2. Basic Notions of Systems and Signals

Signals and systems are basic notions of systems and control theory. Therefore, we briefly summarize the most important concepts of the mathematical description of signals and systems in this introductory chapter in the form they are used later on.

The material in this chapter is presented in two sections:

- *Signals*
  Not only the definition and classification of signals, operations on signals and the notion of signal spaces are given but the most important special signal types are also introduced.
- *Systems*
  The most important system classes are defined based on the abstract notion of systems together with the characterization of their input–output stability.

2.1 Signals

Signals are the basic elements of mathematical systems theory, because the notion of a system depends upon them.

2.1.1 What is a Signal?

Generally, a *signal* is defined as any physical quantity that varies with time, space or any other independent variable(s). A signal can be a function of one or more independent variables.

A longer and more application-oriented definition of signals taken from [50] is the following: "Signals are used to communicate between humans and between humans and machines; they are used to probe our environment to uncover details of structure and state not easily observable; and they are used to control energy and information."

Example 2.1.1 (Simple signals)

The following examples describe different simple signals.

- Let us suppose that the temperature x of a vessel in a process plant is changing with time (measured in seconds) in the following way:

  $$x : \mathbb{R}_0^+ \mapsto \mathbb{R}, \quad x(t) = e^{-t}$$

  We can see that t is the independent variable (time) and x is the dependent variable (see Figure 2.1 (/a)).
- Let us assume that we can observe the temperature x in the first example only at integer time instants (seconds). Let us denote the observed temperature by y, which therefore can be defined as:

  $$y : \mathbb{N}_0^+ \mapsto \mathbb{R}, \quad y[n] = e^{-n}$$

  (see Figure 2.1 (/b)).

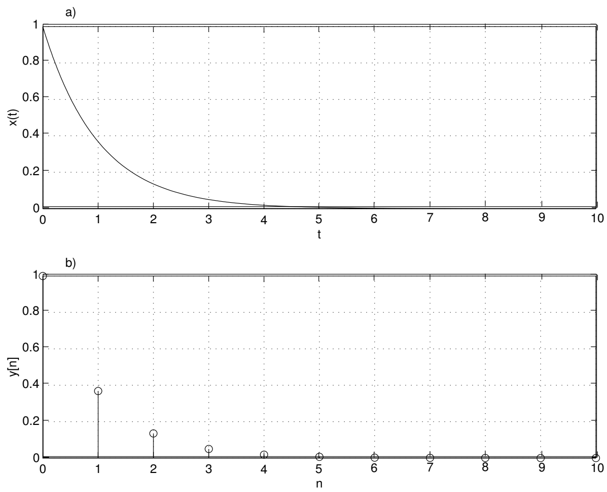

Figure 2.1. (/a) Simple continuous time signal and (/b) its discrete time counterpart

---

- A complex-valued signal with a complex independent variable is, *e.g.*

  $$X : \mathbb{C} \mapsto \mathbb{C}, \quad X(s) = \frac{1}{s+1}$$

  which is actually the Laplace transform of the first signal x (see Subsection 2.1.4).
- Let us denote the temperature of a point (x, y, z coordinates in a Cartesian coordinate system) inside a room at a certain time instant t by $T(x, y, z, t)$. It's easy to see that T is a real–valued function of four independent variables and hence it maps from $\mathbb{R}^4$ to $\mathbb{R}$.
- An example of a vector-valued signal with three independent variables is the picture of a color TV where the intensity functions of the red, green and blue colors (I_r, I_g and I_b) form the vector

  $$I(x, y, t) = \begin{bmatrix} I_r(x, y, t) \\ I_g(x, y, t) \\ I_b(x, y, t) \end{bmatrix}$$

  i.e. $I : \mathbb{R}^3 \mapsto \mathbb{R}^3$. The independent variables in this case are the screen coordinates (x and y) and time (t).

2.1.2 Classification of Signals

Signals are classified according to the properties of their independent and dependent variables.

