

Basic Options

2.1 Asset Price Model and Itô's Lemma

2.1.1 A Model for Asset Prices

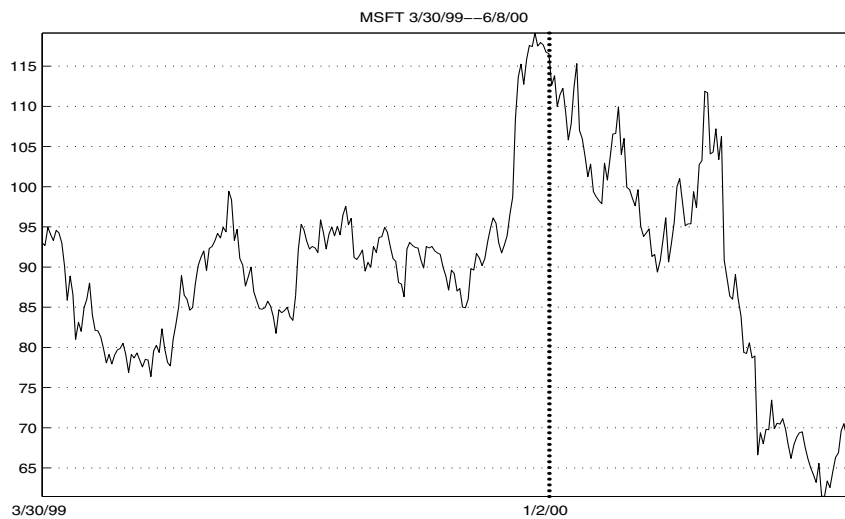


Fig. 2.1. Stock price of Microsoft

As examples, in Figs. 1.1–1.7 we showed how the prices of assets vary with time t . Fig. 2.1 shows the stock price of Microsoft Inc. in the period March 30, 1999 to June 8, 2000. From these figures, we can see the following: the graphs are jagged and the size of the jags changes all the time. This means that we cannot express S as a smooth function of t , and it is difficult to predict exactly the price at time t from the price before time t . It is natural to think

of the price at time t as a random variable. Now let us give a model for such a random variable.

Suppose that at time t the asset price is S . Let us consider a small subsequent time interval dt , during which S changes to $S + dS$. (We use the notation df for the small change in any quantity f over this time interval.) How might we model the corresponding return rate on the asset, dS/S ?

Assume that the return rate on the asset can be described by the following stochastic differential equation:

$$\frac{dS}{S} = \mu(S, t)dt + \sigma(S, t)dX, \quad (2.1)$$

where μ and σ are called the **drift** and the **volatility** respectively, and dX is known as a Wiener process defined by

$$\begin{cases} dX = \phi\sqrt{dt}, \\ \phi \text{ being a standardized normal random variable.} \end{cases}$$

In this model, the first part is an anticipated and deterministic return rate, and the second part is the random return rate of the asset price in response to unexpected events. As we can see, the next asset price $S + dS$ depends solely on today's price. This independence from the past is known as the Markov property. In many situations it is assumed that μ and σ are constants. Due to its simplicity, this is a popular model for asset prices

For a random variable ψ with a probability density function $f(\psi)$ defined on $(-\infty, \infty)$, the expectation of any function $F(\psi)$ is

$$\int_{-\infty}^{\infty} F(\psi)f(\psi)d\psi,$$

i.e., if $E[F(\psi)]$ denotes the expectation of $F(\psi)$, then

$$E[F(\psi)] = \int_{-\infty}^{\infty} F(\psi)f(\psi)d\psi.$$

The variance of $F(\psi)$, $\text{Var}[F(\psi)]$, is defined by

$$\text{Var}[F(\psi)] = E[(F(\psi) - E[F(\psi)])^2].$$

According to these definitions, for any constants a, b, c and random variable W , we have

$$\begin{aligned} E[aW - b] &= aE[W] - b, \\ \text{Var}[W] &= E[(W - E[W])^2] \\ &= E[W^2] - (E[W])^2 \end{aligned}$$

and

$$\text{Var} \left[\frac{W}{c} \right] = \frac{1}{c^2} \text{Var} [W].$$

For a standardized normal random variable ϕ , the probability density function is

$$\frac{1}{\sqrt{2\pi}} e^{-\phi^2/2}, \quad -\infty < \phi < \infty.$$

As a probability density function, this function satisfies¹

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\phi^2/2} d\phi = 1.$$

Therefore we have

$$\text{E} [\phi] = \int_{-\infty}^{\infty} \phi \frac{1}{\sqrt{2\pi}} e^{-\phi^2/2} d\phi = 0$$

and

$$\begin{aligned} \text{Var} [\phi] &= \text{E} [\phi^2] \\ &= \int_{-\infty}^{\infty} \phi^2 \frac{1}{\sqrt{2\pi}} e^{-\phi^2/2} d\phi \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi d(e^{-\phi^2/2}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\phi^2/2} d\phi \\ &= 1. \end{aligned}$$

From these we obtain

$$\text{E} [dX] = \text{E} [\phi] \sqrt{dt} = 0$$

and

$$\text{Var} [dX] = \text{E} [dX^2] = \text{E} [\phi^2] dt = dt.$$

Consequently²

$$\text{E} [dS] = \text{E} [\sigma S dX + \mu S dt] = \mu S dt,$$

and

$$\begin{aligned} \text{Var} [dS] &= \text{E} [dS^2] - (\text{E} [dS])^2 \\ &= \text{E} [\sigma^2 S^2 dX^2 + 2\sigma S^2 \mu dt dX + \mu^2 S^2 dt^2] - \mu^2 S^2 dt^2 \\ &= \sigma^2 S^2 dt. \end{aligned}$$

¹Since $\int_0^\infty e^{-x^2/2} dx \times \int_0^\infty e^{-y^2/2} dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2/2} r dr d\theta = \pi/2$, we have $\int_0^\infty e^{-\phi^2/2} d\phi = \sqrt{\pi/2}$.

²Here dX is a random variable and S is unchanged. In stochastic calculus, it is called conditional expectation (see [45] and [5]).

The square root of the variance is known as the standard deviation. Thus the deviation of the return on the asset is proportional to σ . This means that an asset price with a larger σ would appear more jagged. Typically, for stocks, indices, exchange rates and bonds, the value of σ is in the range 0.05 to 0.4. Usually, the volatility of stocks is greater than indices, exchange rates and bonds, and government bonds have the smallest volatility among these. Among shares, high-tech companies tend to have higher volatility than other companies. For example, assume that the volatility of the price of IBM stock is a constant during 1990–2000, then its value is 0.31. Under the same assumption, for the price of GE stock, $\sigma = 0.23$. For S&P 500, British pound–US dollar exchange rate, Japanese yen–US dollar exchange rate and a five year government bond with coupon 6.5% and maturing on May 31, 2001, $\sigma = 0.10, 0.11, 0.12$ and 0.03 respectively. For the bond, we assume that σ depends upon the time to maturity. Clearly, at maturity σ is zero. The value 0.03 means that the maximum value of σ is 0.03. In practice, the volatility is often quoted as a percentage so that $\sigma = 0.2$ would be 20% volatility.

If $\sigma = 0$, then

$$\frac{dS}{S} = \mu dt$$

and

$$S(t) = S_0 e^{\mu(t-t_0)},$$

where S_0 is the value of the asset at $t = t_0$.

In this asset price model, μ and σ are two parameters. In general, these parameters depend on the asset price S and time t , i.e., $\mu = \mu(S, t)$, $\sigma = \sigma(S, t)$. According to the historical data, we can determine these parameters (or parameter functions) for the past by statistical analysis. If we assume that μ and σ depend on S only, then the functions $\mu(S)$ and $\sigma(S)$ determined by the historical data can be used for the future.

A Wiener process is also referred to as a Brownian motion. There are many excellent books on the Brownian motion. Readers interested in this subject can read, for example, [45]. A basic and very important feature of the Wiener process is that the sum of two independent Wiener processes is also a Wiener process and the variance of the sum is the sum of the two original variances. That is, if $dX_1 = \phi_1 \sqrt{dt_1}$ and $dX_2 = \phi_2 \sqrt{dt_2}$ are two Wiener processes and they are independent, namely, $E[\phi_1 \phi_2] = 0$, then

$$dX_3 = dX_1 + dX_2 = \phi_1 \sqrt{dt_1} + \phi_2 \sqrt{dt_2} = \phi_3 \sqrt{dt_1 + dt_2}, \quad (2.2)$$

where ϕ_3 is also a standardized normal random variable.

2.1.2 Itô's Lemma

There is a practical lower bound for the basic time-step dt of the random walk of an asset price. Thus an asset price is a discrete random variable.

However, sometimes the lower bound is so small that we consider an asset price as a continuous random variable. Also, because it is much more efficient to solve the resulting differential equations than to evaluate options by direct simulation of the random walk on a practical timescale, we will assume that an asset price is a continuous random variable even if the basic time-step is not very small.

Before coming to Itô's lemma, we need one result, which we do not prove. This result is, with probability one,

$$dX^2 = \phi^2 dt \rightarrow dt \quad \text{as} \quad dt \rightarrow 0.$$

This can be explained as follows. Since

$$\mathbb{E}[dX^2] = \mathbb{E}[\phi^2] dt = dt$$

and

$$\text{Var}[dX^2] = \mathbb{E}[dX^4] - (\mathbb{E}[dX^2])^2 = O(dt^2),$$

the variance of dX^2 is very small and the smaller dt becomes, the closer dX^2 comes to being equal to dt .

Assume

$$dS = a(S, t)dt + b(S, t)dX$$

and suppose $f(S, t)$ is a smooth function of a random variable S and time t . We need to find a stochastic differential equation for f . If we vary S and t by a small amount dS and dt , then f also varies by a small amount. From the Taylor series expansion we can write

$$df = \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2} \left(\frac{\partial^2 f}{\partial S^2}dS^2 + 2 \frac{\partial^2 f}{\partial t \partial S}dt dS + \frac{\partial^2 f}{\partial t^2}dt^2 \right) + \dots$$

Since

$$\begin{aligned} dS^2 &= (a(S, t)dt + b(S, t)dX)^2 = \left(adt + b\phi\sqrt{dt}\right)^2 \\ &= a^2(dt)^2 + 2ab\phi(dt)^{3/2} + b^2\phi^2dt \rightarrow b^2dt \quad \text{as} \quad dt \rightarrow 0, \end{aligned}$$

we have

$$df = \frac{\partial f}{\partial S}dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 f}{\partial S^2} \right) dt \quad \text{as} \quad dt \rightarrow 0 \quad (2.3)$$

or in the form of a stochastic differential equation

$$df = b \frac{\partial f}{\partial S}dX + \left(\frac{\partial f}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 f}{\partial S^2} + a \frac{\partial f}{\partial S} \right) dt.$$

This is Itô's lemma. If in the asset price model (2.1), μ and σ are constants, i.e.,

$$dS = \mu Sdt + \sigma SdX,$$

then Itô's lemma is in the form:

$$\begin{aligned} df &= \frac{\partial f}{\partial S} dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt \\ &= \sigma S \frac{\partial f}{\partial S} dX + \left(\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \mu S \frac{\partial f}{\partial S} \right) dt. \end{aligned}$$

2.1.3 Expectation and Variance of Lognormal Random Variables

As a simple example, consider the function $f(S) = \ln S$. Differentiation of this function gives

$$\frac{df}{dS} = \frac{1}{S} \quad \text{and} \quad \frac{d^2 f}{dS^2} = -\frac{1}{S^2}.$$

Suppose that S satisfies (2.1) with constant μ and σ , i.e., $dS = \mu S dt + \sigma S dX$. Using Itô's lemma, for $\ln S$ we have

$$d \ln S = \sigma dX + \left(\mu - \frac{\sigma^2}{2} \right) dt = m dt + \sigma dX, \quad (2.4)$$

where

$$m = \mu - \frac{\sigma^2}{2}. \quad (2.5)$$

It is clear that

$$\mathbb{E}[d \ln S] = \mathbb{E}[m dt + \sigma dX] = m dt$$

and

$$\begin{aligned} \text{Var}[d \ln S] &= \mathbb{E}[(d \ln S)^2] - (\mathbb{E}[d \ln S])^2 \\ &= \mathbb{E}[\sigma^2 dX^2 + 2\sigma m dt dX + m^2 dt^2] - m^2 dt^2 \\ &= \sigma^2 \mathbb{E}[\phi^2 dt] = \sigma^2 dt. \end{aligned}$$

From (2.4) for $d \ln S$ the probability density function is³

$$\frac{1}{\sigma \sqrt{2\pi dt}} e^{-(d \ln S - m dt)^2 / 2\sigma^2 dt}.$$

³ • Here $e^{-(d \ln S - m dt)^2 / 2\sigma^2 dt}$ means $e^{-(d \ln S - m dt)^2 / (2\sigma^2 dt)}$. That is, in the expression $(d \ln S - m dt)^2 / 2\sigma^2 dt$ the division between $(d \ln S - m dt)^2$ and $2\sigma^2 dt$ should be done after $2 \times \sigma^2 \times dt$ is obtained. In the entire book we use such a notation.

• If x is a normal random variable and its mean and variance are a and b^2 , then its probability density function is

$$\frac{1}{b\sqrt{2\pi}} e^{-(x-a)^2 / 2b^2}.$$

Let $d \ln S = \ln S' - \ln S$. Then for $\ln S'$, the probability density function is

$$G_1(\ln S') = \frac{1}{\sigma\sqrt{2\pi dt}} e^{-[\ln S' - \ln S - mdt]^2 / 2\sigma^2 dt}.$$

Here S is the value of the asset at time t and S' is the value of the asset at time $t + dt$ which is a random variable. In Fig. 2.2 the curve of $G_1(\ln S')$ with $\ln S + mdt = 0$ and $\sigma\sqrt{dt} = 0.2$ is shown.

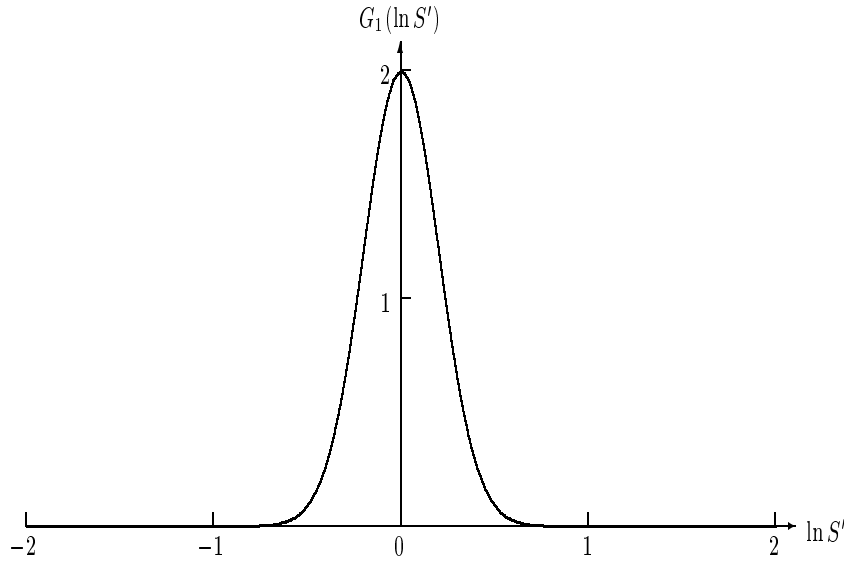


Fig. 2.2. The probability density function for $\ln S'$ with $\ln S + mdt = 0$ and $\sigma\sqrt{dt} = 0.2$

Suppose that for S' the probability density function is $G(S')$. Since⁴

$$G(S') dS' = \frac{1}{\sigma\sqrt{2\pi dt}} e^{-(\ln S' - \ln S - mdt)^2 / 2\sigma^2 dt} d \ln S',$$

we have

⁴If for x the probability density function is $f(x)$, then the probability of $x \in [x, x + dx]$ is $f(x)dx$. If $y = y(x)$ and $y(x)$ is a nondecreasing function, then $x \in [x, x + dx]$ if and only if $y \in [y(x), y(x + dx)] \approx \left[y(x), y(x) + \frac{dy}{dx} dx \right]$. Thus the probability of the event $y \in \left[y(x), y(x) + \frac{dy(x)}{d(x)} dx \right]$ is also $f(x)dx$. If for y the probability density function is $f_1(y)$, then $f_1(y)dy = f(x)dx$, from which we have $f_1(y) = f(x(y)) \frac{dx}{dy}$. If $x = \ln S'$ and $y = S'$, then $f_1(S') = f(x(y)) \frac{dx}{dy} = f(\ln S') \frac{1}{S'}$.

$$G(S') = \frac{1}{S' \sigma \sqrt{2\pi dt}} e^{-(\ln S' - \ln S - mdt)^2 / 2\sigma^2 dt}.$$

In Fig. 2.3 the corresponding curve of $G(S')$ is given. This is called a lognormal distribution since the corresponding distribution for $\ln S'$ is normal.

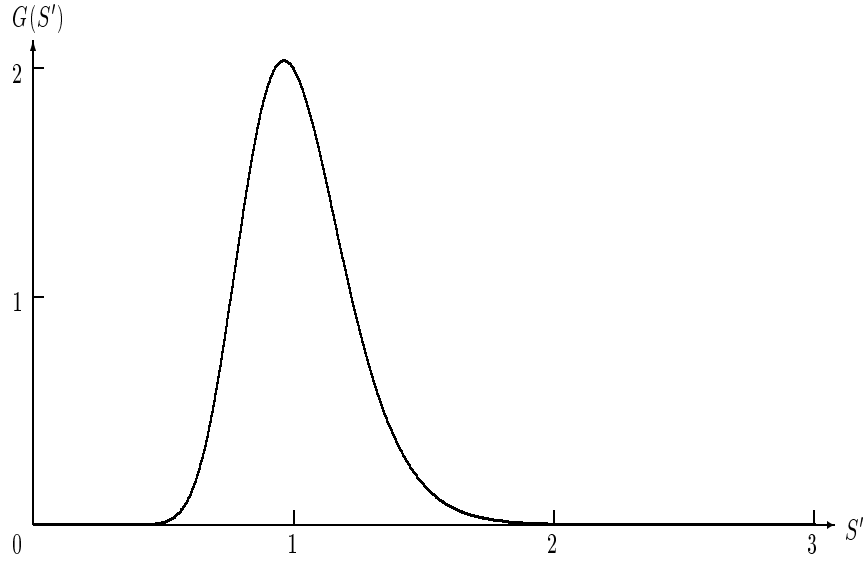


Fig. 2.3. The probability density function for S' with $\ln S + mdt = 0$ and $\sigma\sqrt{dt} = 0.2$

Now the question is what are $E[S']$ and $\text{Var}[S']$. Since we have the probability density function, let

$$y = \frac{\ln S' - \ln S - mdt}{\sigma\sqrt{dt}}$$

and we have

$$\begin{aligned} E[S'] &= \int_0^\infty G(S') S' dS' \\ &= \frac{1}{\sigma\sqrt{2\pi dt}} \int_0^\infty e^{-(\ln S' - \ln S - mdt)^2 / 2\sigma^2 dt} \frac{1}{S'} \times S' dS' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-y^2/2} e^{y\sigma\sqrt{dt} + \ln S + mdt} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-(y - \sigma\sqrt{dt})^2/2} \times e^{\sigma^2 dt/2 + \ln S + mdt} dy \\ &= e^{\sigma^2 dt/2 + \ln S + mdt} = S e^{\mu dt}, \end{aligned}$$

$$\begin{aligned}
E[S'^2] &= \int_0^\infty G(S') S'^2 dS' \\
&= \frac{1}{\sigma\sqrt{2\pi dt}} \int_0^\infty e^{-(\ln S' - \ln S - mdt)^2 / 2\sigma^2 dt} \frac{1}{S'} S'^2 dS' \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-y^2/2} \times e^{2(y\sigma\sqrt{dt} + \ln S + mdt)} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-(y-2\sigma\sqrt{dt})^2/2} e^{2\sigma^2 dt + 2(\ln S + mdt)} dy \\
&= e^{2\sigma^2 dt + \ln S^2 + 2mdt} = S^2 e^{2\mu dt + \sigma^2 dt}
\end{aligned}$$

and

$$\begin{aligned}
\text{Var}[S'] &= S^2 e^{2\mu dt + \sigma^2 dt} - S^2 e^{2\mu dt} \\
&= S^2 e^{2\mu dt} (e^{\sigma^2 dt} - 1),
\end{aligned}$$

where we have used (2.5).

If m and σ in (2.4) are constants, then for a large time period $T - t$, we can have

$$\ln S_T - \ln S = \int_t^T d \ln S = m \int_t^T dt + \sigma \int_t^T dX_t = m(T - t) + \sigma \phi \sqrt{T - t},$$

where S_T is the stock price at time T , S is the stock price at time t and ϕ is a standardized normal random variable. Here we used the relation $\int_t^T dX_t = \phi \sqrt{T - t}$, which can be obtained from (2.2). Therefore, in this case, the probability density function for S_T is

$$G(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi(T - t)}} e^{-[\ln S_T - \ln S - m(T - t)]^2 / 2\sigma^2(T - t)}$$

and

$$\begin{cases} E[S_T] = S e^{\mu(T-t)}, \\ \text{Var}[S_T] = S^2 e^{2\mu(T-t)} (e^{\sigma^2(T-t)} - 1), \end{cases} \quad (2.6)$$

where μ is given by (2.5):

$$\mu = m + \frac{\sigma^2}{2}.$$

2.2 Derivation of the Black–Scholes Equation

2.2.1 Arbitrage Arguments

In the modern world, financial transactions may be done simultaneously in more than one market. Suppose the price of gold is \$324 per ounce in New

York and 37,275 Japanese Yen in Tokyo, while the exchange rate is 1 US dollar for 115 Japanese Yen. An arbitrageur, who is always looking for any arbitrage opportunities to make money, could simultaneously buy 1,000 ounces in New York, sell them in Tokyo to obtain a risk-free profit of

$$37,275 \times 1,000/115 - 324 \times 1,000 = \$130.43$$

if the transaction costs can be ignored. Small investors may not profit from such opportunity due to the transaction costs. However, for large investors the transaction costs might be a small portion of the profit, which makes the arbitrage opportunity very attractive.

Arbitrage opportunities usually cannot last long. As arbitrageurs buy the gold in New York, the price of the gold will rise. Similarly, as they sell the gold in Tokyo, the price of the gold will be driven down. Very quickly, the ratio between the two prices will become closer to the current exchange rate. In practice, due to the existence of arbitrageurs, very few arbitrage opportunities can be observed. Therefore, throughout this book we will assume that there are no arbitrage opportunities for financial transactions.

Let us make the following assumptions: both the borrowing interest rate and the lending interest rate are equal to r , short selling is permitted, the assets and options are divisible, and there is no transaction cost. Then we can conclude that the absence of arbitrage opportunities is equivalent to all risk-free portfolios having the same return rate r .

Let us show this point. Suppose that Π is the value of a portfolio and that during a time step dt the return of the portfolio $d\Pi$ is risk-free. If

$$d\Pi > r\Pi dt,$$

then an arbitrageur could make a risk-free profit $d\Pi - r\Pi dt$ during the time step dt by borrowing an amount Π from a bank to invest in the portfolio. Conversely, if

$$d\Pi < r\Pi dt,$$

then the arbitrageur would short the portfolio and invest Π in a bank and get a net income $r\Pi dt - d\Pi$ during the time step dt without taking any risk. Only when

$$d\Pi = r\Pi dt$$

holds, is it guaranteed that there are no arbitrage opportunities.

In the next subsection, we will derive the equation the prices of derivative securities should satisfy by using the conclusion that all risk-free portfolios have the same return rate r .

2.2.2 The Black–Scholes Equation

Let V denote the value of an option which depends upon the value of the underlying asset S and time t , i.e., $V = V(S, t)$. It is not necessary at this

stage to specify whether V is a call or a put; indeed, V can even be the value of a whole portfolio of various options. For simplicity the reader may think of a simple call or put.

Assume that in a time step dt the underlying asset pays out a dividend SD_0dt , where D_0 is a constant known as the dividend yield.⁵ Suppose S satisfies (2.1):

$$dS = \mu Sdt + \sigma SdX.$$

According to Itô's lemma (2.3), the random walk followed by V is given by

$$dV = \frac{\partial V}{\partial S}dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (2.7)$$

We require V to have at least one t derivative and two S derivatives.

Now construct a portfolio consisting of one option and a number $-\Delta$ of the underlying asset. This number is as yet unspecified. The value of this portfolio is

$$\Pi = V - \Delta S. \quad (2.8)$$

Since the portfolio contains one option and a number $-\Delta$ of the underlying asset, and the owner of the portfolio receives SD_0dt for every asset held, the earnings for the owner of the portfolio during the time step dt is

$$d\Pi = dV - \Delta dS - \Delta SD_0dt.$$

Using (2.7), we find that Π follows the random walk

$$d\Pi = \left(\frac{\partial V}{\partial S} - \Delta \right) dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta SD_0 \right) dt.$$

The random component in this random walk can be eliminated by choosing

$$\Delta = \frac{\partial V}{\partial S}. \quad (2.9)$$

This results in a portfolio whose increment is wholly deterministic:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta SD_0 \right) dt. \quad (2.10)$$

Since for any risk-free portfolio the return should be r , we have

$$r\Pi dt = d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta SD_0 \right) dt. \quad (2.11)$$

⁵This dividend structure is a good model for an index. In this case many discrete dividend payments are paid at many different times and the dividend payment can be approximated by a continuous yield without serious error. Also if the asset is a foreign currency, then the interest rate for the foreign currency plays the role of D_0 .

Substituting (2.8) and (2.9) into (2.11) and dividing by dt we arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV = 0. \quad (2.12)$$

When we take different Π for different S and t , we can conclude that (2.12) holds on a domain. In this book equation (2.12) is called the Black–Scholes partial differential equation or the Black–Scholes equation⁶ even though $D_0 = 0$ in the equation originally given by Black and Scholes (see [10]). With its extensions and variants, it plays the major role in the rest of the book.

About the derivation of this equation and the equation itself, we give more explanation here.

- The key idea of deriving this equation is to eliminate the uncertainty or the risk. $d\Pi$ is not a differential in usual sense. It is the earning of the holder of the portfolio during the time step dt . Therefore $\Delta SD_0 dt$ appear. In the derivation, in order to eliminate any small risk, Δ is chosen before an uncertainty appears and does not depend on the coming risk. Therefore no differential of Δ is needed.
- The linear differential operator given by

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S \frac{\partial}{\partial S} - r$$

has a financial interpretation as a measure of the difference between the return on a hedged option portfolio

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} - D_0 S \frac{\partial}{\partial S}$$

and the return on a bank deposit

$$r \left(1 - S \frac{\partial}{\partial S} \right).$$

Although the difference between the two returns is identically zero for European options, we will later see that the difference between the two returns may be nonzero for American options.

- From the Black–Scholes equation (2.12) we know that the parameter μ in (2.1) does not affect the option price, i.e., the option price determined by this equation is independent of the average change rate of an asset price with respect to time.
- If dividends are paid only on certain dates, then the money the owner of the portfolio will get during the time period $[t, t + dt]$ is

$$dV - \Delta dS - \Delta D(S, t)dt,$$

⁶It is also called Black–Scholes–Merton differential equation (see [39]).

where $D(S, t)$ is a sum of several Dirac delta functions. Suppose that a stock pays dividend $D_1(S)$ at time t_1 and $D_2(S)$ at time t_2 for a share, where $D_1(S) \leq S$ and $D_2(S) \leq S$. Then

$$D(S, t) = D_1(S)\delta(t - t_1) + D_2(S)\delta(t - t_2),$$

where the Dirac delta function⁷ $\delta(t)$ is defined as follows:

$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0, \\ \infty, & \text{if } t = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t) dt = 1.$$

In this case the modified Black–Scholes equation is in the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (rS - D(S, t)) \frac{\partial V}{\partial S} - rV = 0. \quad (2.13)$$

2.2.3 Final Conditions for the Black–Scholes Equation

From the derivation of the Black–Scholes equation (2.13), we know that this partial differential equation holds for any option (or portfolio of options) whose value depends only on S and t . In order to determine a unique solution of the Black–Scholes equation, the solution at the expiry, $t = T$, needs to be given. This condition is called the final condition for the partial differential equation. Different options satisfy the same partial differential equation, but different final conditions. Therefore in order to determine the price of an option, we need to know the value of the option at time T . In what follows, we will derive the final conditions for call and put options.

