

where $\phi_j = \mathbf{1}_{\{T_0 \geq j\}}$ is adapted. For $n \leq N-1$,

$$\begin{aligned} U_{n+1}^{T_0} - U_n^{T_0} &= \phi_{n+1}(U_{n+1} - U_n) \\ &= \mathbf{1}_{\{n+1 \leq T_0\}}(U_{n+1} - U_n). \end{aligned}$$

Now $U_n := \max(Z_n, \mathbb{E}\left(U_{n+1}|\mathcal{F}_n\right))$, and

$$U_n > Z_n \quad \text{on } \{n+1 \leq T_0\}.$$

The Snell Envelope

So from the definition of U_n,

$$U_n = \mathbb{E}\left(U_{n+1}|\mathcal{F}_n\right) \quad \text{on } \{n+1 \le T_0\}.$$

We next prove

$$U^{T_0}_{n+1} - U^{T_0}_n = \mathbf{1}_{\{n+1\le T_0\}}(U_{n+1} - \mathbb{E}\left(U_{n+1}|\mathcal{F}_n\right)). \tag{8}$$

The Snell Envelope

For, suppose first that $T_0 \geq n+1$. Then the left of (8) is $U_{n+1} - U_n$, the right is $U_{n+1} - \mathbb{E}\left(U_{n+1}|\mathcal{F}_n\right)$, and these agree on $\{n+1 \leq T_0\}$ by above. The other possibility is that $T_0 < n+1$, i.e. $T_0 \leq n$. Then the left of (8) is $U_{T_0} - U_{T_0} = 0$, while the right is zero because the indicator is zero, completing the proof of (8).

The Snell Envelope

Now apply $\mathbb{E}(.|\mathcal{F}_n)$ to (8): since $\{n+1 \le T_0\} = \{T_0 \le n\}^c \in \mathcal{F}_n$,

$$\begin{aligned}
&\mathbb{E}\left[(U_{n+1}^{T_0} - U_n^{T_0})|\mathcal{F}_n\right] \\
&= \mathbf{1}_{\{n+1\le T_0\}}\mathbb{E}\left([U_{n+1} - \mathbb{E}(U_{n+1}|\mathcal{F}_n)]|\mathcal{F}_n\right) \\
&= \mathbf{1}_{\{n+1\le T_0)\}}\left[\mathbb{E}(U_{n+1}|\mathcal{F}_n) - \mathbb{E}(U_{n+1}|\mathcal{F}_n)\right] \\
&= 0.
\end{aligned}$$

So $\mathbb{E}(U_{n+1}^{T_0}|\mathcal{F}_n) = U_n^{T_0}$. This says that $U_n^{T_0}$ is a martingale, as required.

■

The Snell Envelope

Write $\mathcal{T}_{n,N}$ for the set of stopping times taking values in $\{n, n+1, \ldots, N\}$ (a finite set, as Ω is finite). We next see that the Snell envelope solves the optimal stopping problem.

T_0 solves the optimal stopping problem for Z:

$$U_0 = \mathbb{E}\left(Z_{T_0}|\mathcal{F}_0\right) = \sup\{\mathbb{E}\left(Z_T|\mathcal{F}_0\right) : T \in \mathcal{T}_{0,N}\}.$$

The Snell Envelope

Proof. To prove the first statement we use that $(U_n^{T_0})$ is a martingale and $U_{T_0} = Z_{T_0}$; then

$$U_0^{T_0} = \mathbb{E}\,(U_N^{T_0}|\mathcal{F}_0) = \mathbb{E}\,(U_{T_0}|F_0) = \mathbb{E}\,(Z_{T_0}|\mathcal{F}_0).$$

Now for any stopping time $T \in \mathcal{T}_{0,N}$, since U is a supermartingale (above), so is the stopped process (U_n^T). Together with the property that (U_n) dominates (Z_n) this yields

$$U_0 = U_0^{T_0} \geq E(U_N^T|\mathcal{F}_0) = E(U_T|\mathcal{F}_0) \geq E(Z_T|\mathcal{F}_0),$$

and this completes the proof. ∎

The Snell Envelope

The same argument, starting at time n rather than time 0, gives

If $T_n := \inf\{j \geq n : U_j = Z_j\}$,

$$U_n = \mathbb{E}\left(Z_{T_n}|\mathcal{F}_n\right) = \sup\{\mathbb{E}\left(Z_T|\mathcal{F}_n\right) : T \in \mathcal{T}_{n,N}\}.$$

As we are attempting to maximise our payoff by stopping $Z = (Z_n)$ at the most advantageous time, the Corollary shows that T_n gives the best stopping time that is realistic: it maximises our expected payoff given only information currently available (it is easy, but irrelevant, to maximise things with hindsight!). We thus call T_0 (or T_n, starting from time n) the *optimal* stopping time for the problem. For textbook accounts of optimal stopping problems, see e.g. (**?**), (Neveu 1975).

Financial Mathematics

Lecture 4

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Aims and Objectives

- Discrete-time models §4.1;
- Fundamental theorems of asset pricing §4.2, §4.3;

The model

We will study so-called finite markets – i.e. discrete-time models of financial markets in which all relevant quantities take a finite number of values. To illustrate the ideas, it suffices to work with a finite probability space $(\Omega, \mathcal{F}, I\!P)$, with a finite number $|\Omega|$ of points ω, each with positive probability: $I\!P(\{\omega\}) > 0$.

We specify a time horizon T, which is the terminal date for all economic activities considered. (For a simple option pricing model the time horizon typically corresponds to the expiry date of the option.)

The model

As before, we use a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0}^{T}$ consisting of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T$: we take $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the trivial σ-field, $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$ (here $\mathcal{P}(\Omega)$ is the power-set of Ω, the class of all $2^{|\Omega|}$ subsets of Ω: we need every possible subset, as they all – apart from the empty set – carry positive probability).

The model

The financial market contains $d+1$ financial assets. The usual interpretation is to assume one risk-free asset (bond, bank account) labelled 0, and d risky assets (stocks, say) labelled 1 to d. While the reader may keep this interpretation as a mental picture, we prefer not to use it directly. The prices of the assets at time t are random variables, $S_0(t,\omega), S_1(t,\omega), \ldots, S_d(t,\omega)$ say, non-negative and $\mathcal{F}_t$-measurable (i.e. adapted: at time t, we know the prices $S_i(t)$). We write

$$S(t) = (S_0(t), S_1(t), \ldots, S_d(t))'$$

for the vector of prices at time t.

The model

Hereafter we refer to the probability space $(\Omega, \mathcal{F}, I\!P)$, the set of trading dates, the price process S and the information structure $\mathbb{F}$, which is typically generated by the price process S, together as a securities market model.

A *numéraire* is a price process $(X(t))_{t=0}^{T}$ (a sequence of random variables), which is strictly positive for all $t \in \{0, 1, \ldots, T\}$.

The model

For the standard approach the risk-free bank account process is used as numéraire. In some applications, however, it is more convenient to use a security other than the bank account and we therefore just use S_0 without further specification as a numéraire. We furthermore take $S_0(0) = 1$ (that is, we reckon in units of the initial value of our numéraire), and define $\beta(t) := 1/S_0(t)$ as a discount factor.

The model

A *trading strategy* (or *dynamic portfolio*) φ is a $\mathbb{R}^{d+1}$ vector stochastic process

$$\varphi = (\varphi(t))_{t=1}^{T}$$

$$= ((\varphi_0(t,\omega), \varphi_1(t,\omega), \ldots, \varphi_d(t,\omega))')_{t=1}^{T}$$

which is predictable (or previsible): each $\varphi_i(t)$ is $\mathcal{F}_{t-1}$-measurable for $t \geq 1$.

The model

Here $\varphi_i(t)$ denotes the number of shares of asset i held in the portfolio at time t – to be determined on the basis of information available *before* time t; i.e. the investor selects his time t portfolio after observing the prices $S(t-1)$. However, the portfolio $\varphi(t)$ must be established before, and held until after, announcement of the prices $S(t)$.

The components $\varphi_i(t)$ may assume negative as well as positive values, reflecting the fact that we allow short sales and assume that the assets are perfectly divisible.

The model

The *value* of the portfolio at time t is the scalar product

$$V_\varphi(t) = \varphi(t) \cdot S(t)$$

$$:= \sum_{i=0}^{d} \varphi_i(t) S_i(t), \ (t = 1, 2, \ldots, T)$$

and

$$V_\varphi(0) = \varphi(1) \cdot S(0).$$

The process $V_\varphi(t, \omega)$ is called the wealth or value process of the trading strategy φ.

The initial wealth $V_\varphi(0)$ is called the *initial investment* or *endowment* of the investor.

The model

Now $\varphi(t) \cdot S(t-1)$ reflects the market value of the portfolio just after it has been established at time $t-1$, whereas $\varphi(t) \cdot S(t)$ is the value just after time t prices are observed, but before changes are made in the portfolio. Hence

$$\varphi(t) \cdot (S(t) - S(t-1)) = \varphi(t) \cdot \Delta S(t)$$

is the change in the market value due to changes in security prices which occur between time $t-1$ and t. This motivates:

The model

The *gains process* G_φ of a trading strategy φ is given by $(t = 1, 2, \ldots, T)$

$$\begin{aligned} G_\varphi(t) &:= \sum_{\tau=1}^{t} \varphi(\tau) \cdot (S(\tau) - S(\tau - 1)) \\ &= \sum_{\tau=1}^{t} \varphi(\tau) \cdot \Delta S(\tau). \end{aligned}$$

Observe the – for now – formal similarity of the gains process G_φ from trading in S following a trading strategy φ to the martingale transform of S by φ.

The model

Define $\tilde{S}(t) = (1, \beta(t)S_1(t), \ldots, \beta(t)S_d(t))'$, the vector of discounted prices, and consider the *discounted value process*

$$\tilde{V}_\varphi(t) = \beta(t)(\varphi(t) \cdot S(t)) = \varphi(t) \cdot \tilde{S}(t),$$

and the *discounted gains process*

$$\begin{aligned} \tilde{G}_\varphi(t) &:= \textstyle\sum_{\tau=1}^{t} \varphi(\tau) \cdot (\tilde{S}(\tau) - \tilde{S}(\tau-1)) \\ &= \sum_{\tau=1}^{t} \varphi(\tau) \cdot \Delta\tilde{S}(\tau). \end{aligned}$$

Observe that the discounted gains process reflects the gains from trading with assets 1 to d only, which in case of the standard model (a bank account and d stocks) are the risky assets.

The model

We will only consider special classes of trading strategies.

The strategy φ is self-financing, $\varphi \in \Phi$, if for $t = 1, 2, \ldots, T-1$

$$\varphi(t) \cdot S(t) = \varphi(t+1) \cdot S(t). \tag{9}$$

When new prices $S(t)$ are quoted at time t, the investor adjusts his portfolio from $\varphi(t)$ to $\varphi(t+1)$, without bringing in or consuming any wealth.

The model

The following result (which is trivial in our current setting, but requires a little argument in continuous time) shows that renormalising security prices (i.e. changing the numéraire) has essentially no economic effects.

Numéraire Invariance Let $X(t)$ be a numéraire. A trading strategy φ is self-financing with respect to $S(t)$ if and only if φ is self-financing with respect to $X(t)^{-1}S(t)$.

The model

Proof. Since $X(t)$ is strictly positive for all $t = 0, 1, \ldots, T$ we have the following equivalence, which implies the claim:

$$\varphi(t) \cdot S(t) = \varphi(t+1) \cdot S(t)$$

$$\Leftrightarrow$$

$$\varphi(t) \cdot X(t)^{-1} S(t) = \varphi(t+1) \cdot X(t)^{-1} S(t).$$

∎

A trading strategy φ is self-financing with respect to $S(t)$ if and only if φ is self-financing with respect to $\tilde{S}(t)$.

The model

We now give a characterisation of self-financing strategies in terms of the discounted processes.

A trading strategy φ belongs to Φ if and only if

$$\tilde{V}_\varphi(t) = V_\varphi(0) + \tilde{G}_\varphi(t), \quad (t = 0, 1, \ldots, T). \tag{10}$$

The model

Proof. Assume $\varphi \in \Phi$. Then using the defining relation (9), the numéraire invariance theorem and the fact that $S_0(0) = 1$

$$
\begin{aligned}
& V_\varphi(0) + \tilde{G}_\varphi(t) \\
= \quad & \varphi(1) \cdot S(0) + \sum_{\tau=1}^{t} \varphi(\tau) \cdot (\tilde{S}(\tau) - \tilde{S}(\tau - 1)) \\
= \quad & \varphi(1) \cdot \tilde{S}(0) + \varphi(t) \cdot \tilde{S}(t) \\
& - \sum_{\tau=1}^{t-1} (\varphi(\tau) - \varphi(\tau + 1)) \cdot \tilde{S}(\tau) - \varphi(1) \cdot \tilde{S}(0) \\
= \quad & \varphi(t) \cdot \tilde{S}(t) = \tilde{V}_\varphi(t).
\end{aligned}
$$

The model

Assume now that (10) holds true. By the numéraire invariance theorem it is enough to show the discounted version of relation (9). Summing up to $t = 2$ (10) is

$$\varphi(2) \cdot \tilde{S}(2) = \varphi(1) \cdot \tilde{S}(0) + \varphi(1) \cdot (\tilde{S}(1) - \tilde{S}(0)) + \varphi(2) \cdot (\tilde{S}(2) - \tilde{S}(1)).$$

Subtracting $\varphi(2) \cdot \tilde{S}(2)$ on both sides gives $\varphi(2) \cdot \tilde{S}(1) = \varphi(1) \cdot \tilde{S}(1)$, which is (9) for $t = 1$. Proceeding similarly – or by induction – we can show $\varphi(t) \cdot \tilde{S}(t) = \varphi(t+1) \cdot \tilde{S}(t)$ for $t = 2, \ldots, T-1$ as required. ■

The model

We are allowed to borrow (so $\varphi_0(t)$ may be negative) and sell short (so $\varphi_i(t)$ may be negative for $i = 1, \ldots, d$). So it is hardly surprising that if we decide what to do about the risky assets and fix an initial endowment, the numéraire will take care of itself, in the following sense.

If $(\varphi_1(t), \ldots, \varphi_d(t))'$ is predictable and V_0 is $\mathcal{F}_0$-measurable, there is a unique predictable process $(\varphi_0(t))_{t=1}^T$ such that $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_d)'$ is self-financing with initial value of the corresponding portfolio $V_\varphi(0) = V_0$.

The model

Proof. If φ is self-financing, then

$$
\begin{aligned}
\tilde{V}_\varphi(t) \\
&= V_0 + \tilde{G}_\varphi(t) \\
&= V_0 + \sum_{\tau=1}^{t} (\varphi_1(\tau)\Delta\tilde{S}_1(\tau) + \ldots + \varphi_d(\tau)\Delta\tilde{S}_d(\tau)).
\end{aligned}
$$

The model

On the other hand,

$$
\begin{aligned}
& \tilde{V}_\varphi(t) \\
= \quad & \varphi(t)\cdot\tilde{S}(t) \\
= \quad & \varphi_0(t)+\varphi_1(t)\tilde{S}_1(t)+\ldots+\varphi_d(t)\tilde{S}_d(t).
\end{aligned}
$$

The model

Equate these:

$$
\begin{aligned}
& \varphi_0(t) \\
= \quad & V_0 + \sum_{\tau=1}^{t} (\varphi_1(\tau)\Delta\tilde{S}_1(\tau) + \ldots + \varphi_d(\tau)\Delta\tilde{S}_d(\tau)) \\
& -(\varphi_1(t)\tilde{S}_1(t) + \ldots + \varphi_d(t)\tilde{S}_d(t)),
\end{aligned}
$$

which defines $\varphi_0(t)$ uniquely.

The model

The terms in $\tilde{S}_i(t)$ are

$$\varphi_i(t)\Delta\tilde{S}_i(t) - \varphi_i(t)\tilde{S}_i(t) = -\varphi_i(t)\tilde{S}_i(t-1),$$

which is $\mathcal{F}_{t-1}$-measurable. So

$$\begin{aligned} & \varphi_0(t) \\ = \quad & V_0 + \sum_{\tau=1}^{t-1}(\varphi_1(\tau)\Delta\tilde{S}_1(\tau) + \ldots + \varphi_d(\tau)\Delta\tilde{S}_d(\tau)) \\ & -(\varphi_1(t)S_1(t-1) + \ldots + \varphi_d(t)\tilde{S}_d(t-1)), \end{aligned}$$

where as $\varphi_1, \ldots, \varphi_d$ are predictable, all terms on the right-hand side are $\mathcal{F}_{t-1}$-measurable, so φ_0 is predictable. ■

The model

The above has a further important consequence: for defining a gains process $\tilde{G}_\varphi$ only the components $(\varphi_1(t), \ldots, \varphi_d(t))'$ are needed. If we require them to be predictable they correspond in a unique way (after fixing initial endowment) to a self-financing trading strategy. Thus for the discounted world predictable strategies and final cash-flows generated by them are all that matters.

The model

We now turn to the modelling of derivative instruments in our current framework. This is done in the following fashion.

A contingent claim X with maturity date T is an arbitrary $\mathcal{F}_T = \mathcal{F}$-measurable random variable (which is by the finiteness of the probability space bounded). We denote the class of all contingent claims by $L^0 = L^0(\Omega, \mathcal{F}, I\!\!P)$.