Dimensionality of the Independent Variable. The independent variable can have one or more dimensions. The most common one-dimensional case is when the independent variable is time.

Dimension of the Dependent Variable (Signal). The signal value evaluated at a certain point in its domain can also be one or more dimensional.

Real–valued and Complex–valued Signals. The value of a signal can be either real or complex. The following question naturally arises: why do we deal with complex signals? The answer is that the magnitude and angle of a complex signal often has clear engineering meaning, which is sometimes analytically simpler to deal with.

Continuous Time and Discrete Time Signals. In systems and control theory, one usually has a one-dimensional independent variable set which is called *time* and denoted by $\mathcal{T}$ in the general case.

Continuous time signals take real or complex (vector) values as a function of an independent variable that ranges over the real numbers, therefore a continuous time signal is a mapping from a subset of $\mathbb{R}$ to $\mathbb{C}^n$, *i.e.* $\mathcal{T} \subseteq \mathbb{R}$.

Discrete time signals take real or complex (vector) values as a function of an independent variable that ranges over the integers, hence a discrete time signal makes a mapping from $\mathbb{N}$ to $\mathbb{C}^n$, *i.e.* $\mathcal{T} \subseteq \mathbb{N}$.

Bounded and Unbounded Signals. A signal $x : \mathbb{R} \mapsto \mathbb{C}$ is bounded if $|x(t)|$ is finite for all t. A signal that does not have this property is unbounded.

Periodic and Aperiodic Signals. A time-dependent real-valued signal $x : \mathbb{R} \mapsto \mathbb{R}$ is periodic with period T if $x(t+T) = x(t)$ for all t. A signal that does not have this property is aperiodic.

Even and Odd Signals. Even signals x_e and odd signals x_o are defined as

$$x_e(t) = x_e(-t)$$
$$x_o(t) = -x_o(-t)$$

Any signal is a sum of unique odd and even signals. Using $x(t) = x_e(t) + x_o(t)$ and $x(-t) = x_e(t) - x_o(t)$ yields $x_e(t) = \frac{1}{2}(x(t) + x(-t))$ and $x_o(t) = \frac{1}{2}(x(t) - x(-t))$.

2.1.3 Signals of Special Importance

Some signals are of theoretical and/or practical importance because they are used as special test signals in dynamic systems analysis.

Definition 2.1.1 (Dirac-δ or unit impulse function)
The Dirac-δ or unit impulse function is not a function in the ordinary sense. The simplest way it can be defined is by the integral relation

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \tag{2.1}$$

where $f : \mathbb{R}_0^+ \mapsto \mathbb{R}$ *is an arbitrary smooth function.*

The unit impulse is not defined in terms of its values, but by how it acts inside an integral when multiplied by a smooth function f. To see that the area of the unit impulse function is 1, we can choose $f(t) = 1$ in the definition.

The unit impulse function is widely used in science and engineering. The following statements illustrate its role from an engineering point of view:

- impulse of current in time delivers a unit charge instantaneously to an electric network,
- impulse of force in time gives an instantaneous momentum to a mechanical system,
- impulse of temperature gives a unit energy, that of pressure gives a unit mass and that of concentration delivers an impulse of component mass to a process system,

- impulse of mass density in space represents a mass-point,
- impulse of charge density in space represents a point charge.

Definition 2.1.2 (Unit step function)
Integration of the unit impulse gives the unit step function

$$\eta(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \tag{2.2}$$

which therefore reads as

$$\eta(t) = \begin{cases} 0 & \textit{if } t < 0 \\ 1 & \textit{if } t \geq 0 \end{cases} \tag{2.3}$$

Example 2.1.2 (Unit impulse and unit step signals)
Unit impulse as the derivative of the unit step

As an example of a method for dealing with generalized functions, consider the following function:

$$x(t) = \frac{d}{dt}\eta(t)$$

with η being the unit step function defined above. Since η is discontinuous, its derivative does not exist as an ordinary function, but it exists as a generalized function. Let's put x in an integral with a smooth testing function f.