Final condition for call options. Let us examine what a holder of a call option will do just at the moment of expiry. If $S > E$ at expiry, it makes financial sense for the holder to exercise the call option, handing over an amount E for an asset worth S . The money earned by the holder from such a transaction is then $S - E$. On the other hand, if $S < E$ at expiry, the holder should not exercise the option because the holder would lose an amount of $E - S$. In this case, the option expires valueless. Thus, the value of the call option at expiry can be written as

⁷It is the limit as $\varepsilon \rightarrow 0$ of the one-parameter family of functions:

$$\delta_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon}, & -\varepsilon \leq x \leq \varepsilon, \\ 0, & |x| > \varepsilon. \end{cases}$$

$$C(S, T) = \max(S - E, 0). \quad (2.14)$$

This function given the value of a call option at expiry is usually called the payoff function of a call option. In Fig. 1.9, we plot $\max(S - E, 0)$ as a function of S , which is usually known as a payoff diagram. A call option with such a payoff is the simplest call option and is known as a vanilla call option.

Final condition for put options. Each option or each portfolio of options has its own payoff at expiry. An argument similar to that given above for the value of a call at expiry leads to the payoff for a put option. At expiry the put option is worthless if $S > E$ but has the value $E - S$ for $S < E$. Thus the payoff function of a put option is

$$P(S, T) = \max(E - S, 0). \quad (2.15)$$

The payoff diagram for a put is shown in Fig. 1.10 where the line shows the payoff function $\max(E - S, 0)$. In order to distinguish this put option from other more complicated put options, sometimes it is referred to as the vanilla put option.

2.2.4 Hedging and Greeks

The way to reduce the sensitivity of a portfolio to the movement of something by taking opposite positions in different financial instruments is called hedging. Hedging is a basic concept in finance. When we derived the Black-Scholes equation in Subsection 2.2.2, we chose the delta to be $\frac{\partial V}{\partial S}$, so that the portfolio Π became risk-free. This gives an important example on how hedging is applied. Let us see another example of hedging which is similar to what we have used in deriving the Black-Scholes equation.

Consider a call option on a stock. Fig. 2.4 shows the relation between the call price and the underlying stock price. When the stock price corresponds to point A, the option price corresponds to point B and the Δ of the call is the slope of the line indicated. As an approximation

$$\Delta = \frac{\delta c}{\delta S},$$

where δS is a small change in the stock price and δc is the corresponding change in the call price.

Assume that the delta of the call option is 0.7 and a writer sold 10,000 shares of call options. Then the writer's position could be hedged by buying $0.7 \times 10,000 = 7,000$ shares of stocks. If the stock price goes up by \$0.50 the writer will earn \$3,500 from the 7,000 shares of stocks held. At the same time, the price of call options will go up approximately $0.7 \times 0.5 = \$0.35$ and he will lose $10,000 \times \$0.35 = \$3,500$ from 10,000 shares of option he sold. Therefore the net profit or loss is about zero. If the price falls down by a small amount,

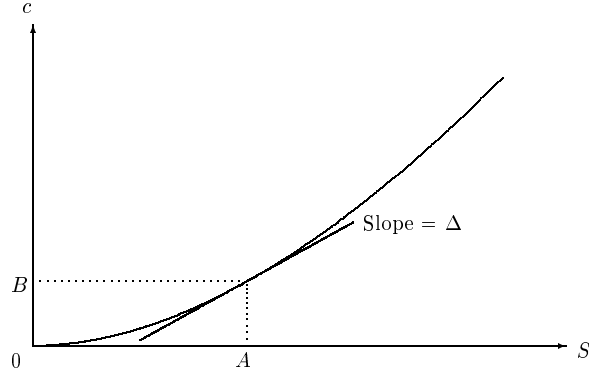


Fig. 2.4. Δ = the slope of a curve

the situation is similar. Consequently the writer's position has been hedged quite well as long as the movement of the price is small. This is called delta hedging.

In the example above, we have considered only a call option. Actually any portfolio can be hedged in this way. If Π denotes the price of option, then the slope is

$$\Delta = \frac{\partial \Pi}{\partial S}.$$

If the movement of the price is not very small, then it might be necessary to use the value of the second derivative of the portfolio with respect to S in order to eliminate most of the risk. The second derivative is known as gamma

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}.$$

When hedging in practice, some other values, for example, $\frac{\partial \Pi}{\partial t}$, $\frac{\partial \Pi}{\partial \sigma}$, $\frac{\partial \Pi}{\partial r}$, $\frac{\partial \Pi}{\partial D_0}$, may need to be known. Usually $\frac{\partial \Pi}{\partial t}$, $\frac{\partial \Pi}{\partial \sigma}$ and $\frac{\partial \Pi}{\partial r}$ are called theta, vega and rho respectively, namely, the following notation is used:

$$\Theta = \frac{\partial \Pi}{\partial t},$$

$$\mathcal{V} = \frac{\partial \Pi}{\partial \sigma}$$

and

$$\rho = \frac{\partial \Pi}{\partial r}.$$

In currency options, D_0 is the interest rate in the foreign country. Thus $\frac{\partial \Pi}{\partial D_0}$ is also known as rho. In order to distinguish $\frac{\partial \Pi}{\partial r}$ and $\frac{\partial \Pi}{\partial D_0}$, here we define

$$\rho_d = \frac{\partial \Pi}{\partial D_0}.$$

These quantities are usually referred to as Greeks.

2.3 Two Transformations on the Black–Scholes Equation

In this section we introduce two transformations. One transformation reduces the Black–Scholes equation to the heat equation. Since Green’s function of the heat equation has an analytic expression, we can obtain an analytic expression of Green’s function for the Black–Scholes equation using the inverse transformation. Based on this, analytic expressions of European call and put option prices can be derived. These are the famous Black–Scholes formulae. When σ depends on S or the payoff function is quite complicated, analytic expressions of option prices may not exist. In this case we have to use numerical methods. Also sometimes (for example, for a call option) the solution is unbounded. It is not convenient to solve a problem numerically on an infinite domain with an unbounded solution. Therefore in Subsection 2.3.2 we also provide a transformation under which the Black–Scholes equation on $[0, \infty)$ becomes an equation on $[0, 1)$ with a bounded solution.

2.3.1 Converting the Black–Scholes Equation into a Heat Equation

The price of a European option is a solution of the following problem

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV = 0, & 0 \leq S, \quad t \leq T, \\ V(S, T) = V_T(S), & 0 \leq S. \end{cases} \quad (2.16)$$

The payoff function $V_T(S)$ is determined by the feature of the option. For example, the payoffs of European calls and puts are given by

$$V(S, T) = \max(\pm(S - 1), 0), \quad 0 \leq S,$$

where $+$ and $-$ in \pm correspond to call and put options respectively. Here the exercise price is 1 because we assume that both the price of the stock and the price of option have been divided by the exercise price. We call a problem with such a payoff a standard put/call problem. Let us set

$$\begin{cases} y = \ln S, \\ \tau = T - t, \\ V(S, t) = e^{-r(T-t)} v(y, \tau). \end{cases} \quad (2.17)$$

Since

$$\begin{aligned}
\frac{\partial V}{\partial t} &= e^{-r(T-t)} \left(rv - \frac{\partial v}{\partial \tau} \right), \\
\frac{\partial V}{\partial S} &= e^{-r(T-t)} \frac{\partial v}{\partial y} \frac{dy}{dS} = e^{-r(T-t)} \frac{1}{S} \frac{\partial v}{\partial y}, \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S} \left(\frac{\partial V}{\partial S} \right) = \frac{\partial}{\partial S} \left(e^{-r(T-t)} \frac{1}{S} \frac{\partial v}{\partial y} \right) \\
&= e^{-r(T-t)} \left(-\frac{1}{S^2} \frac{\partial v}{\partial y} + \frac{1}{S^2} \frac{\partial^2 v}{\partial y^2} \right),
\end{aligned}$$

the Black-Scholes equation is converted into

$$-\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 \left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} \right) + (r - D_0) \frac{\partial v}{\partial y} = 0,$$

and the problem above becomes

$$\begin{cases} \frac{\partial v}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial y^2} + \left(r - D_0 - \frac{1}{2}\sigma^2 \right) \frac{\partial v}{\partial y}, & -\infty < y < \infty, \quad 0 \leq \tau, \\ v(y, 0) = V_T(e^y), & -\infty < y < \infty. \end{cases} \quad (2.18)$$

Furthermore, we let

$$\begin{cases} x = y + \left(r - D_0 - \frac{1}{2}\sigma^2 \right) \tau, \\ \bar{\tau} = \frac{1}{2}\sigma^2 \tau, \\ v(y, \tau) = u(x, \bar{\tau}). \end{cases} \quad (2.19)$$

Noticing the relations

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \frac{1}{2}\sigma^2 \frac{\partial u}{\partial \bar{\tau}} + \left(r - D_0 - \frac{1}{2}\sigma^2 \right) \frac{\partial u}{\partial x}, \\
\frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x}, \\
\frac{\partial^2 v}{\partial y^2} &= \frac{\partial^2 u}{\partial x^2},
\end{aligned}$$

we finally arrive at

$$\begin{cases} \frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \leq \bar{\tau}, \\ u(x, 0) = V_T(e^x), & -\infty < x < \infty. \end{cases} \quad (2.20)$$

The partial differential equation in this problem is usually called the heat or diffusion equation.

Before we go to the next subsection we point out the following:

1. From (2.17) and (2.19), we know

$$\begin{aligned} V(S, t) &= e^{-r(T-t)} u(\ln S + (r - D_0 - \sigma^2/2)(T-t), \sigma^2(T-t)/2) \\ &= e^{-r(T-t)} u\left(\ln \frac{Se^{-D_0(T-t)}}{e^{-r(T-t)}} - \sigma^2(T-t)/2, \sigma^2(T-t)/2\right). \end{aligned}$$

Therefore besides those parameters in the payoff function $V_T(S)$, $V(S, t)$ depends on only three parameters: $Se^{-D_0(T-t)}$, $e^{-r(T-t)}$ and $\sigma^2(T-t)/2$.

2. Actually the transformations (2.17) and (2.19) can be combined into one transformation

$$\begin{cases} x = \ln S + \left(r - D_0 - \frac{1}{2}\sigma^2\right)(T-t), \\ \bar{\tau} = \frac{1}{2}\sigma^2(T-t), \\ V(S, t) = e^{-r(T-t)} u(x, \bar{\tau}). \end{cases} \quad (2.21)$$

That is, through the transformation (2.21), the Black-Scholes equation can be directly converted into the heat equation. The reason we complete the transformation through two steps is to see the function of each single transformation. In fact, from the derivation we know the following:

- Through setting $\tau = T - t$, we change a problem with a final condition to a problem with an initial condition and let the initial time be zero.
- The transformation $y = \ln S$ is to reduce an equation with variable coefficients to one with constant coefficients. This is the transformation by which the Euler equation in ordinary differential equations becomes a differential equation with constant coefficients.
- Letting $V = e^{-r(T-t)} v(y, \tau)$, we eliminate the term rV in the equation.

This is similar to the fact that an equation $\frac{dV}{d\tau} - rV = f$ can be written

as $\frac{d(e^{-r\tau}V)}{d\tau} = e^{-r\tau}f$ after the equation is multiplied by $e^{-r\tau}$. The factor $e^{-r\tau}$ is called an integrating factor for the ordinary differential equation. If r depends on t , then the integrating factor is $e^{-\int_0^\tau r(T-s)ds} = e^{-\int_t^T r(s)ds}$ and the term rV can be eliminated in the same way.

- The transformation $x = y + (r - D_0 - \sigma^2/2)\tau$ is to eliminate the term $(r - D_0 - \sigma^2/2)\frac{\partial v}{\partial y}$. This is similar to reducing the simplest hyperbolic

partial differential equation $\frac{\partial v}{\partial \tau} - a\frac{\partial v}{\partial y} = 0$ to an ordinary differential

equation. For this case, the characteristic equation is $\frac{dy}{d\tau} = -a$ and its solution is $y = -a\tau + c$ or $y + a\tau = c$. Let $x = y + a\tau$ and $v(y, \tau) = u(x, \tau)$, then the hyperbolic partial differential equation becomes $\frac{\partial u(x, \tau)}{\partial \tau} = 0$. If a depends on t , then the solution of the charac-

teristic equation is $y = -\int_0^\tau a(T-s)ds + c = -\int_t^T a(s)ds + c$. Letting $x = y + \int_t^T a(s)ds$ and $v(y, \tau) = u(x, \tau)$, we still have $\frac{\partial u(x, \tau)}{\partial \tau} = 0$.

- In order for the coefficient of $\frac{\partial^2 u}{\partial x^2}$ to be one, we let $\bar{\tau} = \sigma^2 \tau / 2$. If σ depends upon t , then letting $\bar{\tau} = \frac{1}{2} \int_0^\tau \sigma^2(T-s)ds = \frac{1}{2} \int_t^T \sigma^2(s)ds$ can still make the coefficient of $\frac{\partial^2 u}{\partial x^2}$ be one.
3. From the explanation on the function of each single transformation given above, we can see that if r, D_0 and σ are not constant, but depend on t only, then the Black-Scholes equation can still be converted into a heat equation by the following transformation

$$\begin{cases} x = \ln S + \int_t^T (r(s) - D_0(s) - \sigma^2(s)/2) ds, \\ \bar{\tau} = \frac{1}{2} \int_t^T \sigma^2(s)ds, \\ V(S, t) = e^{-\int_t^T r(s)ds} u(x, \bar{\tau}) \end{cases} \quad (2.22)$$

and the solution $V(S, t)$ possesses the following form

$$e^{-\int_t^T r(s)ds} u \left(\ln \frac{S e^{-\int_t^T D_0(s)ds}}{e^{-\int_t^T r(s)ds}} - \frac{1}{2} \int_t^T \sigma^2(s)ds, \frac{1}{2} \int_t^T \sigma^2(s)ds \right), \quad (2.23)$$

where $u(x, \bar{\tau})$ is a solution of the heat equation (see [68]). This is left for the reader as an exercise (Problem 11). There in order to see the function of each part of the transformation, the reader is asked to reduce the Black-Scholes equation with time-dependent parameters to a heat equation through two steps.

4. The transformation to convert the Black-Scholes equation into a heat equation is not unique. In fact we can let $x = \ln S$, $\bar{\tau} = \frac{1}{2} \sigma^2(T-t)$, $V(S, t) = e^{\alpha x + \beta \bar{\tau}} u(x, \bar{\tau})$ and choose constants α and β such that $u(x, \bar{\tau})$ satisfies the heat equation (see [68]).

2.3.2 Transforming the Black-Scholes Equation into an Equation Defined on a Finite Domain

Let us consider the following option problem:

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(S) S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0, \\ \hspace{15em} 0 \leq S < \infty, \quad t \leq T, \\ V(S, T) = V_T(S), \hspace{15em} 0 \leq S < \infty. \end{cases} \quad (2.24)$$

The transformation to be described in this subsection works even when σ , r or D_0 is not constant. For simplicity in the derivation we assume that σ depends on S and r , D_0 are constant. In this case an analytic expression of the solution $V(S, t)$ may not exist and numerical methods may be necessary. Also for a call option, the solution $V(S, t)$ is not bounded. Therefore, we introduce new independent variables and dependent variable through the following transformation:

$$\begin{cases} \xi = \frac{S}{S + P_m}, \\ \tau = T - t, \\ V(S, t) = (S + P_m)\bar{V}(\xi, \tau). \end{cases} \quad (2.25)$$

From (2.25) we have

$$S = \frac{P_m \xi}{1 - \xi}, \quad S + P_m = \frac{P_m}{1 - \xi}$$

and

$$\frac{d\xi}{dS} = \frac{P_m}{(S + P_m)^2} = \frac{(1 - \xi)^2}{P_m}.$$

Since

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{\partial}{\partial t}((S + P_m)\bar{V}(\xi, \tau)) = -(S + P_m)\frac{\partial \bar{V}}{\partial \tau} = -\frac{P_m}{1 - \xi}\frac{\partial \bar{V}}{\partial \tau}, \\ \frac{\partial V}{\partial S} &= \frac{\partial}{\partial S}((S + P_m)\bar{V}(\xi, \tau)) = (S + P_m)\frac{\partial \bar{V}}{\partial \xi}\frac{d\xi}{dS} + \bar{V} = (1 - \xi)\frac{\partial \bar{V}}{\partial \xi} + \bar{V}, \\ \frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial \xi}\left[(1 - \xi)\frac{\partial \bar{V}}{\partial \xi} + \bar{V}\right]\frac{d\xi}{dS} = \frac{(1 - \xi)^3}{P_m}\frac{\partial^2 \bar{V}}{\partial \xi^2}, \end{aligned}$$

and let

$$\bar{\sigma}(\xi) = \sigma(S(\xi)) = \sigma\left(\frac{P_m \xi}{1 - \xi}\right),$$

the original equation becomes

$$\frac{P_m}{1 - \xi}\frac{\partial \bar{V}}{\partial \tau} = \frac{\bar{\sigma}^2(\xi)P_m\xi^2(1 - \xi)}{2}\frac{\partial^2 \bar{V}}{\partial \xi^2} + (r - D_0)P_m\xi\frac{\partial \bar{V}}{\partial \xi} + \frac{(r - D_0)\xi - r}{1 - \xi}P_m\bar{V}$$

or

$$\frac{\partial \bar{V}}{\partial \tau} = \frac{\bar{\sigma}^2(\xi)\xi^2(1 - \xi)^2}{2}\frac{\partial^2 \bar{V}}{\partial \xi^2} + (r - D_0)\xi(1 - \xi)\frac{\partial \bar{V}}{\partial \xi} - [r(1 - \xi) + D_0\xi]\bar{V},$$

$$0 \leq \xi < 1, \quad 0 \leq \tau.$$

Assume that \bar{V} is a smooth function of ξ , then the equation also holds at $\xi = 1$. Since $V(S, T) = (S + P_m)\bar{V}(\xi, 0) = \bar{V}(\xi, 0)\frac{P_m}{1 - \xi}$, the condition $V(S, T) =$

$V_T(S)$ can be rewritten as $\bar{V}(\xi, 0) = V_T \left(\frac{P_m \xi}{1 - \xi} \right) \frac{1 - \xi}{P_m}$. Consequently, the problem (2.24) becomes

$$\begin{cases} \frac{\partial \bar{V}}{\partial \tau} = \frac{1}{2} \bar{\sigma}^2(\xi) \xi^2 (1 - \xi)^2 \frac{\partial^2 \bar{V}}{\partial \xi^2} + (r - D_0) \xi (1 - \xi) \frac{\partial \bar{V}}{\partial \xi} - [r(1 - \xi) + D_0 \xi] \bar{V}, \\ \qquad \qquad \qquad 0 \leq \xi \leq 1, \quad 0 \leq \tau, \\ \bar{V}(\xi, 0) = \frac{1 - \xi}{P_m} V_T \left(\frac{P_m \xi}{1 - \xi} \right), \qquad \qquad \qquad 0 \leq \xi \leq 1. \end{cases} \quad (2.26)$$

Thus the transformation (2.25) converts a problem on an infinite domain into a problem on a finite domain. Usually for a parabolic equation defined on a finite domain to have a unique solution, besides an initial condition, boundary conditions are needed. However in this equation the coefficients of $\frac{\partial^2 \bar{V}}{\partial \xi^2}$ and $\frac{\partial \bar{V}}{\partial \xi}$ at $\xi = 0$ and at $\xi = 1$ are equal to zero, i.e., the equation degenerates to ordinary differential equations at the boundaries. Actually at $\xi = 0$ the equation becomes

$$\frac{\partial \bar{V}(0, \tau)}{\partial \tau} = -r \bar{V}(0, \tau)$$

and the solution is

$$\bar{V}(0, \tau) = \bar{V}(0, 0) e^{-r\tau}. \quad (2.27)$$

Similarly, at $\xi = 1$ the equation reduces to

$$\frac{\partial \bar{V}(1, \tau)}{\partial \tau} = -D_0 \bar{V}(1, \tau),$$

from which we have

$$\bar{V}(1, \tau) = \bar{V}(1, 0) e^{-D_0 \tau}. \quad (2.28)$$

Therefore for this equation, the two solutions of the ordinary differential equations provide the values at the boundaries and no boundary conditions are needed in order for the problem (2.26) to have a unique solution.

Consequently, in order to price an option, we need to solve a problem on a finite domain if this formulation is adopted. From the point view of numerical methods, such a formulation is better. This is its first advantage. Actually the uniqueness of solution for problem (2.26) can easily be proven (see Section 2.9). Indeed, not only the uniqueness can be proven, but the stability of the solution with respect to the initial value can also be shown easily. That is, this formulation makes proof of some theoretical problems easy. This is its other advantage.

For a call option, the payoff is

$$V(S, T) = \max(S - E, 0),$$

So far we assumed that σ depends only on S and r and D_0 are constant. However the result will be the same if σ depends on both S and t , and r and D_0 also depend on S and t .

Finally we would like to point out that from (2.28) we can have an asymptotic expression of the solution of the Black–Scholes equation at infinity. Since at $\xi = 1$ there is an analytic solution (2.28), noticing

$$V(S, t) = (S + P_m)\bar{V}(\xi, \tau),$$

for $S \approx \infty$ we have

$$\begin{aligned} V(S, t) &= (S + P_m)\bar{V}(\xi, \tau) \approx (S + P_m)\bar{V}(1, \tau) \\ &= (S + P_m)\bar{V}(1, 0)e^{-D_0\tau} \\ &\approx V(S, T)e^{-D_0\tau} = V(S, T)e^{-D_0(T-t)}. \end{aligned} \quad (2.31)$$

This is an asymptotic expression of the solution of the Black–Scholes equation at infinity.

2.4 Solutions of European Options

A linear partial differential equation

$$A\frac{\partial^2 u}{\partial t^2} + 2B\frac{\partial^2 u}{\partial t \partial x} + C\frac{\partial^2 u}{\partial x^2} = F\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right)$$

is called a parabolic partial differential equation if $AC - B^2 = 0$, where A, B and C are not all equal to zero. The diffusion equation is the simplest parabolic equation. The Black–Scholes equation is another parabolic equation. In Section 2.3 we reduced the Black–Scholes equation to a diffusion equation. Here we will first find out the analytic expression of the solution of the diffusion equation, then that of the Black–Scholes equation, and finally the Black–Scholes formulae for European options are derived.

2.4.1 The Solutions of Parabolic Equations

In order for a parabolic differential equation to have a unique solution, one has to specify some conditions. For example, the initial value problem for a heat equation

$$\frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad \bar{\tau} \geq 0 \quad (2.32)$$

with

$$u(x, 0) = u_0(x) \quad (2.33)$$

has a unique solution under certain conditions which usually hold in practice.

Let us find the solution of the equation (2.32) with initial condition (2.33). The way to find the solution is not unique. Here we use the following method (see [46]). We first try to find a special solution of (2.32) in the form

$$u(x, \bar{\tau}) = \bar{\tau}^{-1/2} U(\eta),$$

where

$$\eta = \frac{x - \xi}{\sqrt{\bar{\tau}}}, \quad \xi \text{ being a parameter.}$$

Since

$$\begin{aligned} \frac{\partial u}{\partial \bar{\tau}} &= -\frac{\bar{\tau}^{-3/2}}{2} \left(U + \eta \frac{dU}{d\eta} \right) = -\frac{\bar{\tau}^{-3/2}}{2} \frac{d}{d\eta} (\eta U(\eta)), \\ \frac{\partial u}{\partial x} &= \bar{\tau}^{-1/2} \frac{dU}{d\eta} \frac{1}{\sqrt{\bar{\tau}}} = \bar{\tau}^{-1} \frac{dU}{d\eta}, \\ \frac{\partial^2 u}{\partial x^2} &= \bar{\tau}^{-3/2} \frac{d^2 U}{d\eta^2}, \end{aligned}$$

from (2.32) we have

$$-\frac{\bar{\tau}^{-3/2}}{2} \frac{d}{d\eta} (\eta U) = \bar{\tau}^{-3/2} \frac{d^2 U}{d\eta^2},$$

that is,

$$\frac{d^2 U}{d\eta^2} + \frac{1}{2} \frac{d}{d\eta} (\eta U) = 0.$$

Integrating this equation, we have

$$\frac{dU}{d\eta} + \frac{\eta}{2} U = c_1,$$

where c_1 is a constant. Let us choose $c_1 = 0$, so now we have a linear homogeneous equation. The solution of this equation is

$$U(\eta) = ce^{-\eta^2/4},$$

where c is a constant. Thus for the diffusion equation we have a special solution in the form

$$c\bar{\tau}^{-1/2} e^{-(x-\xi)^2/4\bar{\tau}}.$$

If we further require

$$\int_{-\infty}^{\infty} c\bar{\tau}^{-1/2} e^{-(x-\xi)^2/4\bar{\tau}} d\xi = 1,$$

then

$$c = \frac{1}{\int_{-\infty}^{\infty} \bar{\tau}^{-1/2} e^{-(x-\xi)^2/4\bar{\tau}} d\xi} = \frac{1}{\sqrt{2} \int_{-\infty}^{\infty} e^{-\eta^2/2} d\eta} = \frac{1}{2\sqrt{\pi}}$$

and the special solution is

$$\frac{1}{2\sqrt{\pi\bar{\tau}}}e^{-(x-\xi)^2/4\bar{\tau}}.$$

This solution is called the fundamental solution or Green's function for the heat equation (2.32). Let $g(\xi; x, \bar{\tau})$ represent this class of functions with ξ as parameters, i.e., the relation

$$\frac{\partial g(\xi; x, \bar{\tau})}{\partial \bar{\tau}} = \frac{\partial^2 g(\xi; x, \bar{\tau})}{\partial x^2}$$

holds for any ξ . Thus for any $u_0(\xi)$ we have

$$\int_{-\infty}^{\infty} u_0(\xi) \frac{\partial g(\xi; x, \bar{\tau})}{\partial \bar{\tau}} d\xi = \int_{-\infty}^{\infty} u_0(\xi) \frac{\partial^2 g(\xi; x, \bar{\tau})}{\partial x^2} d\xi,$$

i.e.,

$$\frac{\partial \left(\int_{-\infty}^{\infty} u_0(\xi) g(\xi; x, \bar{\tau}) d\xi \right)}{\partial \bar{\tau}} = \frac{\partial^2 \left(\int_{-\infty}^{\infty} u_0(\xi) g(\xi; x, \bar{\tau}) d\xi \right)}{\partial x^2}.$$

Consequently,

$$u(x, \bar{\tau}) = \int_{-\infty}^{\infty} u_0(\xi) \times \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} d\xi \quad (2.34)$$

is also a solution of (2.32). Because

$$\lim_{\bar{\tau} \rightarrow 0} \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} = \begin{cases} 0, & x - \xi \neq 0, \\ \infty, & x - \xi = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} d\xi = 1$$

is true for any $\bar{\tau}$, we have

$$\lim_{\bar{\tau} \rightarrow 0} \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} = \delta(x - \xi)$$

and

$$\lim_{\bar{\tau} \rightarrow 0} \int_{-\infty}^{\infty} u_0(\xi) \times \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} d\xi = u_0(x).$$

Consequently, (2.34) is the solution of the initial-value problem

$$\begin{cases} \frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad \bar{\tau} > 0, \\ u(x, 0) = u_0(x), & -\infty < x < \infty. \end{cases}$$

2.4.2 Solutions of the Black–Scholes Equation

From (2.21), (2.20) and (2.34), we know that the solution of the final value problem

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV = 0, & 0 \leq S < \infty, \quad 0 \leq t \leq T, \\ V(S, T) = V_T(S), & 0 \leq S < \infty \end{cases}$$

is

$$\begin{aligned} V(S, t) &= e^{-r(T-t)} \int_{-\infty}^{\infty} u_0(\xi) \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} d\xi \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} V_T(e^\xi) \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(\xi-x)^2/4\bar{\tau}} d\xi \\ &= e^{-r(T-t)} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \\ &\quad \times \int_0^\infty V_T(S') e^{-[\ln S' - (\ln S + (r-D_0-\sigma^2/2)(T-t))]^2/2\sigma^2(T-t)} \frac{dS'}{S'}. \end{aligned}$$

This result can be written as

$$V(S, t) = e^{-r(T-t)} \int_0^\infty V_T(S') G(S', T; S, t) dS', \quad (2.35)$$

where

$$\begin{aligned} G(S', T; S, t) &= \frac{1}{\sigma\sqrt{2\pi(T-t)}S'} e^{-[\ln S' - (\ln S + (r-D_0-\sigma^2/2)(T-t))]^2/2\sigma^2(T-t)}. \end{aligned} \quad (2.36)$$

From Subsection 2.1.3, this is the probability density function for a lognormal distribution with $m = r - D_0 - \frac{1}{2}\sigma^2$ and according to (2.6) the expectation of S' is

$$E[S'] = S e^{(r-D_0)(T-t)}. \quad (2.37)$$

This function can also be written as

$$G(S', T; S, t) = \frac{1}{\sqrt{2\pi}bS'} e^{-[\ln(S'/a) + b^2/2]^2/2b^2},$$

where

$$a = S e^{(r-D_0)(T-t)}$$

and

$$b = \sigma\sqrt{T-t}.$$

For this function, there are the following useful formulae:

$$\int_c^\infty G(S', T; S, t) dS' = N\left(\frac{\ln(a/c) - b^2/2}{b}\right) \quad (2.38)$$

and

$$\int_c^\infty S' G(S', T; S, t) dS' = a N\left(\frac{\ln(a/c) + b^2/2}{b}\right), \quad (2.39)$$

where $N(z)$ is the cumulative distribution function for the standardized normal variable defined by⁸

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\xi^2/2} d\xi. \quad (2.40)$$

The proof of the two formulae is straightforward. Let

$$\eta(S') = \frac{\ln(S'/a) + b^2/2}{b},$$

i.e.,

$$S' = ae^{b\eta - b^2/2}.$$

Thus

$$dS' = ae^{b\eta - b^2/2} b d\eta = S' b d\eta.$$

Consequently, we have

$$\begin{aligned} & \int_c^\infty \frac{1}{\sqrt{2\pi} b S'} e^{-[\ln(S'/a) + b^2/2]^2 / 2b^2} dS' \\ &= \int_{\eta(c)}^\infty \frac{1}{\sqrt{2\pi} b S'} e^{-\eta^2/2} S' b d\eta \\ &= N(-\eta(c)) \\ &= N\left(-\frac{\ln(c/a) + b^2/2}{b}\right) \\ &= N\left(\frac{\ln(a/c) - b^2/2}{b}\right) \end{aligned}$$

and

⁸The value of this function has to be obtained by numerical methods. If $z \leq 0$, this function can be approximated by

$$\begin{aligned} N(z) = & 0.5t \exp(-x^2 - 1.26551223 + t(1.00002368 + t(0.37409196 + t(0.09678418 \\ & + t(-0.18628806 + t(0.27886807 + t(-1.13520398 + t(1.48851587 \\ & + t(-0.82215223 + t \times 0.17087277))))))))), \end{aligned}$$

where $x = -z \times 0.707106781186550$ and $t = 1.0/(1.0 + 0.5x)$. If $z > 0$, then $N(z) = 1 - N(-z)$. The fractional error is less than 0.6×10^{-7} everywhere. See "NUMERICAL RECIPES IN C: The Art of Scientific Computing, 1988–1992, Cambridge University Press".

$$\begin{aligned}
& \int_c^\infty S' \frac{1}{\sqrt{2\pi b S'}} e^{-[\ln(S'/a) + b^2/2]^2 / 2b^2} dS' \\
&= \int_{\eta(c)}^\infty \frac{1}{\sqrt{2\pi b}} e^{-\eta^2/2} a e^{b\eta - b^2/2} b d\eta \\
&= \frac{a}{\sqrt{2\pi}} \int_{\eta(c)}^\infty e^{-(\eta-b)^2/2} d\eta \\
&= \frac{a}{\sqrt{2\pi}} \int_{\eta(c)-b}^\infty e^{-\xi^2/2} d\xi \\
&= aN\left(-\frac{\ln(c/a) + b^2/2}{b} + b\right) \\
&= aN\left(\frac{\ln(a/c) + b^2/2}{b}\right).
\end{aligned}$$

2.4.3 Prices of Forward Contracts and Delivery Prices

From Subsection 1.2.1 we know that the payoff function for a forward contract is

$$V(S, T) = S - K.$$

Therefore, according to (2.35) and using (2.37), we see that its price is

$$\begin{aligned}
V(S, t) &= e^{-r(T-t)} \int_0^\infty (S' - K) G(S', T; S, t) dS' \\
&= e^{-r(T-t)} (S e^{(r-D_0)(T-t)} - K) \\
&= S e^{-D_0(T-t)} - K e^{-r(T-t)}.
\end{aligned}$$

Since for a forward contract the buyer does not need to pay any premium at $t = 0$, we have

$$V(S, 0) = S e^{-D_0 T} - K e^{-rT} = 0.$$

Consequently, the delivery price should be

$$K = e^{(r-D_0)T} S_0,$$

where in order to make it clear, we use S_0 , instead of S , to denote the price of the underlying asset at the initiation of the contract.