The notation L^0 for contingent claims is motivated by the them being simply random variables in our context (and the functional-analytic spaces used later on).

The model

A typical example of a contingent claim X is an option on some underlying asset S, then (e.g. for the case of a European call option with maturity date T and strike K) we have a functional relation $X = f(S)$ with some function f (e.g. $X = (S(T) - K)^+$). The general definition allows for more complicated relationships which are captured by the $\mathcal{F}_T$-measurability of X (recall that $\mathcal{F}_T$ is typically generated by the process S).

The No-Arbitrage Condition

The central principle in the single period example was the absence of arbitrage opportunities, i.e. the absence investment strategies for making profits without exposure to risk. As mentioned there this principle is central for any market model, and we now define the mathematical counterpart of this economic principle in our current setting.

Let $\tilde{\Phi} \subset \Phi$ be a set of self-financing strategies. A strategy $\varphi \in \tilde{\Phi}$ is called an arbitrage opportunity or arbitrage strategy with respect to $\tilde{\Phi}$ if $I\!P\{V_\varphi(0) = 0\} = 1$, and the terminal wealth of φ satisfies

$$I\!P\{V_\varphi(T) \geq 0\} = 1 \text{ and } I\!P\{V_\varphi(T) > 0\} > 0.$$

The No-Arbitrage Condition

So an arbitrage opportunity is a self-financing strategy with zero initial value, which produces a non-negative final value with probability one and has a positive probability of a positive final value. Observe that arbitrage opportunities are always defined with respect to a certain class of trading strategies.

We say that a security market $\mathcal{M}$ is arbitrage-free if there are no arbitrage opportunities in the class Φ of trading strategies.

The No-Arbitrage Condition

We will allow ourselves to use 'no-arbitrage' in place of 'arbitrage-free' when convenient.

The fundamental insight in the single-period example was the equivalence of the no-arbitrage condition and the existence of risk-neutral probabilities. For the multi-period case we now use probabilistic machinery to establish the corresponding result.

A probability measure $I\!P^*$ on $(\Omega, \mathcal{F}_T)$ equivalent to $I\!P$ is called a *martingale measure* for $\tilde{S}$ if the process $\tilde{S}$ follows a $I\!P^*$-martingale with respect to the filtration $\mathbb{F}$. We denote by $\mathcal{P}(\tilde{S})$ the class of equivalent martingale measures.

The No-Arbitrage Condition

Let $I\!\!P^*$ be an equivalent martingale measure ($I\!\!P^* \in \mathcal{P}(\tilde{S})$) and $\varphi \in \Phi$ any self-financing strategy. Then the wealth process $\tilde{V}_\varphi(t)$ is a $I\!\!P^*$-martingale with respect to the filtration $\mathbb{F}$.

PROOF:

By the self-financing property of φ, (10), we have

$$\tilde{V}_\varphi(t) = V_\varphi(0) + \tilde{G}_\varphi(t) \quad (t = 0, 1, \ldots, T).$$

The No-Arbitrage Condition

So

$$\begin{aligned} & \tilde{V}_\varphi(t+1) - \tilde{V}_\varphi(t) \\ = \quad & \tilde{G}_\varphi(t+1) - \tilde{G}_\varphi(t) \\ = \quad & \varphi(t+1) \cdot (\tilde{S}(t+1) - \tilde{S}(t)). \end{aligned}$$

So for $\varphi \in \Phi$, $\tilde{V}_\varphi(t)$ is the martingale transform of the $I\!P^*$ martingale $\tilde{S}$ by φ and hence a $I\!P^*$ martingale itself. ■

Observe that in our setting all processes are bounded, i.e. the martingale transform theorem is applicable without further restrictions. The next result is the key for the further development.

The No-Arbitrage Condition

If an equivalent martingale measure exists - that is, if $\mathcal{P}(\tilde{S}) \neq \emptyset$ – then the market $\mathcal{M}$ is arbitrage-free.

PROOF:

Assume such a $I\!P^*$ exists. For any self-financing strategy φ, we have as before

$$\tilde{V}_\varphi(t) = V_\varphi(0) + \sum_{\tau=1}^{t} \varphi(\tau) \cdot \Delta\tilde{S}(\tau).$$

Now, $\tilde{S}(t)$ a (vector) $I\!P^*$-martingale implies $\tilde{V}_\varphi(t)$ is a P^*-martingale.

The No-Arbitrage Condition

So the initial and final $I\!P^*$-expectations are the same,

$$\mathbb{E}^*(\tilde{V}_\varphi(T)) = \mathbb{E}^*(\tilde{V}_\varphi(0)).$$

If the strategy is an arbitrage opportunity its initial value – the right-hand side above – is zero. Therefore the left-hand side $\mathbb{E}^*(\tilde{V}_\varphi(T))$ is zero, but $\tilde{V}_\varphi(T) \geq 0$ (by definition). Also each $I\!P^*(\{\omega\}) > 0$ (by assumption, each $I\!P(\{\omega\}) > 0$, so by equivalence each $I\!P^*(\{\omega\}) > 0$). This and $\tilde{V}_\varphi(T) \geq 0$ force $\tilde{V}_\varphi(T) = 0$. So no arbitrage is possible. ■

The No-Arbitrage Condition

If the market $\mathcal{M}$ is arbitrage-free, then the class $\mathcal{P}(\tilde{S})$ of equivalent martingale measures is non-empty.

For the proof (for which we follow (Schachermayer 2003) we need some auxiliary observations.

Recall the definition of arbitrage, in our finite-dimensional setting: a self-financing trading strategy $\varphi \in \Phi$ is an arbitrage opportunity if $V_\varphi(0) = 0$, $V_\varphi(T, \omega) \geq 0 \; \forall \omega \in \Omega$ and there exists a $\omega \in \Omega$ with $V_\varphi(T, \omega) > 0$.

The No-Arbitrage Condition

Now call $L^0 = L^0(\Omega, \mathcal{F}, I\!\!P)$ the set of random variables on $(\Omega, \mathcal{F})$ and $L^0_{++}(\Omega, \mathcal{F}, I\!\!P) := \{X \in L^0 : X(\omega) \geq 0 \; \forall \omega \in \Omega$ and $\exists \, \omega \in \Omega$ such that $X(\omega) > 0\}$. (Observe that L^0_{++} is a *cone* -closed under vector addition and multiplication by positive scalars.) Using L^0_{++} we can write the arbitrage condition more compactly as

$$V_\varphi(0) = \tilde{V}_\varphi(0) = 0$$

$$\Rightarrow \qquad \tilde{V}_\varphi(T) \notin L^0_{++}(\Omega, \mathcal{F}, I\!\!P)$$

for any self-financing strategy φ.

The No-Arbitrage Condition

The next lemma formulates the arbitrage condition in terms of discounted gains processes. The important advantage in using this setting (rather than a setting in terms of value processes) is that we only have to assume predictability of a vector process $(\varphi_1, \ldots, \varphi_d)$. Recall that we can choose a process φ_0 in such a way that the strategy $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_d)$ has zero initial value and is self-financing.

In an arbitrage-free market any predictable vector process $\varphi' = (\varphi_1, \ldots, \varphi_d)$ satisfies

$$\tilde{G}_{\varphi'}(T) \notin L^0_{++}(\Omega, \mathcal{F}, I\!\!P).$$

The No-Arbitrage Condition

PROOF:

There exists a unique predictable process $(\varphi_0(t))$ such that $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_d)$ has zero initial value and is self-financing. Assume $\tilde{G}_{\varphi'}(T) \in L^0_{++}(\Omega, \mathcal{F}, I\!P)$. Then,

$$\begin{aligned} V_\varphi(T) &= \beta(T)^{-1}\tilde{V}_\varphi(T) \\ &= \beta(T)^{-1}(V_\varphi(0) + \tilde{G}_\varphi(T)) \\ &= \beta(T)^{-1}\tilde{G}_{\varphi'}(T) \geq 0, \end{aligned}$$

and is positive somewhere (i.e. with positive probability) by definition of L^0_{++}. Hence φ is an arbitrage opportunity with respect to Φ. This contradicts the assumption that the market is arbitrage-free.

The No-Arbitrage Condition

We now define the space of contingent claims, i.e. random variables on $(\Omega, \mathcal{F})$, which an economic agent may replicate with zero initial investment by pursuing some predictable trading strategy φ.

We call the subspace K of $L^0 = L^0(\Omega, \mathcal{F}, I\!\!P)$ defined by

$$K = \{X \in L^0 : X = \tilde{G}_\varphi(T),\ \varphi \text{ predictable}\}$$

the set of contingent claims attainable at price 0.

The No-Arbitrage Condition

A market is arbitrage-free if and only if

$$K \cap L^0_{++}(\Omega, \mathcal{F}, I\!P) = \emptyset. \tag{11}$$

Proof Since our market model is finite we can use results from Euclidean geometry, in particular we can identify L^0 with $I\!R^{|\Omega|}$). By assumption we have (11), i.e. K and L^0_{++} do not intersect. So K does not meet the subset

$$D := \{X \in L^0_{++} : \sum_{\omega \in \Omega} X(\omega) = 1\}.$$

The No-Arbitrage Condition

Now D is a compact convex set. By the separating hyperplane theorem, there is a vector $\lambda = (\lambda(\omega) : \omega \in \Omega)$ such that for all $X \in D$

$$\lambda \cdot X := \sum_{\omega \in \Omega} \lambda(\omega) X(\omega) > 0, \tag{12}$$

but for all $\tilde{G}_\varphi(T)$ in K,

$$\lambda \cdot \tilde{G}_\varphi(T) = \sum_{\omega \in \Omega} \lambda(\omega) \tilde{G}_\varphi(T)(\omega) = 0. \tag{13}$$

The No-Arbitrage Condition

Choosing each $\omega \in \Omega$ successively and taking X to be 1 on this ω and zero elsewhere, (12) tells us that each $\lambda(\omega) > 0$. So

$$I\!\!P^*(\{\omega\}) := \frac{\lambda(\omega)}{\sum_{\omega' \in \Omega} \lambda(\omega')}$$

defines a probability measure equivalent to $I\!\!P$ (no non-empty null sets). With $\mathbb{E}^*$ as $I\!\!P^*$-expectation, (13) says that

$$\mathbb{E}^*\left(\tilde{G}_\varphi(T)\right) = 0,$$

i.e.

$$\mathbb{E}^*\left(\sum_{\tau=1}^{T} \varphi(\tau) \cdot \Delta \tilde{S}(\tau)\right) = 0.$$

The No-Arbitrage Condition

In particular, choosing for each i to hold only stock i,

$$\mathbb{E}^* \left(\sum_{\tau=1}^{T} \varphi_i(\tau) \Delta \tilde{S}_i(\tau) \right) = 0 \quad (i = 1, \ldots, d).$$

Since this holds for any predictable φ (boundedness holds automatically as Ω is finite), the martingale transform lemma tells us that the discounted price processes $(\tilde{S}_i(t))$ are $I\!P^*$-martingales. ∎

The No-Arbitrage Condition

No-Arbitrage Theorem The market $\mathcal{M}$ is arbitrage-free if and only if there exists a probability measure $I\!P^*$ equivalent to $I\!P$ under which the discounted d-dimensional asset price process $\tilde{S}$ is a $I\!P^*$-martingale.

Risk-Neutral Pricing

We now turn to the main underlying question of this text, namely the pricing of contingent claims (i.e. financial derivatives). As in the one-period setting the basic idea is to reproduce the cash flow of a contingent claim in terms of a portfolio of the underlying assets. On the other hand, the equivalence of the no-arbitrage condition and the existence of risk-neutral probability measures imply the possibility of using risk-neutral measures for pricing purposes. We will explore the relation of these tow approaches in this subsection.

Risk-Neutral Pricing

We say that a contingent claim is *attainable* if there exists a *replicating strategy* $\varphi \in \Phi$ such that

$$V_{\varphi}(T) = X.$$

So the replicating strategy generates the same time T cash-flow as does X. Working with discounted values (recall we use β as the discount factor) we find

$$\beta(T)X = \tilde{V}_{\varphi}(T) = V(0) + \tilde{G}_{\varphi}(T). \tag{14}$$

So the discounted value of a contingent claim is given by the initial cost of setting up a replication strategy and the gains from trading.

Risk-Neutral Pricing

In a highly efficient security market we expect that the law of one price holds true, that is for a specified cash-flow there exists only one price at any time instant. Otherwise arbitrageurs would use the opportunity to cash in a riskless profit. So the no-arbitrage condition implies that for an attainable contingent claim its time t price must be given by the value (inital cost) of any replicating strategy (we say the claim is uniquely replicated in that case). This is the basic idea of the *arbitrage pricing theory*.

Risk-Neutral Pricing

Suppose the market $\mathcal{M}$ is arbitrage-free. Then any attainable contingent claim X is uniquely replicated in $\mathcal{M}$.

Proof. Suppose there is an attainable contingent claim X and strategies φ and ψ such that

$$V_\varphi(T) = V_\psi(T) = X,$$

but there exists a $\tau < T$ such that

$$V_\varphi(u) = V_\psi(u)$$

for every $u < \tau$ and

$$V_\varphi(\tau) \neq V_\psi(\tau).$$

Risk-Neutral Pricing

Define $A := \{\omega \in \Omega : V_\varphi(\tau, \omega) > V_\psi(\tau, \omega)\}$, then $A \in \mathcal{F}_\tau$ and $I\!\!P(A) > 0$ (otherwise just rename the strategies). Define the $\mathcal{F}_\tau$-measurable random variable $Y := V_\varphi(\tau) - V_\psi(\tau)$ and consider the trading strategy ξ defined by

$$\xi(u) = \varphi(u) - \psi(u), u \leq \tau$$

and

$$\xi(u) = \mathbf{1}_{A^c}(\varphi(u) - \psi(u)) + \mathbf{1}_A(Y\beta(\tau), 0, \ldots, 0),$$

for $\tau < u \leq T$. The idea here is to use φ and ψ to construct a self-financing strategy with zero initial investment (hence use their difference ξ) and put any gains at time τ in the savings account (i.e. invest them riskfree) up to time T.

Risk-Neutral Pricing

We need to show formally that ξ satisfies the conditions of an arbitrage opportunity. By construction ξ is predictable and the self-financing condition (9) is clearly true for $t \neq \tau$, and for $t = \tau$ we have using that $\varphi, \psi \in \Phi$

$$\begin{aligned} \xi(\tau) \cdot S(\tau) &= (\varphi(\tau) - \psi(\tau)) \cdot S(\tau) \\ &= V_\varphi(\tau) - V_\psi(\tau) \end{aligned}$$

Risk-Neutral Pricing

and

$$\begin{aligned}
& \xi(\tau+1)\cdot S(\tau) \\
= \quad & \mathbf{1}_{A^c}(\varphi(\tau+1)-\psi(\tau+1))\cdot S(\tau) \\
& +\mathbf{1}_A Y\beta(\tau)S_0(\tau) \\
= \quad & \mathbf{1}_{A^c}(\varphi(\tau)-\psi(\tau))\cdot S(\tau) \\
& +\mathbf{1}_A(V_\varphi(\tau)-V_\psi(\tau))\beta(\tau)\beta^{-1}(\tau) \\
= \quad & V_\varphi(\tau)-V_\psi(\tau).
\end{aligned}$$

Risk-Neutral Pricing

Hence ξ is a self-financing strategy with initial value equal to zero. Furthermore

$$\begin{aligned} V_\xi(T) &= \mathbf{1}_{A^c}(\varphi(T)-\psi(T))\cdot S(T) \\ &\quad +\mathbf{1}_A(Y\beta(\tau),0,\ldots,0)\cdot S(T) \\ &= \mathbf{1}_A Y\beta(\tau)S_0(T)\geq 0 \end{aligned}$$

and

$$I\!P\{V_\xi(T)>0\}=I\!P\{A\}>0.$$

Hence the market contains an arbitrage opportunity with respect to the class Φ of self-financing strategies. But this contradicts the assumption that the market $\mathcal{M}$ is arbitrage-free.

Risk-Neutral Pricing

This uniqueness property allows us now to define the important concept of an arbitrage price process.

Suppose the market is arbitrage-free. Let X be any attainable contingent claim with time T maturity. Then the arbitrage price process $\pi_X(t),\ 0 \leq t \leq T$ or simply arbitrage price of X is given by the value process of any replicating strategy φ for X.

Risk-Neutral Pricing

The construction of hedging strategies that replicate the outcome of a contingent claim (for example a European option) is an important problem in both practical and theoretical applications. Hedging is central to the theory of option pricing. The classical arbitrage valuation models, such as the Black-Scholes model ((Black and Scholes 1973), depend on the idea that an option can be perfectly hedged using the underlying asset (in our case the assets of the market model), so making it possible to create a portfolio that replicates the option exactly. Hedging is also widely used to reduce risk, and the kinds of delta-hedging strategies implicit in the Black-Scholes model are used by participants in option markets. We will come back to hedging problems subsequently.