$$y(t) = \int_{-\infty}^{\infty} f(t) \frac{d}{dt}\eta(t) dt$$

and calculate the integral using the integration-by-parts theorem

$$y(t) = f(t)\eta(t)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \eta(t) \frac{d}{dt} f(t) dt$$

which gives

$$y(t) = f(\infty) - \int_{0}^{\infty} \frac{d}{dt} f(t) dt = f(0)$$

This results in

$$\int_{-\infty}^{\infty} f(t) \frac{d}{dt}\eta(t) dt = f(0)$$

---

which, from the defining Equation (2.1), implies that

$$\delta(t) = \frac{d}{dt}\eta(t)$$

That is, the unit impulse is the derivative of the unit step in a generalized function sense.

2.1.4 Operations on Signals

Operations on signals are used to derive (possibly more complex) signals from elementary signals or to extract some of the important signal properties.

Elementary Operations. Let ν be an n-dimensional vector space with an inner product $< \cdot, \cdot >_\nu$, $\alpha \in \mathbb{R}$ and x, $y : \mathbb{R}_0^+ \mapsto \nu$, *i.e.*

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

Sum of signals. The sum of x and y is defined point-wise, *i.e.*

$$(x + y)(t) = x(t) + y(t), \quad \forall t \in \mathbb{R}_0^+ \tag{2.4}$$

Multiplication by scalar.

$$(\alpha x)(t) = \alpha x(t), \quad \forall t \in \mathbb{R}_0^+ \tag{2.5}$$

Inner product of signals. The inner product of x and y is defined as

$$\langle x, y\rangle_\nu(t) = \langle x(t), y(t)\rangle_\nu, \quad \forall t \in \mathbb{R}_0^+ \tag{2.6}$$

If the inner product on ν for $a, b \in \nu$, $a = [a_1 \ \ldots \ a_n]^T$, $b = [b_1, \ \ldots \ b_n]^T$ is defined as

$$\langle a, b\rangle = \sum_{i=1}^{n} a_i b_i \tag{2.7}$$

then (2.6) simply gives $\langle x, y\rangle_\nu(t) = x^T(t)y(t)$, which is a simple point-wise product if $\dim(\nu) = 1$.

Time shifting. For $a \in \mathbb{R}$, the time shifting of x is defined as

$$\mathbf{T}_a x(t) = x(t - a) \tag{2.8}$$

Causal time shifting. For $a \in \mathbb{R}$, the causal time shifting differs from ordinary time shifting in the fact that the value of the original signal before $t = 0$ is not taken into consideration, *i.e.*

$$\mathbf{T}_a^c x(t) = \eta(t-a)x(t-a) \tag{2.9}$$

where η is the unit step function defined in Equation (2.3).

Truncation. The value of a truncated signal after the truncation time T is zero, *i.e.*

$$x_T(t) = \begin{cases} x(t), & 0 \le t < T \\ 0, & t \ge T \end{cases} \tag{2.10}$$

Convolution. Convolution is a very important binary time-domain operation in linear systems theory as we will see later (*e.g.* it can be used for computing the response of a linear system to a given input).

Definition 2.1.3 (Convolution of signals)
Let $x, y : \mathbb{R}_0^+ \mapsto \mathbb{R}$. The convolution of x and y denoted by $x * y$ is given by

$$(x * y)(t) = \int_0^t x(\tau)y(t-\tau)d\tau, \quad \forall t \ge 0 \tag{2.11}$$

Example 2.1.3 (Convolution of simple signals)

Let us compute the convolution of the signals $x, y : \mathbb{R}_0^+ \mapsto \mathbb{R}$, $x(t) = 1$, $y(t) = e^{-t}$. According to the definition

$$(x * y)(t) = \int_0^t 1 \cdot e^{-(t-\tau)}d\tau = e^{-t}\left[e^{\tau}\right]_0^t = 1 - e^{-t}$$

The signals and their convolution are shown in Figure 2.2.