2.4.4 Derivation of the Black–Scholes Formulae

At $t = T$, the value of a call option is

$$c(S, T) = \max(S - E, 0).$$

According to (2.35), (2.38) and (2.39), the value of a European call is

$$\begin{aligned}
c(S, t) &= e^{-r(T-t)} \int_0^\infty \max(S' - E, 0) G(S', T; S, t) dS' \\
&= e^{-r(T-t)} \int_E^\infty (S' - E) G(S', T; S, t) dS' \\
&= e^{-r(T-t)} \left(\int_E^\infty S' G(S', T; S, t) dS' - \int_E^\infty E G(S', T; S, t) dS' \right) \\
&= e^{-r(T-t)} (S e^{(r-D_0)(T-t)} N(d_1) - E N(d_2)) \\
&= S e^{-D_0(T-t)} N(d_1) - E e^{-r(T-t)} N(d_2), \tag{2.41}
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \left[\ln \frac{S e^{(r-D_0)(T-t)}}{E} + \frac{1}{2} \sigma^2 (T-t) \right] / \left(\sigma \sqrt{T-t} \right) \\
&= \left[\ln \frac{S e^{-D_0(T-t)}}{E e^{-r(T-t)}} + \frac{1}{2} \sigma^2 (T-t) \right] / \left(\sigma \sqrt{T-t} \right), \\
d_2 &= \left[\ln \frac{S e^{(r-D_0)(T-t)}}{E} - \frac{1}{2} \sigma^2 (T-t) \right] / \left(\sigma \sqrt{T-t} \right) \\
&= \left[\ln \frac{S e^{-D_0(T-t)}}{E e^{-r(T-t)}} - \frac{1}{2} \sigma^2 (T-t) \right] / \left(\sigma \sqrt{T-t} \right) \\
&= d_1 - \sigma \sqrt{T-t}.
\end{aligned}$$

For a put, the final value is

$$p(S, T) = \max(E - S, 0).$$

Thus the value of a European put is

$$\begin{aligned}
p(S, t) &= e^{-r(T-t)} \int_0^\infty \max(E - S', 0) G(S', T; S, t) dS' \\
&= e^{-r(T-t)} \int_0^E (E - S') G(S', T; S, t) dS' \\
&= e^{-r(T-t)} \left[E \int_0^E G(S', T; S, t) dS' - \int_0^E S' G(S', T; S, t) dS' \right] \\
&= e^{-r(T-t)} \left[E(1 - N(d_2)) - S e^{(r-D_0)(T-t)} (1 - N(d_1)) \right] \\
&= E e^{-r(T-t)} N(-d_2) - S e^{-D_0(T-t)} N(-d_1). \tag{2.42}
\end{aligned}$$

It is interesting that the values of European call and put options can be expressed in terms of the cumulative distribution function for the standardized normal random variable, $N(z)$. (2.41) and (2.42) give closed-form solutions for European vanilla options and are usually referred to as the Black–Scholes formulae.

When hedging is involved, we not only seek the value of options, but also the value of the first and second derivatives with respect to S , Δ and Γ . For a European call, $\Delta = \frac{\partial c}{\partial S}$ is

$$\begin{aligned}\frac{\partial c}{\partial S} &= e^{-D_0(T-t)} N(d_1) + S e^{-D_0(T-t)} \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \frac{\partial d_1}{\partial S} \\ &\quad - E e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} \frac{\partial d_2}{\partial S} \\ &= e^{-D_0(T-t)} N(d_1) + \frac{1}{\sqrt{2\pi}} \frac{\partial d_1}{\partial S} \left(S e^{-D_0(T-t)-d_1^2/2} - E e^{-r(T-t)-d_2^2/2} \right).\end{aligned}$$

Noticing

$$\begin{aligned}& -r(T-t) - d_2^2/2 \\ &= -r(T-t) - \left[d_1^2 - 2d_1\sigma\sqrt{T-t} + \sigma^2(T-t) \right] / 2 \\ &= -r(T-t) - \left[d_1^2 - 2\ln(S/E) - 2(r - D_0 + \sigma^2/2)(T-t) + \sigma^2(T-t) \right] / 2 \\ &= -d_1^2/2 - D_0(T-t) + \ln(S/E),\end{aligned}$$

i.e.,

$$S e^{-D_0(T-t)-d_1^2/2} = E e^{-r(T-t)-d_2^2/2},$$

we have

$$\frac{\partial c}{\partial S} = e^{-D_0(T-t)} N(d_1).$$

Taking the derivative with respect to S again yields

$$\frac{\partial^2 c}{\partial S^2} = \frac{1}{S\sigma\sqrt{2\pi(T-t)}} e^{-D_0(T-t)-d_1^2/2}.$$

Similarly, for put options

$$\frac{\partial p}{\partial S} = -e^{-D_0(T-t)} N(-d_1) \quad \text{and} \quad \frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}.$$

We need to point out that if the value of an option and the price of the underlying asset are divided by E , then the dimensionless option value V/E and the derivatives of V/E can still be obtained by the same formulae. The only change is to let $E = 1$ and S should have dimensionless value.

What do the functions $c(S, t)$ and $p(S, t)$ look like? The prices of the European call and put options for the case $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$ are shown in Figs. 2.5 and 2.6. Clearly, the curves should approach the payoff functions as $t \rightarrow T$, which can be seen from the two figures. From Fig. 2.6 we can also see that when S is close to zero, the curves approach the payoff from the bottom and when S is large, the curves tend to the payoff from the top.

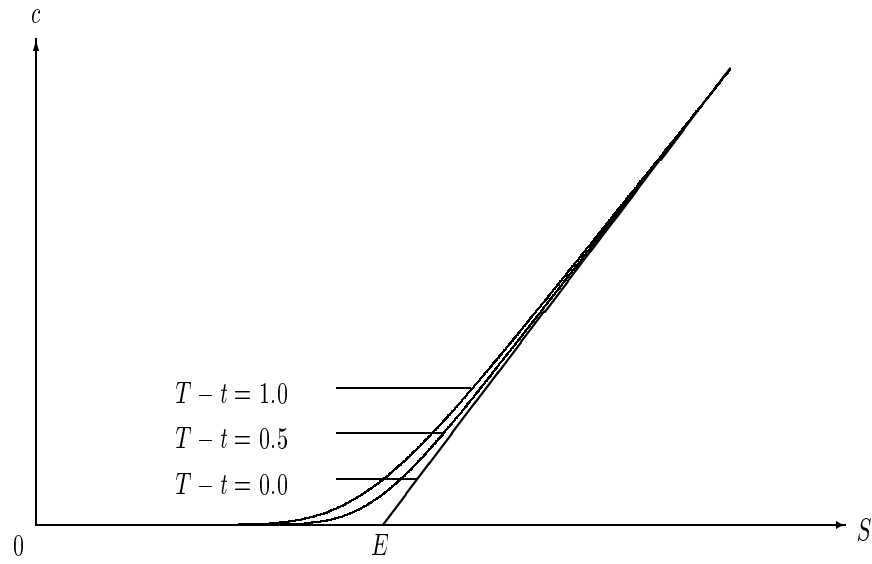


Fig. 2.5. The European call value $c(S, t)$ as a function of S ($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$ and $T - t = 0, 0.5$ and 1.0 .)

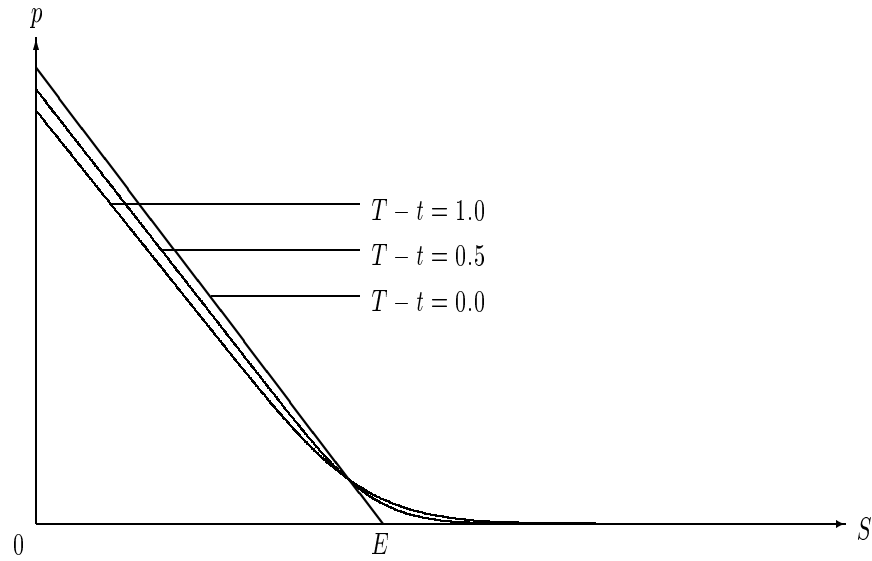


Fig. 2.6. The European put value $p(S, t)$ as a function of S ($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$ and $T - t = 0, 0.5$ and 1.0 .)

That is, $p(S, t)$ is less than the payoff for small S and greater than the payoff for large S . In Sections 2.5 and 2.6, we will see that for American options the price should always be at least the payoff. Because of this, the Black–Scholes equation cannot be used to determine the price of American options in some situations.

When σ , r and D_0 depend upon t , closed-form solutions can still be obtained (see [56] and [68]). Actually, through the transformation (2.22) the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} + (r(t) - D_0(t))S\frac{\partial V}{\partial S} - r(t)V = 0$$

can still be reduced to a diffusion equation. Let

$$\begin{cases} \alpha(t) = \frac{1}{2} \int_t^T \sigma^2(s) ds, \\ \delta(t) = \int_t^T D_0(s) ds, \\ \gamma(t) = \int_t^T r(s) ds, \end{cases}$$

then the solution of the Black–Scholes equation in this case is

$$V(S, t) = e^{-\gamma(t)} \int_{-\infty}^{\infty} V_T(e^x) \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-x)^2/4\bar{\tau}} d\xi,$$

where $x = \ln S + \gamma(t) - \delta(t) - \alpha(t)$ and $\bar{\tau} = \alpha(t)$. Therefore for a call with coefficients $r(t)$, $D_0(t)$ and $\sigma(t)$ the solution should be

$$c(S, t) = Se^{-\delta(t)} N(\bar{d}_1) - Ee^{-\gamma(t)} N(\bar{d}_2),$$

where

$$\begin{aligned} \bar{d}_1 &= \left(\ln \frac{Se^{-\delta(t)}}{Ee^{-\gamma(t)}} + \alpha(t) \right) / (2\alpha(t))^{1/2}, \\ \bar{d}_2 &= \left(\ln \frac{Se^{-\delta(t)}}{Ee^{-\gamma(t)}} - \alpha(t) \right) / (2\alpha(t))^{1/2}. \end{aligned}$$

2.4.5 Put–Call Parity Relation

Although call and put options are superficially different, they can be combined in such a way that they are perfectly correlated. In fact, there is the following relation:

$$c(S, t) - p(S, t) = Se^{-D_0(T-t)} - Ee^{-r(T-t)}, \quad (2.43)$$

which is usually called the put–call parity relation. It can be obtained in different ways. From the Black–Scholes formulae, (2.41) and (2.42), we can have

$$\begin{aligned}
c(S, t) - p(S, t) &= Se^{-D_0(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2) \\
&\quad - Ee^{-r(T-t)}N(-d_2) + Se^{-D_0(T-t)}N(-d_1) \\
&= Se^{-D_0(T-t)} - Ee^{-r(T-t)}.
\end{aligned}$$

This is one way to get it.

We can also find this relation without finding the concrete expressions of $c(S, t)$ and $p(S, t)$. Let us examine the portfolio

$$\Pi = S + p - c - E.$$

The payoff for this portfolio at expiry is

$$\Pi(S, T) = S + \max(E - S, 0) - \max(S - E, 0) - E = 0.$$

According to (2.35),

$$\begin{aligned}
&\Pi(S, t) \\
&= e^{-r(T-t)} \int_0^\infty (S' + \max(E - S', 0) - \max(S' - E, 0) - E) G(S', T; S, t) dS' \\
&= Se^{-D_0(T-t)} + p(S, t) - c(S, t) - Ee^{-r(T-t)} \\
&= 0.
\end{aligned}$$

Here we are actually using the superposition principle of homogeneous linear partial differential equations. From this relation we immediately have the put-call parity. In Section 2.11 we will derive this relation again without using a partial differential equation. Here we need to point out that the put-call parity relation is true only for European options. For American options, the equality becomes an inequality, which will be discussed in Section 2.11.

2.4.6 An Explanation in Terms of Probability

The function $G(S', T; S, t)$ given by (2.36) represents a probability density function of a random variable S' and S' can be interpreted as the random price of a stock at time T . Then we can understand S as the price of the stock at time t since $G(S', T; S, t)$ goes to a Dirac delta function $\delta(S' - S)$ as $T \rightarrow t$. $V_T(S')$ is the value of an option at time T if the price is S' . Therefore

$$\int_0^\infty V_T(S') G(S', T; S, t) dS'$$

is the expectation of the value of the option at time T if the price is S at time t , and

$$e^{-r(T-t)} \int_0^\infty V_T(S') G(S', T; S, t) dS'$$

is the present (or discounted) value of the expectation at time T . That is, the price of an option at time t given by (2.35) is the present value of the

expectation of the option value at time T . This is the explanation of the solution given by (2.35) in terms of probability.

Suppose that S and S' are the prices of a stock at time $T - \Delta t$ and time T respectively and that S' has the probability density function $G(S', T; S, T - \Delta t)$. According to (2.6) we have

$$E[S'] = Se^{(r-D_0)\Delta t}$$

and

$$\text{Var}[S'] = S^2 e^{2(r-D_0)\Delta t} (e^{\sigma^2 \Delta t} - 1) \approx S^2 \sigma^2 \Delta t.$$

Therefore⁹

$$E\left[\frac{S' - S}{S}\right] = \frac{Se^{(r-D_0)\Delta t} - S}{S} \approx (r - D_0)\Delta t$$

and

$$\text{Var}\left[\frac{S' - S}{S}\right] \approx \sigma^2 \Delta t.$$

Consequently

$$\frac{dS}{S} = (r - D_0)dt + \sigma dX.$$

However, in the real world

$$\frac{dS}{S} = \mu dt + \sigma dX.$$

Therefore the random variable in the expression of the solution is a different random variable from that in the real world. Usually we say that the random variable in the expression of the solution is in a “risk-neutral” world. In this case the expected rate of the return on any asset is the difference between the riskless interest rate r and the dividend yield D_0 .

It is clear that if we let

$$\bar{V}(S, t) = e^{r(T-t)} V(S, t),$$

then \bar{V} is the solution of the problem

$$\begin{cases} \frac{\partial \bar{V}}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \bar{V}}{\partial S^2} + (r - D_0) S \frac{\partial \bar{V}}{\partial S} = 0, & 0 \leq S, \quad t \leq T, \\ \bar{V}(S, T) = \bar{V}_T(S), & 0 \leq S \end{cases}$$

and

$$\bar{V}(S, t) = \int_0^\infty \bar{V}_T(S') G(S', T; S, t) dS' = E[\bar{V}_T(S')].$$

In probability theory, when this relation holds, it is said that $\bar{V}(S, t)$ is a martingale under the probability density function $G(S', T; S, t)$.

⁹Here we take a conditional expectation, i.e., S' is a random variable and S is fixed.

2.5 American Option Problems as LC Problems

In the next two sections we will discuss how to determine the value of an American option. In this section we first derive the additional constraints on American call and put options. Then we formulate the American call and put problems as linear complementarity (LC) problems. In the next section we will discuss how to formulate an American option problem as a free-boundary problem (FBP).

2.5.1 Constraints on American Options

Let $C(S, t)$ and $P(S, t)$ denote the prices of American call and put options respectively. As we know from Section 1.2, an American option has the additional feature that it may be exercised at any time during the life of the option. What does this additional feature mean in mathematics? It means that the value of an American call option must satisfy

$$C(S, t) \geq \max(S - E, 0), \quad (2.44)$$

and that the value of an American put option must fulfill

$$P(S, t) \geq \max(E - S, 0). \quad (2.45)$$

Usually $\max(S - E, 0)$ and $\max(E - S, 0)$ are called the intrinsic values of call and put options respectively. Thus satisfying the two inequalities above means that the value of an option must be at least equal to its intrinsic value. Because of this fact, $C(S, t) - \max(S - E, 0)$ and $P(S, t) - \max(E - S, 0)$ must be nonnegative and are usually called the time value of the American call and put options respectively. (2.44) and (2.45) are usually referred to as the constraints on American vanilla options. These conclusions can be proven by arbitrage arguments.

First let us consider an American call option. For $S \leq E$, the condition (2.44) means $C(S, t) \geq 0$. This is always true because a solution of the Black-Scholes equation with a nonnegative payoff function as a final condition is always nonnegative. From the financial point of view, it is also clear that the option price should not be negative because a holder of an option has only rights, no obligation. Thus (2.44) always holds for any $S \in [0, E]$. Suppose that for a price $S > E$, the condition (2.44) is not fulfilled, i.e., $C(S, T) < S - E$. Then an obvious arbitrage opportunity arises: by short selling the asset on the market for S , purchasing the option for C and exercising the call option, a risk-free profit of $S - E - C$ is made. Of course, such an opportunity would not last long before the value of the option was pushed up by the demand of arbitrageurs. We conclude that for a value of an American call we must impose the constraint (2.44). For an American put option the situation is similar. For any $S \geq E$ the condition (2.45) holds naturally. Suppose the option price satisfies $P(S, t) < E - S$ for a price $S < E$. Then by purchasing the option for

P , purchasing the asset from the market for S and exercising the put option, an immediate risk-free profit of $E - P - S$ is made and the demand will push the option price up so that condition (2.45) holds.

Bermudan options are similar to American options, but can be exercised only at several predetermined dates, instead of the entire period $[0, T]$. This means that for a Bermudan option, condition (2.44) or condition (2.45) should be required at several predetermined dates, but not on the entire period $[0, T]$, which is the only difference between American and Bermudan options.

Since an American option can be exercised at any time by expiry, a holder of an American option has more rights than a holder of a European option does. Thus the holder of an American option needs to pay at least as much premium as the holder of a European option with the same parameters does. A Bermudan option can be exercised at several predetermined dates including the expiration date, its holder has less rights than the holder of an American option does and more rights than the holder of a European option does. Thus its premium should be between the premiums of the American and European options with the same parameters.

2.5.2 Formulation of the Linear Complementarity Problem in (S, t) -Plane

As we easily see, at $S = 0$ the Black-Scholes equation degenerates to an ordinary differential equation

$$\frac{\partial V(0, t)}{\partial t} - rV(0, t) = 0$$

and its solution is

$$V(0, t) = V(0, T)e^{-r(T-t)}.$$

For a put, $V(0, T) = E$. Therefore the price of a European put option at $S = 0$ is

$$p(0, t) = Ee^{-r(T-t)} < E$$

for any $t < T$ if $r > 0$. Consequently the price of a European put option will not satisfy the constraint (2.45). Therefore in order to price an American put option, we must modify the method for determining the price of an option in the following way. If the Black-Scholes equation gives a price satisfying the constraint (2.45), then it is the price of the American put option; if not, the value needs to be replaced by $\max(E - S, 0)$. Thus the price of an American option usually is not a solutions of (2.16) but a solution of a so-called linear complementarity (LC) problem. In such a problem, usually in some regions the solution satisfies the partial differential equation (PDE) and in other regions it is not determined by the PDE. Those boundaries between the two types of regions are called free boundaries.

Before we formulate the American option problems as linear complementarity problems, we would like to show the following theorem.

Theorem 2.1

Let $\mathbf{L}_{\mathbf{s},t}$ be an operator in an option problem in the form:

$$\mathbf{L}_{\mathbf{s},t} = a(S, t) \frac{\partial^2}{\partial S^2} + b(S, t) \frac{\partial}{\partial S} + c(S, t)$$

and $G(S, t)$ be the constraint function for an American option. Furthermore we assume that $\frac{\partial G}{\partial t} + \mathbf{L}_{\mathbf{s},t}G$ exists. Suppose $V(S, t^*) = G(S, t^*)$ on an open interval (A_1, B_1) on the S -axis. Let $t = t^* - \Delta t$, where Δt is a sufficiently small positive number and let (A, B) be an open interval in (A_1, B_1) . Show the following conclusions: If for any $S \in (A, B)$,

$$\frac{\partial G}{\partial t}(S, t^*) + \mathbf{L}_{\mathbf{s},t^*}G(S, t^*) \geq 0,$$

then the value $V(S, t)$ determined by the equation

$$\frac{\partial V}{\partial t}(S, t) + \mathbf{L}_{\mathbf{s},t}V(S, t) = 0$$

satisfies the condition $V(S, t) - G(S, t) \geq 0$ on (A, B) ; and if for any $S \in (A, B)$,

$$\frac{\partial G}{\partial t}(S, t^*) + \mathbf{L}_{\mathbf{s},t^*}G(S, t^*) < 0,$$

then the equation

$$\frac{\partial V}{\partial t}(S, t) + \mathbf{L}_{\mathbf{s},t}V(S, t) = 0$$

cannot give a solution satisfying the condition $V(S, t) - G(S, t) \geq 0$ for any $S \in (A, B)$.

Proof. Since $V(S, t^*) = G(S, t^*)$, the fact that $V(S, t) - G(S, t) \geq 0$ holds for any $t = t^* - \Delta t$, Δt being a sufficiently small positive number, is equivalent to that at time t^* , $V(S, t) - G(S, t)$ is a non-increasing function with respect to t , that is,

$$\frac{\partial V}{\partial t}(S, t^*) - \frac{\partial G}{\partial t}(S, t^*) \leq 0.$$

If

$$\frac{\partial G}{\partial t}(S, t^*) + \mathbf{L}_{\mathbf{s},t^*}G(S, t^*) \geq 0$$

and

$$\frac{\partial V}{\partial t}(S, t^*) + \mathbf{L}_{\mathbf{s},t^*}V(S, t^*) = \frac{\partial V}{\partial t}(S, t^*) + \mathbf{L}_{\mathbf{s},t^*}G(S, t^*) = 0,$$

then

$$\frac{\partial G}{\partial t}(S, t^*) \geq -\mathbf{L}_{\mathbf{s},t^*}G(S, t^*) = \frac{\partial V}{\partial t}(S, t^*)$$

or

$$\frac{\partial V}{\partial t}(S, t^*) - \frac{\partial G}{\partial t}(S, t^*) \leq 0.$$

Therefore in this case we can use the equation

$$\frac{\partial V}{\partial t}(S, t) + \mathbf{L}_{S,t}V(S, t) = 0$$

to get a solution satisfying the condition $V(S, t) - G(S, t) \geq 0$.

If

$$\frac{\partial G}{\partial t}(S, t^*) + \mathbf{L}_{S,t^*}G(S, t^*) < 0$$

and

$$\frac{\partial V}{\partial t}(S, t^*) + \mathbf{L}_{S,t^*}V(S, t^*) = \frac{\partial V}{\partial t}(S, t^*) + \mathbf{L}_{S,t^*}G(S, t^*) = 0,$$

then

$$\frac{\partial G}{\partial t}(S, t^*) < -\mathbf{L}_{S,t^*}G(S, t^*) = \frac{\partial V}{\partial t}(S, t^*)$$

or

$$\frac{\partial V}{\partial t}(S, t^*) - \frac{\partial G}{\partial t}(S, t^*) > 0,$$

which will cause $V(S, t) - G(S, t) < 0$ for any $t = t^* - \Delta t$. Therefore we cannot get the solution by using the equation

$$\frac{\partial V}{\partial t}(S, t) + \mathbf{L}_{S,t}V(S, t) = 0.$$

Instead we have to let $V(S, t) - G(S, t) = 0$ in order to get a solution satisfying the condition $V(S, t) - G(S, t) \geq 0$. ■

About this theorem, we would like to make the following remarks.

- Let \mathbb{D}_{ge} denote the open domain where $\frac{\partial G}{\partial t}(S, t) + \mathbf{L}_{S,t}G(S, t) \geq 0$ and \mathbb{D}_l the open domain where $\frac{\partial G}{\partial t}(S, t) + \mathbf{L}_{S,t}G(S, t) < 0$. From this theorem we see that there are two possibilities for a point to be on a free boundary. One possibility is: in a neighbourhood of the point, $V(S, t) = G(S, t)$ and some portion of the neighbourhood belongs to \mathbb{D}_{ge} and another portion of the neighbourhood belongs to \mathbb{D}_l . The another possibility is: in some portion of a neighbourhood of the point $V(S, t) > G(S, t)$ and in another portion of the neighbourhood $V(S, t) = G(S, t)$ and it belongs to \mathbb{D}_l . Therefore a free boundary cannot appear in the open domain \mathbb{D}_{ge} . A free boundary will appear only in the open domain \mathbb{D}_l and on the boundary of \mathbb{D}_l . If $V(S, T) = G(S, T)$, then a free boundary will start at a point between the open domains \mathbb{D}_{ge} and \mathbb{D}_l . If $V(S, T) > G(S, T)$ on some portion of the interval $[0, \infty)$ and $V(S, T) = G(S, T)$ on the another portion, then a free boundary will start at a boundary between an open interval belonging to \mathbb{D}_l and an open interval where $V(S, T) > G(S, T)$. Later a free boundary may move but never move into the open domain \mathbb{D}_{ge} .