Risk-Neutral Pricing

Analysing the arbitrage-pricing approach we observe that the derivation of the price of a contingent claim doesn't require any specific preferences of the agents other than nonsatiation, i.e. agents prefer more to less, which rules out arbitrage. So, the pricing formula for any attainable contingent claim must be independent of all preferences that do not admit arbitrage. In particular, an economy of risk-neutral investors must price a contingent claim in the same manner. This fundamental insight, due to Cox and Ross (Cox and Ross 1976) and to Harrison and Kreps (Harrison and Kreps 1979), simplifies the pricing formula enormously. In its general form the price of an attainable simple contingent claim is just the expected value of the discounted payoff with respect to an equivalent martingale measure.

Risk-Neutral Pricing

The arbitrage price process of any attainable contingent claim X is given by the **risk-neutral valuation formula**

$$\pi_X(t) = \beta(t)^{-1}\mathbb{E}^*\left(X\beta(T)|\mathcal{F}_t\right) \tag{15}$$

where $\mathbb{E}^*$ is the expectation operator with respect to an equivalent martingale measure $I\!P^*$.

Risk-Neutral Pricing

Proof Since we assume the the market is arbitrage-free there exists (at least) an equivalent martingale measure $I\!P^*$. Also the discounted value process $\tilde{V}_\varphi$ of any self-financing strategy φ is a $I\!P^*$-martingale. So for any contingent claim X with maturity T and any replicating trading strategy $\varphi \in \Phi$ we have for each $t = 0, 1, \ldots, T$

$$
\begin{aligned}
\pi_X(t) &= V_\varphi(t) = \beta(t)^{-1}\tilde{V}_\varphi(t) \\
&= \beta(t)^{-1}\mathbb{E}^*(\tilde{V}_\varphi(T)|\mathcal{F}_t) \\
&= \beta(t)^{-1}\mathbb{E}^*(\beta(T)V_\varphi(T)|\mathcal{F}_t) \\
&= \beta(t)^{-1}\mathbb{E}^*(\beta(T)X|\mathcal{F}_t),
\end{aligned}
$$

use $\tilde{V}_\varphi(t)$ is a martingale and φ is replicating X.

Complete Markets

The last section made clear that attainable contingent claims can be priced using an equivalent martingale measure. In this section we will discuss the question of the circumstances under which all contingent claims are attainable. This would be a very desirable property of the market $\mathcal{M}$, because we would then have solved the pricing question (at least for contingent claims) completely. Since contingent claims are merely $\mathcal{F}_T$-measurable random variables in our setting, it should be no surprise that we can give a criterion in terms of probability measures. We start with:

Complete Markets

A market $\mathcal{M}$ is *complete* if every contingent claim is attainable, i.e. for every $\mathcal{F}_T$-measurable random variable $X \in L^0$ there exists a *replicating* self-financing strategy $\varphi \in \Phi$ such that $V_\varphi(T) = X$.

In the case of an arbitrage-free market $\mathcal{M}$ one can even insist on replicating nonnegative contingent claims by an admissible strategy $\varphi \in \Phi_a$. Indeed, if φ is self-financing and $I\!P^*$ is an equivalent martingale measure under which discounted prices $\tilde{S}$ are $I\!P^*$-martingales (such $I\!P^*$ exist since $\mathcal{M}$ is arbitrage-free and we can hence use the no-arbitrage theorem, $\tilde{V}_\varphi(t)$ is also a $I\!P^*$-martingale, being the martingale transform of the martingale $\tilde{S}$ by φ. So

$$\tilde{V}_\varphi(t) = E^*(\tilde{V}_\varphi(T)|\mathcal{F}_t) \quad (t = 0, 1, \ldots, T).$$

Complete Markets

If φ replicates X, $V_\varphi(T) = X \geq 0$, so discounting, $\tilde{V}_\varphi(T) \geq 0$, so the above equation gives $\tilde{V}_\varphi(t) \geq 0$ for each t. Thus all the values at each time t are non-negative – not just the final value at time T – so φ is admissible.

Completeness Theorem An arbitrage-free market $\mathcal{M}$ is complete if and only if there exists a unique probability measure $I\!P^*$ equivalent to $I\!P$ under which discounted asset prices are martingales.

Complete Markets

Proof. '$\Rightarrow$': Assume that the arbitrage-free market $\mathcal{M}$ is complete. Then for any $\mathcal{F}_T$-measurable random variable X (contingent claim), there exists an admissible (so self-financing) strategy φ replicating X: $X = V_\varphi(T)$. As φ is self-financing,

$$\beta(T)X = \tilde{V}_\varphi(T) = V_\varphi(0) + \sum_{\tau=1}^{T} \varphi(\tau) \cdot \Delta\tilde{S}(\tau).$$

Complete Markets

We know by the no-arbitrage theorem that an equivalent martingale measure $I\!P^*$ exists; we have to prove uniqueness. So, let $I\!P_1, I\!P_2$ be two such equivalent martingale measures. For $i = 1, 2$, $(\tilde{V}_\varphi(t))_{t=0}^T$ is a $I\!P_i$-martingale. So,

$$\mathbb{E}_i(\tilde{V}_\varphi(T)) = \mathbb{E}_i(\tilde{V}_\varphi(0)) = V_\varphi(0),$$

since the value at time zero is non-random $(\mathcal{F}_0 = \{\emptyset, \Omega\})$ and $\beta(0) = 1$. So

$$\mathbb{E}_1(\beta(T)X) = \mathbb{E}_2(\beta(T)X).$$

Since X is arbitrary, $\mathbb{E}_1, \mathbb{E}_2$ have to agree on integrating all integrands. Now $\mathbb{E}_i$ is expectation (i.e. integration) with respect to the measure $I\!P_i$, and measures that agree on integrating all integrands must coincide. So $I\!P_1 = I\!P_2$, giving uniqueness as required.

Complete Markets

'$\Leftarrow$': Assume that the arbitrage-free market $\mathcal{M}$ is incomplete: then there exists a non-attainable $\mathcal{F}_T$-measurable random variable X (a contingent claim). We may confine attention to the risky assets $S_1, \ldots, S_d$, as these suffice to tell us how to handle the numéraire S_0.

Consider the following set of random variables:

$$\tilde{K} := \left\{ Y \in L^0 : Y = Y_0 + \sum_{t=1}^{T} \varphi(t) \cdot \Delta\tilde{S}(t), \right.$$

$$\left. Y_0 \in I\!R\,,\ \varphi \ \text{ predictable} \right\}.$$

(Recall that Y_0 is $\mathcal{F}_0$-measurable and set $\varphi = ((\varphi_1(t), \ldots, \varphi_d(t))')_{t=1}^{T}$ with predictable components.)

Complete Markets

Then by the above reasoning, the discounted value $\beta(T)X$ does not belong to $\tilde{K}$, so $\tilde{K}$ is a *proper* subset of the set L^0 of all random variables on Ω (which may be identified with $\mathbb{R}^{|\Omega|}$). Let $\mathbb{P}^*$ be a probability measure equivalent to $\mathbb{P}$ under which discounted prices are martingales (such $\mathbb{P}^*$ exist by the no-arbitrage theorem. Define the scalar product

$$(Z,Y) \to \mathbb{E}^*(ZY)$$

on random variables on Ω. Since $\tilde{K}$ is a proper subset, there exists a non-zero random variable Z orthogonal to $\tilde{K}$ (since Ω is finite, $\mathbb{R}^{|\Omega|}$ is Euclidean: this is just Euclidean geometry).

Complete Markets

That is,

$$\mathbb{E}^*(ZY) = 0, \quad \forall\, Y \in \tilde{K}.$$

Choosing the special $Y = 1 \in \tilde{K}$ given by
$\varphi_i(t) = 0, t = 1, 2, \ldots, T;\ i = 1, \ldots, d$ and $Y_0 = 1$ we find

$$\mathbb{E}^*(Z) = 0.$$

Write $\|X\|_\infty := \sup\{|X(\omega)| : \omega \in \Omega\}$, and define $I\!P^{**}$ by

$$I\!P^{**}(\{\omega\}) = \left(1 + \frac{Z(\omega)}{2\,\|Z\|_\infty}\right) I\!P^*(\{\omega\}).$$

By construction, $I\!P^{**}$ is equivalent to $I\!P^*$ (same null sets - actually, as $I\!P^* \sim I\!P$ and $I\!P$ has no non-empty null sets, neither do $I\!P^*, I\!P^{**}$). From $\mathbb{E}^*(Z) = 0$, we see that $\sum I\!P^{**}(\omega) = 1$, i.e. is a probability measure. As Z is non-zero, $I\!P^{**}$ and $I\!P^*$ are *different*.

Complete Markets

Now

$$
\begin{aligned}
&\mathbb{E}^{**}\left(\sum_{t=1}^{T}\varphi(t)\cdot\Delta\tilde{S}(t)\right)\\
=\;&\sum_{\omega\in\Omega}I\!P^{**}(\omega)\left(\sum_{t=1}^{T}\varphi(t,\omega)\cdot\Delta\tilde{S}(t,\omega)\right)\\
=\;&\sum_{\omega\in\Omega}\left(1+\frac{Z(\omega)}{2\left\|Z\right\|_{\infty}}\right)I\!P^{*}(\omega)\left(\sum_{t=1}^{T}\varphi(t,\omega)\cdot\Delta\tilde{S}(t,\omega)\right).
\end{aligned}
$$

The '1' term on the right gives

$$
\mathbb{E}^{*}\left(\sum_{t=1}^{T}\varphi(t)\cdot\Delta\tilde{S}(t)\right),
$$

which is zero since this is a martingale transform of the $\tilde{S}(t)$.

Complete Markets

The 'Z' term gives a multiple of the inner product

$$(Z, \sum_{t=1}^{T} \varphi(t) \cdot \Delta\tilde{S}(t)),$$

which is zero as Z is orthogonal to $\tilde{K}$ and $\sum_{t=1}^{T} \varphi(t) \cdot \Delta\tilde{S}(t) \in \tilde{K}$. By the martingale transform lemma, $\tilde{S}(t)$ is a $I\!P^{**}$-martingale since φ is an arbitrary predictable process. Thus $I\!P^{**}$ is a second equivalent martingale measure, different from $I\!P^{*}$. So incompleteness implies non-uniqueness of equivalent martingale measures, as required. ∎

Financial Mathematics

Lecture 5

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Aims and Objectives

- Cox-Ross-Rubinstein model §4.5;
- Binomial Approximation §4.6.

The Cox-Ross-Rubinstein Model

We take $d = 1$, that is, our model consists of two basic securities. Recall that the essence of the relative pricing theory is to take the price processes of these basic securities as given and price secondary securities in such a way that no arbitrage is possible.

Our time horizon is T and the set of dates in our financial market model is $t = 0, 1, \ldots, T$. Assume that the first of our given basic securities is a (riskless) bond or bank account B, which yields a riskless rate of return $r > 0$ in each time interval $[t, t + 1]$, i.e.

$$B(t + 1) = (1 + r)B(t), \quad B(0) = 1.$$

So its price process is

$$B(t) = (1 + r)^t, \quad t = 0, 1, \ldots, T.$$

The CRR Model

Furthermore, we have a risky asset (stock) S with price process

$$S(t+1) = \begin{cases} (1+u)S(t) & \text{with prob} \quad p, \\ (1+d)S(t) & \text{with prob} \quad 1-p, \end{cases}$$

with $-1 < d < u, S_0 \in I\!R_0^+$ and for $t = 0, 1, \ldots, T-1$.

The CRR Model

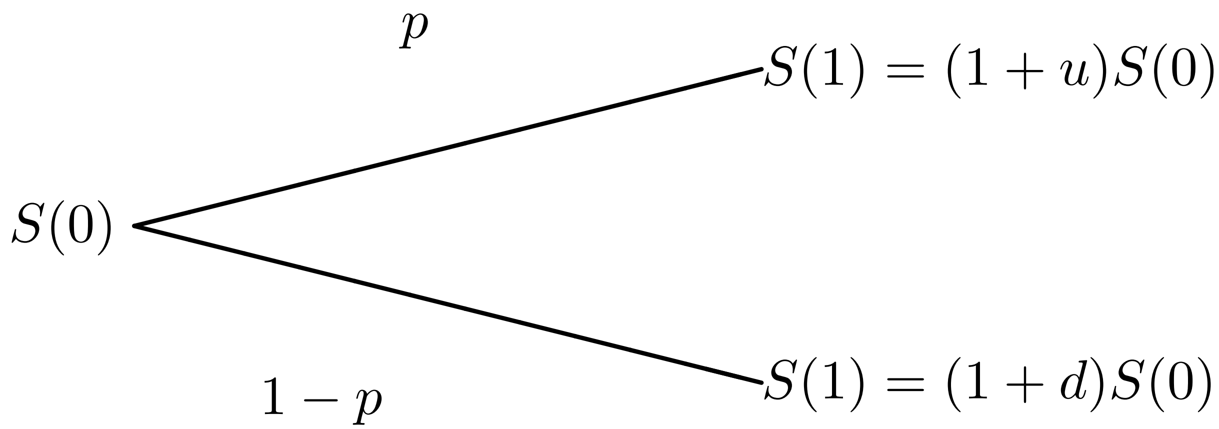

Figure 2: One-step tree diagram

Alternatively we write this as

$$Z(t+1) := \frac{S(t+1)}{S(t)} - 1, \quad t = 0, 1, \ldots, T-1.$$

The CRR Model

We set up a probabilistic model by considering the $Z(t),\ t = 1, \ldots, T$ as random variables defined on probability spaces $(\tilde{\Omega}_t, \tilde{\mathcal{F}}_t, \tilde{I\!P}_t)$ with

$$\begin{aligned} \tilde{\Omega}_t &= \tilde{\Omega} &&= \{d, u\}, \\ \tilde{\mathcal{F}}_t &= \tilde{\mathcal{F}} &&= \mathcal{P}(\tilde{\Omega}) = \{\emptyset, \{d\}, \{u\}, \tilde{\Omega}\}, \\ \tilde{I\!P}_t &= \tilde{I\!P} \end{aligned}$$

with $\tilde{I\!P}(\{u\}) = p,\ \tilde{I\!P}(\{d\}) = 1 - p,\ p \in (0, 1)$. On these probability spaces we define

$$Z(t, u) = u \quad \text{and} \quad Z(t, d) = d, \quad t = 1, 2, \ldots, T.$$

The CRR Model

Our aim, of course, is to define a probability space on which we can model the basic securities (B, S). Since we can write the stock price as

$$S(t) = S(0) \prod_{\tau=1}^{t} (1 + Z(\tau)), \quad t = 1, 2, \ldots, T,$$

the above definitions suggest using as the underlying probabilistic model of the financial market the *product space* $(\Omega, \mathcal{F}, I\!\!P)$ (see e.g. (Williams 1991) ch. 8), i.e.

$$\Omega = \tilde{\Omega}_1 \times \ldots \times \tilde{\Omega}_T = \tilde{\Omega}^T = \{d, u\}^T,$$

with each $\omega \in \Omega$ representing the successive values of $Z(t),\ t = 1, 2, \ldots, T$.

The CRR Model

Hence each $\omega \in \Omega$ is a T-tuple $\omega = (\tilde{\omega}_1, \ldots, \tilde{\omega}_T)$ and $\tilde{\omega}_t \in \tilde{\Omega} = \{d, u\}$. For the σ-algebra we use $\mathcal{F} = \mathcal{P}(\Omega)$ and the probability measure is given by

$$\begin{aligned} I\!P(\{\omega\}) &= \tilde{I\!P}_1(\{\omega_1\}) \times \ldots \times \tilde{I\!P}_T(\{\omega_T\}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ &= \tilde{I\!P}(\{\omega_1\}) \times \ldots \times \tilde{I\!P}(\{\omega_T\}) \end{aligned}$$

The role of a product space is to model independent replication of a random experiment. The $Z(t)$ above are two-valued random variables, so can be thought of as tosses of a biased coin; we need to build a probability space on which we can model a succession of such independent tosses.

The CRR Model

Now we redefine (with a slight abuse of notation) the $Z(t),\ t = 1, \ldots, T$ as random variables on $(\Omega, \mathcal{F}, I\!P)$ as (the tth projection)

$$Z(t, \omega) = Z(t, \omega_t).$$

Observe that by this definition (and the above construction) $Z(1), \ldots, Z(T)$ are independent and identically distributed with

$$I\!P(Z(t) = u) = p = 1 - I\!P(Z(t) = d).$$

To model the flow of information in the market we use the obvious filtration with

$$\begin{aligned} \mathcal{F}_0 &= \{\emptyset, \Omega\} \\ \mathcal{F}_t &= \sigma(Z(1), \ldots, Z(t)) = \sigma(S(1), \ldots, S(t)), \\ \mathcal{F}_T &= \mathcal{F} = \mathcal{P}(\Omega). \end{aligned}$$

The CRR Model

This construction emphasises again that a multi-period model can be viewed as a sequence of single-period models. Indeed, in the Cox-Ross-Rubinstein case we use identical and independent single-period models. As we will see in the sequel this will make the construction of equivalent martingale measures relatively easy. Unfortunately we can hardly defend the assumption of independent and identically distributed price movements at each time period in practical applications.

The CRR Model

We now turn to the pricing of derivative assets in the Cox-Ross-Rubinstein market model. To do so we first have to discuss whether the Cox-Ross-Rubinstein model is arbitrage-free and complete.