Laplace Transformation. The main use of Laplace transformation in linear systems theory is to transform linear differential equations into algebraic ones. Moreover, the transformation allows us to interpret signals and linear systems in the frequency domain.

Definition 2.1.4 (The domain of Laplace transformation)
The domain Λ of Laplace transformation is the set of integrable complex-valued functions mapping from the set of real numbers whose absolute value is not increasing faster than exponentially, i.e.

$$\begin{aligned} \Lambda = \{\ f \mid f : \mathbb{R}_0^+ \mapsto \mathbb{C}, f \textit{ is integrable on } [0, a]\ \forall a > 0 \textit{ and} \\ \exists A_f \ge 0, a_f \in \mathbb{R} \textit{ such that } |f(x)| \le A_f e^{a_f x}\ \forall x \ge 0\ \} \end{aligned} \tag{2.12}$$

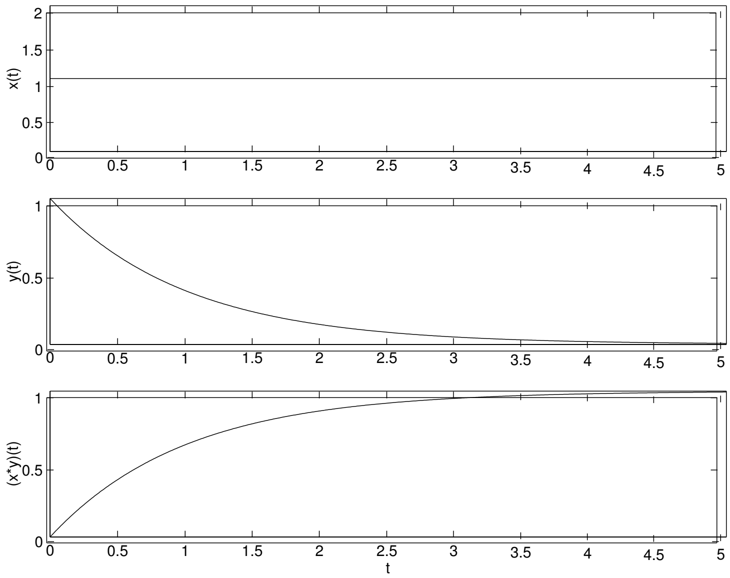

Figure 2.2. Two signals ($x(t) = 1$, $y(t) = e^{-t}$) and their convolution

Definition 2.1.5 (Laplace transformation)
With the domain defined above, the Laplace transformation of a signal f is defined as

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt, \quad f \in \Lambda,\ s \in \mathbb{C} \tag{2.13}$$

The most important properties (among others) of Laplace transformation that we will use later are the following:

Let $f, g \in \Lambda$. Then the following equalities hold

1. *Linearity*

$$\mathcal{L}\{c \cdot f\} = c \cdot \mathcal{L}\{f\}, \quad c \in \mathbb{C} \tag{2.14}$$
$$\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\} \tag{2.15}$$

2. *Laplace transform of the derivative*

$$\mathcal{L}\{f'\}(s) = s \cdot \mathcal{L}\{f\}(s) - f(0^+) \tag{2.16}$$

3. *Convolution theorem*

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\} \tag{2.17}$$

4. *Time shifting*

$$\mathcal{L}\{\mathbf{T}_a^c(f)\} = e^{-as}\mathcal{L}\{f\}(s), \quad a \in \mathbb{R} \tag{2.18}$$

5. *Modulation*

$$\mathcal{L}\{e_\lambda f\}(s) = \mathcal{L}\{s - \lambda\} \tag{2.19}$$

where $\lambda \in \mathbb{C}$ and $e_\lambda(t) = e^{\lambda t}$.

2.1.5 L_q Signal Spaces and Signal Norms

Signals with similar mathematical properties belong to signal spaces which are usually equipped by suitable signal norms. L_q spaces are the most frequently used general signal spaces described below.