- If in a linear complementarity problem, the value of the solution at time $t = 0$ is given and the solution for $t > 0$ needs to be determined, i.e., if $V(S, t)$ is given and we want to determine $V(S, t + \Delta t)$ from $t = 0$ to $t = T$ successively, then the theorem and the remark above are still true if the condition

$$\frac{\partial G}{\partial t}(S, t) + \mathbf{L}_{\mathbf{s}, \mathbf{t}} G(S, t) \geq 0$$

is changed into

$$\frac{\partial G}{\partial t}(S, t) + \mathbf{L}_{\mathbf{s}, \mathbf{t}} G(S, t) \leq 0$$

and the condition

$$\frac{\partial G}{\partial t}(S, t) + \mathbf{L}_{\mathbf{s}, \mathbf{t}} G(S, t) < 0$$

is changed into

$$\frac{\partial G}{\partial t}(S, t) + \mathbf{L}_{\mathbf{s}, \mathbf{t}} G(S, t) > 0.$$

Now let us formulate the problem the price of the American option should satisfy. Let us assume that at time t we have obtained $P(S, t)$ satisfying (2.45) and we need to determine $P(S, t - \Delta t)$ satisfying (2.45), where Δt is a sufficiently small positive number. Define $G_p(S, t) = \max(E - S, 0)$. For simplicity, we assume that the entire interval consists of three open intervals plus their boundaries. On the first open interval, $P(S, t) > G_p(S, t)$. For any point in this interval, $P(S, t - \Delta t)$ must be still greater than $G_p(S, t)$ if Δt is small enough. Therefore at any point in this open interval

$$\begin{cases} \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0)S \frac{\partial P}{\partial S} - rP = 0, \\ P(S, t) > G_p(S, t). \end{cases}$$

On the second open interval $P(S, t) = G_p(S, t)$ and

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S \frac{\partial}{\partial S} - r \right) G_p(S, t) \geq 0$$

and on the third open interval $P(S, t) = G_p(S, t)$ and

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S \frac{\partial}{\partial S} - r \right) G_p(S, t) < 0.$$

According to theorem 2.1, for a point (S, t) in the second open interval the Black-Scholes can be used to determine $P(S, t - \Delta t)$ and the following is true:

$$\begin{cases} \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0)S \frac{\partial P}{\partial S} - rP = 0, \\ P(S, t) = G_p(S, t) \end{cases}$$

On the third interval the Black–Scholes equation cannot be used to determine $P(S, t - \Delta t)$. Instead $P(S, t - \Delta t)$ should equal $G_p(S, t - \Delta t)$. In this situation

$$\frac{P(S, t) - P(S, t - \Delta t)}{\Delta t} = \frac{G_p(S, t) - G_p(S, t - \Delta t)}{\Delta t} \rightarrow \frac{\partial G_p(S, t)}{\partial t}$$

as $\Delta t \rightarrow 0$ and we have

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r \right) P \\ = \left(\frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r \right) G_p(S, t) < 0, \\ P(S, t) = G_p(S, t). \end{cases}$$

Since $P(S, T) = G_p(S, T)$, we can use this argument from T to 0. Putting all the cases together, for $S \in [0, \infty)$ and $t \leq T$ we have

$$\begin{cases} \left(\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0) S \frac{\partial P}{\partial S} - rP \right) (P - G_p) = 0, \\ \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0) S \frac{\partial P}{\partial S} - rP \leq 0, \\ P(S, t) - G_p(S, t) \geq 0, \\ P(S, T) = G_p(S, T), \end{cases} \quad (2.46)$$

where $G_p(S, t) = \max(E - S, 0)$. Here we use the fact that (2.46) is also true in some sense at the boundary points of these open intervals. This problem is called the linear complementarity problem for an American put option.

Similarly for an American call option, the corresponding linear complementarity problem is

$$\begin{cases} \left(\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} - rC \right) (C - G_c) = 0, \\ \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} - rC \leq 0, \\ C(S, t) - G_c(S, t) \geq 0, \\ C(S, T) = G_c(S, T), \end{cases} \quad (2.47)$$

where $0 \leq S < \infty$, $t \leq T$ and $G_c(S, t) = \max(S - E, 0)$. From the derivation of (2.46), when σ, r, D_0 depend on S and t , the formulations are still correct.

In the last subsection we concluded that the price of an American option is at least as much as the price of a European option with the same parameters by using the financial reason. Here we assume that σ, r and D_0 are constants and explain this conclusion by using mathematical tools. Let $V(S, t), v(S, t)$

denote the prices of the American and European options respectively and let $G_v(S, t)$ be the constraint for the American option satisfying the condition $G_v(S, t^*) \geq G_v(S, t^{**})$ if $t^* \leq t^{**}$. Set $\Delta t = T/N$, where N is a positive integer. Let $\tilde{V}(S, T) = G_v(S, T)$ and for $t_n = n\Delta t$, $n = N-1, N-2, \dots, 0$, define

$$\tilde{V}(S, t_n) = \max \left(e^{-r\Delta t} \int_0^\infty \tilde{V}(S', t_{n+1}) G(S', t_{n+1}; S, t_n) dS', G_v(S, t_n) \right),$$

where $G(S', t_{n+1}; S, t_n)$ is given by (2.36). Suppose $\tilde{V}(S, t_{n+1}) \geq v(S, t_{n+1})$. From (2.35) we know

$$\begin{aligned} v(S, t_n) &= e^{-r\Delta t} \int_0^\infty v(S', t_{n+1}) G(S', t_{n+1}; S, t_n) dS' \\ &\leq e^{-r\Delta t} \int_0^\infty \tilde{V}(S', t_{n+1}) G(S', t_{n+1}; S, t_n) dS' \\ &\leq \tilde{V}(S, t_n). \end{aligned}$$

At $t = t_N = N\Delta t = T$, the condition $\tilde{V}(S, t_N) = \tilde{V}(S, T) \geq v(S, T) = v(S, t_N)$ holds. Therefore using the induction method we can prove $\tilde{V}(S, t_n) \geq v(S, t_n)$ for $n = N-1, N-2, \dots, 0$ successively. Letting $N \rightarrow \infty$ and noticing that $\tilde{V}(S, t)$ generates $V(S, t)$ as $N \rightarrow \infty$, we can have the conclusion we need.

The price of an American option has another property: $V(S, t^*) \geq V(S, t^{**})$ if $t^* \leq t^{**}$. Let us explain this fact by using mathematical tools. Suppose $\tilde{V}(S, t_n) \geq \tilde{V}(S, t_{n+1})$. According to the definition of $\tilde{V}(S, t_n)$, we have

$$\begin{aligned} \tilde{V}(S, t_n) &= \max \left(e^{-r\Delta t} \int_0^\infty \tilde{V}(S', t_{n+1}) G(S', t_{n+1}; S, t_n) dS', G_v(S, t_n) \right) \\ &\leq \max \left(e^{-r\Delta t} \int_0^\infty \tilde{V}(S', t_n) G(S', t_n; S, t_{n-1}) dS', G_v(S, t_{n-1}) \right) \\ &= \tilde{V}(S, t_{n-1}). \end{aligned}$$

Here we have used the facts

$$G(S', t_{n+1}; S, t_n) = G(S', t_n; S, t_{n-1}) \quad \text{and} \quad G_v(S, t_n) \leq G_v(S, t_{n-1}).$$

Since

$$\begin{aligned} \tilde{V}(S, t_{N-1}) &= \max \left(e^{-r\Delta t} \int_0^\infty \tilde{V}(S', t_N) G(S', t_N; S, t_{N-1}) dS', G_v(S, t_{N-1}) \right) \\ &\geq G_v(S, t_{N-1}) \geq G_v(S, t_N) = \tilde{V}(S, t_N), \end{aligned}$$

we can prove

$$\tilde{V}(S, t_n) \geq \tilde{V}(S, t_{n+1}) \quad \text{for} \quad n = N-2, N-3, \dots, 0$$

successively. This means

$$\tilde{V}(S, t_n) \geq \tilde{V}(S, t_m) \quad \text{for} \quad n \leq m \leq N.$$

Letting $N \rightarrow \infty$ and noticing that $\tilde{V}(S, t)$ generates $V(S, t)$ as $N \rightarrow \infty$, we arrive at the conclusion

$$V(S, t^*) \geq V(S, t^{**}) \quad \text{if} \quad t^* \leq t^{**}.$$

2.5.3 Formulation of the Linear Complementarity Problem in $(x, \bar{\tau})$ -Plane

As we know from Subsection 2.3.1, if we set

$$\begin{cases} x = \ln S + \left(r - D_0 - \frac{1}{2}\sigma^2\right)(T - t), \\ \bar{\tau} = \frac{1}{2}\sigma^2(T - t), \\ V(S, t) = e^{-r(T-t)}u(x, \bar{\tau}), \end{cases}$$

then

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV$$

becomes

$$-\frac{1}{2}\sigma^2 e^{-r(T-t)} \left(\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} \right).$$

Thus,

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial P}{\partial S^2} + (r - D_0)S \frac{\partial P}{\partial S} - rP < 0$$

is equivalent to

$$\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} > 0$$

and the Black-Scholes equation holds if and only if

$$\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} = 0.$$

Let us define

$$g_p(x, \bar{\tau}) = \max \left(e^{2r\bar{\tau}/\sigma^2} - e^{x+(2D_0/\sigma^2+1)\bar{\tau}}, 0 \right),$$

then

$$\begin{aligned} P - G_p &= P(S, t) - \max(1 - S, 0) \\ &= e^{-r(T-t)}u(x, \bar{\tau}) - \max \left(1 - e^{x-(r-D_0-\sigma^2/2)(T-t)}, 0 \right) \\ &= e^{-r(T-t)} \left[u(x, \bar{\tau}) - \max \left(e^{r(T-t)} - e^{x+(D_0+\sigma^2/2)(T-t)}, 0 \right) \right] \\ &= e^{-r(T-t)}[u(x, \bar{\tau}) - g_p(x, \bar{\tau})], \end{aligned}$$

where we suppose $E = 1$ for simplicity. Thus $P - G_p > 0$ is equivalent to

$$u(x, \bar{\tau}) - g_p(x, \bar{\tau}) > 0$$

and $P - G_p = 0$ if and only if

$$u(x, \bar{\tau}) - g_p(x, \bar{\tau}) = 0.$$

Therefore the American put option is the solution of the following problem:

$$\begin{cases} \left(\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} \right) (u(x, \bar{\tau}) - g_p(x, \bar{\tau})) = 0, \\ \frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} \geq 0, \\ u(x, \bar{\tau}) - g_p(x, \bar{\tau}) \geq 0, \\ u(x, 0) = g_p(x, 0), \end{cases} \quad (2.48)$$

where $x \in (-\infty, \infty)$ and $0 \leq \bar{\tau}$. Similarly, for American call options we have

$$\begin{cases} \left(\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} \right) (u(x, \bar{\tau}) - g_c(x, \bar{\tau})) = 0, \\ \frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} \geq 0, \\ u(x, \bar{\tau}) - g_c(x, \bar{\tau}) \geq 0, \\ u(x, 0) = g_c(x, 0), \end{cases} \quad (2.49)$$

where $x \in (-\infty, \infty)$, $0 \leq \bar{\tau}$ and

$$g_c(x, \bar{\tau}) = \max \left(e^{x + (2D_0/\sigma^2 + 1)\bar{\tau}} - e^{2r\bar{\tau}/\sigma^2}, 0 \right).$$

The derivation of (2.49) is almost identical to the American put. The only difference is that instead of using $P - G_p = e^{-r(T-t)} [u(x, \bar{\tau}) - g_p(x, \bar{\tau})]$, we need to use the relation

$$\begin{aligned} C - G_c &= C(S, t) - \max(S - 1, 0) \\ &= e^{-r(T-t)} u(x, \bar{\tau}) - \max \left(e^{x - (r - D_0 - \sigma^2/2)(T-t)} - 1, 0 \right) \\ &= e^{-r(T-t)} \left[u(x, \bar{\tau}) - \max \left(e^{x + (D_0 + \sigma^2/2)(T-t)} - e^{r(T-t)}, 0 \right) \right] \\ &= e^{-r(T-t)} [u(x, \bar{\tau}) - g_c(x, \bar{\tau})], \end{aligned}$$

where we also assume $E = 1$.

It is clear that if r , D_0 and σ depend on t , then similar results hold. However, if σ depends on S , then we may not be able to convert the problems (2.46) and (2.47) into (2.48) and (2.49) by a simple transformation.

2.5.4 Formulation of the Linear Complementarity Problem on a Finite Domain

Generally speaking, r , D_0 and σ are not constants. For simplicity, we assume that σ depends on S in this subsection even though the derivation is almost the same when r , D_0 and σ all are not constants.

From Subsection (2.3.2) we know that through the transformation

$$\begin{cases} \xi = \frac{S}{S+E}, \\ \tau = T - t, \\ V(S, t) = (S+E)\bar{V}(\xi, \tau) = \frac{E}{1-\xi}\bar{V}(\xi, \tau), \end{cases}$$

the operator

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r$$

is converted into

$$\frac{-E}{1-\xi}\left(\frac{\partial}{\partial \tau} - \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} - (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} + [r(1-\xi) + D_0\xi]\right),$$

where $\bar{\sigma}(\xi) = \sigma(\xi E/(1-\xi))$, and the function $\max(\pm(S-E), 0)$ becomes

$$\frac{E}{1-\xi}\max(\pm(2\xi-1), 0).$$

Therefore the problem (2.46) can be rewritten as

$$\begin{cases} \left(\frac{\partial \bar{V}}{\partial \tau} - \mathbf{L}_\xi \bar{V}\right)(\bar{V}(\xi, \tau) - \max(1-2\xi, 0)) = 0, \\ \frac{\partial \bar{V}}{\partial \tau} - \mathbf{L}_\xi \bar{V} \geq 0, \\ \bar{V}(\xi, \tau) - \max(1-2\xi, 0) \geq 0, \\ \bar{V}(\xi, 0) = \max(1-2\xi, 0), \end{cases} \quad (2.50)$$

where $0 \leq \xi \leq 1$, $0 \leq \tau$ and

$$\mathbf{L}_\xi = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi) + D_0\xi].$$

This is the American put option problem reformulated as a linear complementarity problem on a finite domain. Similarly from (2.47) we know that the American call option problem can be reformulated as the following linear complementarity problem

$$\left\{ \begin{array}{l} \left(\frac{\partial \bar{V}}{\partial \tau} - \mathbf{L}_\xi \bar{V} \right) (\bar{V}(\xi, \tau) - \max(2\xi - 1, 0)) = 0, \\ \frac{\partial \bar{V}}{\partial \tau} - \mathbf{L}_\xi \bar{V} \geq 0, \\ \bar{V}(\xi, \tau) - \max(2\xi - 1, 0) \geq 0, \\ \bar{V}(\xi, 0) = \max(2\xi - 1, 0), \end{array} \right. \quad (2.51)$$

where

$$0 \leq \xi \leq 1 \quad \text{and} \quad 0 \leq \tau.$$

In this section, an American option is reduced to a linear complementarity problem. Such a problem usually needs to be solved numerically. Here we need to point out that the version given in Subsection 2.5.3 can be applied only if σ does not depend upon S and that the other two versions can be applied for any case. However the version given in Subsection 2.5.3 has the simplest equation. Also, if an implicit scheme is used, then for the versions given in Subsections 2.5.2 and 2.5.3, artificial boundary conditions are needed at the boundaries because numerical methods have to be performed on a finite domain. However the version given in this subsection does not have such a problem.

2.5.5 More General Form of the Linear Complementarity Problems

From the three previous subsections, we see that a linear complementarity problem could be in the form:

$$\left\{ \begin{array}{l} \left(\frac{\partial V(S, t)}{\partial t} + \mathbf{L}_{\mathbf{s}, \mathbf{t}} V(S, t) \right) (V(S, t) - G(S, t)) = 0, \\ \frac{\partial V(S, t)}{\partial t} + \mathbf{L}_{\mathbf{s}, \mathbf{t}} V(S, t) \leq 0, \\ V(S, t) - G(S, t) \geq 0, \\ V(S, T) = G(S, T), \end{array} \right.$$

where¹⁰

$$S_l \leq S \leq S_u, t \leq T$$

and

$$\mathbf{L}_{\mathbf{s}, \mathbf{t}} = a(S, t) \frac{\partial^2}{\partial S^2} + b(S, t) \frac{\partial}{\partial S} + c(S, t).$$

However, a linear complementarity problem could have a more general form such as

¹⁰If $S_l = -\infty$, then the first “ \leq ” needs to be changed into “ $<$ ” and if $S_u = \infty$, then the second “ \leq ” needs to be changed into “ $<$ ”.

$$\begin{cases} \left(\frac{\partial V(S, t)}{\partial t} + \mathbf{L}_{\mathbf{s}, t} V(S, t) + d(S, t) \right) (V(S, t) - G(S, t)) = 0, \\ \frac{\partial V(S, t)}{\partial t} + \mathbf{L}_{\mathbf{s}, t} V(S, t) + d(S, t) \leq 0, \\ V(S, t) - G(S, t) \geq 0, \\ V(S, T) = G_1(S) \geq G(S, T), \end{cases} \quad (2.52)$$

where¹¹ $S_l \leq S \leq S_u, t \leq T$. For example, the linear complementarity problem for one-factor convertible bonds has such a form. For two-factor convertible bonds, the form of the linear complementarity problem is similar but the operator $\mathbf{L}_{\mathbf{s}, t}$ is two-dimensional (see Chapter 4).

2.6 American Option Problems as FBPs

2.6.1 Free Boundaries

From the last section, we discovered that there are some regions where the Black–Scholes equation cannot be used. Therefore there exist two different types of regions, one where the Black–Scholes equation is valid, and the other where the Black–Scholes equation cannot be used and the solution is equal to the constraint. Because we do not know *a priori* the location of the boundaries between the two types of different regions, these boundaries are called free boundaries. Since in some regions the solution is known, we only need to determine the price in other regions and the locations of these free boundaries. In order to do that, we reformulate the American option problems as so-called free-boundary problems (FBPs).

Let us first discuss how to find the locations of the free boundaries at time T . Using Theorem 2.1, we can easily determine the locations of free boundaries at time T , namely, the starting points of free boundaries. We will show that for an American put option with $r > 0$, there is a free boundary starting from the point $(\min(E, rE/D_0), T)$ on the (S, t) -plane. If $r = 0$, then there is no free boundary. This implies that the Black–Scholes equation is valid everywhere and that the prices of the American and European put options are the same. For an American call option the situation is similar. If $D_0 > 0$, then there is a free boundary starting from the point $(\max(E, rE/D_0), T)$ on the (S, t) -plane. If $D_0 = 0$, then there is no free boundary, implying that an American call option is the same as a European call option.

First let us consider an American put option and let $P(S, t)$ denote its value as we did in Subsection 2.5.1. In this case

$$G(S, t) = \max(E - S, 0) = \begin{cases} E - S, & \text{for } S < E, \\ 0, & \text{for } S \geq E \end{cases}$$

¹¹The last footnote can also be applied in this case.

and the operator $\mathbf{L}_{\mathbf{s},t}$ in this case does not depend on t and is equal to

$$\mathbf{L}_{\mathbf{s}} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S \frac{\partial}{\partial S} - r.$$

For $S < \min(E, rE/D_0)$, $G(S, t) = E - S$ and

$$\begin{aligned} & \frac{\partial G}{\partial t}(S, T) + \mathbf{L}_{\mathbf{s}}G(S, T) \\ &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + (r - D_0)S \frac{\partial G}{\partial S} - rG \\ &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}(E - S) + (r - D_0)S \frac{\partial}{\partial S}(E - S) - r(E - S) \\ &= D_0S - rE < 0 \end{aligned}$$

because $S < rE/D_0$. Consequently, for $S < \min(E, rE/D_0)$, we cannot get the solution by using the Black-Scholes equation at time T . Instead, the price will be pushed up to satisfy $P(S, T) - G(S, T) = 0$, so $P(S, T) = E - S$ and

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0)S \frac{\partial P}{\partial S} - rP < 0.$$

For $S > E$, $G(S, t) = 0$ and

$$\frac{\partial G}{\partial t}(S, T) + \mathbf{L}_{\mathbf{s}}G(S, T) = 0.$$

If $rE/D_0 < E$, then we have an interval $(rE/D_0, E)$ and for $S \in (rE/D_0, E)$, $G(S, t) = E - S$ and

$$\frac{\partial G}{\partial t}(S, T) + \mathbf{L}_{\mathbf{s}}G(S, T) = D_0S - rE > 0$$

because $S > rE/D_0$. Thus for $S > \min(E, rE/D_0)$ the Black-Scholes equation gives a solution satisfying

$$P(S, T - \Delta t) \geq G(S, t) = \max(E - S, 0)$$

for any sufficiently small positive Δt .

Therefore at $t = T$, if $r > 0$, then the S -axis is divided into two parts: $[0, \min(E, rE/D_0))$ where the Black-Scholes equation cannot be used and $(\min(E, rE/D_0), \infty)$ where the Black-Scholes equation gives the price of the American put option. Thus at time T there is a free boundary at $S = \min(E, rE/D_0)$. If $r = 0$, then $\min(E, rE/D_0) = 0$, so in the entire interval $(0, \infty)$ the Black-Scholes equation can be used and there is no free boundary.

Now let us explain that in the case $r > 0$ no new free boundary can appear at any time $t < T$, so the free boundary starting from the point

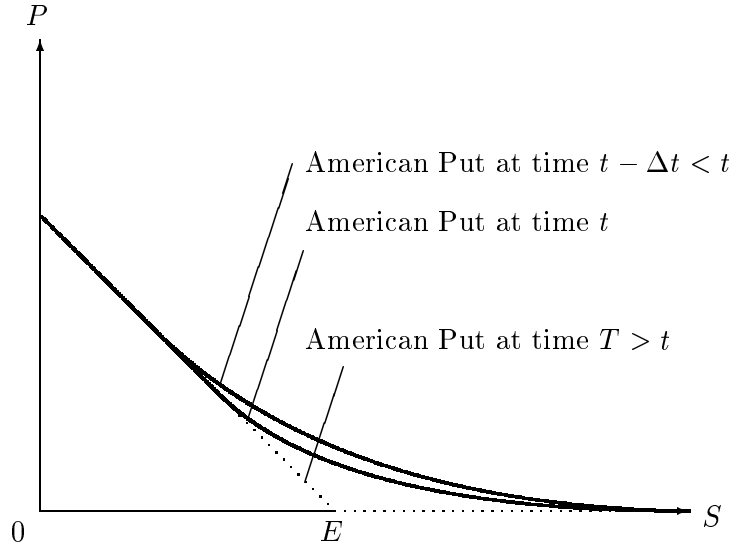


Fig. 2.7. $P(S, t - \Delta t) \geq P(S, t)$ for any positive Δt

$(\min(E, rE/D_0), T)$ is the only free boundary in this problem. Let $S_f(t)$ denote this free boundary. Since it starts from the point $(\min(E, rE/D_0), T)$, we have

$$S_f(T) = \min\left(E, \frac{rE}{D_0}\right). \quad (2.53)$$

Since a holder of a long-life American put option has more exercise opportunities than a holder of a short-life American put option, we have $P(S, t - \Delta t) \geq P(S, t)$ for any positive Δt . Therefore if $P(S, t) > \max(E - S, 0)$ for time t , then $P(S, t - \Delta t) \geq P(S, t) > \max(E - S, 0)$ for any $\Delta t > 0$ (see Fig. 2.7). Consequently, at time t there is no chance for a new free boundary to appear in the interval $(S_f(t), \infty)$ where $P(S, t) > \max(E - S, 0)$ and $S_f(t)$ must be less than $S_f(T) = \min(E, rE/D_0)$ for any t . Because of $S_f(t) \leq \min(E, rE/D_0)$, we have $P(S, t) = E - S$ at any point (S, t) with $S \in (0, S_f(t))$. From Theorem 2.1 we know that it is impossible for a new free boundary to appear at the point because at a neighbourhood of the point

$$\frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + (r - D_0)S \frac{\partial G}{\partial S} - rG < 0.$$

Therefore no new free boundary can appear anywhere. However if a point is close enough to the point $(S_f(t), t)$, there is a chance for the point to be on the free boundary at some time $t - \Delta t$. Thus the free boundary starting from the point $(\min(E, rE/D_0), T)$ is movable.

Consequently if $r > 0$, then there is a unique free boundary and the entire domain is divided into two regions (see Fig. 2.8), one region is $[0, S_f(t)) \times [0, T]$,

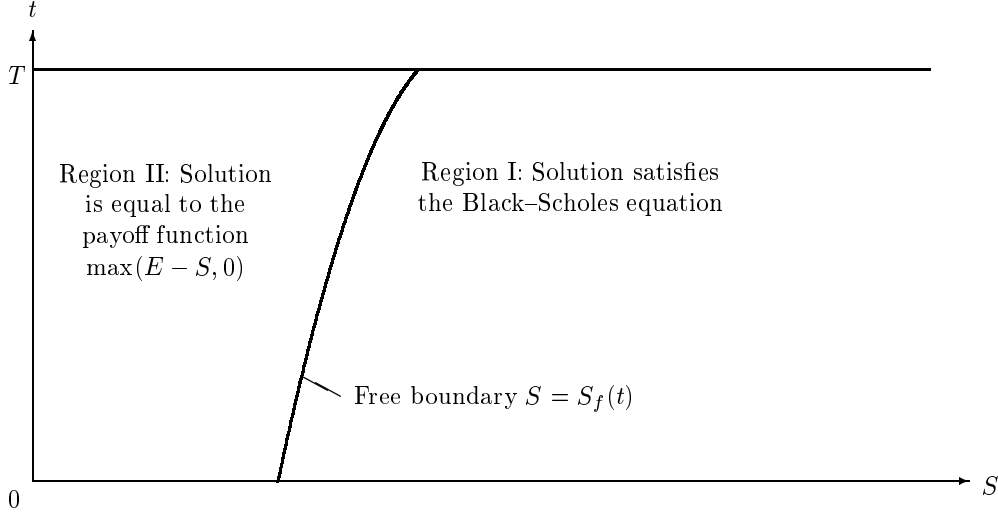


Fig. 2.8. Structure of solution to American put options ($r > 0$)

where

$$\begin{cases} P = E - S = \max(E - S, 0), \\ \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0)S \frac{\partial P}{\partial S} - rP < 0 \end{cases}$$

and the other is $(S_f(t), \infty) \times [0, T]$, where

$$\begin{cases} P > \max(E - S, 0), \\ \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0)S \frac{\partial P}{\partial S} - rP = 0 \end{cases}$$

if $t < T$.

Before going further, it is necessary to explain the meaning of the inequality

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0)S \frac{\partial P}{\partial S} - rP < 0.$$

As pointed out in Subsection 2.2.2, this means that the difference between the return from the portfolio and the return from an equivalent bank deposit is negative. Hence it is optimal to exercise the portfolio. If $P(S, t) > \max(E - S, 0)$, one should hold the option, since one should not give up a higher value (the option) for a lower value (the payoff). Therefore, the free boundary is the optimal exercise price which divides the exercise region and the non-exercise region.

As pointed out above, if $P(S, t) > \max(E - S, 0)$, then $P(S, t - \Delta t) > P(S, t)$ for any positive Δt . From this fact, we can further see that the following

inequality holds (see Fig. 2.7):

$$S_f(t) > S_f(t - \Delta t), \quad \Delta t > 0,$$

implying that $S_f(t)$ is an increasing function of t (see Fig. 2.8).

Now let us consider an American call option. As we know, at very large S , the solution of the Black–Scholes equation with final condition $V(S, t) = \max(S - E, 0)$ has the following asymptotic expression

$$V(S, t) \approx V(S, T)e^{-D_0(T-t)} = \max(S - E, 0)e^{-D_0(T-t)},$$

so if $D_0 > 0$, then $V(S, t) < \max(S - E, 0)$ for any $t < T$. Therefore if $D_0 > 0$, the American call problem is a free-boundary problem. Now let us show that the free-boundary problem has only one free boundary, which is also denoted by $S_f(t)$ in what follows, and determine the location of the free boundary at $t = T$ from the constraint condition $C(S, t) \geq G(S, t)$.