To answer these questions we have, according to our fundamental theorems, to understand the structure of equivalent martingale measures in the Cox-Ross-Rubinstein model. In trying to do this we use (as is quite natural and customary) the bond price process $B(t)$ as numéraire.

The CRR Model

Our first task is to find an equivalent martingale measure $\mathbb{Q}$ such that the $Z(1), \ldots, Z(T)$ remain independent and identically distributed, i.e. a probability measure $\mathbb{Q}$ defined as a product measure via a measure $\tilde{\mathbb{Q}}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ such that $\tilde{\mathbb{Q}}(\{u\}) = q$ and $\tilde{\mathbb{Q}}(\{d\}) = 1 - q$. We have:

The CRR Model

(i) A martingale measure $\mathbb{Q}$ for the discounted stock price $\tilde{S}$ exists if and only if

$$d < r < u. \tag{16}$$

(ii) If equation (16) holds true, then there is a unique such measure in $\mathcal{P}$ characterised by

$$q = \frac{r-d}{u-d}. \tag{17}$$

The CRR Model

Proof Since $S(t) = \tilde{S}(t)B(t) = \tilde{S}(t)(1+r)^t$, we have $Z(t+1) = S(t+1)/S(t) - 1 = (\tilde{S}(t+1)/\tilde{S}(t))(1+r) - 1$. So, the discounted price $(\tilde{S}(t))$ is a $\mathbb{Q}$-martingale if and only if for all t

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}}[\tilde{S}(t+1)|\mathcal{F}_t] = \tilde{S}(t) \\ \Leftrightarrow \quad & \mathbb{E}^{\mathbb{Q}}[(\tilde{S}(t+1)/\tilde{S}(t))|\mathcal{F}_t] = 1 \\ \Leftrightarrow \quad & \mathbb{E}^{\mathbb{Q}}[Z(t+1)|\mathcal{F}_t] = r. \end{aligned}$$

But $Z(1), \ldots, Z(T)$ are mutually independent and hence $Z(t+1)$ is independent of $\mathcal{F}_t = \sigma(Z(1), \ldots, Z(t))$. So

$$r = \mathbb{E}^{\mathbb{Q}}(Z(t+1)|\mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(Z(t+1)) = uq + d(1-q)$$

is a weighted average of u and d; this can be r if and only if $r \in [d, u]$. As $\mathbb{Q}$ is to be *equivalent* to $I\!P$ and $I\!P$ has no non-empty null sets, $r = d, u$ are excluded and (16) is proved.

The CRR Model

To prove uniqueness and to find the value of q we simply observe that under (16)

$$u \times q + d \times (1-q) = r$$

has a unique solution. Solving it for q leads to the above formula. ■

From now on we assume that (16) holds true. Using the above we immediately get:

The Cox-Ross-Rubinstein model is arbitrage-free.

The Cox-Ross-Rubinstein model is complete.

The CRR Model

One can translate this result – on uniqueness of the equivalent martingale measure – into financial language. Completeness means that all contingent claims can be replicated. If we do this in the large, we can do it in the small by restriction, and conversely, we can build up our full model from its constituent components. To summarize:

The multi-period model is complete if and only if every underlying single-period model is complete.

The CRR Model

We can now use the risk-neutral valuation formula to price *every* contingent claim in the Cox-Ross-Rubinstein model.

The arbitrage price process of a contingent claim X in the Cox-Ross-Rubinstein model is given by

$$\pi_X(t) = B(t)\mathbb{E}^*\left(X/B(T)|\mathcal{F}_t\right) \quad \forall t = 0, 1, \ldots, T,$$

where $\mathbb{E}^*$ is the expectation operator with respect to the unique equivalent martingale measure $I\!P^*$ characterised by $p^* = (r-d)/(u-d)$.

The CRR Model

We now give simple formulas for pricing (and hedging) of European contingent claims $X = f(S_T)$ for suitable functions f (in this simple framework all functions $f : I\!R \to I\!R$). We use the notation

$$\begin{aligned} & F_\tau(x,p) \\ := \quad & \sum_{j=0}^{\tau} \binom{\tau}{j} p^j (1-p)^{\tau-j} f\left(x(1+u)^j(1+d)^{\tau-j}\right) \end{aligned} \tag{18}$$

Observe that this is just an evaluation of $f(S(j))$ along the probability-weighted paths of the price process. Accordingly, j, $\tau - j$ are the numbers of times $Z(i)$ takes the two possible values d, u.

The CRR Model

Consider a European contigent claim with expiry T given by $X = f(S_T)$. The arbitrage price process $\pi_X(t),\ t = 0, 1, \ldots, T$ of the contingent claim is given by (set $\tau = T - t$)

$$\pi_X(t) = (1+r)^{-\tau} F_\tau(S_t, p^*). \tag{19}$$

Proof Recall that

$$S(t) = S(0) \prod_{j=1}^{t} (1 + Z(j)), \quad t = 1, 2, \ldots, T.$$

The CRR Model

By the risk-neutral valuation principle the price $\pi_X(t)$ of a contingent claim $X = f(S_T)$ at time t is

$$\begin{aligned}
&\pi_X(t) \\
= \quad & (1+r)^{-(T-t)}\mathbb{E}^*[f(S(T))|\mathcal{F}_t] \\
= \quad & (1+r)^{-(T-t)}\mathbb{E}^*\left[f\left(S(t) \prod_{i=t+1}^{T} (1+Z(i)) \right) \middle| \mathcal{F}_t \right] \\
= \quad & (1+r)^{-(T-t)}\mathbb{E}^*\left[f\left(S(t) \prod_{i=t+1}^{T} (1+Z(i)) \right) \right] \\
= \quad & (1+r)^{-\tau} F_\tau(S(t), p^*).
\end{aligned}$$

The CRR Model

We used the role of independence property of conditional expectations in the next-to-last equality. It is applicable since $S(t)$ is $\mathcal{F}_t$-measurable and $Z(t+1), \ldots, Z(T)$ are independent of $\mathcal{F}_t$. ■

An immediate consequence is the pricing formula for the European call option, i.e. $X = f(S_T)$ with $f(x) = (x - K)^+$.

The CRR Model

Consider a European call option with expiry T and strike price K written on (one share of) the stock S. The arbitrage price process $\Pi_C(t),\ t = 0, 1, \ldots, T$ of the option is given by (set $\tau = T - t$)

$$
\begin{aligned}
& \Pi_C(t) \\
= \quad & (1+r)^{-\tau} \sum_{j=0}^{\tau} \binom{\tau}{j} p^{*j} (1-p^*)^{\tau-j} \\
& (S(t)(1+u)^j (1+d)^{\tau-j} - K)^+ .
\end{aligned} \tag{20}
$$

For a European put option, we can either argue similarly or use put-call parity.

Binomial Approximations

Suppose we observe financial assets during a continuous time period $[0, T]$. To construct a stochastic model of the price processes of these assets (to, e.g. value contingent claims) one basically has two choices: one could model the processes as continuous-time stochastic processes (for which the theory of stochastic calculus is needed) or one could construct a sequence of discrete-time models in which the continuous-time price processes are approximated by discrete-time stochastic processes in a suitable sense. We describe the the second approach now by examining the asymptotic properties of a sequence of Cox-Ross-Rubinstein models.

Binomial Approximations

We assume that all random variables subsequently introduced are defined on a suitable probability space $(\Omega, \mathcal{F}, I\!\!P)$. We want to model two assets, a riskless bond B and a risky stock S, which we now observe in a continuous-time interval $[0, T]$. To transfer the continuous-time framework into a binomial structure we make the following adjustments.

Binomial Approximations

Looking at the nth Cox-Ross-Rubinstein model in our sequence, there is a prespecified number k_n of trading dates. We set $\Delta_n = T/k_n$ and divide $[0, T]$ in k_n subintervals of length Δ_n, namely $I_j = [j\Delta_n, (j+1)\Delta_n],\ j = 0, \ldots, k_n - 1$. We suppose that trading occurs only at the equidistant time points $t_{n,j} = j\Delta_n,\ j = 0, \ldots, k_n - 1$. We fix r_n as the riskless interest rate over each interval I_j, and hence the bond process (in the nth model) is given by

$$B(t_{n,j}) = (1 + r_n)^j, \quad j = 0, \ldots, k_n.$$

Binomial Approximations

In the continuous-time model we compound continuously with spot rate $r \geq 0$ and hence the bond price process $B(t)$ is given by $B(t) = e^{rt}$. In order to approximate this process in the discrete-time framework, we choose r_n such that

$$1 + r_n = e^{r\Delta_n}. \tag{21}$$

With this choice we have for any $j = 0, \dots, k_n$ that $(1 + r_n)^j = \exp(rj\Delta_n) = \exp(rt_{n,j})$. Thus we have approximated the bond process exactly at the time points of the discrete model.

Binomial Approximations

Next we model the one-period returns $S(t_{n,j+1})/S(t_{n,j})$ of the stock by a family of random variables $Z_{n,i}; i = 1, \ldots, k_n$ taking values $\{d_n, u_n\}$ with

$$I\!P(Z_{n,i} = u_n) = p_n = 1 - I\!P(Z_{n,i} = d_n)$$

for some $p_n \in (0,1)$ (which we specify later). With these $Z_{n,j}$ we model the stock price process S_n in the nth Cox-Ross-Rubinstein model as

$$S_n(t_{n,j}) = S_n(0) \prod_{i=1}^{j} (1 + Z_{n,i}), \quad j = 0, 1, \ldots, k_n.$$

Binomial Approximations

With the specification of the one-period returns we get a complete description of the discrete dynamics of the stock price process in each Cox-Ross-Rubinstein model. We call such a finite sequence $Z_n = (Z_{n,i})_{i=1}^{k_n}$ a *lattice* or *tree*. The parameters u_n, d_n, p_n, k_n differ from lattice to lattice, but remain constant throughout a specific lattice. In the triangular array $(Z_{n,i}),\ i = 1, \ldots, k_n;\ n = 1, 2, \ldots$ we assume that the random variables are row-wise independent (but we allow dependence between rows). The approximation of a continuous-time setting by a sequence of lattices is called the lattice approach.

Binomial Approximations

It is important to stress that for each n we get a different discrete stock price process $S_n(t)$ and that in general these processes do not coincide on common time points (and are also different from the price process $S(t)$).

Turning back to a specific Cox-Ross-Rubinstein model, we now have a discrete-time bond and stock price process. We want arbitrage-free financial market models and therefore have to choose the parameters u_n, d_n, p_n accordingly. An arbitrage-free financial market model is guaranteed by the existence of an equivalent martingale measure, and the (necessary and) sufficient condition for that is

$$d_n < r_n < u_n.$$

Binomial Approximations

The risk-neutrality approach implies that the expected (under an equivalent martingale measure) one-period return must equal the one-period return of the riskless bond and hence we get

$$p_n^* = \frac{r_n - d_n}{u_n - d_n}. \tag{22}$$

So the only parameters to choose freely in the model are u_n and d_n. In the next sections we consider some special choices.

Binomial Approximations

We now choose the parameters in the above lattice approach in a special way. Assuming the risk-free rate of interest r as given, we have by (21) $1 + r_n = e^{r\Delta_n}$, and the remaining degrees of freedom are resolved by choosing u_n and d_n. We use the following choice:

$$1 + u_n = e^{\sigma\sqrt{\Delta_n}},$$

and

$$1 + d_n = (1 + u_n)^{-1} = e^{-\sigma\sqrt{\Delta_n}}.$$

Binomial Approximations

By condition (22) the risk-neutral probabilities for the corresponding single period models are given by

$$p_n^* = \frac{r_n - d_n}{u_n - d_n} = \frac{e^{r\Delta_n} - e^{-\sigma\sqrt{\Delta_n}}}{e^{\sigma\sqrt{\Delta_n}} - e^{-\sigma\sqrt{\Delta_n}}}.$$

Binomial Approximations

We can now price contingent claims in each Cox-Ross-Rubinstein model using the expectation operator with respect to the (unique) equivalent martingale measure characterised by the probabilities p_n^*. In particular we can compute the price $\Pi_C(t)$ at time t of a European call on the stock S with strike K and expiry T by formula (20). Let us reformulate this formula slightly. We define

$$\begin{aligned} a_n \quad = \quad & \min\{j \in \mathbb{N}_0 | \\ & S(0)(1+u_n)^j(1+d_n)^{k_n-j} > K\}. \end{aligned} \tag{23}$$

Binomial Approximations

Then we can rewrite the pricing formula (20) for $t = 0$ in the setting of the nth Cox-Ross-Rubinstein model as

$$\Pi_C(0) = (1+r_n)^{-k_n}$$

$$\sum_{j=a_n}^{k_n} \binom{k_n}{j} {p_n^*}^j (1-p_n^*)^{k_n-j}$$

$$(S(0)(1+u_n)^j(1+d_n)^{k_n-j} - K)$$

Binomial Approximations

So

$$\begin{aligned}\Pi_C(0) &= S(0)\left[\sum_{j=a_n}^{k_n}\binom{k_n}{j}\left(\frac{p_n^*(1+u_n)}{1+r_n}\right)^j\right.\\ &\qquad\left.\left(\tfrac{(1-p_n^*)(1+d_n)}{1+r_n}\right)^{k_n-j}\right]\\ &\quad -(1+r_n)^{-k_n}K\left[\sum_{j=a_n}^{k_n}\binom{k_n}{j}p_n^{*j}(1-p_n^*)^{k_n-j}\right].\end{aligned}$$

Binomial Approximations

Denoting the binomial cumulative distribution function with parameters (n, p) as $B^{n,p}(.)$ we see that the second bracketed expression is just

$$\bar{B}^{k_n,p_n^*}(a_n) = 1 - B^{k_n,p_n^*}(a_n).$$

Also the first bracketed expression is $\bar{B}^{k_n,\hat{p}_n}(a_n)$ with

$$\hat{p}_n = \frac{p_n^*(1+u_n)}{1+r_n}.$$

That $\hat{p}_n$ is indeed a probability can be shown straightforwardly. Using this notation we have in the nth Cox-Ross-Rubinstein model for the price of a European call at time $t = 0$ the following formula:

$$\begin{aligned} \Pi_C^{(n)}(0) &= S_n(0)\bar{B}^{k_n,\hat{p}_n}(a_n) \\ &\quad -K(1+r_n)^{-k_n}\bar{B}^{k_n,p_n^*}(a_n). \end{aligned} \tag{24}$$

Binomial Approximations

We have the following limit relation:

$$\lim_{n\to\infty} \Pi_C^{(n)}(0) = \Pi_C^{BS}(0)$$

with $\Pi_C^{BS}(0)$ given by the Black-Scholes formula (we use $S = S(0)$ to ease the notation)

$$\Pi_C^{BS}(0) = SN(d_1(S,T)) - Ke^{-rT}N(d_2(S,T)). \tag{25}$$

Binomial Approximations

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$
\begin{aligned}
d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\
d_2(s,t) &= d_1(s,t) - \sigma\sqrt{t} \\
&= \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}
\end{aligned}
$$

and $N(.)$ is the standard normal cumulative distribution function.

Binomial Approximations

The above is the famous Black-Scholes European call price formula.

PROOF:

Since $S_n(0) = S$ (say) all we have to do to prove the proposition is to show

$$\begin{array}{llcl} \text{(i)} & \lim\limits_{n\to\infty} \bar{B}^{k_n,\hat{p}_n}(a_n) & = & N(d_1(S,T)), \\ \text{(ii)} & \lim\limits_{n\to\infty} \bar{B}^{k_n,p^*_n}(a_n) & = & N(d_2(S,T)). \end{array}$$

Binomial Approximations

These statements involve the convergence of distribution functions.

To show (i) we interpret

$$\bar{B}^{k_n,\hat{p}_n}(a_n) = I\!P\left(a_n \le Y_n \le k_n\right)$$

with (Y_n) a sequence of random variables distributed according to the binomial law with parameters $(k_n, \hat{p}_n)$.

Binomial Approximations

We normalise Y_n to

$$\begin{aligned}\tilde{Y}_n &= \frac{Y_n - \mathbb{E}\left(Y_n\right)}{\sqrt{Var(Y_n)}} \\ &= \frac{Y_n - k_n \hat{p}_n}{\sqrt{k_n \hat{p}_n (1-\hat{p}_n)}} \\ &= \frac{\sum_{j=1}^{k_n} (B_{j,n} - \hat{p}_n)}{\sqrt{k_n \hat{p}_n (1-\hat{p}_n)}},\end{aligned}$$

where $B_{j,n},\ j = 1, \ldots, k_n;\ n = 1, 2, \ldots$ are row-wise independent Bernoulli random variables with parameter $\hat{p}_n$.