Definition 2.1.6 (L_q spaces, scalar case)
For $q = 1, 2, \ldots$ the signal space $L_q[0, \infty)$ contains the functions $f : \mathbb{R}_0^+ \mapsto \mathbb{R}$, which are Lebesgue-measurable (i.e. their generalized integral exists in the Lebesgue sense) and satisfy

$$\int_0^\infty |f(t)|^q dt \leq \infty \tag{2.20}$$

The magnitude of functions in an L_q space is measured using norms (see Section A.1 in the Appendix for the defining properties of norms).

Definition 2.1.7 (q-norm, scalar case)
Let $f \in L_q[0, \infty)$ for $q = 1, 2, \ldots$. The q-norm of f denoted by $\|f\|_q$ is defined as

$$\|f\|_q = \left(\int_0^\infty |f(t)|^q dt \right)^{\frac{1}{q}} \tag{2.21}$$

It is known that $L_q[0, \infty)$ are complete normed linear spaces (Banach spaces) with respect to the q-norms.

L_q spaces can be extended further in the following way:

Definition 2.1.8 (L_{qe} spaces, scalar case)
For $q = 1, 2, \ldots$ the signal space L_{qe} consists of the functions $f : \mathbb{R}_0^+ \mapsto \mathbb{R}$, which are Lebesgue-measurable and $f_T \in L_q$ for all T, $0 \leq T < \infty$.

Note that the L_{qe} spaces are not normed spaces.

It is easy to see that $L_q \subset L_{qe}$. For this, consider a function $f \in \text{Ł}_q$. Then for any $0 < T < \infty$ $\int_0^\infty |f_T(t)|^q dt \leq \infty$ and therefore $f \in L_{qe}$. For the opposite direction, consider the signal $g(t) = \frac{1}{t+1}$ and $q = 1$. It's clear that for $T < \infty$, $\int_0^T |g(t)| dt = \ln(T + 1) - \ln(1)$, but the L_1 norm of g is not finite since $\lim_{t\to\infty} \ln(t) = \infty$. Therefore $g \in \text{Ł}_{1e}$ but $g \notin L_1$.

For the treatment of multi-input–multi-output systems, we define L_q spaces, q-norms and L_{qe} spaces for vector-valued signals, too. For this, consider a finite dimensional normed linear space ν equipped with a norm $\| \cdot \|_\nu$.

Definition 2.1.9 (L_q spaces, vector case)
For $q = 1, 2, \ldots$ the signal space $L_q(\nu)$ contains the functions $f : \mathbb{R}_0^+ \mapsto \nu$, which are Lebesgue-measurable and satisfy

$$\int_0^\infty \|f(t)\|_\nu^q dt < \infty \tag{2.22}$$

Definition 2.1.10 (q-norm, vector case)
Let $f \in L_q(\nu)$ for $q = 1, 2, \ldots$. The q-norm of f denoted by $\|f\|_q$ is defined as

$$\|f\|_q = \left(\int_0^\infty \|f(t)\|_\nu^q dt \right)^{\frac{1}{q}} \tag{2.23}$$

Definition 2.1.11 (L_{qe} spaces, vector case)
For $q = 1, 2, \ldots$ the signal space $L_{qe}(\nu)$ consists of the functions $f : \mathbb{R}_0^+ \mapsto \nu$, which are Lebesgue-measurable and $f_T \in L_q(\nu)$ for all T, $0 \leq T < \infty$.

Special Cases. Two important special cases for the q-norms are given below (compare with Subsection A.1.2).