In the case of an American call option,

$$G(S, t) = \max(S - E, 0) = \begin{cases} S - E, & S > E, \\ 0, & S \leq E. \end{cases}$$

Let $S > \max(E, rE/D_0)$. In this case

$$G(S, t) = S - E$$

and

$$\begin{aligned} & \frac{\partial G}{\partial t}(S, T) + \mathbf{L}_S G(S, T) \\ &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + (r - D_0)S \frac{\partial G}{\partial S} - rG \\ &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 (S - E)}{\partial S^2} + (r - D_0)S \frac{\partial (S - E)}{\partial S} - r(S - E) \\ &= rS - D_0S - rS + rE = -D_0S + rE < 0 \end{aligned}$$

because $S > rE/D_0$. Therefore the Black–Scholes equation cannot hold in this case and $C(S, T - \Delta t)$ should be equal to $S - E$ for $S > \max(E, rE/D_0)$. For $S < \max(E, rE/D_0)$, the Black–Scholes equation can hold. Thus a free boundary starts at $S = \max(E, rE/D_0)$, i.e.,

$$S_f(T) = \max\left(E, \frac{rE}{D_0}\right). \quad (2.54)$$

Using the same argument we have used for an American put option, we can show that the free boundary starting from the point $(\max(E, rE/D_0), T)$ is the only free boundary since no new free boundary can appear at time $t < T$. Just like the put case, the entire domain is divided into two parts by the free boundary. However, the situation is a little different from the American put.

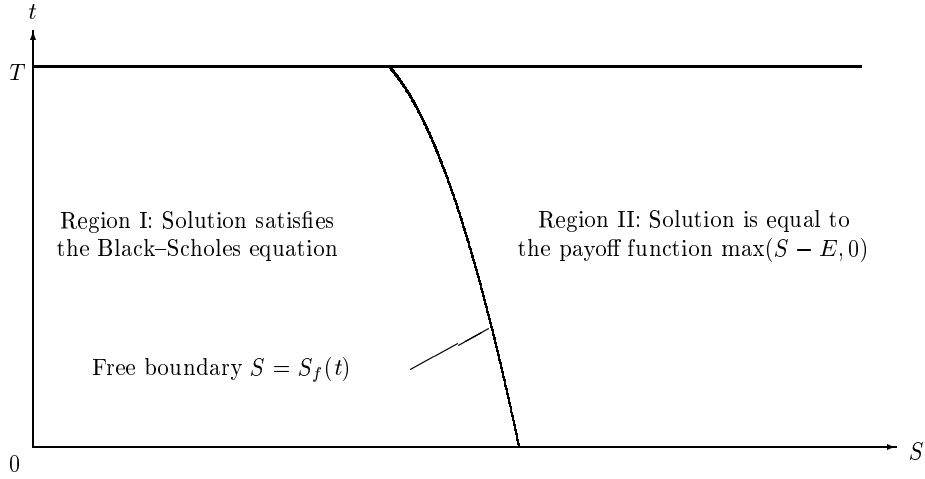


Fig. 2.9. Structure of solution to American call options ($D_0 > 0$)

Here in the region $[0, S_f(t)) \times [0, T]$, the Black-Scholes equation holds, while in the region $(S_f(t), \infty) \times [0, T]$, the Black-Scholes equation cannot be used. In other words, for $S \in [0, S_f(t))$ and $t < T$,

$$\begin{cases} C(S, t) \geq \max(S - E, 0), \\ \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0)S \frac{\partial C}{\partial S} - rC = 0, \end{cases}$$

where the equal sign in $C(S, t) \geq \max(S - E, 0)$ holds only at $S = 0$; while for $S \in (S_f(t), \infty)$,

$$\begin{cases} C(S, t) = S - E = \max(S - E, 0), \\ \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0)S \frac{\partial C}{\partial S} - rC < 0 \end{cases}$$

and the option should be exercised (see Fig. 2.9). It can also be shown that for an American call option, the free boundary $S_f(t)$ is a decreasing function of t , as graphed in Fig. 2.9, and that the price of an American call option is the same as a European call if $D_0 = 0$.

2.6.2 Free-Boundary Problems

In this subsection we will describe the formulation of American option problems as free-boundary problems. In order to give a complete formulation, we

need to give the conditions on the free boundary. For an initial-boundary value problem of a parabolic equation on a finite interval, if the locations of the boundaries are given and if the coefficient of the second derivative at the boundaries is not equal to zero, one boundary condition at each boundary is needed in order for the problem to have a unique solution. However, the location of the free boundary is unknown, so two conditions are needed at the free boundary in order for the problem to have a unique solution. One boundary condition determines the option value on the free boundary and the other boundary condition determines the location of the free boundary. Now the question is what the two conditions should be. For some other linear complementarity problems, it has been proved that on the free boundary the value and the first derivative are continuous (see [28]). For this problem, from the proof given by Badea and Wang (see [3] and [4]) we know that the situation is still the same. Therefore the two conditions on the free boundary are: both the value and the derivative with respect to S are continuous.

For an American put option, in the region $[0, S_f(t))$,

$$P(S, t) = E - S \quad \text{and} \quad \frac{\partial P}{\partial S} = -1.$$

Therefore the boundary conditions on the free boundary $S_f(t)$ are

$$P(S_f(t), t) = E - S_f(t) \tag{2.55}$$

and

$$\frac{\partial P}{\partial S}(S_f(t), t) = -1. \tag{2.56}$$

It is clear that when (2.56) holds, the gradient $\frac{\partial P}{\partial S}$ must be continuous at $S = S_f$ (see Fig. 2.10).

Now we can formulate the American put option problem. On the domain $[0, S_f(t)) \times [0, T]$, $P(S, t) = E - S$, while on the domain $[S_f(t), \infty) \times [0, T]$, $P(S, t)$ is the solution of the free-boundary problem for American put options

$$\left\{ \begin{array}{ll} \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0)S \frac{\partial P}{\partial S} - rP = 0, & S_f(t) \leq S, \quad 0 \leq t \leq T, \\ P(S, T) = \max(E - S, 0), & S_f(T) \leq S, \\ P(S_f(t), t) = E - S_f(t), & t \leq T, \\ \frac{\partial P(S_f(t), t)}{\partial S} = -1, & t \leq T, \\ S_f(T) = \min\left(E, \frac{rE}{D_0}\right). \end{array} \right. \tag{2.57}$$

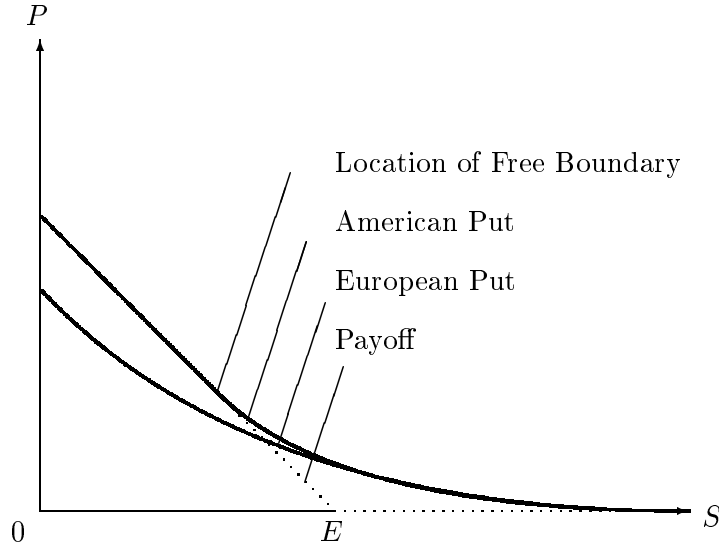


Fig. 2.10. The price of an American put option before expiry

Similarly, for call options we need two boundary conditions on the free boundary. One is

$$C(S, t) = S_f(t) - E \quad (2.58)$$

and the other still can be obtained by requiring the continuity of the slope of the solution at $S = S_f(t)$. In this case, the condition is

$$\frac{\partial C(S_f(t), t)}{\partial S} = 1. \quad (2.59)$$

Therefore for the American call option, the formulation is as follows. On the domain $[0, S_f(t)] \times [0, T]$, $C(S, t)$ is the solution of the free-boundary problem for American call options

$$\left\{ \begin{array}{ll} \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} - rC = 0, & 0 \leq S \leq S_f(t), \quad 0 \leq t \leq T, \\ C(S, T) = \max(S - E, 0), & 0 \leq S \leq S_f(T), \\ C(S_f(t), t) = S_f(t) - E, & 0 \leq t \leq T, \\ \frac{\partial C}{\partial S}(S_f(t), t) = 1, & 0 \leq t \leq T, \\ S_f(T) = \max\left(E, \frac{rE}{D_0}\right); \end{array} \right. \quad (2.60)$$

while on the domain $(S_f(t), \infty) \times [0, T]$, $C(S, t) = S - E$. In Fig. 2.11, the value of an American call option is plotted, from which we can see how the two parts of solution are connected smoothly. The parameters of the problem are $E = 100$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$ and $T = 1$ year.

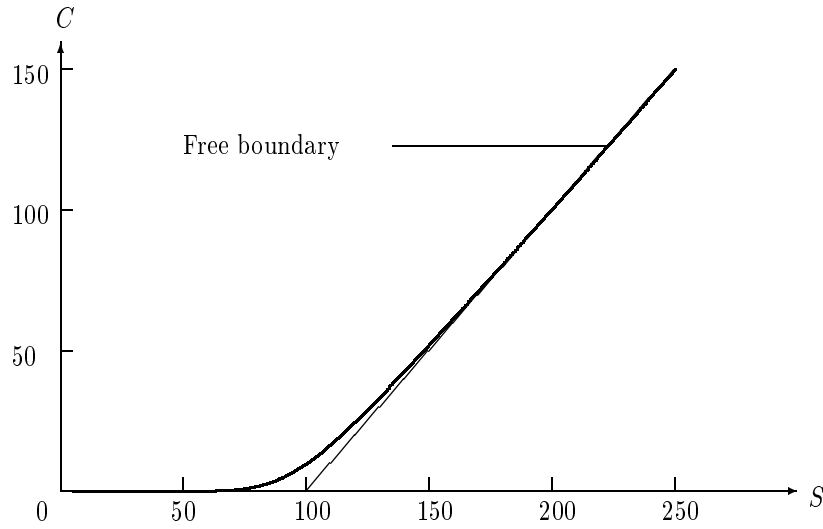


Fig. 2.11. Numerically calculated solution of an American call problem with $E = 100$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$ and $T = 1$ year

Finally we need to point out that $S_f(T)$ is determined by the partial differential operator and the final condition. Therefore in a free-boundary problem the starting location of the free boundary is not arbitrary and should be consistent with the partial differential operator and the final condition.

2.6.3 Put–Call Symmetry Relations

From Subsection 2.3.2, we know that under the transformation

$$\begin{cases} \xi = \frac{S}{S + E}, \\ \tau = T - t, \\ c(S, t) = (S + E)\bar{c}(\xi, \tau), \end{cases} \quad (2.61)$$

a call option problem with a constant σ is converted into the following problem

$$\begin{cases} \frac{\partial \bar{c}}{\partial \tau} = \frac{1}{2} \sigma^2 \xi^2 (1 - \xi)^2 \frac{\partial^2 \bar{c}}{\partial \xi^2} + (r - D_0) \xi (1 - \xi) \frac{\partial \bar{c}}{\partial \xi} - [r(1 - \xi) + D_0 \xi] \bar{c}, \\ \bar{c}(\xi, 0) = \max(2\xi - 1, 0), \end{cases} \quad \begin{matrix} 0 \leq \xi \leq 1, & 0 \leq \tau, \\ 0 \leq \xi \leq 1. \end{matrix}$$

If we introduce the transformation

$$\begin{cases} \xi = 1 - \frac{S}{S + E} = \frac{E}{S + E}, \\ \tau = T - t, \\ p(S, t) = (S + E) \tilde{p}(\xi, \tau), \end{cases} \quad (2.62)$$

the put option with a constant σ is reduced to

$$\begin{cases} \frac{\partial \tilde{p}}{\partial \tau} = \frac{1}{2} \sigma^2 \xi^2 (1 - \xi)^2 \frac{\partial^2 \tilde{p}}{\partial \xi^2} + (D_0 - r) \xi (1 - \xi) \frac{\partial \tilde{p}}{\partial \xi} - [D_0(1 - \xi) + r\xi] \tilde{p}, \\ \tilde{p}(\xi, 0) = \max(2\xi - 1, 0), \end{cases} \quad \begin{matrix} 0 \leq \xi \leq 1, & 0 \leq \tau, \\ 0 \leq \xi \leq 1. \end{matrix}$$

Comparing the two problems above, we know that if $r = a$ and $D_0 = b$ in the first problem and $r = b$ and $D_0 = a$ in the second problem, then the two solutions should be equal:

$$\bar{c}(\xi, \tau; a, b) = \tilde{p}(\xi, \tau; b, a).$$

Here the first and second arguments after the semicolon in \bar{c} , \tilde{p} are the values of the interest rate and dividend yield respectively and in this section for $c, p, C, P, S_{pf}, S_{cf}$ etc., the same notation will be used. Since

$$\bar{c}(\xi, \tau; a, b) = \frac{c\left(\frac{E\xi}{1-\xi}, t; a, b\right)(1-\xi)}{E}$$

and

$$\tilde{p}(\xi, \tau; b, a) = \frac{p\left(\frac{E(1-\xi)}{\xi}, t; b, a\right)\xi}{E},$$

we can rewrite the above relation between \bar{c} and \tilde{p} as

$$\frac{c\left(\frac{E\xi}{1-\xi}, t; a, b\right)(1-\xi)}{E} = \frac{p\left(\frac{E(1-\xi)}{\xi}, t; b, a\right)\xi}{E}$$

and further have

$$\begin{cases} c(S, t; a, b) = p\left(\frac{E^2}{S}, t; b, a\right) S/E, & \text{or} \\ p(S, t; b, a) = c\left(\frac{E^2}{S}, t; a, b\right) S/E \end{cases} \quad (2.63)$$

and its special case

$$p(E, t; b, a) = c(E, t; a, b).$$

Unlike the put-call parity, the relation (2.63) still holds for American options. In fact, using the transformation (2.61) an American call problem can be written as

$$\begin{cases} \left(\frac{\partial \bar{C}}{\partial \tau} - \frac{1}{2} \sigma^2 \xi^2 (1 - \xi)^2 \frac{\partial^2 \bar{C}}{\partial \xi^2} - (r - D_0) \xi (1 - \xi) \frac{\partial \bar{C}}{\partial \xi} + [r(1 - \xi) + D_0 \xi] \bar{C} \right) \\ \times (\bar{C}(\xi, \tau) - \max(2\xi - 1, 0)) = 0, \\ \frac{\partial \bar{C}}{\partial \tau} - \frac{1}{2} \sigma^2 \xi^2 (1 - \xi)^2 \frac{\partial^2 \bar{C}}{\partial \xi^2} - (r - D_0) \xi (1 - \xi) \frac{\partial \bar{C}}{\partial \xi} + [r(1 - \xi) + D_0 \xi] \bar{C} \geq 0, \\ \bar{C}(\xi, \tau) - \max(2\xi - 1, 0) \geq 0, \\ \bar{C}(\xi, 0) = \max(2\xi - 1, 0), \end{cases}$$

where $\xi \in [0, 1]$ and $0 \leq \tau$. Under the transformation (2.62) an American put option can be converted into

$$\begin{cases} \left(\frac{\partial \tilde{P}}{\partial \tau} - \frac{1}{2} \sigma^2 \xi^2 (1 - \xi)^2 \frac{\partial^2 \tilde{P}}{\partial \xi^2} - (D_0 - r) \xi (1 - \xi) \frac{\partial \tilde{P}}{\partial \xi} + [D_0(1 - \xi) + r\xi] \tilde{P} \right) \\ \times (\tilde{P}(\xi, \tau) - \max(2\xi - 1, 0)) = 0, \\ \frac{\partial \tilde{P}}{\partial \tau} - \frac{1}{2} \sigma^2 \xi^2 (1 - \xi)^2 \frac{\partial^2 \tilde{P}}{\partial \xi^2} - (D_0 - r) \xi (1 - \xi) \frac{\partial \tilde{P}}{\partial \xi} + [D_0(1 - \xi) + r\xi] \tilde{P} \geq 0, \\ \tilde{P}(\xi, \tau) - \max(2\xi - 1, 0) \geq 0, \\ \tilde{P}(\xi, 0) = \max(2\xi - 1, 0), \end{cases}$$

where $\xi \in [0, 1]$ and $0 \leq \tau$. Just as in the European case, if $r = a, D_0 = b$ in the former problem and $r = b, D_0 = a$ in the latter problem, then the two solutions should be the same and

$$\begin{cases} C(S, t; a, b) = P\left(\frac{E^2}{S}, t; b, a\right) S/E, & \text{or} \\ P(S, t; b, a) = C\left(\frac{E^2}{S}, t; a, b\right) S/E \end{cases} \quad (2.64)$$

holds. Also the location of free boundary in the former problem, $\xi_{cf}(\tau; a, b)$, must be the same as the location of free boundary of the latter problem, $\xi_{pf}(\tau; b, a)$, i.e.,

$$\xi_{cf}(\tau; a, b) = \xi_{pf}(\tau; b, a)$$

or

$$\frac{S_{cf}(t; a, b)}{E + S_{cf}(t; r, b)} = \frac{E}{E + S_{pf}(t; b, a)}.$$

Therefore

$$E S_{cf}(t; a, b) + S_{cf}(t; a, b) S_{pf}(t; b, a) = E^2 + E S_{cf}(t; a, b),$$

which can be reduced to

$$S_{cf}(t; a, b) \times S_{pf}(t; b, a) = E^2. \quad (2.65)$$

The relations (2.63)-(2.65) are called the put-call symmetry relations. If σ depends upon S , then similar relations exist. In this case we have

$$\begin{cases} C(S, t; a, b, \sigma(S)) = P\left(\frac{E^2}{S}, t; b, a, \sigma(S)\right) S/E, & \text{or} \\ P(S, t; b, a, \sigma(S)) = C\left(\frac{E^2}{S}, t; a, b, \sigma(S)\right) S/E \end{cases}$$

and

$$S_{cf}(t; a, b, \sigma(S)) \times S_{pf}(t; b, a, \sigma(E^2/S)) = E^2.$$

Here the third argument after the semicolon is the function for the volatility. The proof is left for the reader as an exercise (Problem 34).

They can also be derived in another way, which gives the financial meaning of these relations. Suppose that S is the price of a British pound in U.S. dollars and that ξ is the price of a U.S. dollar in British pounds. Then $\xi = 1/S$. Let P be the value of a put option on S in U.S. dollars with exercise price E and \tilde{C} be the value of a call option on ξ in British pounds with an exercise price $\frac{1}{E}$. The holder of the put option has the right to sell one pound for E U.S. dollars even if $S \leq E$. The holder of E shares of the call option has the right to buy E dollars by paying one British pound even if $\xi \geq \frac{1}{E}$. The condition $S \leq E$ is equivalent to $\xi \geq \frac{1}{E}$. Thus both the holder of one share of the put option and the holder of E shares of the call options have the right to exchange one British pound for E U.S. dollars even if $S < E$, i.e., even if the price of a British pound on the market is less than E U.S. dollars. The two holders have the same rights, so the value of one share of the put option and the value of E shares of the call option in U.S. dollars should be equal, i.e.,

$$P = SE\tilde{C},$$

Here we need to notice that P and \tilde{C} have different but related volatilities, interest rates, dividend yields and exercise prices. According to Itô's lemma, if

$$dS = \mu S dt + \sigma S dX,$$

then

$$d\xi = (-\mu + \sigma^2)\xi dt - \sigma\xi dX.$$

Hence the volatilities of S and $\xi = 1/S$ are the same in this case. Consequently the volatilities of the two problems are the same if the volatilities are constants.

Suppose that σ, r and D_0 are constant and that the interest rates of the British pound and the U.S. dollar are a and b respectively. Then for the call, $r = a$ and $D_0 = b$ and for the put $r = b$ and $D_0 = a$, and the volatilities are the same. In this case the relation above can be written as

$$P(S, t; b, a, E) = SE\tilde{C}\left(\frac{1}{S}, t; a, b, 1/E\right),$$

where the third argument after the semicolon in P and \tilde{C} represents the exercise price. Let $\eta = E^2\xi$ and $C(\eta, t) = E^2\tilde{C}$. It is clear that for C the corresponding partial differential equation is

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 C}{\partial \eta^2} + (a-b)\eta\frac{\partial C}{\partial \eta} - aC = 0$$

and the final condition and the constraint for C are

$$C(\eta, T) = E^2 \max(\xi - 1/E, 0) = \max(\eta - E, 0),$$

and

$$C(\eta, t) = E^2\tilde{C}(\xi, t) \geq E^2 \max(\xi - 1/E, 0) = \max(\eta - E, 0).$$

C represents the value of a call option with an exercise price E . From $P = SE\tilde{C}$ and $C = E^2\tilde{C}$, we have $P = CS/E$, i.e.,

$$P(S, t; b, a) = C\left(\frac{E^2}{S}, t; a, b\right) S/E.$$

Here both the put option P and the call option C have the same exercise price E , so the dependence on E is suppressed. Let $S_{pf}(t; b, a, E)$ be the position of the free boundary corresponding to $P(S, t; b, a, E)$ and $\xi_{cf}(t; a, b, 1/E)$ be the position of the free boundary corresponding to $\tilde{C}(1/S, t; a, b, 1/E)$. From the above, $\xi_{cf}(t; a, b, 1/E)$ is equal to $1/S_{pf}(t; b, a, E)$. Suppose that corresponding to $C(E^2/S, t; a, b, E)$, the position of the free boundary is $\eta_{cf}(t; a, b, E)$. Since $\eta_{cf}(t; a, b, E)$ is equal to $E^2\xi_{cf}(t; a, b, 1/E)$, we have

$$S_{pf}(t; b, a, E) \times \eta_{cf}(t; a, b, E) = E^2.$$

Here the exercise prices for both C and P are E and the dependence on the exercise price can also be suppressed. Moreover, if we use S , instead of η , as the state variable for the call option C , then the relation above can be written as

$$S_{pf}(t; b, a) \times S_{cf}(t; a, b) = E^2.$$

Since

$$\max(E - S, 0) = \frac{S}{E} \max\left(\frac{E^2}{S} - E, 0\right),$$

when we take, instead of S and P , E^2/S and PE/S as the new variables, a vanilla put option problem is converted into a call option problem even if the parameters depend on S and t . Hence, if one has a code to price all the vanilla call option problems, then one can also evaluate any vanilla put option problem by using the code. The converse is also true. If one already has a code which can deal with both American call and put options, then the symmetry relations can be used for checking the accuracy of the numerical results. Since the numerical results have errors, they will not exactly satisfy the symmetry relation and can be used as indicators to show how accurate the numerical results are if the values of a call and the corresponding put options have been obtained. For details, see the paper [83] by Zhu, Ren and Xu. For more about symmetry relations and similar results, see [48], [55], [47] and [23].

2.7 Equations for Some Greeks

Here for American options we would like to derive the equations and boundary conditions $\mathcal{V} = \frac{\partial \Pi}{\partial \sigma}$, $\rho = \frac{\partial \Pi}{\partial r}$ and $\rho_d = \frac{\partial \Pi}{\partial D_0}$ should satisfy. Let us first consider American call options and write the dependence of C and S_f on r, D_0 and σ explicitly, that is, instead of $C(S, t)$ and $S_f(t)$, we use $C(S, t; r, D_0, \sigma)$ and $S_f(t; r, D_0, \sigma)$ to denote the price of American call options and the free boundary in what follows. Differentiating the partial differential equation in (2.60) with respect to r, D_0 or σ yields the equations for $\frac{\partial C}{\partial r}$, $\frac{\partial C}{\partial D_0}$ or $\frac{\partial C}{\partial \sigma}$.

For example, for $\frac{\partial C}{\partial \sigma}$ we have

$$\frac{\partial C_\sigma}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C_\sigma}{\partial S^2} + (r - D_0) S \frac{\partial C_\sigma}{\partial S} - r C_\sigma + \sigma S^2 \frac{\partial^2 C}{\partial S^2} = 0,$$

where C_σ denotes the partial derivative of the call option with respect to σ . The final condition for the price of American call options is

$$C(S, T; r, D_0, \sigma) = \max(S - E, 0).$$

Therefore

$$\frac{\partial C}{\partial \sigma} = 0$$

at $t = T$. The boundary conditions on the free boundary are

$$C(S_f(t; r, D_0, \sigma), t; r, D_0, \sigma) = S_f(t; r, D_0, \sigma) - E \quad (2.66)$$

and

$$\frac{\partial C(S_f(t; r, D_0, \sigma), t; r, D_0, \sigma)}{\partial S} = 1. \quad (2.67)$$

From (2.66) we have

$$\frac{\partial C}{\partial S} \frac{\partial S_f}{\partial \sigma} + \frac{\partial C}{\partial \sigma} = \frac{\partial S_f}{\partial \sigma}$$

on the free boundary. Noticing (2.67), we have

$$\frac{\partial C}{\partial \sigma} = 0$$

at the free boundary. Consequently, $\frac{\partial C}{\partial \sigma}$ is the solution of the following final-boundary value problem

$$\begin{cases} \frac{\partial C_\sigma}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_\sigma}{\partial S^2} + (r - D_0)S \frac{\partial C_\sigma}{\partial S} - rC_\sigma + \sigma S^2 \frac{\partial^2 C}{\partial S^2} = 0, \\ \quad \quad \quad 0 \leq S \leq S_f(t), \quad 0 \leq t \leq T, \\ C_\sigma(S, T) = 0, \quad \quad \quad 0 \leq S \leq S_f(T), \\ C_\sigma(S_f(t), t) = 0, \quad \quad 0 \leq t \leq T, \end{cases} \quad (2.68)$$

where $\frac{\partial^2 C}{\partial S^2}$ and $S_f(t)$ are known functions obtained from the solution of problem (2.60).

For $\frac{\partial C}{\partial r}$ and $\frac{\partial C}{\partial D_0}$, we can derive the same final and boundary conditions as $\frac{\partial C}{\partial \sigma}$, namely,

$$\frac{\partial C}{\partial r} = \frac{\partial C}{\partial D_0} = 0 \quad (2.69)$$

at $t = T$ and

$$\frac{\partial C}{\partial r} = \frac{\partial C}{\partial D_0} = 0 \quad (2.70)$$

at the free boundary. The only difference is the equation. Differentiating the partial differential equation in (2.60) with respect to r and D_0 yields

$$\frac{\partial C_r}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_r}{\partial S^2} + (r - D_0)S \frac{\partial C_r}{\partial S} - rC_r + S \frac{\partial C}{\partial S} - C = 0 \quad (2.71)$$

and

$$\frac{\partial C_{D_0}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{D_0}}{\partial S^2} + (r - D_0)S \frac{\partial C_{D_0}}{\partial S} - rC_{D_0} - S \frac{\partial C}{\partial S} = 0 \quad (2.72)$$

respectively, where C_r stands for $\frac{\partial C}{\partial r}$ and C_{D_0} for $\frac{\partial C}{\partial D_0}$.

For American put options, the Greeks are solutions of similar problems. This is left for the reader to show as Problem 36 of this chapter.