Binomial Approximations

Now using the central limit theorem we know that for $\alpha_n \to \alpha,\ \beta_n \to \beta$ we have

$$\lim_{n\to\infty} I\!P(\alpha_n \le \tilde{Y}_n \le \beta_n) = N(\beta) - N(\alpha).$$

By definition we have

$$I\!P\left(a_n \le Y_n \le k_n\right) = I\!P\left(\alpha_n \le \tilde{Y}_n \le \beta_n\right)$$

with

$$\alpha_n = \frac{a_n - k_n\hat{p}_n}{\sqrt{k_n\hat{p}_n(1-\hat{p}_n)}} \quad \text{and } \beta_n = \frac{k_n(1-\hat{p}_n)}{\sqrt{k_n\hat{p}_n(1-\hat{p}_n)}}.$$

Binomial Approximations

Observe the following limiting relations:

$$\lim_{n\to\infty} \hat{p}_n = \frac{1}{2}$$

and

$$\lim_{n\to\infty} k_n(1-2\hat{p}_n)\sqrt{\Delta_n} = -T\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right),$$

Binomial Approximations

From the defining relation for a_n, formula (23), we get

$$\begin{aligned}
\lim_{n\to\infty} \alpha_n &= \lim_{n\to\infty} \frac{\dfrac{\log(K/S)+k_n\sigma\sqrt{\Delta_n}}{2\sigma\sqrt{\Delta_n}} - k_n\hat{p}_n}{\sqrt{k_n\hat{p}_n(1-\hat{p}_n)}} \\
&= \lim_{n\to\infty} \frac{\log(K/S)+\sigma k_n\sqrt{\Delta_n}(1-2\hat{p}_n)}{2\sigma\sqrt{k_n\Delta_n\hat{p}_n(1-\hat{p}_n)}} \\
&= \frac{\log(K/S)-(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = -d_1(S,T).
\end{aligned}$$

Binomial Approximations

Furthermore we have

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \sqrt{k_n \hat{p}_n^{-1}(1-\hat{p}_n)} = +\infty.$$

So $N(\beta_n) \to 1, N(\alpha_n) \to N(-d_1) = 1 - N(d_1)$, completing the proof of (i).

Binomial Approximations

To prove (ii) we can argue in very much the same way and arrive at parameters α_n^* and β_n^* with $\hat{p}_n$ replaced by p_n^*. Using the following limiting relations:

$$\lim_{n\to\infty} p_n^* = \frac{1}{2}, \quad \lim_{n\to\infty} k_n(1-2p_n^*)\sqrt{\Delta_n} = T\left(\frac{\sigma}{2} - \frac{r}{\sigma}\right),$$

we get

$$\begin{aligned}\lim_{n\to\infty} \alpha_n^* &= \lim_{n\to\infty} \frac{\log(K/S) + \sigma n\sqrt{\Delta_n}(1-2p_n^*)}{2\sigma\sqrt{n\Delta_n p_n^*(1-p_n^*)}} \\ &= \frac{\log(K/S) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = -d_2(s,T).\end{aligned}$$

Binomial Approximations

For the upper limit we get

$$\lim_{n\to\infty} \beta_n^* = \lim_{n\to\infty} \sqrt{k_n (p_n^*)^{-1}(1-p_n^*)} = +\infty,$$

whence (ii) follows similarly. ■

By the above proposition we have derived the classical Black-Scholes European call option valuation formula as an asymptotic limit of option prices in a sequence of Cox-Ross-Rubinstein type models with a special choice of parameters. We will therefore call these models discrete Black-Scholes models.

Financial Mathematics

Lecture 6

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Aims and Objectives

- American Option §4.7
- American Options in the Cox-Ross-Rubinstein setting §4.7
- A three-period example §4.8.

American Options

Consider a general multi-period framework. The holder of an American derivative security can 'exercise' in any period t and receive payment $f(S_t)$ (or more general a non-negative payment f_t). In order to hedge such an option, we want to construct a self-financing trading strategy φ_t such that for the corresponding value process $V_\varphi(t)$

$$\begin{aligned} V_\varphi(0) &= x \text{ initial capital} \\ V_\varphi(t) &\geq f(S_t), \quad \forall t. \end{aligned} \tag{26}$$

Such a hedging portfolio is minimal, if for a stopping time τ

$$V_\varphi(\tau) = f(S_\tau).$$

Our aim in the following will be to discuss existence and construction of such a stopping time.

American Options

We assume now that we work in a market model $(\Omega, \mathcal{F}, \mathbb{F}, I\!\!P)$, which is complete with $I\!\!P^*$ the unique martingale measure.

Then for any hedging strategy φ we have that under $I\!\!P^*$

$$M(t) = \tilde{V}_\varphi(t) = \beta(t)V_\varphi(t) \tag{27}$$

is a martingale. Thus we can use the stopping time principle to find for any stopping time τ

$$V_\varphi(0) = M_0 = \mathbb{E}^*(\tilde{V}_\varphi(\tau)). \tag{28}$$

Since we require $V_\varphi(\tau) \geq f_\tau(S)$ for any stopping time we find for the required initial capital

$$x \geq \sup_{\tau \in \mathcal{T}} \mathbb{E}^*(\beta(\tau)f_\tau(S)). \tag{29}$$

American Options

Suppose now that τ^* is such that $V_\varphi(\tau^*) = f_{\tau^*}(S)$ then the strategy φ is minimal and since $V_\varphi(t) \geq f_t(S)$ for all t we have

$$x = \mathbb{E}^*(\beta(\tau^*) f_{\tau^*}(S)) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^*(\beta(\tau) f_\tau(S)) \tag{30}$$

Thus (30) is a necessary condition for the existence of a minimal strategy φ. We will show that it is also sufficient and call the price in (30) the rational price of an American contingent claim.

American Options

Now consider the problem of the option writer to construct such a strategy φ. At time T the hedging strategy needs to cover f_T, i.e. $V_\varphi(T) \geq f_T$ is required (We write short f_t for $f_t(S)$). At time $T-1$ the option holder can either exercise and receive f_{T-1} or hold the option to expiry, in which case $B(T-1)\mathbb{E}^*(\beta(T)f_T|F_{T-1})$ needs to be covered. Thus the hedging strategy of the writer has to satisfy

$$V_\varphi(T-1) = \max\{f_{T-1}, B(T-1)\mathbb{E}^*(\beta(T)f_T|\mathcal{F}_{T-1})\} \tag{31}$$

American Options

Using a backwards induction argument we can show that

$$V_\varphi(t-1) = \max\{f_{t-1}, B(t-1)\mathbb{E}^*(\beta(t)V_\varphi(t)|\mathcal{F}_{t-1})\}. \tag{32}$$

Considering only discounted values this leads to

$$\tilde{V}_\varphi(t-1) = \max\{\tilde{f}_{t-1}, \mathbb{E}^*(\tilde{V}_\varphi(t)|\mathcal{F}_{t-1})\}. \tag{33}$$

Thus we see that $\tilde{V}_\varphi(t)$ is the Snell envelope Z_t of $\tilde{f}_t$.

American Options

In particular we know that

$$Z_t = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}^*(\tilde{f}_\tau | \mathcal{F}_t) \tag{34}$$

and the stopping time $\tau^* = \min\{s \geq t : Z_s = \tilde{f}_s\}$ is optimal. So

$$Z_t = \mathbb{E}^*(\tilde{f}_{\tau^*} | \mathcal{F}_t) \tag{35}$$

In case $t = 0$ we can use $\tau_0^* = \min\{s \geq 0 : Z_s = \tilde{f}_s\}$ and then

$$x = Z_0 = \mathbb{E}^*(\tilde{f}_{\tau_0^*}) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}^*(\tilde{f}_\tau) \tag{36}$$

is the rational option price.

American Options

We still need to construct the strategy φ. To do this recall that Z is a supermartingale and so the Doob decomposition yields

$$Z = \tilde{M} - \tilde{A} \tag{37}$$

with a martingale $\tilde{M}$ and a predictable, increasing process $\tilde{A}$. We write $M_t = \tilde{M}_t B_t$ and $A_t = \tilde{A}_t B_t$. Since the market is complete we know that there exists a self-financing strategy $\bar{\varphi}$ such that

$$\tilde{M}_t = \tilde{V}_{\bar{\varphi}}(t). \tag{38}$$

American Options

Also using (37) we find $Z_t B_t = V_{\bar{\varphi}}(t) - A_t$. Now on $C = \{(t,\omega) : 0 \le t < \tau^*(\omega)\}$ we have that Z is a martingale and thus $A_t(\omega) = 0$. Thus we obtain from $\tilde{V}_{\bar{\varphi}}(t) = Z_t$ that

$$\tilde{V}_{\bar{\varphi}}(t) = \sup_{t \le \tau \le T} \mathbb{E}^*(\tilde{f}_\tau | \mathcal{F}_t) \quad \forall\, (t,\omega) \in C. \tag{39}$$

Now τ^* is the smallest exercise time and $\tilde{A}_{\tau^*(\omega)} = 0$. Thus

$$\tilde{V}_{\bar{\varphi}}(\tau^*(\omega),\omega) = Z_{\tau^*(\omega)}(\omega) = \tilde{f}_{\tau^*(\omega)}(\omega) \tag{40}$$

Undoing the discounting we find

$$V_{\bar{\varphi}}(\tau^*) = f_{\tau^*} \tag{41}$$

and therefore $\bar{\phi}$ is a minimal hedge.

American Options

Now consider the problem of the option holder, how to find the optimal exercise time. We observe that the optimal exercise time must be an optimal stopping time, since for any other stopping time σ

$$\tilde{V}_{\varphi}(\sigma) = Z_{\sigma} > \tilde{f}_{\sigma} \tag{42}$$

and holding the asset longer would generate a larger payoff. Thus the holder needs to wait until $Z_{\sigma} = \tilde{f}_{\sigma}$.

American Options

On the other hand with ν the largest stopping time, we see that $\sigma \leq \nu$. This follows since using $\bar{\phi}$ after ν with initial capital from exercising will always yield a higher portfolio value than the strategy of exercising later. To see this recall that $V_{\bar{\varphi}} = Z_t B_t + A_t$ with $A_t > 0$ for $t > \nu$. So we must have $\sigma \leq \nu$ and since $A_t = 0$ for $t \leq \nu$ we see that Z^σ is a martingale. Thus both criteria of the characterisation of optimality are true and σ is thus optimal. So

American Options

A stopping time $\sigma \in \mathcal{T}_t$ is an optimal exercise time for the American option (f_t) if and only if

$$\mathbb{E}^*(\beta(\sigma) f_\sigma) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}^*(\beta(\tau) f_\tau) \tag{43}$$

American Options in the CRR model

We now consider how to evaluate an American put option in a standard CRR model. We assume that the time interval $[0, T]$ is divided into N equal subintervals of length Δ say. Assuming the risk-free rate of interest r (over [0,T]) as given, we have $1 + \rho = e^{r\Delta}$ (where we denote the risk-free rate of interest in each subinterval by ρ). The remaining degrees of freedom are resolved by choosing u and d as follows:

$$1 + u = e^{\sigma\sqrt{\Delta}}, \quad \text{and} \quad 1 + d = (1 + u)^{-1} = e^{-\sigma\sqrt{\Delta}}.$$

The risk-neutral probabilities for the corresponding single period models are given by

American Options in the CRR model

$$p^* = \frac{\rho - d}{u - d} = \frac{e^{r\Delta} - e^{-\sigma\sqrt{\Delta}}}{e^{\sigma\sqrt{\Delta}} - e^{-\sigma\sqrt{\Delta}}}.$$

Thus the stock with initial value $S = S(0)$ is worth $S(1+u)^i(1+d)^j$ after i steps up and j steps down. Consequently, after N steps, there are $N+1$ possible prices, $S(1+u)^i(1+d)^{N-i}$ $(i = 0, \ldots, N)$. There are 2^N possible paths through the tree.

American Options

It is common to take N of the order of 30, for two reasons:

(i) typical lengths of time to expiry of options are measured in months (9 months, say); this gives a time step around the corresponding number of days,
(ii) 2^{30} paths is about the order of magnitude that can be comfortably handled by computers (recall that $2^{10} = 1,024$, so 2^{30} is somewhat over a billion).

American Options in the CRR model

We can now calculate both the value of an American put option and the optimal exercise strategy by working backwards through the tree (this method of backward recursion in time is a form of the dynamic programming (DP) technique, due to Richard Bellman, which is important in many areas of optimisation and Operational Research).

American Options in the CRR model

1. Draw a binary tree showing the initial stock value and having the right number, N, of time intervals.
2. Fill in the stock prices: after one time interval, these are $S(1+u)$ (upper) and $S(1+d)$ (lower); after two time intervals, $S(1+u)^2$, S and $S(1+d)^2 = S/(1+u)^2$; after i time intervals, these are $S(1+u)^j(1+d)^{i-j} = S(1+u)^{2j-i}$ at the node with j 'up' steps and $i-j$ 'down' steps (the '(i,j)' node).

3. Using the strike price K and the prices at the terminal nodes, fill in the payoffs $f^A_{N,j} = \max\{K - S(1+u)^j(1+d)^{N-j}, 0\}$ from the option at the terminal nodes underneath the terminal prices.

American Options in the CRR model

4. Work back down the tree, from right to left. The no-exercise values f_{ij} of the option at the (i, j) node are given in terms of those of its upper and lower right neighbours in the usual way, as discounted expected values under the risk-neutral measure:

$$f_{ij} = e^{-r\Delta}[p^* f^A_{i+1,j+1} + (1 - p^*) f^A_{i+1,j}].$$

The intrinsic (or early-exercise) value of the American put at the (i, j) node – the value there if it is exercised early – is

$$K - S(1 + u)^j (1 + d)^{i-j}$$

(when this is non-negative, and so has any value).

American Options in the CRR model

The value of the American put is the higher of these:

$$
\begin{aligned}
f_{ij}^A &= \max\{f_{ij}, K - S(1+u)^j(1+d)^{i-j}\} \\
&= \max\Big\{e^{-r\Delta}(p^* f_{i+1,j+1}^A + (1-p^*) f_{i+1,j}^A)\,, \\
&\qquad\qquad K - S(1+u)^j(1+d)^{i-j}\Big\}.
\end{aligned}
$$

5. The initial value of the option is the value f_0^A filled in at the root of the tree.

6. At each node, it is optimal to exercise early if the early-exercise value there exceeds the value f_{ij} there of expected discounted future payoff.

A Three-period Example

Assume we have two basic securities: a risk-free bond and a risky stock. The one-year risk-free interest rate (continuously compounded) is $r = 0.06$ and the volatility of the stock is 20%. We price calls and puts in three-period Cox-Ross-Rubinstein model. The up and down movements of the stock price are given by

$$1 + u = e^{\sigma\sqrt{\Delta}} = 1.1224$$

and

$$1 + d = (1 + u)^{-1} = e^{-\sigma\sqrt{\Delta}} = 0.8910,$$

with $\sigma = 0.2$ and $\Delta = 1/3$. We obtain risk-neutral probabilities

$$p^* = \frac{e^{r\Delta} - d}{u - d} = 0.5584.$$

A Three-period Example

We assume that the price of the stock at time $t = 0$ is $S(0) = 100$. To price a European call option with maturity one year ($N = 3$) and strike $K = 10$) we can either use the explicit valuation formula or work our way backwards through the tree. Prices of the stock and the call are given below.