1. $q = 2$
   For $f \in L_2$

   $$\|f\|_2 = \left(\int_0^\infty f^2(t) \right)^{\frac{1}{2}} \tag{2.24}$$

   which can be associated with the inner product

   $$\langle f, f \rangle = \int_0^\infty f^2(t) dt \tag{2.25}$$

   Similarly, in the case of $f \in L_2(\nu)$

   $$\|f\|_2 = \left(\int_0^\infty \langle f(t), f(t) \rangle_\nu dt \right)^{\frac{1}{2}} = \langle f, f \rangle^{\frac{1}{2}} \tag{2.26}$$

2. $q = \infty$
   For $f \in L_\infty$

   $$\|f\|_\infty = \sup_{t \in \mathbb{R}_0^+} |f(t)| \tag{2.27}$$

   and for $f \in L_\infty(\nu)$

   $$\|f\|_\infty = \sup_{t \in \mathbb{R}_0^+} \|f(t)\|_\nu \tag{2.28}$$

Example 2.1.4 (Signal spaces and their relations)

Besides the L_q-spaces, we can define many other signal spaces. A few examples are given below.

$\mathcal{F}(T)$ vector space: let V be the set of functions mapping from a set T to the set of real or complex numbers, where the operations are defined point-wise, *i.e.* for $x, y \in V$ and $\alpha \in \mathbb{K}$

$$(x+y)(t) = x(t) + y(t), \quad \forall t \in T$$
$$(\alpha x)(t) = \alpha \cdot x(t), \quad \forall t \in T$$

The following vector spaces are the subspaces of $\mathcal{F}(T)$:

$\mathcal{B}(T)$ vector space: $\mathcal{B}(T)$ is the set of bounded functions mapping from T to the set of real or complex numbers.

$\mathcal{C}([a,b])$ vector space: $\mathcal{C}([a,b])$ is the set of functions which are continuous on the closed interval $[a,b]$.

$\mathcal{C}^{(k)}([a,b])$ vector space: let $k \in \mathbb{N}$, $k > 1$. The set of functions which are k times continuously differentiable on the interval $[a,b]$ is a vector space and is denoted by $\mathcal{C}^{(k)}([a,b])$. At the endpoints of the interval a and b, the left and right derivatives of the functions are considered respectively.

It is clear that the following relations hold in the above examples in the case when $T = [a,b]$:

$$\mathcal{C}^{(k)}[a,b] \subset \mathcal{C}[a,b] \subset \mathcal{B}([a,b]) \subset \mathcal{F}([a,b]) \tag{2.29}$$

2.2 Systems

A system can be defined as a physical or logical device that performs an operation on a signal. Therefore we can say that systems process input signals to produce output signals.

A system is a set of objects linked by different interactions and relationships. The elements and boundaries of a system are determined by the interactions and mutual relationships, that are taken into consideration.

The nature and outcome of the interactions in physical systems is governed by certain laws. These laws can be used as *a priori* information in further examinations. The information about the system should be used in a well-defined form.

A special group of information with major importance is the one that gives the current *state* of the system. The state of a system is the collection of all

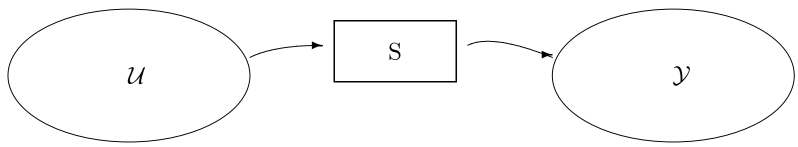

Figure 2.3. System mapping the elements of the input signal space to the output signal space

the information that describes the relations between the different interactions of the system at a given time instant (compare with Section 3.1 later). Two important kinds of information must be known in order to determine the state of the system: firstly, the *structure* of the system and secondly, the *parameters* of the system. A system can be described structurally by defining the *system topology, the rules of interconnection of the elements* and the *functional descriptions of the elements (constitutive relations).*

Based on the above, a system can be considered as an abstract operator mapping from the input signal space to the output signal space (see Figure 2.3). The notation of this is

$$y = \mathbf{S}[u], \quad u \in \mathcal{U}, \; y \in \mathcal{Y} \tag{2.30}$$

where $\mathbf{S}$ is the system operator, u is the input, y is the output, and $\mathcal{U}$ and $\mathcal{Y}$ denote the input and output signal spaces respectively.