2.8 Perpetual Options

If an option does not have an expiry date but rather an infinite time horizon, then the option is called a perpetual option. In order to determine the price of such an option, we can let T be fixed and find the value of the option as $t \rightarrow -\infty$. When $t \rightarrow -\infty$, the solution does not depend on t anymore and it is a solution of an ordinary differential equation. Let

$$C_\infty(S) = \lim_{t \rightarrow -\infty} C(S, t).$$

This is the price of a perpetual American call option. As $t \rightarrow -\infty$, the problem (2.60) for an American call option becomes

$$\begin{cases} \frac{1}{2}\sigma^2 S^2 \frac{d^2 C_\infty}{dS^2} + (r - D_0)S \frac{dC_\infty}{dS} - rC_\infty = 0, & 0 \leq S \leq S_f, \\ C_\infty(S_f) = S_f - E, \\ \frac{dC_\infty(S_f)}{dS} = 1. \end{cases} \quad (2.73)$$

Thus $C_\infty(S)$ is the solution of this free-boundary problem. Let

$$C_\infty(S) = S^\alpha,$$

then

$$\frac{dC_\infty}{dS} = \alpha S^{\alpha-1}, \quad \frac{d^2 C_\infty}{dS^2} = \alpha(\alpha-1)S^{\alpha-2}.$$

Substitute these into the ordinary differential equation in (2.73) we get

$$\frac{1}{2}\sigma^2 \alpha^2 + \left(r - D_0 - \frac{1}{2}\sigma^2\right) \alpha - r = 0.$$

The two roots of this equation are

$$\alpha_\pm = \frac{1}{\sigma^2} \left[-\left(r - D_0 - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(r - D_0 - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r} \right].$$

Thus

$$C_\infty(S) = C_+ S^{\alpha_+} + C_- S^{\alpha_-}.$$

It is clear that $\alpha_+ > 0$ and $\alpha_- < 0$. In order to guarantee the solution to be bounded at $S = 0$, C_- should equal zero. Consequently, we arrive at

$$C_\infty(S) = C_+ S^{\alpha_+}.$$

From the free-boundary conditions in (2.73) we obtain

$$\begin{aligned} C_+ S_f^{\alpha_+} &= S_f - E, \\ C_+ \alpha_+ S_f^{\alpha_+ - 1} &= 1. \end{aligned}$$

Solving these two equations we get

$$S_f = \frac{E}{1 - 1/\alpha_+} \quad \text{and} \quad C_+ = \frac{1}{\alpha_+ S_f^{\alpha_+ - 1}}.$$

Thus, the solution of problem (2.73) is

$$C_\infty(S) = \frac{S_f}{\alpha_+} \left(\frac{S}{S_f} \right)^{\alpha_+}. \quad (2.74)$$

For an American put option, as $t \rightarrow -\infty$, the free-boundary problem (2.57) becomes

$$\begin{cases} \frac{1}{2} \sigma^2 S^2 \frac{d^2 P_\infty}{dS^2} + (r - D_0) S \frac{dP_\infty}{dS} - r P_\infty = 0, & S_f \leq S, \\ P_\infty(S_f) = E - S_f, \\ \frac{dP_\infty(S_f)}{dS} = -1. \end{cases}$$

Similar to the call option, the price of a perpetual American put option is

$$P_\infty(S) = \frac{-S_f}{\alpha_-} \left(\frac{S}{S_f} \right)^{\alpha_-}, \quad (2.75)$$

where $S_f = \frac{E}{1 - 1/\alpha_-}$.

2.9 General Equations for Derivatives

Generally speaking, a financial derivative could depend upon several random variables and a random variable may not represent a price of an asset which can be traded on the market. For example, a derivative could depend on prices of several assets and interest rates, while volatilities may need to be treated as random variables. As we know, both interest rates and volatilities are not prices of assets. In this section, we will derive the general partial differential equations satisfied by derivatives, where there exist several state variables and a state variable may not be a price of an asset traded on the market. The derivation of general equations can be found from other books, for example, the books by Hull^[39], and Wilmott, Dewynne and Howison^[68].

2.9.1 Models for Random Variables

Suppose a financial derivative depends upon time t and n random state variables, namely, S_1, S_2, \dots, S_n . Each of them satisfies a stochastic differential equation

$$dS_i = a_i dt + b_i dX_i, \quad i = 1, 2, \dots, n, \quad (2.76)$$

where a_i, b_i are functions of S_1, S_2, \dots, S_n and t , and $dX_i = \phi_i \sqrt{dt}$ are Wiener processes. In addition, the random variables ϕ_i and ϕ_j could be correlated. Let us suppose

$$E[\phi_i \phi_j] = \rho_{ij}, \quad (2.77)$$

where $-1 \leq \rho_{ij} \leq 1$. If $\rho_{ij} = 0$, then ϕ_i and ϕ_j are not correlated. If $\rho_{ij} = \pm 1$, then ϕ_i and ϕ_j are completely correlated. It is clear that $\rho_{ii} = 1$.

In practice, a random variable always has a lower bound and an upper bound. How do we model a random variable with such a property? For simplicity, we consider problems with only one random variable S . Suppose that we want a random variable S to have a lower boundary S_l , i.e., if $S \geq S_l$ at time t , then we want to guarantee that S is still greater than or equal to S_l after time t even though the movement of S possesses some uncertainty. In this case, we need to require that $a(S, t)$ and $b(S, t)$ at $S = S_l$ satisfy either the condition:

$$\begin{cases} a(S_l, t) - b(S_l, t) \frac{\partial}{\partial S} b(S_l, t) \geq 0, & 0 \leq t \leq T, \\ b(S_l, t) = 0, & 0 \leq t \leq T \end{cases} \quad (2.78)$$

or when $b(S, t)$ is differentiable, the condition

$$\begin{cases} a(S_l, t) \geq 0, & 0 \leq t \leq T, \\ b(S_l, t) = 0, & 0 \leq t \leq T. \end{cases}$$

In Subsection 2.9.5 we will see that if (2.78) holds at $S = S_l$, then a unique solution of the corresponding partial differential equation can be determined by a final condition on $[S_l, \infty)$ without any boundary conditions at $S = S_l$. Therefore what happens at $S = S_l$ will not affect the solution at $t = 0$ for any S . This fact can be interpreted as follows. If (2.78) holds for any $t \in [t_0, T]$, then for any such time t , S will be greater than or equal to S_l if $S > S_l$ at $t = t_0$. That is, S is either reflected into the region or is absorbed by the boundary in the event S hits the lower bound S_l at some time $t \in [t_0, T]$. This is because if there are paths which pass through a point (S_l, t) and go to outside of $[S_l, \infty)$, then the solution at the point $(S, 0)$ should depend on the value of the option at the point (S_l, t) . For example, in the popular model

$$dS = \mu S dt + \sigma S dX,$$

we have $a = \mu S$ and $b = \sigma S$. Therefore at $S = 0$, the condition (2.78) holds and S is always greater than or equal to zero. In the Cox–Ingersoll–Ross interest rate model (see [22])

$$dr = (\bar{\mu} - \bar{\gamma}r)dt + \sqrt{\alpha r}dX, \quad \bar{\mu}, \bar{\gamma}, \alpha > 0,$$

which will be discussed in Chapter 4, $a = \bar{\mu} - \bar{\gamma}r$, $b = \sqrt{\alpha r}$ and the condition (2.78) is reduced to $\bar{\mu} - \alpha/2 \geq 0$ if the lower bound is zero. This means that if $\bar{\mu} - \alpha/2 \geq 0$, then at $r = 0$, no boundary condition is needed. In fact, if $\bar{\mu} - \alpha/2 \geq 0$, the upward drift is sufficiently large to make the origin inaccessible (see [22]). Therefore no boundary condition at $r = 0$ is related to inaccessibility to the origin.

Actually, S_l may not be zero and at $S = S_u > S_l$, a similar condition

$$\begin{cases} a(S_u, t) - b(S_u, t) \frac{\partial}{\partial S} b(S_u, t) \leq 0, & 0 \leq t \leq T, \\ b(S_u, t) = 0, & 0 \leq t \leq T \end{cases} \quad (2.79)$$

can also be required so that S will always be in $[S_l, S_u]$. If $a(S_l, t) \geq 0$ and $a(S_u, t) \leq 0$, then it is usually said that the model has a mean reversion property. However if $b(S_l, t) \neq 0$ or $b(S_u, t) \neq 0$, then there is still a chance for S to become less than S_l or greater than S_u . If the conditions (2.78) and (2.79) hold, then we say that the model really has a reversion property because S will always be in $[S_l, S_u]$. In this book, the conditions (2.78) and (2.79) will be referred to as the reversion conditions.

The two random variables given above as examples are defined on $[0, \infty)$. In what follows, we will show that they can be converted into new random variables whose domains can be naturally extended to $[0, 1]$ and for them the reversion conditions hold at both the lower and upper boundaries.

Let us introduce a new random variable

$$\xi = \frac{S}{S + P_m},$$

where P_m is a positive parameter. From this relation we can have

$$S = \frac{P_m \xi}{1 - \xi}, \quad S + P_m = \frac{P_m}{1 - \xi},$$

and

$$\frac{d\xi}{dS} = \frac{P_m}{(S + P_m)^2} = \frac{(1 - \xi)^2}{P_m}, \quad \frac{d^2 \xi}{dS^2} = \frac{-2P_m}{(S + P_m)^3} = \frac{-2(1 - \xi)^3}{P_m^2}.$$

According to Itô's lemma, if S satisfies $dS = \mu S dt + \sigma S dX$, then for ξ the stochastic differential equation is

$$\begin{aligned}
d\xi &= \frac{(1-\xi)^2}{P_m} dS - \frac{(1-\xi)^3}{P_m^2} \sigma^2 S^2 dt \\
&= (\mu\xi(1-\xi) - \sigma^2 \xi^2(1-\xi)) dt + \sigma\xi(1-\xi) dX.
\end{aligned}$$

Consequently for ξ , the conditions (2.78) and (2.79) are fulfilled at $\xi = 0$ and $\xi = 1$ respectively.

Similarly for the Cox–Ingersoll–Ross interest rate model, let

$$\xi = \frac{r}{r + P_m},$$

then we get

$$d\xi = \left[\frac{(1-\xi)^2}{P_m} \left(\bar{\mu} - \frac{\bar{\gamma} P_m \xi}{1-\xi} \right) - \frac{\alpha \xi (1-\xi)^2}{P_m} \right] dt + \frac{\sqrt{\alpha} \xi^{1/2} (1-\xi)^{3/2}}{P_m^{1/2}} dX.$$

Thus both (2.78) and (2.79) hold. In what follows, we always assume that the reversion conditions hold. If the domain of the random variable is infinite, then after converting it into another random variable defined on a finite domain, the reversion conditions are fulfilled.

2.9.2 Generalization of Itô's Lemma

Let $V = V(S_1, S_2, \dots, S_n, t)$. According to the Taylor expansion, we have

$$\begin{aligned}
dV &= V(S_1 + dS_1, S_2 + dS_2, \dots, S_n + dS_n, t + dt) - V(S_1, S_2, \dots, S_n, t) \\
&= \sum_{i=1}^n \frac{\partial V}{\partial S_i} dS_i + \frac{\partial V}{\partial t} dt + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 V}{\partial S_i \partial S_j} dS_i dS_j \\
&\quad + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 V}{\partial S_i \partial t} dS_i dt + \dots.
\end{aligned}$$

Since

$$\lim_{dt \rightarrow 0} dS_i dS_j / dt = b_i b_j \rho_{ij}$$

and $dS_i dt$ is a quantity of order $(dt)^{3/2}$, the relation above as $dt \rightarrow 0$ becomes

$$dV = f dt + \sum_{i=1}^n \frac{\partial V}{\partial S_i} dS_i, \quad (2.80)$$

where

$$f = \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 V}{\partial S_i \partial S_j} b_i b_j \rho_{ij}.$$

This is called the generalized Itô's lemma.

2.9.3 Derivation of Equations for Financial Derivatives

Suppose that there are $n+1$ distinct financial derivatives dependent on S_1, S_2, \dots, S_n and t . They could have different expiries or different exercise prices. Even some of the derivatives may depend on only some of the random variables S_i . Let V_k stand for the value of the k -th derivative. According to the generalized Itô's lemma in Subsection 2.9.2, we have

$$dV_k = f_k dt + \sum_{i=1}^n \nu_{i,k} dS_i,$$

where

$$f_k = \frac{\partial V_k}{\partial t} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 V_k}{\partial S_i \partial S_j} b_i b_j \rho_{ij}$$

and

$$\nu_{i,k} = \frac{\partial V_k}{\partial S_i}.$$

Furthermore, we suppose that the holder of a derivative might receive some cash payment, such as a coupon, during the life of the derivative. Let the cash payment for the k -th derivative during the time interval $[t, t+dt]$ be $K_k dt$, K_k being a known function that may depend on S_1, S_2, \dots, S_n and t . Consider a portfolio consisting of the $n+1$ derivatives:

$$\Pi = \sum_{k=1}^{n+1} \Delta_k V_k,$$

where Δ_k is the amount of the k -th derivative in the portfolio. During the time interval $[t, t+dt]$, the holder of this portfolio will earn

$$\begin{aligned} & \sum_{k=1}^{n+1} \Delta_k (dV_k + K_k dt) \\ &= \sum_{k=1}^{n+1} \Delta_k \left(f_k dt + \sum_{i=1}^n \nu_{i,k} dS_i + K_k dt \right) \\ &= \sum_{k=1}^{n+1} \Delta_k (f_k + K_k) dt + \sum_{i=1}^n \left(\sum_{k=1}^{n+1} \Delta_k \nu_{i,k} \right) dS_i. \end{aligned}$$

Let us choose Δ_k such that

$$\sum_{k=1}^{n+1} \Delta_k \nu_{i,k} = 0, \quad i = 1, 2, \dots, n.$$

In this case the portfolio is risk-free, so its return rate is r , i.e.,

$$\sum_{k=1}^{n+1} \Delta_k (f_k + K_k) dt = r \Pi dt = r \sum_{k=1}^{n+1} \Delta_k V_k dt$$

or

$$\sum_{k=1}^{n+1} \Delta_k (f_k + K_k - rV_k) = 0.$$

This relation and the relations which the chosen Δ_k satisfy can be written together in a matrix form as follows:

$$\begin{bmatrix} \nu_{1,1} & \nu_{1,2} & \cdots & \nu_{1,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \nu_{n,1} & \nu_{n,2} & \vdots & \nu_{n,n+1} \\ g_1 & g_2 & \cdots & g_{n+1} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_{n+1} \end{bmatrix} = 0,$$

where $g_k = f_k + K_k - rV_k$, $k = 1, 2, \dots, n+1$.

In order for the system to have a non-trivial solution, the determinant of the matrix must be zero, or the $n+1$ row vectors of the matrix must be linearly dependent. Therefore, we let the last row be expressed as a linear combination of the other rows with coefficients $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n$,

$$\begin{bmatrix} f_1 + K_1 - rV_1 \\ f_2 + K_2 - rV_2 \\ \vdots \\ f_{n+1} + K_{n+1} - rV_{n+1} \end{bmatrix} = \tilde{\lambda}_1 \begin{bmatrix} \nu_{1,1} \\ \nu_{1,2} \\ \vdots \\ \nu_{1,n+1} \end{bmatrix} + \tilde{\lambda}_2 \begin{bmatrix} \nu_{2,1} \\ \nu_{2,2} \\ \vdots \\ \nu_{2,n+1} \end{bmatrix} + \cdots + \tilde{\lambda}_n \begin{bmatrix} \nu_{n,1} \\ \nu_{n,2} \\ \vdots \\ \nu_{n,n+1} \end{bmatrix}.$$

This relation can be rewritten as

$$f_k + K_k - rV_k - \sum_{i=1}^n \tilde{\lambda}_i \nu_{i,k} = 0, \quad k = 1, 2, \dots, n+1,$$

which means that any derivative satisfies an equation of the form

$$f + K - rV - \sum_{i=1}^n \tilde{\lambda}_i \nu_i = 0.$$

Usually $\tilde{\lambda}_i$ is written in the form:

$$\tilde{\lambda}_i = \lambda_i b_i - a_i$$

so the equation above can be written as

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n b_i b_j \rho_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^n (a_i - \lambda_i b_i) \frac{\partial V}{\partial S_i} - rV + K = 0. \quad (2.81)$$

This equation is called the general partial differential equation for derivatives. In (2.81) a_i and b_i are given functions in the model of dS_i , ρ_{ij} is the correlation coefficient between dS_i and dS_j , and K depends on the individual derivative security. For each i , λ_i is an unknown function, which could depend on S_1, S_2, \dots, S_n and t , and is called the market price of risk for S_i . The reason is as follows. According to Itô's lemma and using the equation (2.81), we have

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 V}{\partial S_i \partial S_j} b_i b_j \rho_{ij} \right) dt + \sum_{i=1}^n \frac{\partial V}{\partial S_i} dS_i \\ &= \left(\sum_{i=1}^n (\lambda_i b_i - a_i) \frac{\partial V}{\partial S_i} + rV - K \right) dt + \sum_{i=1}^n \frac{\partial V}{\partial S_i} (a_i dt + b_i dX_i) \end{aligned}$$

or

$$dV + Kdt - rVdt = \sum_{i=1}^n \frac{\partial V}{\partial S_i} b_i (dX_i + \lambda_i dt).$$

Here $dV + Kdt$ is the return for the derivative including the cash payment, such as a coupon, and $rVdt$ is the return if the investment is risk-free. Therefore $dV + Kdt - rVdt$ is the excess return above the risk-free rate during the time interval $[t, t + dt]$. This equals the right hand side of the equation. Its expectation is $\sum_{i=1}^n \frac{\partial V}{\partial S_i} b_i \lambda_i dt$ because $E[dX_i] = 0$, $i = 1, 2, \dots, n$. Therefore,

the term $\frac{\partial V}{\partial S_i} b_i \lambda_i dt$ may be interpreted as an excess return above the risk-free return for taking the risk dX_i . Consequently λ_i is a price of risk for S_i which is associated with dX_i and is often called the market price of risk for S_i .

2.9.4 Three Types of State Variables

There are three types of state variables, for which the term $a_i - \lambda_i b_i$ in (2.81) will be determined in different ways.

Suppose S_i is an asset price which can be traded on the market. For example, S_i is a stock price per share. In this case the stock itself can be considered as a derivative security. Suppose the stock pays a dividend continuously with dividend yield D_{0i} . In this case the price of this derivative security should be $S_i e^{-D_{0i}(T-t)}$ (see Problem 9), i.e., $S_i e^{-D_{0i}(T-t)}$ should be a solution of the equation (2.81). Substituting $V = S_i e^{-D_{0i}(T-t)}$ into (2.81) yields $e^{-D_{0i}(T-t)}(D_{0i}S_i + a_i - \lambda_i b_i - rS_i) = 0$. Therefore, for this case

$$a_i - \lambda_i b_i = (r - D_{0i})S_i. \quad (2.82)$$

We obtain the same result as we had when the Black–Scholes equation for continuous dividend-paying assets was derived. If the dividend is paid discretely, the situation is similar:

$$a_i - \lambda_i b_i = rS_i - D_i(S_i, t) \quad (2.83)$$

because if V depends only on S_i and t , then (2.81) should become (2.13). Here $D_i(S_i, t)dt$ is the dividend paid during the time period $[t, t + dt]$.

A state variable S_i with $b_i = 0$ in (2.76) is another type of state variable. From $b_i = 0$, we have

$$a_i - \lambda_i b_i = a_i, \quad (2.84)$$

so λ_i disappears in the equation (2.81). As we will see from Chapter 3, if S_i is the maximum, minimum or average price of the stock during a time period, then $dS_i = a_i dt$.

If S_i is the spot interest rate, then in order to determine λ_i , we have to solve an inverse problem. We will discuss this problem in detail in Chapter 4. This is an example of the third type of state variable. Besides the interest rate, the random volatility also falls into this type of state variable.

2.9.5 Uniqueness of Solutions

The equation (2.81) is a parabolic equation. If $b_i = 0$ at $S_i = S_{i,l}$ and $S_i = S_{i,u}$, $i = 1, 2, \dots, n$, then we say that the equation is a degenerate parabolic partial differential equation. In this subsection we are going to discuss when a degenerate equation has a unique solution. The conclusion expected is that if for any i ,

$$a_i(S_{i,l}, t) - b_i(S_{i,l}, t) \frac{\partial}{\partial S_i} b_i(S_{i,l}, t) \geq 0 \quad (2.85)$$

and

$$a_i(S_{i,u}, t) - b_i(S_{i,u}, t) \frac{\partial}{\partial S_i} b_i(S_{i,u}, t) \leq 0 \quad (2.86)$$

hold,¹² the solution of the degenerate parabolic equation on a rectangular domain with a final condition at $t = T$ is unique.¹³ If

$$a_i(S_{i,l}, t) - b_i(S_{i,l}, t) \frac{\partial}{\partial S_i} b_i(S_{i,l}, t) < 0$$

¹² a_i and b_i could also depend on $S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n$. Here the dependence of a_i and b_i on them is suppressed and the two relations hold for $S_j \in [S_{j,l}, S_{j,u}]$, $j = 1, \dots, i-1, i+1, \dots, n$.

¹³For a parabolic equation defined on a general domain, the conditions for a parabolic partial differential equation to be degenerate and the conditions for the solution of its initial-value problem to be unique, see the paper [78] by Zhu.

or

$$a_i(S_{i,u}, t) - b_i(S_{i,u}, t) \frac{\partial}{\partial S_i} b_i(S_{i,u}, t) > 0,$$

then a boundary condition at $S_i = S_{i,l}$ or $S_i = S_{i,u}$ needs to be imposed besides the final condition in order to have a unique solution. We now prove this conclusion for the one-dimensional case.

In the case $n = 1$, (2.81) simplifies to

$$\frac{\partial V}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} + (a - \lambda b) \frac{\partial V}{\partial S} - rV + K = 0.$$

Here the sign of the coefficient of the second derivative is opposite of the coefficient of the second derivative in the heat equation. We say that such a parabolic equation has an “anti-directional” time. For a heat equation an initial condition is given at $t = 0$ and the solution for $t \geq 0$ needs to be determined. Therefore for the equation with an “anti-directional” time, a final condition should be given at $t = T$ and the solution for $t \leq T$ is needed to be determined. Consequently, we consider the following problem:

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} + (a - \lambda b) \frac{\partial V}{\partial S} - rV + K = 0, \\ \qquad \qquad \qquad 0 \leq t \leq T, \quad S_l \leq S \leq S_u, \\ V(S, T) = f(S), \quad S_l \leq S \leq S_u, \\ V(S_l, t) \left\{ \begin{array}{l} \text{needs not to be given if (2.85) holds,} \\ = f_l(t) \text{ if (2.85) does not hold,} \end{array} \right. \\ V(S_u, t) \left\{ \begin{array}{l} \text{needs not to be given if (2.86) holds,} \\ = f_u(t) \text{ if (2.86) does not hold.} \end{array} \right. \end{array} \right. \quad (2.87)$$

It is not difficult to convert (2.87) into a problem in the form:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial \tau} = f_1(x, \tau) \frac{\partial^2 u}{\partial x^2} + f_2(x, \tau) \frac{\partial u}{\partial x} + f_3(x, \tau) u + g(x, \tau), \\ \qquad \qquad \qquad 0 \leq x \leq 1, \quad 0 \leq \tau \leq T, \\ u(x, 0) = f(x), \quad 0 \leq x \leq 1, \\ u(0, \tau) \left\{ \begin{array}{l} \text{needs not to be given if } f_2(0, \tau) - \frac{\partial f_1(0, \tau)}{\partial x} \geq 0, \\ = f_l(\tau) \text{ if } f_2(0, \tau) - \frac{\partial f_1(0, \tau)}{\partial x} < 0, \end{array} \right. \\ u(1, \tau) \left\{ \begin{array}{l} \text{needs not to be given if } f_2(1, \tau) - \frac{\partial f_1(1, \tau)}{\partial x} \leq 0, \\ = f_u(\tau) \text{ if } f_2(1, \tau) - \frac{\partial f_1(1, \tau)}{\partial x} > 0, \end{array} \right. \end{array} \right. \quad (2.88)$$

where $f_1(0, \tau) = f_1(1, \tau) = 0$ and $f_1(x, \tau) \geq 0$. Thus if we can prove the uniqueness of the solution of (2.88), then we have the uniqueness of the solution of (2.87). The third and fourth relations in (2.88) are the boundary conditions for degenerate parabolic equations. For parabolic equations there is always a boundary condition at any boundary, that is, the number of boundary conditions for parabolic equations is always one. However for degenerate parabolic equations, sometimes there is a boundary condition and sometimes there is not, depending on the value of $f_2(x, \tau) - \frac{\partial f_1(x, \tau)}{\partial x}$ at the boundary. For (2.88), we have the following theorem (see [65]).

Theorem 2.2 *Suppose that the solution of (2.88) exists and is bounded¹⁴ and that there exist a constant c_1 and two bounded functions $c_2(\tau)$ and $c_3(\tau)$ such that*

$$1 + \max_{0 \leq x \leq 1, 0 \leq \tau \leq \tau} \left(\left| \frac{\partial^2 f_1(x, \tau)}{\partial x^2} - \frac{\partial f_2(x, \tau)}{\partial x} + 2f_3(x, \tau) \right| \right) \leq c_1,$$

$$- \min \left(0, f_2(0, \tau) - \frac{\partial f_1(0, \tau)}{\partial x} \right) \leq c_2(\tau)$$

and

$$\max \left(0, f_2(1, \tau) - \frac{\partial f_1(1, \tau)}{\partial x} \right) \leq c_3(\tau).$$

In this case its solution is unique and stable with respect to the initial value $f(x)$, inhomogeneous term $g(x, \tau)$ and the boundary values $f_l(\tau), f_u(\tau)$ if there are any.

Proof. Since the partial differential equation in (2.88) can be rewritten as

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial x} \left(f_1(x, \tau) \frac{\partial u}{\partial x} \right) + \left[f_2(x, \tau) - \frac{\partial f_1(x, \tau)}{\partial x} \right] \frac{\partial u}{\partial x} + f_3(x, \tau)u + g(x, \tau),$$

multiplying that equation by $2u$, we have

$$\begin{aligned} \frac{\partial(u^2)}{\partial \tau} &= 2 \frac{\partial}{\partial x} \left(f_1 u \frac{\partial u}{\partial x} \right) + \left(f_2 - \frac{\partial f_1}{\partial x} \right) \frac{\partial(u^2)}{\partial x} - 2f_1 \left(\frac{\partial u}{\partial x} \right)^2 + 2f_3 u^2 + 2gu \\ &= 2 \frac{\partial}{\partial x} \left(f_1 u \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial x} \left[\left(f_2 - \frac{\partial f_1}{\partial x} \right) u^2 \right] - 2f_1 \left(\frac{\partial u}{\partial x} \right)^2 \\ &\quad + \left(\frac{\partial^2 f_1}{\partial x^2} - \frac{\partial f_2}{\partial x} + 2f_3 \right) u^2 + 2gu. \end{aligned}$$

Integrating this equality with respect to x on the interval $[0, 1]$, we obtain the second equality

¹⁴This is proved in the paper [6] by Behboudi.

$$\begin{aligned}
& \frac{d}{d\tau} \int_0^1 u^2(x, \tau) dx \\
&= 2 \left(f_1 u \frac{\partial u}{\partial x} \right) \Big|_{x=0}^1 + \left[\left(f_2 - \frac{\partial f_1}{\partial x} \right) u^2 \right] \Big|_{x=0}^1 - 2 \int_0^1 f_1 \left(\frac{\partial u}{\partial x} \right)^2 dx \\
& \quad + \int_0^1 \left(\frac{\partial^2 f_1}{\partial x^2} - \frac{\partial f_2}{\partial x} + 2f_3 \right) u^2 dx + 2 \int_0^1 g u dx.
\end{aligned}$$

Since

$$\begin{aligned}
& \left[\left(f_2 - \frac{\partial f_1}{\partial x} \right) u^2 \right] \Big|_{x=0}^1 \\
&= \left[f_2(1, \tau) - \frac{\partial f_1(1, \tau)}{\partial x} \right] u^2(1, \tau) - \left[f_2(0, \tau) - \frac{\partial f_1(0, \tau)}{\partial x} \right] u^2(0, \tau) \\
&\leq \max \left(0, f_2(1, \tau) - \frac{\partial f_1(1, \tau)}{\partial x} \right) f_u^2(\tau) - \min \left(0, f_2(0, \tau) - \frac{\partial f_1(0, \tau)}{\partial x} \right) f_l^2(\tau),
\end{aligned}$$

from the equality above and the relations $f_1(0, \tau) = f_1(1, \tau) = 0$ and $f_1(x, \tau) \geq 0$, we have

$$\begin{aligned}
& \frac{d}{d\tau} \int_0^1 u^2(x, \tau) dx \\
&\leq c_1 \int_0^1 u^2(x, \tau) dx + \int_0^1 g^2(x, \tau) dx + c_2(\tau) f_l^2(\tau) + c_3(\tau) f_u^2(\tau).
\end{aligned}$$

Based on this inequality and by the Gronwall inequality, we arrive at

$$\begin{aligned}
& \int_0^1 u^2(x, \tau) dx \\
&\leq e^{c_1 \tau} \left\{ \int_0^1 f^2(x) dx + \int_0^\tau \left[\int_0^1 g^2(x, s) dx + c_2(s) f_l^2(s) + c_3(s) f_u^2(s) \right] ds \right\}, \\
& \quad t \in [0, T].
\end{aligned}$$

From the last inequality, we know that the solution is stable with respect to $f(x)$ and $g(x, \tau)$. Also if

$$f_2(0, \tau) - \frac{\partial f_1(0, \tau)}{\partial x} \geq 0$$

and

$$f_2(1, \tau) - \frac{\partial f_1(1, \tau)}{\partial x} \leq 0$$

hold and $f(x) \equiv 0$, $g(x, \tau) \equiv 0$, then the solution of (2.88) must be zero. Hence the functions $f(x)$ and $g(x, \tau)$ determine the solution uniquely. If

$$f_2(0, \tau) - \frac{\partial f_1(0, \tau)}{\partial x} < 0$$

and

$$f_2(1, \tau) - \frac{\partial f_1(1, \tau)}{\partial x} \leq 0$$

hold, then the solution is determined by $f(x)$, $g(x, \tau)$ and $f_l(\tau)$ uniquely. The situation for other cases are similar. Therefore we may conclude that if the solution of (2.88) exists, then it is unique and stable with respect to the initial value $f(x)$, the inhomogeneous term $g(x, \tau)$ and the boundary values $f_l(\tau)$, $f_u(\tau)$ if there are any. This completes the proof and gives an explanation on when a boundary condition is necessary. ■

Here we give some remarks.