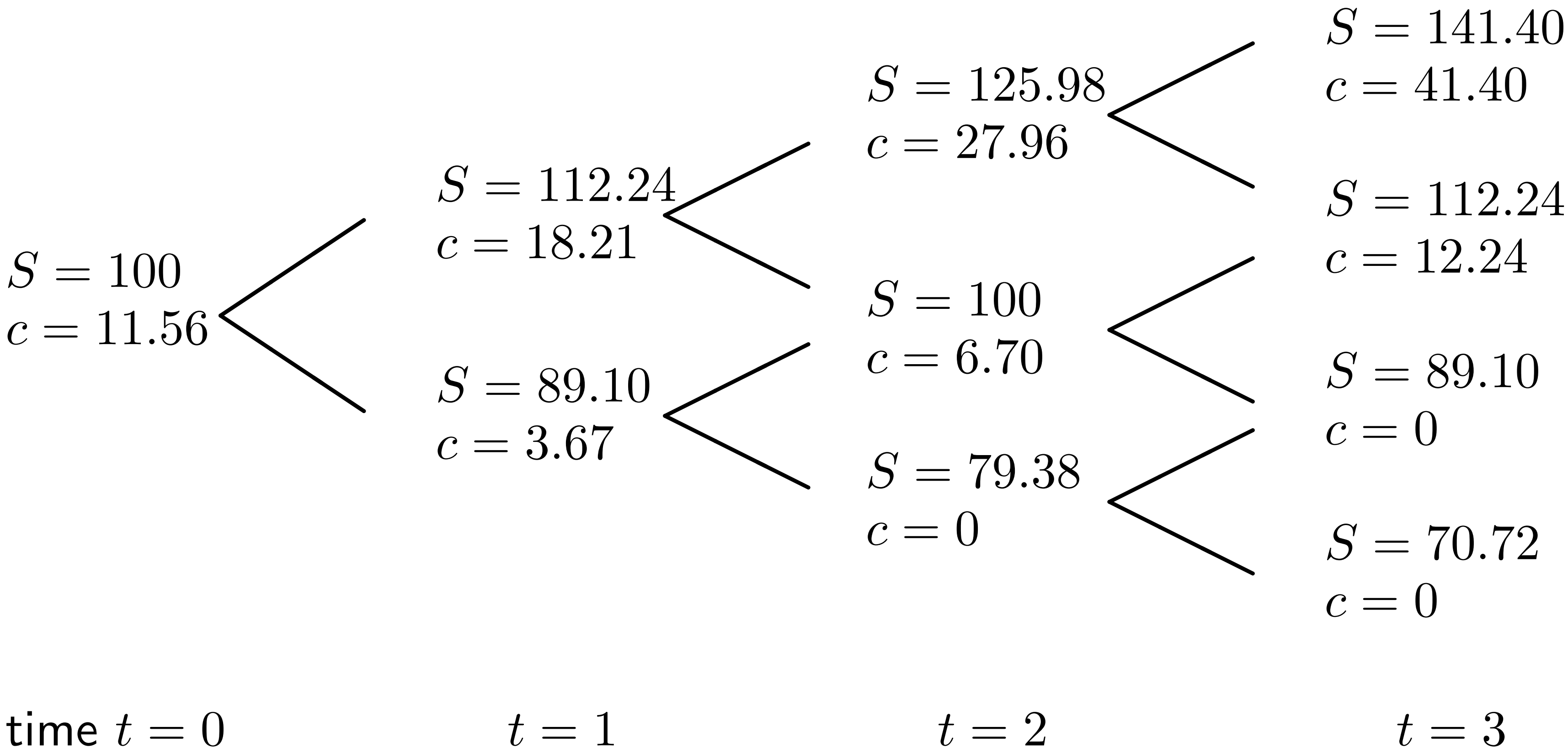
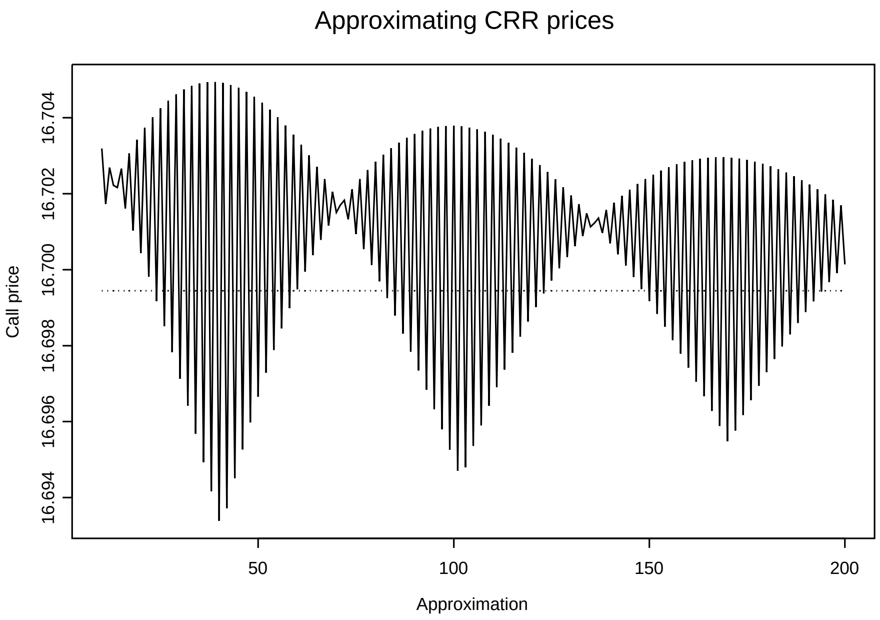


Figure 3: Stock and European call prices

A Three-period Example

One can implement the simple evaluation formulae for the CRR- and the BS-models and compare the values. The figure is for $S = 100, K = 90, r = 0.06, \sigma = 0.2, T = 1.$


Approximating CRR prices
Call price
16.694
16.696
16.698
16.700
16.702
16.704
50
100
150
200
Approximation

A Three-period Example

To price a European put, with price process denoted by $p(t)$, and an American put, $P(t)$, (maturity $N = 3$, strike 100), we can for the European put either use the put-call parity, the risk-neutral pricing formula, or work backwards through the tree. For the prices of the American put we use the technique outlined above.

We indicate the early exercise times of the American put in bold type. Recall that the discrete-time rule is to exercise if the intrinsic value $K - S(t)$ is larger than the value of the corresponding European put.

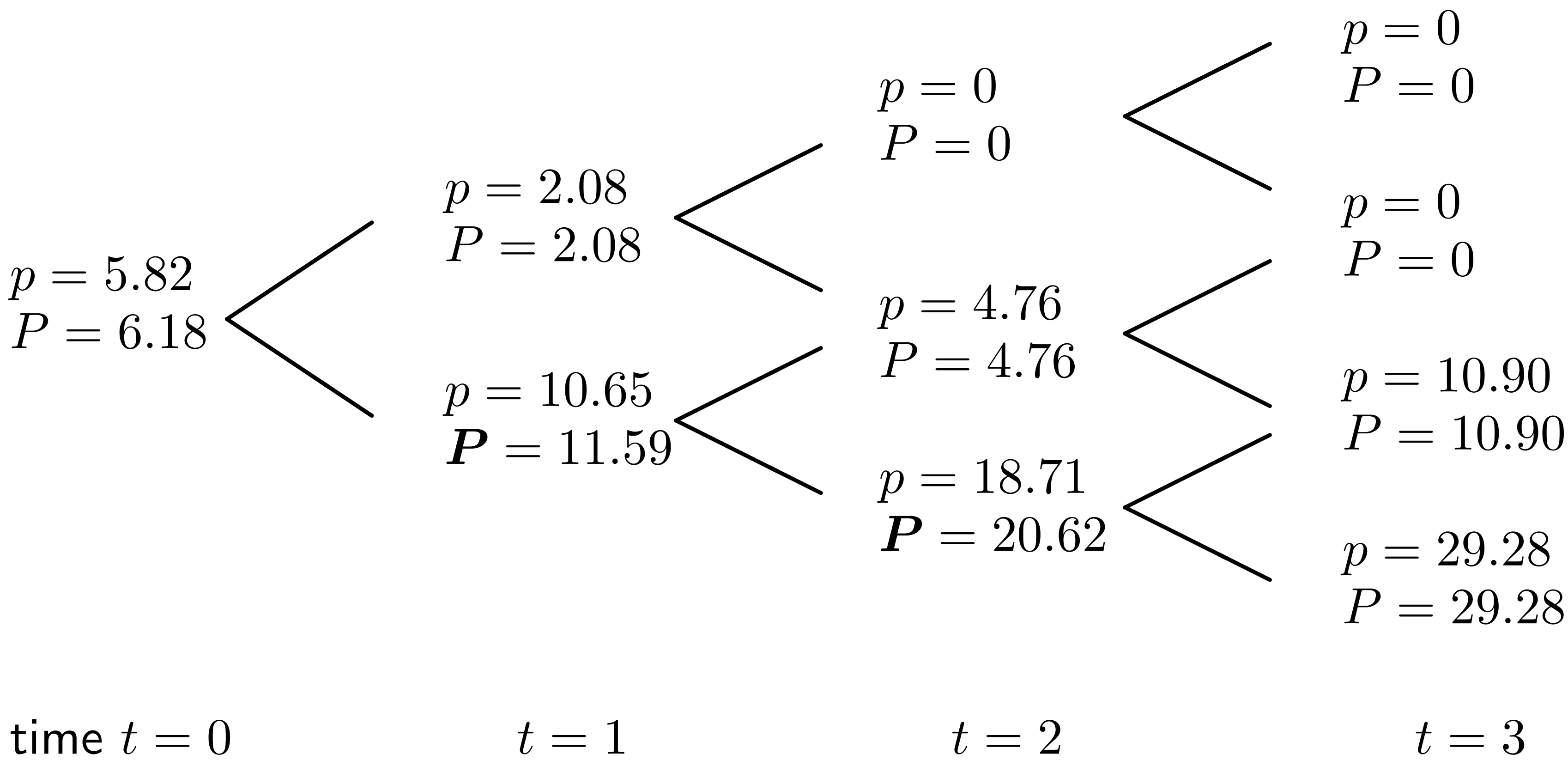


Figure 5: European $p(.)$ and American $P(.)$ put prices

Financial Mathematics

Lecture 7

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Aims and Objectives

- Review of Itô's formula (§5.6);
- Main Theorems from Stochastic Analysis (§5.7)
- The Financial Market Model (§6.1)
- Equivalent Martingale Measures (§6.1)

Itô Processes

$$X(t) := x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s)$$

defines a stochastic process X with $X(0) = x_0$.

We express such an equation symbolically in differential form, in terms of the stochastic differential equation

$$dX(t) = b(t)dt + \sigma(t)dW(t), \quad X(0) = x_0.$$

For $f \in C^2$ we want to give meaning to the stochastic differential $df(X(t))$ of the process $f(X(t))$.

Multiplication rules

These are just shorthand for the corresponding properties of the quadratic variations.

	dt	dW
dt	0	0
dW	0	dt

We find

$$\begin{aligned} d\langle X\rangle &= (bdt + \sigma dW)^2 \\ &= \sigma^2 dt + 2b\sigma dt dW + b^2(dt)^2 = \sigma^2 dt \end{aligned}$$

Basic Itô Formula

If X is a Itô Process and $f \in C^2$, then $f(X)$ has stochastic differential

$$\begin{aligned} df(X(t)) &= f'(X(t))dX(t) \\ &+ \frac{1}{2} f''(X(t))d\langle X\rangle(t), \end{aligned}$$

or writing out the integrals,

$$\begin{aligned} f(X(t)) &= f(x_0) + \int_0^t f'(X(u))dX(u) \\ &+ \frac{1}{2}\int_0^t f''(X(u))d\langle X\rangle(u). \end{aligned}$$

Itô Formula

If $X(t)$ is an Itô process and $f \in C^{1,2}$ then $f = f(t, X(t))$ has stochastic differential

$$df = \left(f_t + b f_x + \frac{1}{2}\sigma^2 f_{xx} \right) dt + \sigma f_x dW.$$

That is, writing f_0 for $f(0, x_0)$, the initial value of f,

$$f = f_0 + \int_0^t (f_t + b f_x + \frac{1}{2}\sigma^2 f_{xx}) dt + \int_0^t \sigma f_x dW.$$

Example: GBM

The SDE for GBM has the unique solution

$$S(t) = S(0)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma dW(t)\right\}.$$

For, writing

$$f(t,x) := \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma x\right\},$$

we have

$$f_t = \left(\mu - \frac{1}{2}\sigma^2\right)f, \quad f_x = \sigma f, \quad f_{xx} = \sigma^2 f,$$

and with $x = W(t)$, one has

$$dx = dW(t), \quad (dx)^2 = dt.$$

Example: GBM

Thus Itô's lemma gives

$$\begin{aligned} df &= f_t dt + f_x dW + \frac{1}{2} f_{xx} (dW)^2 \\ &= f\left(\left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW + \frac{1}{2}\sigma^2 dt\right) \\ &= f(\mu dt + \sigma dW). \end{aligned}$$

Girsanov's Theorem

Consider independent $N(0,1)$ random variables $Z_1, \ldots, Z_n$ on $(\Omega, \mathcal{F}, I\!\!P)$.

Given $\gamma = (\gamma_1, \ldots, \gamma_n)$, consider a new probability measure $\tilde{I\!\!P}$ on $(\Omega, \mathcal{F})$ defined by

$$\tilde{I\!\!P}(d\omega) = \exp\left\{\sum_{i=1}^{n} \gamma_i Z_i(\omega) - \frac{1}{2}\sum_{i=1}^{n} \gamma_i^2\right\} I\!\!P(d\omega).$$

Girsanov's Theorem

Then

$$\begin{aligned}
&\tilde{I\!P}(Z_i \in dz_i, \forall i) \\
= \quad & e^{\left\{\sum_{i=1}^n \gamma_i Z_i - \frac{1}{2}\sum_{i=1}^n \gamma_i^2\right\}} I\!P(Z_i \in dz_i, \forall i) \\
= \quad & \frac{1}{(2\pi)^{\frac{n}{2}}} e^{\left\{\sum_{i=1}^n \gamma_i z_i - \frac{1}{2}\sum_{i=1}^n \gamma_i^2 - \frac{1}{2}\sum_{i=1}^n z_i^2\right\}} \prod_{i=1}^n dz_i \\
= \quad & \frac{1}{(2\pi)^{\frac{n}{2}}} e^{\left\{-\frac{1}{2}\sum_{i=1}^n (z_i - \gamma_i)^2\right\}} dz_1 \dots dz_n.
\end{aligned}$$

So the Z_i are independent $N(\gamma_i, 1)$ under $\tilde{I\!P}$.

Girsanov's Theorem

Let $W = (W_1, \dots W_d)$ be a d-dimensional BM on$(\Omega, \mathcal{F}, I\!\!P, \mathbb{F})$.

With $(\gamma(t)$ a suitable d-dimensional process

$$L(t) = \exp\left\{ - \int_0^t \gamma(s)' dW(s) - \frac{1}{2} \int_0^t \|\gamma(s)\|^2 \, ds \right\}.$$

Girsanov Define

$$\tilde{W}_i(t) := W_i(t) + \int_0^t \gamma_i(s) ds.$$

Under the equivalent probability measure $\tilde{I\!\!P}$ with Radon-Nikodým derivative

$$\frac{d\tilde{I\!\!P}}{dI\!\!P} = L(T),$$

the process $\tilde{W} = (\tilde{W}_1, \dots, \tilde{W}_d)$ is d-dimensional Brownian motion.

Girsanov's Theorem

For $\gamma(t) = \gamma$, change of measure by the Radon-Nikodým derivative

$$\exp\left\{-\gamma W(t) - \frac{1}{2}\gamma^2 t\right\}$$

corresponds to a change of drift from c to $c - \gamma$.

If $\mathbb{F} = (\mathcal{F}_t)$ is the Brownian filtration any pair of equivalent probability measures $\mathbb{Q} \sim I\!P$ on $\mathcal{F} = \mathcal{F}_T$ is a Girsanov pair, i.e.

$$\left.\frac{d\tilde{\mathbb{Q}}}{dI\!P}\right|_{\mathcal{F}_t} = L(t)$$

with L defined as above.

Representation Theorem

Let $M = (M(t))_{t\geq 0}$ be a martingale with respect to the Brownian filtration $(\mathcal{F}_t)$. Then

$$M(t) = M(0) + \int_0^t H(s)dW(s),\ t \geq 0$$

with $H = (H(t))_{t\geq 0}$ a progressively measurable process such that $\int_0^t H(s)^2 ds < \infty,\ t \geq 0$ with probability one.

All Brownian martingales may be represented as stochastic integrals with respect to Brownian motion.

Let C be an $\mathcal{F}_T$-measurable random variable with $\mathbb{E}\left(|C|\right) < \infty$; then there exists a process H as abone such that

$$C = \mathbb{E}\, C + \int_0^T H(s)dW(s).$$

Feynman-Kac formula

Consider a SDE,

$$dX(t) = \mu(t, X(t))ds + \sigma(t, X(t))dW(t),$$

with initial condition

$$X(t_0) = x.$$

Let $X = X(t)$ be the unique solution and consider a smooth function $F(t, X(t))$ of it. By Itô's lemma,

$$dF = F_t dt + F_x dX + \frac{1}{2} F_{xx} d\langle X \rangle,$$

and as $d\langle X \rangle = \langle \mu dt + \sigma dW \rangle = \sigma^2 dt$, this is

$$\begin{aligned} dF &= F_t dt + F_x(\mu dt + \sigma dW) + \frac{1}{2}\sigma^2 F_{xx} dt \\ &= \left(F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}\right) dt + \sigma F_x dW. \end{aligned}$$

Feynman-Kac formula

Now suppose that F satisfies the partial differential equation

$$F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx} = 0$$

with boundary condition,

$$F(T, x) = h(x).$$

Then the above expression for dF gives

$$dF = \sigma F_x dW.$$

Feynman-Kac formula

This can be written in stochastic-integral rather than stochastic-differential form as $F_0 = F(t_0, X(t_0))$

$$F(s, X_s) = F_0 + \int_{t_0}^{s} \sigma(u, X_u) F_x(u, X_u) dW_u.$$

The stochastic integral on the right is a martingale with constant expectation $= 0$. Then

$$F(t_0, x) = \mathbb{E}\left(F(s, X(s)) \middle| X(t_0) = x \right).$$

Feynman-Kac formula

In the time-homogeneous case $\mu(t,x) = \mu(x)$ and $\sigma(t,x) = \sigma(x)$, with μ and σ Lipschitz, and $h \in C_0^2$ the solution $F = F(t,x)$ to the PDE

$$F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx} = 0$$

with final condition $F(T,x) = h(x)$ has the stochastic representation

$$F(t,x) = \mathbb{E}\left[h(X(T))\right| X(t) = x],$$

where X satisfies the stochastic differential equation

$$dX(s) = \mu(X(s))ds + \sigma(X(s))dW(s)$$

with initial condition $X(t) = x$.

The Feynman-Kac formula gives a stochastic representation to solutions of partial differential equations.

Financial Market Model

$T > 0$ is a fixed a planning horizon.

Uncertainty in the financial market is modelled by a probability space $(\Omega, \mathcal{F}, I\!P)$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions of right-continuity and completeness.

There are $d+1$ primary traded assets, whose price processes are given by stochastic processes $S_0, \ldots, S_d$, which represent the prices of some traded assets.

A numéraire is a price process $X(t)$ almost surely strictly positive for each $t \in [0, T]$.

'Historically' money market account $B(t) = e^{r(t)}$ with a positive deterministic process $r(t)$ and $r(0) = 0$, was used as a numéraire.