2.2.1 Classification of Systems: Important System Properties

Causality. The "present" in a causal system does not depend on the "future" but only on the past. This applies for every signal that belongs to the system and to the system operator $\mathbf{S}$ as well.

Definition 2.2.1 (Causal system)
A system is called causal if $y(t)$ does not depend on $u(t+\tau)$, $\forall t \geq 0, \tau > 0$.

Physical systems where time is the independent variable are causal systems. However, there are some systems that are not causal, *e.g.*

- some optical systems where the independent variables are the space coordinates,
- many off-line signal processing filters when the whole signal to be processed is previously recorded.

Linearity. A property of special interest is linearity.

Definition 2.2.2 (Linear system)
A system $\mathbf{S}$ is called linear if it responds to a linear combination of its possible input functions with the same linear combination of the corresponding output functions. Thus for the linear system we note that:

$$\mathbf{S}[c_1u_1 + c_2u_2] = c_1\mathbf{S}[u_1] + c_2\mathbf{S}[u_2] \tag{2.31}$$

with $c_1, c_2 \in \mathbb{R}$, $u_1, u_2 \in \mathcal{U}$, $y_1, y_2 \in \mathcal{Y}$ *and* $\mathbf{S}[u_1] = y_1, \quad \mathbf{S}[u_2] = y_2$.

Continuous Time and Discrete Time Systems. We may classify systems according to the time variable $t \in \mathcal{T}$ we apply to their description (see Subsection 2.1.2 for the definition of continuous and discrete time signals). There are continuous time systems where time is an open interval of the real line ($\mathcal{T} \subseteq \mathcal{R}$). Discrete time systems have an ordered set $\mathcal{T} = \{\cdots, t_0, t_1, t_2, \cdots\}$ as their time variable set.

SISO and MIMO Systems. Here the classification is determined by the number of input and output variables. The input and output of a single input–single output (SISO) system is a scalar value at each time instant, while multi input–multi output (MIMO) systems process and produce vector-valued signals.

Time-invariant and Time-varying Systems. The second interesting class of systems are time-invariant systems. A system **S** is *time-invariant* if its response to a given input is invariant under time shifting. Loosely speaking, time-invariant systems do not change their system properties in time. If we were to repeat an experiment under the same circumstances at some later time we get the same response as originally observed. This situation is depicted in Figure 2.4 below.

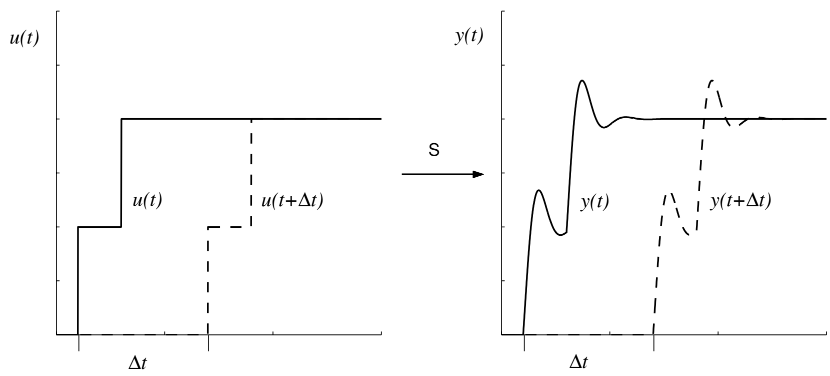

Figure 2.4. The notion of time invariance

The system parameters of a time-invariant system are constants, i.e. they do not depend on time.

We can define the notion of time invariance in a rigorous mathematical way using the shift operator defined in Equation (2.8).

Definition 2.2.3 (Time-invariant system)
A system is called time-invariant if its system operator commutes with the time shift operator, i.e.

$$\mathbf{T}_a \circ \mathbf{S} = \mathbf{S} \circ \mathbf{T}_a, \quad \forall a \in \mathbb{R} \tag{2.32}$$

2.2.2 Input–output Stability: L_q-stability and L_q-gain

As we will see later in Chapter 7, it is of interest from the viewpoint of stability how a system operator changes the norm of the input signals. This property is expressed in the L_q-stability and L_q-gain of a system.