- From the probabilistic point of view, a boundary condition on a boundary is needed if and only if there are paths reaching the boundary from a point $x \in (0, 1)$ and $t = 0$. Therefore on whether or not a random variable can reach a boundary from the interior, there are similar conclusions (see [30]).
- For the case $n = 2$ and on a rectangular finite domain, if $b_1(S_1, S_2, t)$ and $b_2(S_1, S_2, t)$ are analytic, the uniqueness is also proved (see [82]) and the idea can be generalized to the case with $n > 2$. On a general finite three dimensional domain, a similar result is also obtained (see [78]). Therefore a degenerate parabolic equation at boundaries is similar to a hyperbolic equation.¹⁵ Due to this fact, roughly speaking, we might say that the parabolic equation degenerates into a hyperbolic-parabolic equation (a hyperbolic equation for one-dimensional case) at the boundaries. When conditions (2.85) and (2.86) hold, incoming information is not needed at boundaries, that is, the value of V at the boundaries at $t = t^*$ is determined by the value V on the region: $S_{i,l} \leq S_i \leq S_{i,u}$, $i = 1, 2, \dots, n$ and $t^* \leq t \leq T$. Therefore, in this case, in order for a degenerate parabolic equation to have a unique solution, only the final condition is needed. If b_i is not analytic at boundaries, this conclusion has not been proved for $n > 1$. However it is expected that the conclusion is still true.
- When the domain of S_i is not finite, a final condition is still enough for such an equation to have a unique solution if S_i can be converted into a random variable for which the reversion conditions hold. The reason is that a final condition can determine a unique solution if the new random variable is used. However, a transformation will not change the nature of the problem. If the problem has a unique solution as a function of a random variable, the problem will also have a unique solution as a function of another random variable associated by a transformation. Applying this theorem to problem (2.26), we know that it has a unique solution and the solution is stable with respect to the initial value. Problem (2.26) is obtained through a transformation from the European option problem (2.24). Therefore the European option problem (2.24) also has a unique solution.

¹⁵When $f_1(x, t) \equiv 0$, the partial differential equation in (2.88) is called a hyperbolic equation.

2.10 Jump Conditions

2.10.1 Hyperbolic Equations with a Dirac Delta Function

Consider the following linear hyperbolic partial differential equation

$$\frac{\partial u}{\partial t} + f_1(x_1, x_2, \dots, x_K, t) \frac{\partial u}{\partial x_1} + \dots + f_K(x_1, x_2, \dots, x_K, t) \frac{\partial u}{\partial x_K} = 0.$$

Let C be a curve defined by the system of ordinary differential equations

$$\begin{aligned} \frac{dx_1(t)}{dt} &= f_1(x_1, x_2, \dots, x_K, t), \\ &\vdots \\ \frac{dx_K(t)}{dt} &= f_K(x_1, x_2, \dots, x_K, t) \end{aligned}$$

with initial conditions

$$x_1(0) = \xi_1, x_2(0) = \xi_2, \dots, x_K(0) = \xi_K.$$

Along the curve we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} \frac{dx_1}{dt} + \dots + \frac{\partial u}{\partial x_K} \frac{dx_K}{dt} = 0.$$

Therefore, u is a constant along the curve:

$$u(x_1(t^*), x_2(t^*), \dots, x_K(t^*), t^*) = u(x_1(t^{**}), x_2(t^{**}), \dots, x_K(t^{**}), t^{**}),$$

where t^* and t^{**} are any two times. If

$$f_k(x_1, x_2, \dots, x_K, t) = F_k(x_1, x_2, \dots, x_K, t) \delta(t - t_i),$$

where $\delta(t - t_i)$ is the Dirac delta function, then

$$\begin{aligned} x_k(t_i^+) - x_k(t_i^-) &= \int_{t_i^-}^{t_i^+} F_k(x_1, x_2, \dots, x_K, t) \delta(t - t_i) dt \\ &= F_k(x_1(t_i^-), x_2(t_i^-), \dots, x_K(t_i^-), t_i^-) \end{aligned}$$

and

$$\begin{aligned} &u(x_1(t_i^-), x_2(t_i^-), \dots, x_K(t_i^-), t_i^-) \\ &= u(x_1(t_i^+), x_2(t_i^+), \dots, x_K(t_i^+), t_i^+) \\ &= u(x_1(t_i^-) + F_{1i}^-, x_2(t_i^-) + F_{2i}^-, \dots, x_K(t_i^-) + F_{Ki}^-, t_i^+), \end{aligned} \quad (2.89)$$

where t_i^- and t_i^+ denote the time just before and after t_i respectively, and

$$F_{ki}^- \equiv F_k(x_1(t_i^-), x_2(t_i^-), \dots, x_K(t_i^-), t_i^-).$$

For such a jump condition, a similar derivation is given in the book [68] by Wilmott, Dewynne and Howison.

2.10.2 Jump Conditions for Options with Discrete Dividends and Discrete Sampling

From (2.89), jump conditions of various options can be derived. Here we give two examples. One is simple and the other is quite complicated. Jump conditions for other options will be given when they are discussed.

Suppose $V(S, t)$ is the value of an option on a stock, which pays a dividend D_i at time t_i , $i = 1, 2, \dots, I$. Here we assume that $t_i \leq T$, T being expiry. From Section 2.2, we know that $V(S, t)$ satisfies (2.13):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (rS - D(S, t)) \frac{\partial V}{\partial S} - rV = 0,$$

where

$$D(S, t) = \sum_{i=1}^I D_i(S) \delta(t - t_i), \quad \text{with } D_i(S) \leq S.$$

This means that at $t \neq t_i$, $i = 1, 2, \dots, I$, V satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

and at $t = t_i$, $i = 1, 2, \dots$, or I , the equation

$$\frac{\partial V}{\partial t} - D_i(S) \delta(t - t_i) \frac{\partial V}{\partial S} = 0$$

holds. According to (2.89), at $t = t_i$ we have

$$V(S, t_i^-) = V(S - D_i(S), t_i^+). \quad (2.90)$$

This is the jump condition for options on stocks with discrete dividends. We now explain the financial meaning of this relation. At $t = t_i$ the stock pays a dividend D_i , so the stock price will drop by D_i . If the price is S at t_i^- , then the price is $S - D_i$ at t_i^+ . However the price of the option is unchanged at time t_i because the holder of the option does not receive any money at time t_i .

The second example involves several independent variables. Suppose the stock price is measured discretely and let S_1, S_2, \dots, S_N be the first N largest sampled stock prices until time t and $S_1 \geq S_2 \geq \dots \geq S_N$. Assume that the value of option V depends on S, S_1, \dots, S_N, t . From Subsection 3.4.6, we will see that if sampling occurs at $t = t_i$, then

$$\frac{dS_n}{dt} = \begin{cases} [\max(S, S_1(t_i^-)) - S_1(t_i^-)] \delta(t - t_i), & \text{if } n = 1, \\ [\max(\min(S, S_{n-1}(t_i^-)), S_n(t_i^-)) \\ - S_n(t_i^-)] \delta(t - t_i), & \text{if } n = 2, 3, \dots, N; \end{cases}$$

otherwise

$$\frac{dS_n}{dt} = 0.$$

According to Section 2.9, in this case, the option price is the solution of

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_1} \frac{dS_1}{dt} + \frac{\partial V}{\partial S_2} \frac{dS_2}{dt} + \cdots + \frac{\partial V}{\partial S_N} \frac{dS_N}{dt} \\ + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0. \end{aligned}$$

Consequently, at $t = t_i$, V satisfies

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_1} \frac{dS_1}{dt} + \frac{\partial V}{\partial S_2} \frac{dS_2}{dt} + \cdots + \frac{\partial V}{\partial S_N} \frac{dS_N}{dt} = 0.$$

From (2.89) we know when $t = t_i$, the jump condition

$$\begin{aligned} V(S, S_1^-, S_2^-, \dots, S_N^-, t_i^-) = V(S, \max(S, S_1^-), \max(\min(S, S_1^-), S_2^-), \\ \dots, \max(\min(S, S_{N-1}^-), S_N^-), t_i^+) \end{aligned} \quad (2.91)$$

holds, where S_n^- denotes $S_n(t_i^-)$ for brevity.

It is clear how to use such a jump condition when a European-style derivative is evaluated. When the price of an American-style derivative needs to be calculated, such a condition should be used on the solution obtained by the PDE. After that, taking the maximum between the new solution and the constraint yields the solution for the American derivative.

2.11 More Arbitrage Theory

In Section 2.2 we derived the Black–Scholes equation by using arbitrage arguments. Here we will further use arbitrage arguments to obtain some properties of option prices. Similar materials can be found in the book [39] by Hull.

2.11.1 Three Conclusions and Some Portfolios

Consider two portfolios \mathbf{X} and \mathbf{Y} , whose values depend on a stock price S and time t . Let $\mathbf{X}(S, t)$ and $\mathbf{Y}(S, t)$ denote the values of portfolios \mathbf{X} and \mathbf{Y} respectively. \mathbf{X} and \mathbf{Y} could involve options and all their expiries are T . By using arbitrage arguments, we can have three conclusions, which are written in the form of theorems.

Theorem 2.3 *If only European options are involved and $\mathbf{X}(S, T) \geq \mathbf{Y}(S, T)$ for any S , then for any $t \leq T$, $\mathbf{X}(S, t)$ must be greater than or equal to $\mathbf{Y}(S, t)$. This result also implies that if $\mathbf{X}(S, T) \leq \mathbf{Y}(S, T)$, then for any $t \leq T$, $\mathbf{X}(S, t) \leq \mathbf{Y}(S, t)$.*

Proof. Suppose that at time \bar{t} the value of portfolio \mathbf{X} is less than the value of portfolio \mathbf{Y} and the latter is higher than the former by an amount of $Z(\bar{t})$. In this case an arbitrageur can earn at least $Z(\bar{t})e^{r(T-\bar{t})}$ at time T by doing the following: sell \mathbf{Y} , buy \mathbf{X} and invest $Z(\bar{t})$ into a bank at an interest rate r at time \bar{t} , and get $\mathbf{X}(S, T)$ from portfolio \mathbf{X} , pay $\mathbf{Y}(S, T)$ for portfolio \mathbf{Y} and obtain $Z(\bar{t})e^{r(T-\bar{t})}$ from the risk-free investment at time T . Since $\mathbf{X}(S, T) \geq \mathbf{Y}(S, T)$ for any S , the arbitrageur will always earn at least $Z(\bar{t})e^{r(T-\bar{t})}$ at the time T , which means that the earning is risk-free. Thus everyone will do such a thing. Since so many people sell \mathbf{Y} and buy \mathbf{X} , the price of \mathbf{Y} will drop and the price of \mathbf{X} will rise and will be immediately equal to or greater than the price of \mathbf{Y} . Therefore Theorem 2.3 holds. ■

From this result, assuming $\mathbf{X}(S, T) \leq \mathbf{Y}(S, T)$, we can immediately get that for any time $t \leq T$, $\mathbf{X}(S, t) \leq \mathbf{Y}(S, t)$ and furthermore we can have

Theorem 2.4 *If $\mathbf{X}(S, T) = \mathbf{Y}(S, T)$ for any S , then for any $t \leq T$, $\mathbf{X}(S, t)$ must be equal to $\mathbf{Y}(S, t)$ for any S .*

Proof. Since $\mathbf{X}(S, T) = \mathbf{Y}(S, T)$ means $\mathbf{X}(S, T) \geq \mathbf{Y}(S, T)$ and $\mathbf{X}(S, T) \leq \mathbf{Y}(S, T)$, from the conclusion above we have for any t

$$\mathbf{X}(S, t) \geq \mathbf{Y}(S, t) \quad \text{and} \quad \mathbf{X}(S, t) \leq \mathbf{Y}(S, t),$$

which means

$$\mathbf{X}(S, t) = \mathbf{Y}(S, t).$$

Thus we have Theorem 2.4. ■

We can also have the following conclusion.

Theorem 2.5 *Suppose that portfolio \mathbf{Y} involves only one American option and no European option and that portfolio \mathbf{X} involves only European options. If $\mathbf{X}(S, T) \geq \mathbf{Y}(S, T)$ at time T and if the amount of cash and the number of stocks in \mathbf{X} is greater than or equal to the amount of cash and the number of stocks the holder of \mathbf{Y} has when the American option is exercised at time $\bar{t} < T$, then $\mathbf{X}(S, t) \geq \mathbf{Y}(S, t)$ for any time t .*

Proof. The argument is similar to the argument for proving Theorem 2.3. Suppose $\mathbf{X}(S, t) < \mathbf{Y}(S, t)$ at time $t < T$. Then an arbitrageur can purchase \mathbf{X} , sell \mathbf{Y} and earn some money. Later when the American option is exercised early at time $\bar{t} < T$, the arbitrageur will never lose money because the amount of cash and the number of stocks in \mathbf{X} are greater than or equal to the amount of cash and the number of stocks the holder of \mathbf{Y} has. When the American option is not exercised before time T , the arbitrageur will also never lose any money since the value of \mathbf{X} is greater than or equal to the value of \mathbf{Y} at time T . Therefore the earning is risk-free, which means $\mathbf{X}(S, t)$ should not be less than $\mathbf{Y}(S, t)$ at any time. ■

Before applying these conclusions, we define some portfolios and find their values at time T along with what their holders will have if American options are exercised at time $\bar{t} < T$.

Portfolio A: An amount of cash equal to $Ee^{-r(T-t)}$ invested at an interest rate r . It is clear that its value at time T is E .

Portfolio B: $e^{-D_0(T-t)}$ shares of a stock with dividends being reinvested in the stock if the stock pays the dividend continuously or one share of a stock plus a loan $D(t)$ if the stock pays cash dividends discretely. Here $D(t)$ is equal to the present value of these dividends to be paid from time t to time T and the money will be returned to the loaner as soon as the stock pays a dividend. Obviously, its value at time T is the price of the stock S .

Portfolio C: One European call option plus portfolio **A**. The value of this portfolio at time T is $\max(S - E, 0) + E = \max(S, E)$.

Portfolio D: One European put option plus portfolio **B**. Its value at time T is $\max(E - S, 0) + S = \max(S, E)$.

Portfolio E: One American call option plus portfolio **A**. If the American call option is not exercised before time T , its value at time T is $\max(S - E, 0) + E = \max(S, E)$. If at some time $\bar{t} < T$, the stock price S is greater than E and the American option is exercised, then the holder of the portfolio has one share plus a loan of $(1 - e^{-r(T-\bar{t})})E$.

Portfolio F: One American put option plus portfolio **B**. $\max(S, E)$ is its value at time T if the put option is not exercised before time T ; while its holder has an amount of cash E minus $(1 - e^{-D_0(T-\bar{t})})$ shares or an amount of cash $E - D(\bar{t})$ if at some time $\bar{t} < T$ the stock price S is less than E and the put option is exercised.

Portfolio G: One European call option plus E . Its value at time T is equal to $\max(S, E)$.

Portfolio H: One European put option plus one share. Its value is equal to $\max(S, E)$ at expiry.

2.11.2 Bounds of Option Prices

Consider a European call option and portfolio **B**. At time T , $c(S, T) = \max(S - E, 0) \leq \mathbf{B}(S, T) = S$. From Theorem 2.3, we have

$$c(S, t) \leq Se^{-D_0(T-t)}$$

or

$$c(S, t) \leq S - D(t).$$

Now let us compare portfolio **C** with portfolio **B**. Since at time T

$$\mathbf{C}(S, T) = \max(S, E) \geq \mathbf{B}(S, T) = S,$$

we have

$$c(S, t) + Ee^{-r(T-t)} \geq Se^{-D_0(T-t)}$$

or

$$c(S, t) + Ee^{-r(T-t)} \geq S - D(t).$$

Clearly, $c(S, t) \geq 0$ for any case. Therefore for a European call option we have

$$\max(Se^{-D_0(T-t)} - Ee^{-r(T-t)}, 0) \leq c(S, t) \leq Se^{-D_0(T-t)} \quad (2.92)$$

or

$$\max(S - D(t) - Ee^{-r(T-t)}, 0) \leq c(S, t) \leq S - D(t). \quad (2.93)$$

Consequently, the lower bound of $c(S, t)$ is $\max(Se^{-D_0(T-t)} - Ee^{-r(T-t)}, 0)$ or $\max(S - D(t) - Ee^{-r(T-t)}, 0)$ and the upper bound is $Se^{-D_0(T-t)}$ or $S - D(t)$. Here we assume that $S - D(t)$ is always greater than zero. If $S < D(t)$ at time t , then any person will buy one share of the stock by finding a loan of amount S at time t and returning the loan as soon as the stock pays a dividend. In this way the person gets one share and some cash free at time T . Therefore the price must rise until $S \geq D(t)$.

Since $C(S, t) \geq c(S, t)$, we require that $C(S, t)$ is greater than or equal to the lower bound of $c(S, t)$. Also $C(S, t)$ needs to be greater than or equal to the constraint $\max(S - E, 0)$. Thus

$$\max(Se^{-D_0(T-t)} - Ee^{-r(T-t)}, S - E, 0)$$

or

$$\max(S - D(t) - Ee^{-r(T-t)}, S - E, 0)$$

is a lower bound. Clearly S is an upper bound for an American call option. Consequently, for the price of an American call option, we have

$$\max(Se^{-D_0(T-t)} - Ee^{-r(T-t)}, S - E, 0) \leq C(S, t) \leq S \quad (2.94)$$

or

$$\max(S - D(t) - Ee^{-r(T-t)}, S - E, 0) \leq C(S, t) \leq S. \quad (2.95)$$

Now let us compare a European put option with portfolio **A**. At time T ,

$$p(S, T) = \max(E - S, 0) \leq \mathbf{A}(S, T) = E.$$

Thus

$$p(S, t) \leq Ee^{-r(T-t)}.$$

In order to get a lower bound of $p(S, t)$, let us look at portfolios **D** and **A**. Since at time T , $\mathbf{D}(S, T) = \max(S, E) \geq \mathbf{A}(S, T) = E$, we arrive at

$$p(S, t) + Se^{-D_0(T-t)} \geq Ee^{-r(T-t)}$$

or

$$p(S, t) + S - D(t) \geq Ee^{-r(T-t)}.$$

Also $p(S, t)$ must be nonnegative. Therefore we have

$$\max(Ee^{-r(T-t)} - Se^{-D_0(T-t)}, 0) \leq p(S, t) \leq Ee^{-r(T-t)} \quad (2.96)$$

or

$$\max(Ee^{-r(T-t)} - S + D(t), 0) \leq p(S, t) \leq Ee^{-r(T-t)}. \quad (2.97)$$

These give the lower and upper bounds of European put options.

For an American put option, we can also get the lower and upper bounds. Since $P(S, t) \geq p(S, t)$, we have

$$P(S, t) \geq \max(Ee^{-r(T-t)} - Se^{-D_0(T-t)}, 0)$$

or

$$P(S, t) \geq \max(Ee^{-r(T-t)} - S + D(t), 0).$$

Also $P(S, t)$ must be greater than or equal to $\max(E - S, 0)$. Therefore we further obtain

$$P(S, t) \geq \max(Ee^{-r(T-t)} - Se^{-D_0(T-t)}, E - S, 0)$$

or

$$P(S, t) \geq \max(Ee^{-r(T-t)} - S + D(t), E - S, 0).$$

E is an upper bound of $P(S, t)$, consequently we have

$$\max(Ee^{-r(T-t)} - Se^{-D_0(T-t)}, E - S, 0) \leq P(S, t) \leq E \quad (2.98)$$

or

$$\max(Ee^{-r(T-t)} - S + D(t), E - S, 0) \leq P(S, t) \leq E. \quad (2.99)$$

From the proofs we know that if one of these relations is not true, then we can find an arbitrage opportunity to earn some money. This means that the lower bound is the greatest lower bound and that the upper bound is the least upper bound. From Subsection 1.2.4, we know that the price of an option is an increasing function of the volatility. Therefore if the lower bound is the greatest lower bound, then as the volatility approaches zero, the limit of option should be the lower bound. Similarly, if the upper bound is the least upper bound, then as the volatility approaches infinity, the limit of the option should be the upper bound. When r, D_0 and σ are constant, the European option price is given by the Black-Scholes formulae in Subsection 2.4.4:

$$c(S, t) = Se^{-D_0(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$$

and

$$p(S, t) = Ee^{-r(T-t)}N(-d_2) - Se^{-D_0(T-t)}N(-d_1),$$

where

$$d_1 = \left[\ln \frac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t) \right] / (\sigma\sqrt{T-t})$$

and

$$d_2 = \left[\ln \frac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} - \frac{1}{2}\sigma^2(T-t) \right] / (\sigma\sqrt{T-t}).$$

Therefore we have

$$\left\{ \begin{array}{l} \lim_{\sigma \rightarrow 0} c(S, t) = \begin{cases} 0, & \text{if } Se^{-D_0(T-t)} < Ee^{-r(T-t)}, \\ Se^{-D_0(T-t)} - Ee^{-r(T-t)}, & \text{if } Se^{-D_0(T-t)} > Ee^{-r(T-t)}, \end{cases} \\ \lim_{\sigma \rightarrow \infty} c(S, t) = Se^{-D_0(T-t)}, \\ \lim_{\sigma \rightarrow 0} p(S, t) = \begin{cases} 0, & \text{if } Ee^{-r(T-t)} < Se^{-D_0(T-t)}, \\ Ee^{-r(T-t)} - Se^{-D_0(T-t)}, & \text{if } Ee^{-r(T-t)} > Se^{-D_0(T-t)}, \end{cases} \\ \lim_{\sigma \rightarrow \infty} p(S, t) = Ee^{-r(T-t)}. \end{array} \right.$$

That is, (2.92) and (2.96) truly provide the least upper and greatest lower bounds of European options respectively.

Here we give an example to show that if the price of an option does not satisfy a related condition, then there exists an arbitrage opportunity. More examples are given as problems for the reader to study.

Example 1. Consider a European call option on a dividend-paying stock. Suppose the following: $S = \$102$, $E = \$100$, $c = \$8.50$, $r = 0.1$, the time to maturity is 9 months and the present value of the dividend $D(t)$ is \$0.50. Is there any arbitrage opportunity?

Solution: As we know, the price of a call option has to satisfy the condition (2.93):

$$\max(S - D(t) - Ee^{-r(T-t)}, 0) \leq c(S, t) \leq S - D(t).$$

In this case

$$\max(S - D(t) - Ee^{-r(T-t)}, 0) = \max(102 - 0.5 - 100e^{-0.9/12}, 0) = 8.73.$$

Therefore the price of the call option is less than the lower bound. In this case if we own one share of the stock or if you can borrow one share of the stock for the period $[t, T]$, then we should take a long position in a portfolio **C** and a short position in a portfolio **B**. In other words, buy one call option, sell one share and deposit $Ee^{-r(T-t)} + D(t)$ in a bank at time t . In this case we will get $-8.5 + 102 - 100e^{-0.9/12} - 0.5 = \0.23 at time t and this is a risk-free earning. This is because we can get the money from the bank to pay the dividends on the stock during the time interval $[t, T]$ and get E from the bank at time T . If $S \geq E$ at time T , we can exercise the call option and get one share. If $S < E$, we can have one share of the stock which is bought from the market and an amount of cash $E - S$. In any case we have one share plus at least \$0.23. That is, we can get one share back or return one share to the borrower and earn at least \$0.23 free at time T .

2.11.3 Relations Between Call and Put Prices

Let us look at portfolios **C** and **D**. Since $\mathbf{C}(S, T) = \mathbf{D}(S, T)$. According to Theorem 2.4, we have

$$c(S, t) + Ee^{-r(T-t)} = p(S, t) + Se^{-D_0(T-t)} \quad (2.100)$$

or

$$c(S, t) + Ee^{-r(T-t)} = p(S, t) + S - D(t). \quad (2.101)$$

These are called put–call parities of European options. For stocks with continuous dividends, we obtained such a relation through a very long procedure in Section 2.4. However the derivation here is so simple. This shows that arbitrage theory is a very powerful tool.

The put–call parity relations hold only for European options. For American options they are not true, but the following inequalities on the difference between the American call and put option prices are fulfilled

$$Se^{-D_0(T-t)} - E \leq C(S, t) - P(S, t) \leq S - Ee^{-r(T-t)} \quad (2.102)$$

or

$$S - D(t) - E \leq C(S, t) - P(S, t) \leq S - Ee^{-r(T-t)}. \quad (2.103)$$

The two inequalities can also be written as

$$\begin{cases} Se^{-D_0(T-t)} - E + P(S, t) \leq C(S, t) \leq S - Ee^{-r(T-t)} + P(S, t), \\ C(S, t) - S + Ee^{-r(T-t)} \leq P(S, t) \leq C(S, t) - Se^{-D_0(T-t)} + E \end{cases}$$

or

$$\begin{cases} S - D(t) - E + P(S, t) \leq C(S, t) \leq S - Ee^{-r(T-t)} + P(S, t), \\ C(S, t) - S + Ee^{-r(T-t)} \leq P(S, t) \leq C(S, t) - S + D(t) + E, \end{cases}$$

which gives the lower and upper bounds of an American call (put) option if the price of the corresponding American put (call) option is known.

First let us prove the left portions of the inequalities (2.102) and (2.103). Consider portfolios **G** and **F**. Since **G** contains European options only and **F** contains only one American option, it is possible to use Theorem 2.5. According to Theorem 2.5, the value of **G** is always greater than or equal to the value of **F** if we can prove the two things:

1. The value of **G** is greater than or equal to the value of **F** at time T ;
2. The amount of cash and the number of stocks in **G** is greater than or equal to the amount of cash and the number of stocks in **F** when the American option is exercised at time $\bar{t} < T$.

At time T the value of **G** is equal to the value of **F**. At any time $\bar{t} < T$, there is an amount of cash E and no stock in **G**. If the American put option in **F** is exercised before time T , **F** contains an amount of cash E and $-(1 - e^{-D_0(T-\bar{t})})$ shares or an amount of cash $E - D(t)$. Therefore both the amount of cash and the number of stocks in **G** is greater than or equal to those in **F** if the American option in **F** is exercised at some time $\bar{t} < T$. Consequently, according

to Theorem 2.5 the value of \mathbf{G} is greater than or equal to the value of \mathbf{F} for any case, so

$$P(S, t) + Se^{-D_0(T-t)} \leq c(S, t) + E$$

or

$$P(S, t) + S - D(t) \leq c(S, t) + E.$$

Since $C(S, t) \geq c(S, t)$, we further have

$$Se^{-D_0(T-t)} - E \leq C(S, t) - P(S, t)$$

or

$$S - D(t) - E \leq C(S, t) - P(S, t).$$

In order to prove the right portions of the relations, we need to look at portfolios \mathbf{H} and \mathbf{E} . In \mathbf{H} there is only one European option and in \mathbf{E} the American option is the only option, so we can use Theorem 2.5 again. When the American call option in \mathbf{E} is exercised before time T , the amount of cash and the number of stocks in \mathbf{H} is greater than or equal to those in \mathbf{E} . When it is not exercised before expiry, the value of \mathbf{H} is equal to the value of \mathbf{E} at time T . Therefore

$$C(S, t) + Ee^{-r(T-t)} \leq p(S, t) + S.$$

Noticing $P(S, t) \geq p(S, t)$, we have

$$C(S, t) - P(S, t) \leq S - Ee^{-r(T-t)}.$$

This completes our proof.

Example 2. Suppose there are an American call option and an American put option on the same stock. The stock pays dividends continuously and $D_0 = 0.05$. For both options $E = \$100$ and $T = 1$ month. At present $r = 0.1$, $S = \$103$ and $C = \$5.50$. Find the upper and lower bounds for the price of the American put option by using (2.102). How do we take the arbitrage opportunity if the price of the American put option is greater than the calculated upper bound?