Trading Strategies

We call an $\mathbb{R}^{d+1}$-valued predictable process

$$\varphi(t) = (\varphi_0(t), \ldots, \varphi_d(t)), \quad t \in [0, T]$$

a trading strategy (or dynamic portfolio process).

Here $\varphi_i(t)$ denotes the number of shares of asset i held in the portfolio at time t - to be determined on the basis of information available *before* time t; i.e. the investor selects his time t portfolio after observing the prices $S(t-)$.

Trading Strategies

- The value of the portfolio φ at time t is given by

$$V_\varphi(t) := \varphi(t) \cdot S(t) = \sum_{i=0}^{d} \varphi_i(t) S_i(t).$$

$V_\varphi(t)$ is called the value process, or wealth process, of the trading strategy φ.

- The gains process $G_\varphi(t)$ is

$$G_\varphi(t) := \sum_{i=0}^{d} \int_0^t \varphi_i(u) dS_i(u).$$

- A trading strategy φ is called self-financing if the wealth process $V_\varphi(t)$ satisfies

$$V_\varphi(t) = V_\varphi(0) + G_\varphi(t) \quad \text{for all } t \in [0, T].$$

Discounted Processes

The discounted price process is

$$\tilde{S}(t) := \frac{S(t)}{S_0(t)} = (1, \tilde{S}_1(t), \ldots \tilde{S}_d(t))$$

with $\tilde{S}_i(t) = S_i(t)/S_0(t),\ i = 1, 2, \ldots, d$. The discounted wealth process $\tilde{V}_\varphi(t)$ is

$$\tilde{V}_\varphi(t) := \frac{V_\varphi(t)}{S_0(t)} = \varphi_0(t) + \sum_{i=1}^{d} \varphi_i(t)\tilde{S}_i(t)$$

and the discounted gains process $\tilde{G}_\varphi(t)$ is

$$\tilde{G}_\varphi(t) := \sum_{i=1}^{d} \int_0^t \varphi_i(t) d\tilde{S}_i(t).$$

Self-Financing

φ is self-financing if and only if

$$\tilde{V}_\varphi(t) = \tilde{V}_\varphi(0) + \tilde{G}_\varphi(t).$$

Thus a self-financing strategy is completely determined by its initial value and the components $\varphi_1, \ldots, \varphi_d$. Any set of predictable processes $\varphi_1, \ldots, \varphi_d$ such that the stochastic integrals $\int \varphi_i d\tilde{S}_i$ exist can be uniquely extended to a self-financing strategy φ with specified initial value $\tilde{V}_\varphi(0) = v$ by setting the cash holding as

$$\varphi_0(t) = v + \sum_{i=1}^{d} \int_0^t \varphi_i(u) d\tilde{S}_i(u) - \sum_{i=1}^{d} \varphi_i(t)\tilde{S}_i.$$

Arbitrage Opportunities

A self-financing trading strategy φ is called an arbitrage opportunity if the wealth process V_φ satisfies the following set of conditions:

$$V_\varphi(0) = 0, \quad I\!\!P(V_\varphi(T) \geq 0) = 1,$$

and

$$I\!\!P(V_\varphi(T) > 0) > 0.$$

Martingale Measure

A probability measure $\mathbb{Q}$ defined on $(\Omega, \mathcal{F})$ is an equivalent martingale measure (EMM) if:

(i) $\mathbb{Q}$ is equivalent to $I\!\!P$,
(ii) the discounted price process $\tilde{S}$ is a $\mathbb{Q}$ martingale.

Assume $S_0(t) = B(t) = e^{r(t)}$, then $\mathbb{Q} \sim I\!\!P$ is a martingale measure if and only if every asset price process S_i has price dynamics under $\mathbb{Q}$ of the form

$$dS_i(t) = r(t)S_i(t)dt + dM_i(t),$$

where M_i is a $\mathbb{Q}$-martingale.

EMMs and Arbitrage

Assume $\mathbb{Q}$ is an EMM. Then the market model contains no arbitrage opportunities.

Proof. Under $\mathbb{Q}$ we have that $\tilde{V}_\varphi(t)$ is a martingale. That is,

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)|\mathcal{F}_u\right) = \tilde{V}_\varphi(u), \quad \text{for all} \quad u \le t \le T.$$

For $\varphi \in \Phi$ to be an arbitrage opportunity we must have $\tilde{V}_\varphi(0) = V_\varphi(0) = 0$. Now

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)\right) = 0, \quad \text{for all } 0 \le t \le T.$$

EMMs and Arbitrage

Now $\tilde{V}_\varphi(t)$ is a martingale, so

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)\right) = 0,\ 0 \le t \le T,$$

in particular $\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(T)\right) = 0$.

For an arbitrage opportunity φ we have $I\!P\left(V_\varphi(T) \ge 0\right) = 1$, and since $\mathbb{Q} \sim I\!P$, this means $\mathbb{Q}\left(V_\varphi(T) \ge 0\right) = 1$.

Both together yield

$$\mathbb{Q}\left(V_\varphi(T) > 0\right) = I\!P\left(V_\varphi(T) > 0\right) = 0,$$

and hence the result follows. ∎

Financial Mathematics

Lecture 8

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Aims and Objectives

- Risk-Neutral Pricing (§6.1);
- Black-Scholes Model (§6.2)
- Barrier Options (§6.3)

Admissible Strategies

A SF strategy φ is called ($I\!\!P^*$-) admissible if

$$\tilde{G}_\varphi(t) = \int_0^t \varphi(u) d\tilde{S}(u)$$

is a ($I\!\!P^*$-) martingale.

By definition $\tilde{S}$ is a martingale, and $\tilde{G}$ is the stochastic integral with respect to $\tilde{S}$.

The financial market model $\mathcal{M}$ contains no arbitrage opportunities wrt admissible strategies.

Contingent Claims

A contingent claims X is a random variable such that $X/S_0(T) \in L^1(\mathcal{F}, I\!P^*)$.

- A contingent claim X is called attainable if there exists at least one admissible trading strategy such that

$$V_\varphi(T) = X.$$

  We call such a trading strategy φ a replicating strategy for X.

- The financial market model $\mathcal{M}$ is said to be complete if any contingent claim is attainable.

No-Arbitrage Price

If a contingent claim X is attainable, X can be replicated by a portfolio $\varphi \in \Phi(I\!\!P^*)$. This means that holding the portfolio and holding the contingent claim are equivalent from a financial point of view. In the absence of arbitrage the (arbitrage) price process $\Pi_X(t)$ of the contingent claim must therefore satisfy

$$\Pi_X(t) = V_\varphi(t).$$

Risk-Neutral Valuation

The arbitrage price process of any attainable claim is given by the risk-neutral valuation formula

$$\Pi_X(t) = S_0(t)\mathbb{E}_{I\!P^*}\left[\left.\frac{X}{S_0(T)}\right|\mathcal{F}_t\right].$$

Thus, for any two replicating portfolios $\varphi, \psi \in \Phi(I\!P^*)$

$$V_\varphi(t) = V_\psi(t).$$

Risk-Neutral Valuation

Proof. Since X is attainable, there exists a replicating strategy $\varphi \in \Phi(I\!\!P^*)$ such that $V_\varphi(T) = X$ and $\Pi_X(t) = V_\varphi(t)$. Since $\varphi \in \Phi(I\!\!P^*)$ the discounted value process $\tilde{V}_\varphi(t)$ is a martingale, and hence

$$\begin{aligned}
\Pi_X(t) &= V_\varphi(t) = S_0(t)\tilde{V}_\varphi(t) \\
&= S_0(t)\mathbb{E}_{I\!\!P^*}\left[\tilde{V}_\varphi(T)\middle|\mathcal{F}_t\right] \\
&= S_0(t)\mathbb{E}_{I\!\!P^*}\left[\frac{V_\varphi(T)}{S_0(T)}\middle|\mathcal{F}_t\right] \\
&= S_0(t)\mathbb{E}_{I\!\!P^*}\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right].
\end{aligned}$$

Black-Scholes Model

The classical Black-Scholes model is

$$\begin{aligned} dB(t) &= rB(t)dt, & B(0) &= 1, \\ dS(t) &= S(t)\,(bdt + \sigma dW(t)), & S(0) &= p, \end{aligned}$$

with constant coefficients $b \in I\!R,\ r, \sigma \in I\!R_+$.

We use the bank account being the natural numéraire) and get from Itô's formula

$$d\tilde{S}(t) = \tilde{S}(t)\,\{(b-r)dt + \sigma dW(t)\}.$$

EMM in BS-model

Any EMM is a Girsanov pair

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L(t)$$

with

$$L(t) = \exp\left\{-\int_0^t \gamma(s)dW(s) - \frac{1}{2}\int_0^t \gamma(s)^2 ds\right\}.$$

By Girsanov's theorem

$$dW(t) = d\tilde{W}(t) - \gamma(t)dt,$$

where $\tilde{W}$ is a $\mathbb{Q}$-BM. Thus the $\mathbb{Q}$-dynamics for $\tilde{S}$ are

$$d\tilde{S}(t) = \tilde{S}(t)\left\{(b - r - \sigma\gamma(t))dt + \sigma d\tilde{W}(t)\right\}.$$

EMM in BS-model

Since $\tilde{S}$ has to be a martingale under $\mathbb{Q}$ we must have

$$b - r - \sigma\gamma(t) = 0 \quad t \in [0, T],$$

and so we must choose

$$\gamma(t) \equiv \gamma = \frac{b - r}{\sigma},$$

this argument leads to a unique martingale measure. The $\mathbb{Q}$-dynamics of S are

$$dS(t) = S(t)\left\{r dt + \sigma d\tilde{W}\right\}.$$

Pricing Contingent Claims

By the risk-neutral valuation principle

$$\Pi_X(t) = e^{\{-r(T-t)\}}\mathbb{E}^*[X|\mathcal{F}_t],$$

with $\mathbb{E}^*$ given via the Girsanov density

$$L(t) = \exp\left\{-\left(\frac{b-r}{\sigma}\right)W(t) - \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 t\right\}.$$

Pricing Contingent Claims

For a European call $X = (S(T) - K)^+$ and we can evaluate the above expected value

The Black-Scholes price process of a European call is given by

$$C(t) = S(t)N(d_1(S(t), T-t)) - Ke^{-r(T-t)}N(d_2(S(t), T-t)).$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$d_1(s,t) = \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}},$$

$$d_2(s,t) = \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

Hedging Contingent Claims

From the risk-neutral valuation principle

$$M(t) = \exp\{-rT\}\,\mathbb{E}^*\left[X|\,\mathcal{F}_t\right].$$

By Itô's lemma we find for the dynamics of the $I\!P^*$-martingale $M(t) = G(t, S(t))$:

$$dM(t) = \sigma S(t) G_s(t, S(t)) d\tilde{W}(t).$$

Hedging Contingent Claims

Using this representation, we get for the stock component of the replicating portfolio

$$h(t) = \sigma S(t) G_s(t, S(t)).$$

Now for the discounted assets the stock component is

$$\varphi_1(t) = G_s(t, S(t)) B(t),$$

and using the self-financing condition the cash component is

$$\varphi_0(t) = G(t, S(t)) - G_s(t, S(t)) S(t).$$

Hedging Contingent Claims

To transfer this portfolio to undiscounted values we multiply it by the discount factor, i.e $F(t, S(t)) = B(t)G(t, S(t))$, and obtain.

The replicating strategy in the classical Black-Scholes model is given by

$$\begin{aligned} \varphi_0 &= \frac{F(t, S(t)) - F_s(t, S(t))S(t)}{B(t)}, \\ \varphi_1 &= F_s(t, S(t)). \end{aligned}$$

BS by Arbitrage

Consider a self-financing portfolio which has dynamics

$$\begin{aligned} dV_\varphi(t) &= \varphi_0(t)dB(t) + \varphi_1(t)dS(t) \\ &= (\varphi_0 rB + \varphi_1 \mu S)dt + \varphi_1 \sigma S dW. \end{aligned}$$

Assume that $V_\varphi(t) = V(t) = f(t, S(t))$. Then by Itô's formula

$$\begin{aligned} dV &= (f_t + f_x S\mu + \frac{1}{2} S^2 \sigma^2 f_{xx})dt \\ &\quad + f_x \sigma S dW. \end{aligned}$$

BS by Arbitrage

We match coefficients and find

$$\varphi_1 = f_x \text{ and } \varphi_0 = \frac{1}{rB}(f_t + \frac{1}{2}\sigma^2 S_t^2 f_{xx}).$$

So $f(t,x)$ must satisfy the Black-Scholes PDE

$$f_t + rxf_x + \frac{1}{2}\sigma^2 x^2 f_{xx} - rf = 0$$

and initial condition $f(T,x) = (x-K)^+$.

Barrier Options

One-barrier options specify a stock-price level, H say, such that the option pays ('knocks in') or not ('knocks out') according to whether or not level H is attained, from below ('up') or above ('down'). There are thus four possibilities: 'up and in', 'up and out', 'down and in' and 'down and out'. Since barrier options are path-dependent (they involve the behaviour of the path, rather than just the current price or price at expiry), they may be classified as exotic; alternatively, the four basic one-barrier types above may be regarded as 'vanilla barrier' options, with their more complicated variants, described below, as 'exotic barrier' options.

Down-and-out Call

Consider a down-and-out call option with strike K and barrier H. The payoff is

$$(S(T) - K)^+ \mathbf{1}_{\{\min S(.) \geq H\}}$$

$$= \quad (S(T) - K)\mathbf{1}_{\{S(T) \geq K, \min S(.) \geq H\}},$$

so by risk-neutral pricing the value of the option $DOC_{K,H}$ is

$$\mathbb{E}^* \left[e^{-rT} (S(T) - K) \mathbf{1}_{\{S(T) \geq K, \min S(.) \geq H\}} \right],$$

where S is geometric Brownian motion.

max and min of BM

Write $c := \mu - \frac{1}{2}\sigma^2/\sigma$; then

$$\min S(.) \geq H$$

iff

$$\min(ct + W(t)) \geq \sigma^{-1}\log(H/p_0).$$

Writing X for $X(t) := ct + W(t)$ – drifting Brownian motion with drift c, m, M for its minimum and maximum processes

$$m(t) := \min\{X(s) : s \in [0,t]\},$$

$$M(t) := \max\{X(s) : s \in [0,t]\},$$

the payoff function involves the bivariate process (X, m), and the option price involves the joint law of this process.

Reflection Principle

Consider $c = 0$. We require the joint law of standard BM and its maximum (or minimum), (W, M).

We choose a level $b > 0$, and run the process until the *first-passage time*

$$\tau(b) := \inf\{t \geq 0 : W(t) \geq b\}.$$

This is a stopping time, and we may use the strong Markov property for W at time $\tau(b)$. The process now begins afresh at level b, and by symmetry the probabilistic properties of its further evolution are invariant under *reflection* in the level b. This *reflection principle* leads to Lévy's joint density formula $(x < y)$

$$\begin{aligned} & \mathbb{P}_0\left(W(t) \in dx, M(t) \in dy\right) \\ = \quad & \frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{1}{2}(2y-x)^2/t\right\}. \end{aligned}$$

Density of $(X(t), M(t))$

Lévy's formula for the joint density of $(W(t), M(t))$ may be extended to the case of general drift c by the usual method for changing drift, Girsanov's theorem. The general result is

$$\begin{aligned} & I\!P_0 \left(X(t) \in dx, M(t) \in dy \right) \\ = \quad & \frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp \left\{ -\frac{(2y-x)^2}{2t} + cx - \frac{1}{2} c^2 t \right\}. \end{aligned}$$

Here as before $0 \leq x \leq y$.

Valuation Formula

It is convenient to decompose the price $DOC_{K,H}$ of the down-and-out call into the (Black-Scholes) price of the corresponding vanilla call, C_K say, and the *knockout discount*, $KOD_{K,H}$ say, by which the knockout barrier at H lowers the price:

$$DOC_{K,H} = C_K - KOD_{K,H}.$$

The option formula is, writing $\lambda := r - \frac{1}{2}\sigma^2$,

$$\begin{aligned} KOD_{K,H} &= p_0(H/p_0)^{2+2\lambda/\sigma^2} N(c_1) \\ &\quad -Ke^{-rT}(H/p_0)^{2\lambda/\sigma^2} N(c_2), \end{aligned}$$

where c_1, c_2 are given by

$$c_{1,2}(p,t) = \frac{\log(H^2/pK) + (r \pm \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

Financial Mathematics

Lecture 9

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Aims and Objectives

- The Bond Market (§8.1);
- Short Rate Models (§8.2)

Bonds

A zero-coupon bond with maturity date T, also called a T-bond, is a contract that guarantees the holder a cash payment of one unit on the date T. The price at time t of a bond with maturity date T is denoted by $p(t,T)$.

Coupon bonds are bonds with regular interest payments, called coupons, plus a principal repayment at maturity. Let c_j be the payments at times t_j, $j = 1, \ldots, n$, F be the face value paid at time t_n. Then the price of the coupon bond B_c must satisfy

$$B_c = \sum_{j=1}^{n} c_j p(0, t_j) + F p(0, t_n).$$

Hence, we see that a coupon bond is equivalent to a portfolio of zero-coupon bonds.