For the following definitions, assume that $\mathcal{U}$ and $\mathcal{Y}$ are finite dimensional linear spaces of the input and output signals respectively, and $\mathbf{S}$ is a system operator mapping from $L_{qe}(\mathcal{U})$ to $L_{qe}(\mathcal{Y})$.

Definition 2.2.4 (L_q-stability)
$\mathbf{S}$ *is called L_q-stable if*

$$u \in L_q(\mathcal{U}) \ \Rightarrow G(u) \in L_q(\mathcal{Y}) \tag{2.33}$$

Definition 2.2.5 (Finite L_q-gain)
$\mathbf{S}$ *is said to have finite L_q-gain if there exist finite constants γ_q and b_q such that*

$$\|(\mathbf{S}[u])_T\|_q \leq \gamma_q \|u_T\|_q + b_q, \quad \forall T \geq 0, \ \forall u \in L_{qe}(\mathcal{U}) \tag{2.34}$$

It is said that $\mathbf{S}$ *has finite L_q-gain with zero bias if b_q can be zero in (2.34).*

Definition 2.2.6 (L_q-gain)
Let $\mathbf{S}$ *have finite L_q-gain. The L_q-gain of* $\mathbf{S}$ *is defined as*

$$\gamma_q(\mathbf{S}) = \inf\{\gamma_q \ | \exists b_q \ \textit{such that}\ (2.34)\ \textit{holds}\} \tag{2.35}$$

2.3 Summary

Systems are described as abstract operators acting on signal spaces in this basic introductory chapter. The definition, classification and basic operations on both signals and systems are summarized, which will be extensively used throughout the whole book.

2.4 Questions and Exercises

Exercise 2.4.1. Give examples of signals of special importance. What is the relationship between the unit impulse and the unit step signal?

Exercise 2.4.2. What are the operations that are defined on signals? Characterize the elementary- and integral-type operations. Compute the convolution of an arbitrary signal

$$x : \mathbb{R}_0^+ \mapsto \mathbb{R}$$

with the shifted Dirac-δ function δ_τ which has its singular point at $t = \tau$.

Exercise 2.4.3. Give the most important system classes. Define the class of continuous time linear time-invariant (LTI) systems.

Exercise 2.4.4. Give the definitions of L_q and L_{qe} spaces and their underlying norms both in the scalar and vector case. Compare your definitions with the special cases of signal norms in Section A.1.2 in the Appendix.

Exercise 2.4.5. Consider the following signal:

$$f : \mathbb{R}_0^+ \mapsto \mathbb{R}, \quad f(t) = \exp(-t^2), \; t \geq 0$$

1. Show that $f \in L_1$ and $f \in L_\infty$.
2. Calculate the L_1 and L_∞ norm of f.

Exercise 2.4.6. The following two-dimensional signal is given:

$$g : \mathbb{R}_0^+ \mapsto \mathbb{R}^2, \quad f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \cdot \exp(-2t) \end{bmatrix}, \; t \geq 0$$

1. Is $g \in L_2$?
2. Is $g \in L_{2e}$?
3. Let $\| \cdot \|_\nu$ be the normal Euclidean norm on $\mathbb{R}^2$. Calculate $\|g(t)\|_\nu$ for $t \geq 0$ and $\|g\|_\infty$.

Exercise 2.4.7. Calculate the convolution of f_1 and f_2 in Exercise 2.4.6 and check the convolution theorem for the Laplace transform of f_1 and f_2.

Springer

http://www.springer.com/978-1-85233-600-4

Analysis and Control of Nonlinear Process Systems
Hangos, K.M.; Bokor, J.; Szederkényi, G.
2004, XXVI, 310 p., Softcover
ISBN: 978-1-85233-600-4