Solution: According to (2.102), the lower bound of $P(S, t)$ is

$$C(S, t) - S + Ee^{-r(T-t)} = 5.5 - 103 + 100e^{-0.1/12} = 1.67$$

and the upper bound is

$$C(S, t) - Se^{-D_0(T-t)} + E = 5.5 - 103e^{-0.05/12} + 100 = 2.93.$$

Suppose that on the market $P(S, t) = \$3.50$ and that we have $e^{-0.05/12}$ shares in hand or can borrow $e^{-0.05/12}$ shares for a period $[t, t + 1/12]$. Now we describe how to take advantage of the arbitrage opportunity. At time t , we can sell the American put option and $e^{-0.05/12}$ shares, purchase one European call option which is less than or equal to $\$5.50$ and hold at least an amount

of cash $3.5 + 103e^{-0.05/12} - 5.5 = \100.57 . If we want, it can be deposited into a bank. At any time $\bar{t} \in [t, T)$, the holder of the American option wants to exercise the option, we pay \$100 and get one share. In this case we have one share of stock and at least an amount of cash equal to \$0.57. If the holder of the American option does not exercise the option before time T , we will always have at least \$0.57 in cash plus one share of stock at time T . The reason is that if $S > E$ we can exercise the European option and get one share, while if $S \leq E$ we can purchase one share from the market. Since we have $e^{-0.05/12}$ shares of stocks at time t , we should have one share of stock in order to have no loss on stocks or we need to return one share to the borrower at time T . Therefore the risk-free earning in this case is at least \$0.57.

Problems

1. a) Suppose that S_1 and S_2 are two independent normal random variables. The mean and variance of S_1 are μ_1 and σ_1^2 and for S_2 they are μ_2 and σ_2^2 . Show that $S_1 + S_2$ is a normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.
- b) Suppose $\Delta t = t/n$ and $\phi_i, i = 1, 2, \dots, n$ are independent standardized normal random variables. Show that

$$X(t) = \lim_{n \rightarrow \infty} (\phi_1 \sqrt{\Delta t} + \phi_2 \sqrt{\Delta t} + \dots + \phi_n \sqrt{\Delta t})$$

is a normal random variable with mean zero and variance t .

- c) Define $dX = X(t + dt) - X(t)$. Show that it is a normal random variable with mean zero and variance dt .
- d) Suppose $S(t) = e^{\mu t + \sigma X(t)}$. Show $d \ln S(t) \equiv \ln S(t + dt) - \ln S(t) = \mu dt + \sigma dX$.
2. ^{*16} Suppose

$$dS = a(S, t)dt + b(S, t)dX,$$

where dX is a Wiener process. Let f be a function of S and t . Show that

$$\begin{aligned} df &= \frac{\partial f}{\partial S} dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial S^2} \right) dt \\ &= b \frac{\partial f}{\partial S} dX + \left(\frac{\partial f}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial S^2} + a \frac{\partial f}{\partial S} \right) dt. \end{aligned}$$

3. Suppose that a random variable satisfies

$$dS = \mu S dt + \sigma S dX,$$

¹⁶A problem with * in this book means that you can find the answer in this book. It is suggested that a student should first read and understand the corresponding material and then do the problem without looking at the book.

where dX is a Wiener process. Find the stochastic equation for $\xi = \frac{1}{S}$ by using Itô's lemma and determine the mean and variance of $\frac{d\xi}{\xi}$.

4. Suppose that S satisfies

$$dS = \mu S dt + \sigma S dX, \quad 0 \leq S < \infty,$$

where μ, σ are positive constants and dX is a Wiener process. Let

$$\xi = \frac{S}{S + P_m},$$

where P_m is a positive constant. The range of ξ is $[0, 1)$. The stochastic differential equation for ξ is in the form:

$$d\xi = a(\xi)dt + b(\xi)dX.$$

Find the concrete expressions for $a(\xi)$ and $b(\xi)$ by Itô's lemma and show

$$\begin{cases} a(0) = 0, \\ b(0) = 0, \end{cases}$$

and

$$\begin{cases} a(1) = 0, \\ b(1) = 0. \end{cases}$$

5. Consider a random variable r satisfying the stochastic differential equation

$$dr = (\mu - \gamma r)dt + w dX, \quad -\infty < r < \infty,$$

where μ, γ, w are positive constants and dX is a Wiener process. Define

$$\xi = \frac{r}{|r| + P_m}, \quad P_m > 0,$$

which transforms the domain $(-\infty, \infty)$ for r into $(-1, 1)$ for ξ . Suppose the stochastic equation for the new random variable ξ is

$$d\xi = a(\xi)dt + b(\xi)dX.$$

Find the concrete expressions of $a(\xi)$ and $b(\xi)$ and show that $a(\xi)$ and $b(\xi)$ fulfill the conditions

$$\begin{cases} a(-1) = 0, \\ b(-1) = 0, \end{cases}$$

and

$$\begin{cases} a(1) = 0, \\ b(1) = 0. \end{cases}$$

6. Suppose that S has the probability density function

$$G(S) = \frac{1}{\sqrt{2\pi bS}} e^{-(\ln(S/a) + b^2/2)^2 / 2b^2}.$$

Let $\xi = \frac{1}{S}$. Find the probability density function for ξ , $E[\xi]$ and $\text{Var}[\xi]$.

7. a) * Show that if an investment is risk-free, then theoretically its return rate must be the spot interest rate.
 b) * Using this fact and Itô's lemma, derive the Black-Scholes equation.
8. Find the solution of the form
 a) $V(S, t) = V(S)$,
 b) $V(S, t) = A(t)B(S)$
 for the Black-Scholes equation.
9. Show by substitution that
 a) $V(S, t) = Se^{-D_0(T-t)}$,
 b) $V(S, t) = Ee^{-r(T-t)}$
 are solutions of the Black-Scholes equation. What do these solutions represent?
10. *Suppose $V(S, t)$ is the solution of the problem

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV = 0, \\ 0 \leq S, \quad t \leq T, \\ V(S, T) = V_T(S), \quad 0 \leq S. \end{cases}$$

Let $x = \ln S + (r - D_0 - \sigma^2/2)(T - t)$, $\bar{\tau} = \sigma^2(T - t)/2$ and $V(S, t) = e^{-r(T-t)}u(x, \bar{\tau})$. Show that $u(x, \bar{\tau})$ is the solution of the problem

$$\begin{cases} \frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \leq \bar{\tau}, \\ u(x, 0) = V_T(e^x), & -\infty < x < \infty. \end{cases}$$

11. Consider the problem **A**

$$\begin{cases} \frac{\partial V}{\partial t} + a(t)S^2 \frac{\partial^2 V}{\partial S^2} + b(t)S \frac{\partial V}{\partial S} - r(t)V = 0, & 0 \leq S, \quad t \leq T, \\ V(S, T) = V_T(S), & 0 \leq S \end{cases}$$

and define

$$\alpha(t) = \int_t^T a(s)ds,$$

$$\beta(t) = \int_t^T b(s)ds$$

and

$$\gamma(t) = \int_t^T r(s)ds.$$

Show that

- a) Let $x = \ln S + \beta(t) - \alpha(t)$, $\bar{\tau} = \alpha(t)$ and $V(S, t) = e^{-\gamma(t)}u(x, \bar{\tau})$, then $u(x, \bar{\tau})$ is the solution of the problem:

$$\begin{cases} \frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \leq \bar{\tau}, \\ u(x, 0) = V_T(e^x), & -\infty < x < \infty. \end{cases}$$

- b) $V(S, t)$ must be in a form

$$V(S, t) = e^{-\gamma(t)}u(\ln S + \beta(t) - \alpha(t), \alpha(t))$$

or

$$V(S, t) = e^{-\gamma(t)}\bar{u}(Se^{\beta(t)}, \alpha(t)).$$

- c) If

$$V(S, t) = e^{-r(T-t)}\bar{u}(Se^{b(T-t)}, a(T-t))$$

is the solution of the problem **A** with constant coefficients, then

$$V(S, t) = e^{-\gamma(t)}\bar{u}(Se^{\beta(t)}, \alpha(t)).$$

is the solution of problem **A** with time-dependent coefficients.

12. *Suppose $V(S, t)$ is the solution of the problem

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0, & 0 \leq S, \quad t \leq T, \\ V(S, T) = V_T(S), & 0 \leq S. \end{cases}$$

Let $\xi = \frac{S}{S + P_m}$, $\tau = T - t$ and $V(S, t) = (S + P_m)\bar{V}(\xi, \tau)$, where P_m is a positive constant.

- a) Show that $\bar{V}(\xi, \tau)$ is the solution of the problem

$$\begin{cases} \frac{\partial \bar{V}}{\partial \tau} = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2 \bar{V}}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\frac{\partial \bar{V}}{\partial \xi} \\ \quad - [r(1-\xi) + D_0\xi]\bar{V}, & 0 \leq \xi \leq 1, \quad 0 \leq \tau, \\ \bar{V}(\xi, 0) = \frac{1-\xi}{P_m}V_T\left(\frac{P_m\xi}{1-\xi}\right), \end{cases}$$

where $\bar{\sigma}(\xi) = \sigma\left(\frac{P_m\xi}{1-\xi}\right)$.

- b) What are the advantages of reformulating the problem on a finite domain?
13. *Find an integral expression of the solution of the following problem

$$\begin{cases} \frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \leq \bar{\tau}, \\ u(x, 0) = u_0(x), & -\infty < x < \infty. \end{cases}$$

14. Suppose that S is a random variable which is defined on $[0, \infty)$ and whose probability density function is

$$G(S) = \frac{1}{\sqrt{2\pi}bS} e^{-(\ln(S/a) + b^2/2)^2 / 2b^2},$$

a and b being positive numbers. Show that

a)

$$\int_0^c G(S) dS = N\left(\frac{\ln(c/a) + b^2/2}{b}\right);$$

b)

$$\int_0^c SG(S) dS = aN\left(\frac{\ln(c/a) - b^2/2}{b}\right);$$

c) for any real number n

$$\int_0^c S^n G(S) dS = a^n e^{(n^2-n)b^2/2} N\left(\frac{\ln(c/a) + b^2/2}{b} - nb\right);$$

d) for any real number n

$$E[S^n] = a^n e^{(n^2-n)b^2/2};$$

e) for any real number n

$$\int_c^\infty S^n G(S) dS = a^n e^{(n^2-n)b^2/2} N\left(-\frac{\ln(c/a) + b^2/2}{b} + nb\right),$$

where

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\xi^2/2} d\xi.$$

15. Using the results given in Problems 10, 13 and 14, derive the Black–Scholes formula for a European put option.

16. Verify that the Black–Scholes formula for a put option is the solution of the following problem:

$$\begin{cases} \frac{\partial p}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + (r - D_0)S \frac{\partial p}{\partial S} - rp = 0, & 0 \leq S, \quad 0 \leq t \leq T, \\ p(S, T) = \max(E - S, 0), & 0 \leq S. \end{cases}$$

(Hint: Show the identity $Ee^{-r(T-t)-d_2^2/2} = Se^{-D_0(T-t)-d_1^2/2}$ first.)

17. Using the Black–Scholes formula for a put option and the results in Problem 11, find the formula for the price of a put option with time-dependent parameters.
18. Consider the following problem

$$\begin{cases} \frac{\partial c_b}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c_b}{\partial S^2} + (r - D_0)S \frac{\partial c_b}{\partial S} - rc_b = 0, \\ \quad 0 \leq S < \infty, \quad 0 < t < T, \\ c_b(S, T) = \begin{cases} 0, & \text{if } 0 \leq S < S^{**}, \\ f(S), & \text{if } S^{**} \leq S < S^*, \\ S - E, & \text{if } S^* \leq S < \infty, \end{cases} \end{cases}$$

where

$$f(S) = a_0 + a_1 S + \cdots + a_J S^J.$$

Show that it has a solution in the following closed form:

$$\begin{aligned} c_b(S, t) = & \sum_{n=0}^J \left\{ a_n S^n e^{[(n-1)r - nD_0 + (n-1)n\sigma^2/2](T-t)} \right. \\ & \times \left[N(d^* - n\sigma\sqrt{T-t}) - N(d^{**} - n\sigma\sqrt{T-t}) \right] \Big\} \\ & + Se^{-D_0(T-t)} \left[1 - N(d^* - \sigma\sqrt{T-t}) \right] - Ee^{-r(T-t)} [1 - N(d^*)], \end{aligned}$$

where

$$\begin{aligned} d^* &= \left[\ln(S^*/S) - \left(r - D_0 - \frac{1}{2}\sigma^2 \right) (T-t) \right] / \left(\sigma\sqrt{T-t} \right), \\ d^{**} &= \left[\ln(S^{**}/S) - \left(r - D_0 - \frac{1}{2}\sigma^2 \right) (T-t) \right] / \left(\sigma\sqrt{T-t} \right). \end{aligned}$$

19. Consider a European call option on a non-dividend paying stock. Use the Black–Scholes formula to find the option price when the stock price is \$63, the strike price is \$60, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months.

20. Consider a European put option on a dividend paying stock. Use the Black–Scholes formula to find the option price when the stock price is \$55, the strike price is \$60, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, the dividend yield is 3% per annum, and the time to maturity is six months.
21. Consider a European call option on a non-dividend paying stock. The option price is \$4.5, the stock price is \$86, the exercise price is \$92, the risk-free interest rate is 5% per annum, and the time to maturity is 3 months. Use the Black–Scholes formula for a call option to find what the corresponding volatility should be. (This volatility is usually referred to as the implied volatility associated with the given option price.)
22. Consider a European option on a non-dividend paying stock. The stock price is \$37, the exercise price is \$34, the risk-free interest rate is 5% per annum, the volatility is 30% per annum, and the time to maturity is six months. Find the put and call option prices by using the Black–Scholes formulae and verify that the put–call parity holds.
23. *Suppose that $c(S, t)$ and $p(S, t)$ are the prices of European call and put options with the same parameters respectively. Show the put–call parity

$$c(S, t) - p(S, t) = Se^{-D_0(T-t)} - Ee^{-r(T-t)}.$$

without using the Black–Scholes formulae.

24. By using the put–call parity relation of European options

$$c(S, t) - p(S, t) = Se^{-D_0(T-t)} - Ee^{-r(T-t)},$$

show that the following relations hold:

$$\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - e^{-D_0(T-t)}, \quad \frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}$$

and

$$\frac{\partial^2 p}{\partial S \partial \sigma} = \frac{\partial^2 c}{\partial S \partial \sigma}, \quad \frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma}, \quad \frac{\partial^2 p}{\partial \sigma^2} = \frac{\partial^2 c}{\partial \sigma^2}.$$

25. a) Show that the solution of the problem

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV = 0, & 0 \leq S, \quad t \leq T, \\ V(S, T) = \max(S, E), & 0 \leq S \end{cases}$$

is

$$V(S, t) = Ee^{-r(T-t)}N(-d_2) + Se^{-D_0(T-t)}N(d_1),$$

where

$$d_1 = \left[\ln \frac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t) \right] / (\sigma\sqrt{T-t}),$$

$$d_2 = \left[\ln \frac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} - \frac{1}{2}\sigma^2(T-t) \right] / (\sigma\sqrt{T-t}).$$

- b) Let $\bar{S} = E^2/S$ and $\bar{V}(\bar{S}, t) = EV(S, t)/S$. Show that $\bar{V}(\bar{S}, t)$ is the solution of the following problem:

$$\begin{cases} \frac{\partial \bar{V}}{\partial t} + \frac{1}{2}\sigma^2\bar{S}^2\frac{\partial^2 \bar{V}}{\partial \bar{S}^2} + (D_0 - r)\bar{S}\frac{\partial \bar{V}}{\partial \bar{S}} - D_0\bar{V} = 0, & 0 \leq \bar{S}, \quad t \leq T, \\ \bar{V}(\bar{S}, T) = \max(\bar{S}, E), & 0 \leq \bar{S} \end{cases}$$

and find the expression of $\bar{V}(\bar{S}, t)$.

- c) Let

$$G(S', T; S, t, a_1) = \frac{1}{\sigma\sqrt{2\pi(T-t)}S'} e^{-(\ln(S'/a) + \sigma^2(T-t)/2)^2 / 2\sigma^2(T-t)},$$

where $a = Se^{a_1(T-t)}$. Show that $N(d_1)$ and $N(-d_1)$ are the probabilities of the events $\bar{S}' \leq E$ ($S' \geq E$) and $\bar{S}' \geq E$ ($S' \leq E$) respectively in the world where the probability density function of \bar{S}' is

$$G(\bar{S}', T; \bar{S}, t, D_0 - r),$$

and that $N(d_2)$ and $N(-d_2)$ are the probabilities of the events $S' \geq E$ and $S' \leq E$ respectively in the world where the probability density function of S' is

$$G(S', T; S, t, r - D_0).$$

- d) Replacing $Se^{-D_0(T-t)}$ by $Ee^{-r(T-t)}$ and $Ee^{-r(T-t)}$ by $Se^{-D_0(T-t)}$, we can obtain the second (first) term in the solution in (a) from the first (second) term and the entire solution is unchanged. Why does the solution have this symmetry?
26. Explain why an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date by financial reasoning and by mathematical tools.
27. Explain why an American option is always worth at least as much as its intrinsic value. What does the time value of an American option represent?
28. Let $\mathbf{L}_{S,t}$ be an operator in an option problem in the form:

$$\mathbf{L}_{S,t} = a(S, t)\frac{\partial^2}{\partial S^2} + b(S, t)\frac{\partial}{\partial S} + c(S, t)$$

and $G(S, t)$ be the constraint function for an American option. Furthermore we assume that $\frac{\partial G}{\partial t} + \mathbf{L}_{S,t}G$ exists. Suppose $V(S, t^*) = G(S, t^*)$ on an open interval (A_1, B_1) on the S -axis. Let $t = t^* - \Delta t$, where Δt is a sufficiently small positive number and let (A, B) be an open interval in (A_1, B_1) . Show the following conclusions: If for any $S \in (A, B)$,

$$\frac{\partial G}{\partial t}(S, t^*) + \mathbf{L}_{S,t^*}G(S, t^*) \geq 0,$$

then the value $V(S, t)$ determined by the equation

$$\frac{\partial V}{\partial t}(S, t) + \mathbf{L}_{S,t}V(S, t) = 0$$

satisfies the condition $V(S, t) - G(S, t) \geq 0$ on (A, B) ; and if for any $S \in (A, B)$,

$$\frac{\partial G}{\partial t}(S, t^*) + \mathbf{L}_{S,t^*}G(S, t^*) < 0,$$

then the equation

$$\frac{\partial V}{\partial t}(S, t) + \mathbf{L}_{S,t}V(S, t) = 0$$

cannot give a solution satisfying the condition $V(S, t) - G(S, t) \geq 0$ for any $S \in (A, B)$.

29. A European option is the solution of the problem

$$\begin{cases} \frac{\partial V}{\partial t} + \mathbf{L}_S V = 0, & 0 \leq S, \quad t \leq T, \\ V(S, T) = V_T(S), & 0 \leq S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S \frac{\partial}{\partial S} - r.$$

For an American option, the constraint is that the inequality

$$V(S, t) \geq G(S, t)$$

holds for any S and t , where $G(S, T) = V_T(S)$. Derive the linear complementarity problem for the American option.

30. Let $V(S, t)$ be the price of a vanilla American option. Explain why $V(S, t - \Delta t) \geq V(S, t)$ is always true by financial reasoning and by mathematical tools, where $\Delta t > 0$.
31. The American call option is the solution of the following linear complementarity problem on a finite domain:

$$\begin{cases} \left(\frac{\partial \bar{V}}{\partial \tau} - \mathbf{L}_\xi \bar{V} \right) (\bar{V}(\xi, 0) - \max(2\xi - 1, 0)) = 0, \\ \frac{\partial \bar{V}}{\partial \tau} - \mathbf{L}_\xi \bar{V} \geq 0, \\ \bar{V}(\xi, \tau) - \max(2\xi - 1, 0) \geq 0, \\ \bar{V}(\xi, 0) = \max(2\xi - 1, 0), \end{cases}$$

where

$$0 \leq \xi \leq 1, \quad 0 \leq \tau$$

and

$$\mathbf{L}_\xi = \frac{1}{2}\sigma^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial\xi^2} + (r-D_0)\xi(1-\xi)\frac{\partial}{\partial\xi} - [r(1-\xi) + D_0\xi].$$

Reformulate this problem as a free-boundary problem if

$$D_0 > 0.$$

32. The American put option is the solution of the following linear complementarity problem:

$$\begin{cases} \left(\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} \right) (u(x, \bar{\tau}) - g_p(x, \bar{\tau})) = 0, \\ \frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} \geq 0, \\ u(x, \bar{\tau}) - g_p(x, \bar{\tau}) \geq 0, \\ u(x, 0) = g_p(x, 0), \end{cases}$$

where

$$-\infty < x < \infty, \quad 0 \leq \bar{\tau}$$

and

$$g_p(x, \bar{\tau}) = \max(e^{2r\bar{\tau}/\sigma^2} - e^{x+(2D_0/\sigma^2+1)\bar{\tau}}, 0).$$

Find the domain where a free boundary may appear and the domain where it is impossible for a free boundary to appear, show that there is only one free boundary at $\bar{\tau} = 0$ and give the starting location of this free boundary.

33. The price of a one-factor convertible bond is the solution of the linear complementarity problem

$$\begin{cases} \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D_0)S \frac{\partial V}{\partial S} - rV \right) (V(S, t) - nS) = 0, \\ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D_0)S \frac{\partial V}{\partial S} - rV \leq 0, \\ V(S, t) - nS \geq 0, \\ V(S, T) = \max(Z, nS) \geq nS, \end{cases}$$

where $0 \leq S < \infty$, $0 \leq t \leq T$, σ , r and D_0 are constants. Show that if $D_0 > 0$, then the solution of a one-factor convertible bond must involve a free boundary and its location at $t = T$ is $S = Z/n$. Also derive the corresponding free boundary problem.

34. a) Suppose $\sigma = \sigma(S, t)$, $r = r(t)$ and $D_0 = D_0(S, t)$. Show that the problem of pricing a put option can always be converted into a problem of pricing a call option and explain how to use this conclusion when we write codes in order to reduce the amount of work.

- b) Let the exercise price be E . Suppose that r , D_0 are constants and $\sigma = \sigma(S)$. Show

$$P(S, t; b, a, \sigma(S)) = C(E^2/S, t; a, b, \sigma(S)) S/E,$$

$$C(S, t; a, b, \sigma(E^2/S)) = P(E^2/S, t; b, a, \sigma(E^2/S)) S/E$$

and

$$S_{cf}(t; a, b, \sigma(E^2/S)) \times S_{pf}(t; b, a, \sigma(S)) = E^2.$$

Here the first, second and third parameters after the semicolon in P , C , S_{pf} and S_{cf} are the interest rate, the dividend yield and the volatility function respectively.

35. Suppose that σ , r , D_0 are constants. In this case we have the following symmetry relation for European options

$$p(S, t; b, a) = c\left(\frac{E^2}{S}, t; a, b\right) S/E,$$

where the first and second arguments after the semicolon in p and c are the values of the interest rate and the dividend yield respectively. For a European call option, the price is

$$c(S, t) = S e^{-D_0(T-t)} N(d_1) - E e^{-r(T-t)} N(d_2),$$

where

$$d_1 = \frac{\ln \frac{S e^{-D_0(T-t)}}{E e^{-r(T-t)}} + \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}},$$

$$d_2 = \frac{\ln \frac{S e^{-D_0(T-t)}}{E e^{-r(T-t)}} - \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}}.$$

Find the price of a European put option by using the symmetry relation.

36. Derive the formulation of the problem for $\frac{\partial P}{\partial r}$ and write down the formulation of the problems for $\frac{\partial P}{\partial \sigma}$ and $\frac{\partial P}{\partial D_0}$, where P is the price of an American put option.
37. Find an expression for the value of a perpetual American put option.
38. *Describe the reversion conditions of a stochastic process and give the intuitive meaning of the conditions.
39. *Describe and derive the generalized Itô's lemma
40. *Describe and derive the general equations for derivative securities.
41. Suppose $f_1(r, t) \geq 0$ and $f_1(r_l, t) = \frac{\partial f_1(r_l, t)}{\partial r} = f_1(r_u, t) = \frac{\partial f_1(r_u, t)}{\partial r} = 0$, and $f_2(r_l, t) < 0$, $f_2(r_u, t) > 0$. Explain why problem **A**

$$\begin{cases} \frac{\partial V}{\partial t} = f_1 \frac{\partial^2 V}{\partial r^2} + f_2 \frac{\partial V}{\partial r} + f_3 V, & r_l \leq r \leq r_u, \quad 0 \leq t, \\ V(r, 0) = V_0(r), & r_l \leq r \leq r_u, \\ V(r_l, t) = f_l(t), & 0 \leq t, \\ V(r_u, t) = f_u(t), & 0 \leq t \end{cases}$$

and problem **B**

$$\begin{cases} \frac{\partial V}{\partial t} = -f_1 \frac{\partial^2 V}{\partial r^2} + f_2 \frac{\partial V}{\partial r} + f_3 V, & r_l \leq r \leq r_u, \quad t \leq T, \\ V(r, T) = V_T(r), & r_l \leq r \leq r_u \end{cases}$$

have unique solutions.

42. Suppose that S is the price of a dividend-paying stock and satisfies

$$dS = \mu(S, t)Sdt + \sigma SdX_1, \quad 0 \leq S < \infty,$$

where dX_1 is a Wiener process and σ is another random variable. Let the dividend paid during the time period $[t, t + dt]$ be $D(S, t)dt$. Assume that for σ , the stochastic equation

$$d\sigma = p(\sigma, t)dt + q(\sigma, t)dX_2, \quad \sigma_l \leq \sigma \leq \sigma_u$$

holds. Here $p(\sigma, t)$ and $q(\sigma, t)$ are differentiable functions and satisfy the reversion conditions. dX_2 is another Wiener process correlated with dX_1 , and the correlation coefficient between them is ρdt . Derive the equation for options on such a stock directly (without using the general PDE for derivatives) which contains only the unknown market price of risk for the volatility. (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1 + \Delta_2 V_2 + S$, where V_1 and V_2 are two different options.)

43. Explain the financial meaning of the jump conditions for option values.
44. *Use arbitrage arguments to show the put-call parity of European options for the following two cases.
a) When the dividend is paid continuously, the put-call parity is

$$c(S, t) - p(S, t) = Se^{-D_0(T-t)} - Ee^{-r(T-t)};$$

- b) when the dividend is paid discretely, the put-call parity is

$$c(S, t) - p(S, t) = S - D(t) - Ee^{-r(T-t)},$$

where $D(t)$ is the value of dividend at time t .

45. *Use arbitrage arguments to show the inequalities of American options for the following two cases.

- a) When the dividend is paid continuously, there is the inequality

$$Se^{-D_0(T-t)} - E \leq C(S, t) - P(S, t) \leq S - Ee^{-r(T-t)}$$

between American put option $P(S, t)$ and American call option $C(S, t)$ with the same parameters.

- b) When the dividend is paid discretely, there is the inequality

$$S - D(t) - E \leq C(S, t) - P(S, t) \leq S - Ee^{-r(T-t)}$$

between American put option $P(S, t)$ and American call option $C(S, t)$ with the same parameters.

46. Consider a European call option with $T = 6$ months and $E = \$80$ on a dividend-paying stock. The dividend is paid continuously with a dividend yield $D_0 = 0.05$. Today $t = 0$, $r = 0.1$ and $S = \$82$.
- Find the lower bound of the call option.
 - What are the least profits we could make at time T by arbitrage if the call option price today is \$0.10 less than the lower bound and why?
47. Consider a European put option with $T = 3$ months and $E = \$60$ on a dividend-paying stock. Today $t = 0$, $r = 0.05$ and $S = \$55$. The dividends are paid discretely and the total present value of them is $D(0) = \$0.30$.
- Find the lower bound of the put option.
 - What are the least profits we could make at time T by arbitrage if the put option price today is \$0.20 less than the lower bound and why?
48. Suppose that there are an American call option and an American put option on the same stock which pays dividends discretely. For both of them $E = \$90$ and $T = 3$ months. At time $t = 0$, the stock price is \$93 and the present value of dividend payments during the period $[0, T]$ is $D(0) = \$0.50$. Assume that $r = 0.1$ and $P(S, 0) = \$2.50$.
- Find the upper and lower bounds of the price of the American call option.
 - What are the risk-free profits we could make today by arbitrage if the price of the call option today is \$0.10 greater than the calculated upper bound and why?
49. Suppose that $c_1(S, t)$ and $c_2(S, t)$ are the prices of European call options with strikes E_1 and E_2 respectively, where $E_1 < E_2$. Also assume that the two options have the same maturity T and that the interest rate r is a constant. Show

$$0 \leq c_1(S, t) - c_2(S, t) \leq (E_2 - E_1)e^{-r(T-t)}.$$

50. Suppose that p_1 , p_2 and p_3 are the prices of European put options with strike prices E_1 , E_2 and E_3 respectively, where $E_2 = \frac{1}{2}(E_1 + E_3)$. All the options have the same maturity. Show

$$p_2 \leq \frac{1}{2}(p_1 + p_3).$$



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