Rates

Given three dates $t < T_1 < T_2$ the basic question is: what is the risk-free rate of return, determined at the contract time t, over the interval $[T_1, T_2]$ of an investment of 1 at time T_1?

t	T_1	T_2
Sell T_1-bond Buy $\frac{p(t,T_1)}{p(t,T_2)}$ T_2-bonds	Pay out 1	Receive $\frac{p(t,T_1)}{p(t,T_2)}$
0	-1	$+\frac{p(t,T_1)}{p(t,T_2)}$

Rates

To exclude arbitrage opportunities, the equivalent constant rate of interest R over this period (we pay out 1 at time T_1 and receive $e^{R(T_2-T_1)}$ at T_2) has thus to be given by

$$e^{R(T_2-T_1)} = \frac{p(t,T_1)}{p(t,T_2)}.$$

Rates

1. The forward rate for the period $[T_1, T_2]$ as seen at time at t is
$$R(t; T_1, T_2) = -\frac{\log p(t, T_2) - \log p(t, T_1)}{T_2 - T_1}.$$
2. The spot rate $R(T_1, T_2)$, for the period $[T_1, T_2]$ is
$$R(T_1, T_2) = R(T_1; T_1, T_2).$$
3. The instantaneous forward rate with maturity T, at time t, is
$$f(t, T) = -\frac{\partial \log p(t, T)}{\partial T}.$$
4. The instantaneous short rate at time t is
$$r(t) = f(t, t).$$

Simple Relations

The money account process is defined by

$$B(t) = \exp\left\{\int_0^t r(s)ds\right\}.$$

The interpretation of the money market account is a strategy of instantaneously reinvesting at the current short rate.

For $t \le s \le T$ we have

$$p(t,T) = p(t,s)\exp\left\{-\int_s^T f(t,u)du\right\},$$

and in particular

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

Process Dynamics

Short-rate Dynamics:

$$dr(t) = a(t)dt + b(t)dW(t),$$

Bond-price Dynamics:

$$dp(t,T) = p(t,T)\left\{m(t,T)dt + v(t,T)dW(t)\right\}$$

Forward-rate Dynamics:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t).$$

EMMs and RNV

A measure $\mathbb{Q} \sim I\!P$ defined on $(\Omega, \mathcal{F}, I\!P)$ is an equivalent martingale measure for the bond market, if for every fixed $0 \leq T \leq T^*$ the process

$$\frac{p(t,T)}{B(t)}, \quad 0 \leq t \leq T$$

is a $\mathbb{Q}$-martingale.

Consider a T-contingent claim X. Then the price process is

$$\Pi_X(t) = \mathbb{E}_{\mathbb{Q}}\left[X e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right].$$

In particular, the price process of a zero-coupon bond with maturity T is given by

$$p(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right].$$

Short-rate model

We now fix an equivalent martingale measure $\mathbb{Q}$ and model the short rate as

$$dr(t) = a(t, r(t))dt + b(t, r(t))dW(t).$$

If the contingent claim is of the form $X = \Phi(r(T))$ its arbitrage-free price process is given by $\Pi_X(t) = F(t, r(t))$, where F is the solution of the partial differential equation

$$F_t + aF_r + \frac{b^2}{2}F_{rr} - rF = 0$$

with terminal condition $F(T, r) = \Phi(r)$ for all $r \in I\!R$. In particular, T-bond prices are given by $p(t, T) = F(t, r(t); T)$, with F solving the PDE and terminal condition $F(T, r; T) = 1$.

Contingent Claim Pricing

We want to evaluate the price of a European call option with maturity S and strike K on an underlying T-bond. This means we have to price the S-contingent claim

$$X = \max\{p(S,T) - K, 0\}.$$

We first have to find the price process $p(t,T) = F(t,r;T)$ by solving the PDE with terminal condition $F(T,r;T) = 1$. Secondly, we use the risk-neutral valuation principle to obtain $\Pi_X(t) = G(t,r)$, with G solving

$$G_t + aG_r + \frac{b^2}{2} G_{rr} - rG = 0$$

and

$$G(S,r) = \max\{F(S,r;T) - K, 0\}, \ \forall r \in I\!R.$$

Affine Term Structure

If bond prices are given as

$$p(t,T) = \exp\{A(t,T) - B(t,T)r\},$$

with $A(t,T)$ and $B(t,T)$ deterministic functions, we say that the model possesses an affine term structure. For

$$a(t,r) = \alpha(t) - \beta(t)r$$

and

$$b(t,r) = \sqrt{\gamma(t) + \delta(t)r},$$

Affine Term Structure

we find that A and B are given as solutions of ODEs

$$\begin{aligned} A_t - \alpha(t)B + \frac{\gamma(t)}{2}B^2 &= 0, \\ (1 + B_t) - \beta(t)B - \frac{\delta(t)}{2}B^2 &= 0, \end{aligned}$$

with $A(T,T) = B(T,T) = 0$. The equation for B is a Riccati equation, which can be solved analytically.

Affine Term Structure

1. *Vasicek model:*
   $dr = (\alpha - \beta r)dt + \gamma dW$;
2. *Cox-Ingersoll-Ross (CIR) model:*
   $dr = (\alpha - \beta r)dt + \delta\sqrt{r}dW$;
3. *Ho-Lee model:*
   $dr = \alpha(t)dt + \gamma dW$;
4. *Hull-White (extended Vasicek) model:*
   $dr = (\alpha(t) - \beta(t)r)dt + \gamma(t)dW$;
5. *Hull-White (extended CIR) model:*
   $dr = (\alpha(t) - \beta(t)r)dt + \delta(t)\sqrt{r}dW$.

Financial Mathematics

Lecture 10

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Aims and Objectives

- HJM models (§8.3)
- Contingent Claims (§8.4)

Heath-Jarrow-Morton (HJM) model

The Heath-Jarrow-Morton model uses the entire forward rate curve as (infinite-dimensional) state variable. The dynamics of the instantaneous, continuously compounded forward rates $f(t,T)$ are *exogenously* given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t).$$

For any fixed maturity T, the initial condition of the stochastic differential equation (**??**) is determined by the current value of the empirical (observed) forward rate for the future date T which prevails at time 0.

Heath-Jarrow-Morton (HJM) model

The exogenous specification of the family of forward rates $\{f(t,T); T > 0\}$ is equivalent to a specification of the entire family of bond prices $\{p(t,T); T > 0\}$. Furthermore, the dynamics of the bond-price processes are

$$dp(t,T) = p(t,T)\,\{m(t,T)dt + S(t,T)dW(t)\},$$

where

$$m(t,T) = r(t) + A(t,T) + \frac{1}{2}\,\|S(t,T)\|^2\,,$$

with

$$A(t,T) = -\int_t^T \alpha(t,s)ds$$

and

$$S(t,T) = -\int_t^T \sigma(t,s)ds.$$

HJM Drift Condition

We want to find an EMM equivalent measure (the *risk-neutral martingale measure*) such that

$$Z(t,T) = \frac{p(t,T)}{B(t)}$$

is a martingale for every $0 \le T \le T^*$.

A risk-neutral EMM exists iff there exists a process $\lambda(t)$, with

1. $$L(t) = e^{-\int_0^t \lambda dW - \frac{1}{2}\int_0^t \|\lambda\|^2 du}.$$

   defines a Girsanov pair and

2. for all $0 \le T \le T^*$ and for all $t \le T$, we have

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds + \sigma(t,T)\lambda(t).$$

Forward Risk-neutral Martingale Measures

For many valuation problems in the bond market it is more suitable to use the bond price process $p(t, T^*)$ as numéraire.

One needs an equivalent probability measure $\mathbb{Q}^*$ such that the auxiliary process

$$Z^*(t,T) = \frac{p(t,T)}{p(t,T^*)}, \quad \forall t \in [0,T],$$

is a martingale under $\mathbb{Q}^*$ for $T \le T^*$.

This measure is called such a measure *forward risk-neutral martingale measure*.

Forward Risk-neutral Martingale Measures (FRN-EMM)

Bond price dynamics under the original probability measure $I\!P$ are given as

$$dp(t,T) = p(t,T)\left\{m(t,T)dt + S(t,T)dW(t)\right\},$$

with $m(t,T)$ from the HJM-drift condition.

Application of Itô's formula to the quotient $p(t,T)/p(t,T^*)$ yields

$$\begin{aligned} dZ^*(t,T) &= Z^*(t,T)\left\{\tilde{m}(t,T)dt \right.\\ &\quad \left. + (S(t,T) - S(t,T^*))dW(t)\right\}, \end{aligned}$$

with

$$\begin{aligned} \tilde{m}(t,T) = &\; m(t,T) - m(t,T^*) \\ &- S(t,T^*)(S(t,T) - S(t,T^*)). \end{aligned}$$

FRN-EMM

The drift coefficient of $Z^*(t,T)$ under any EMM $\mathbb{Q}^*$ is given as

$$\tilde{m}(t,T) - (S(t,T) - S(t,T^*))\gamma(t).$$

For $Z^*(t,T)$ to be a $\mathbb{Q}^*$-martingale this coefficient has to be zero, and replacing $\tilde{m}$ with its definition we get

$$\begin{aligned} & (A(t,T) - A(t,T^*)) \\ + \quad & \tfrac{1}{2}\left(\|S(t,T)\|^2 - \|S(t,T^*)\|^2\right) \\ = \quad & (S(t,T^*) + \gamma(t))\,(S(t,T) - S(t,T^*))\,. \end{aligned}$$

FRN-EMM

Written in terms of the coefficients of the forward-rate dynamics, this identity simplifies to

$$\begin{aligned}&\int_T^{T^*}\alpha(t,s)ds+\frac{1}{2}\left\|\int_T^{T^*}\sigma(t,s)ds\right\|^2\\ = \quad &\gamma(t)\int_T^{T^*}\sigma(t,s)ds.\end{aligned}$$

Taking the derivative with respect to T, we obtain

$$\alpha(t,T)+\sigma(t,T)\int_T^{T^*}\sigma(t,s)ds=\gamma(t)\sigma(t,T).$$

FRN-EMM

There exists a forward risk-neutral martingale measure if and only if there exists an adapted process $\gamma(t)$ such that for all $0 \le t \le T \le T^*$

$$\alpha(t,T) = \sigma(t,T)\left(S(T,T^*) + \gamma(t)\right)$$

Gaussian HJM Framework

Assume that the dynamics of the forward rate are given under a risk-neutral martingale measure $\mathbb{Q}$ by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\tilde{W}(t)$$

with deterministic forward rate volatility. Then

$$\begin{aligned} f(t,t) &= r(t) \\ &= f(0,t) + \int_0^t (-\sigma(u,t)S(u,t))du \\ &\quad + \int_0^t \sigma(u,t)d\tilde{W}(u), \end{aligned}$$

which implies that the short-rate as well as the forward rates $f(t,T)$ have Gaussian probability laws.

Options on Bonds

Consider a European call C on a T^*-bond with maturity $T \leq T^*$ and strike K. So we consider the T-contingent claim

$$X = (p(T, T^*) - K)^+.$$

Its price at time $t = 0$ is

$$C(0) = p(0, T^*)\mathbb{Q}^*(A) - Kp(0, T)\mathbb{Q}^T(A),$$

with $A = \{\omega : p(T, T^*) > K\}$ and $\mathbb{Q}^T$ resp. $\mathbb{Q}^*$ the T- resp. T^*-forward risk-neutral measure.

Options on Bonds

$$\tilde{Z}(t,T) = \frac{p(t,T^*)}{p(t,T)}$$

has $\mathbb{Q}$-dynamics

$$d\tilde{Z} = \tilde{Z}\left\{S(S-S^*)dt - (S-S^*)d\tilde{W}(t)\right\},$$

so a deterministic variance coefficient. Now

$$\begin{aligned} & \mathbb{Q}^*(p(T,T^*) \geq K) \\ = \quad & \mathbb{Q}^*\left(\frac{p(T,T^*)}{p(T,T)} \geq K\right) \\ = \quad & \mathbb{Q}^*(\tilde{Z}(T,T) \geq K). \end{aligned}$$

Options on Bonds

Since $\tilde{Z}(t,T)$ is a $\mathbb{Q}^T$-martingale with $\mathbb{Q}^T$-dynamics

$$d\tilde{Z}(t,T) = -\tilde{Z}(t,T)(S(t,T) - S(t,T^*))dW^T(t),$$

we find that under $\mathbb{Q}^T$

$$\begin{aligned}\tilde{Z}(T,T) &= \frac{p(0,T^*)}{p(0,T)} \exp\left\{-\int_0^T (S - S^*)dW_t^T\right\} \\ &\quad \times \exp\left\{-\frac{1}{2}\int_0^T (S - S^*)^2 dt\right\}\end{aligned}$$

The stochastic integral in the exponential is Gaussian with zero mean and variance

$$\Sigma^2(T) = \int_0^T (S(t,T) - S(t,T^*))^2 dt.$$

Options on Bonds So

$$\begin{aligned}
& \mathbb{Q}^T(p(T,T^*) \geq K) \\
= \quad & \mathbb{Q}^T(\tilde{Z}(T,T) \geq K) = N(d_2)
\end{aligned}$$

with

$$d_2 = \frac{\log\left(\frac{p(0,T)}{Kp(0,T^*)}\right) - \frac{1}{2}\Sigma^2(T)}{\sqrt{\Sigma^2(T)}}.$$

Repeat the argument to get

The price of the call option is given by

$$C(0) = p(0,T^*)N(d_2) - Kp(0,T)N(d_1),$$

with parameters given as above.

Swaps Consider the case of a *forward swap settled in arrears* characterized by:

- a fixed time t, the contract time,
- dates $T_0 < T_1, \ldots < T_n$, equally distanced $T_{i+1} - T_i = \delta$,
- R, a prespecified fixed rate of interest,
- K, a nominal amount.

Swaps

A swap contract S with K and R fixed for the period $T_0, \ldots T_n$ is a sequence of payments, where the amount of money paid out at $T_{i+1},\ i = 0, \ldots, n-1$ is defined by

$$X_{i+1} = K\delta(L(T_i, T_i) - R).$$

The floating rate over $[T_i, T_{i+1}]$ observed at T_i is a simple rate defined as

$$p(T_i, T_{i+1}) = \frac{1}{1 + \delta L(T_i, T_i)}.$$

Swaps

Using the risk-neutral pricing formula we obtain (we may use $K = 1$),

$$
\begin{aligned}
\Pi(t, S) &= \sum_{i=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \delta(L(T_i, T_i) - R) \middle| \mathcal{F}_t \right] \\
&= \sum_{i=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{T_{i-1}}^{T_i} r(s)ds} \middle| \mathcal{F}_{T_{i-1}} \right] \right. \\
&\qquad \left. \times\ e^{-\int_t^T r(s)ds} \left(\frac{1}{p(T_{i-1}, T_i)} - (1 + \delta R) \right) \middle| \mathcal{F}_t \right] \\
&= \sum_{i=1}^{n} \left(p(t, T_{i-1}) - (1 + \delta R) p(t, T_i) \right) \\
&= p(t, T_0) - \sum_{i=1}^{n} c_i p(t, T_i),
\end{aligned}
$$

with $c_i = \delta R, i = 1, \ldots, n-1$ and $c_n = 1 + \delta R$. So a swap is a linear combination of zero-coupon bonds, and we obtain its price accordingly.

Caps An interest cap is a contract where the seller of the contract promises to pay a certain amount of cash to the holder of the contract if the interest rate exceeds a certain predetermined level (the cap rate) at some future date. A cap can be broken down in a series of caplets.

A caplet is a contract written at t, in force between $[T_0, T_1]$, $\delta = T_1 - T_0$, the nominal amount is K, the cap rate is denoted by R. The relevant interest rate (LIBOR, for instance) is observed in T_0 and defined by

$$p(T_0, T_1) = \frac{1}{1 + \delta L(T_0, T_0)}.$$

Caplets

A caplet C is a T_1-contingent claim with payoff

$$X = K\delta(L(T_0, T_0) - R)^+.$$

Writing $L = L(T_0, T_0), p = p(T_0, T_1), R^* = 1 + \delta R$, we have $L = (1-p)/(\delta p)$, (assuming $K = 1$) and

$$\begin{aligned} X &= \delta(L - R)^+ = \delta\left(\frac{1-p}{\delta p} - R\right)^+ \\ &= \left(\frac{1}{p} - (1 + \delta R)\right)^+ = \left(\frac{1}{p} - R^*\right)^+. \end{aligned}$$

Caplets

The risk-neutral pricing formula leads to

$$
\begin{aligned}
\Pi_C(t) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_1} r(s)ds} \left(\tfrac{1}{p} - R^*\right)^+ \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r(s)ds} \Big| \mathcal{F}_{T_0} \right] e^{-\int_t^{T_0} r(s)ds} \left(\frac{1}{p} - R^*\right)^+ \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[p(T_0, T_1)\, e^{-\int_t^{T_0} r(s)ds} \left(\frac{1}{p} - R^*\right)^+ \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_0} r(s)ds} \left(1 - pR^*\right)^+ \Big| \mathcal{F}_t \right] \\
&= R^* \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_0} r(s)ds} \left(\frac{1}{R^*} - p\right)^+ \Big| \mathcal{F}_t \right].
\end{aligned}
$$

So a caplet is equivalent to R^* put options on a T_1-bond with maturity T_0 and strike $1/R^*$.

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