Preface

A classical problem in the calculus of variations is the investigation of critical points of a C^1-functional $\mathcal{L} : V \to \mathbb{R}$ on a normed space V. Typical examples are $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$ with $\Omega \subset \mathbb{R}^n$ and V a space of admissible functions $u : \Omega \to \mathbb{R}^k$. A large variety of methods has been invented to obtain existence of critical points of $\mathcal{L}$. The present work addresses a different question:

> *Under what conditions on the Lagrangian L, the domain Ω and the set of admissible functions V does $\mathcal{L}$ have at most one critical point?*

The following sufficient condition for uniqueness is presented in this work: the functional $\mathcal{L}$ has at most one critical point u_0 if a differentiable one-parameter group $G = \{g_\epsilon\}_{\epsilon \in \mathbb{R}}$ of transformations $g_\epsilon : V \to V$ exists, which strictly reduces the values of $\mathcal{L}$, i.e. $\mathcal{L}[g_\epsilon u] < \mathcal{L}[u]$ for all $\epsilon > 0$ and all $u \in V \setminus \{u_0\}$. If G is not differentiable the uniqueness result is recovered under the extra assumption that the Lagrangian is a convex function of ∇u (ellipticity condition). This approach to uniqueness is called "the method of transformation groups".

The interest for uniqueness results in the calculus of variations comes from two sources:

1) In applications to physical problems uniqueness is often considered as supporting the validity of a model.
2) For semilinear boundary value problems like $\Delta u + \lambda u + |u|^{p-1}u = 0$ in Ω with $u = 0$ on $\partial\Omega$ uniqueness means that $u \equiv 0$ is the only solution. Conditions on Ω, p, λ ensuring uniqueness may be compared with those conditions guaranteeing the existence of nontrivial solutions. E.g., if Ω is bounded and $1 < p < \frac{n+2}{n-2}$, then nontrivial solutions exist for all λ. If, in turn, one can prove uniqueness for $p \geq \frac{n+2}{n-2}$ and certain λ and Ω, then the restriction on p made for existence is not only sufficient but also necessary.

A very important uniqueness theorem for semilinear problems was found in 1965 by S.I. Pohožaev [75]. If Ω is star-shaped with respect to the origin,

$p \geq \frac{n+2}{n-2}$ and $\lambda \leq 0$, then uniqueness of the trivial solution follows. In his proof Pohožaev tested the equation with $x \cdot \nabla u$ and u. The resulting integral identity admits only the zero-solution. A crucial role is played by the vector-field x. The motivation of the present work was to exhibit arguments within the calculus of variations which explain Pohožaev's result and, in particular, explain the role of the vector-field x.

Chapter 1 provides two examples illustrating the method of transformation groups in an elementary way.

In Chapter 2 we develop the general theory of uniqueness of critical points for abstract functionals $\mathcal{L} : V \to \mathbb{R}$ on a normed space V. The notion of a differentiable one-parameter transformation group $g_\epsilon : \operatorname{dom} g_\epsilon \subset V \to V$ is developed and the following fundamental uniqueness result is shown: if $\mathcal{L}[g_\epsilon u] < \mathcal{L}[u]$ for all $\epsilon > 0$ and all $u \in V \setminus \{u_0\}$ then u_0 is the only possible critical point of $\mathcal{L}$. We mention two applications: 1) a strictly convex functional has at most one critical point and 2) the first eigenvalue of a linear elliptic divergence-operator with zero Dirichlet or Neumann boundary conditions is simple.

As a generalization the concept of non-differentiable one-parameter transformation groups is developed in Chapter 3. Its interaction with first order variational functionals $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$ is studied. Under the extra assumption of rank-one convexity of L w.r.t. ∇u, a uniqueness result in the presence of energy reducing transformation groups is proved, which is a suitable generalization of the one in Chapter 2. In particular, Pohožaev's identity will emerge as two ways of computing the rate of change of the functional $\mathcal{L}$ under the action of the one-parameter transformation group.

In Chapter 4 the semilinear Dirichlet problem $\Delta u + \lambda u + |u|^{p-1}u = 0$ in Ω, $u = 0$ on $\partial\Omega$ is treated, where Ω is a domain on a Riemannian manifold M. An exponent $p^* \geq \frac{n+2}{n-2}$ is associated with Ω such that $u \equiv 0$ is the only solution provided $p \geq p^*$ and λ is sufficiently small. On more special manifolds better results can be achieved. If M possesses a one-parameter group $\{\Phi_t\}_{t\in\mathbb{R}}$ of conformal self-maps $\Phi_t : M \to M$, then a complete analogue of the Euclidean vector-field x is given by the so-called conformal vector-field $\boldsymbol{\xi}(x) := \frac{d}{dt}\Phi_t(x)|_{t=0}$. In the presence of conformal vector-fields one can show that the critical Sobolev exponent $\frac{n+2}{n-2}$ is the true barrier for existence/non-existence of non-trivial solutions. Generalizations of the semilinear Dirichlet problem to nonlinear Neumann boundary value problems are also considered.

In Chapter 5 and 6 we study variational problems in Euclidean $\mathbb{R}^n$. Examples of non-starshaped domains are given, for which Pohožaev's original result still holds. A number of boundary value problems for semilinear and quasilinear equations is studied. Uniqueness results for trivial/non-trivial solutions of supercritical problems as well as L^∞-bounds from below for solutions of subcritical problems are investigated. Uniqueness questions from the theory of elasticity (boundary displacement problem) and from geometry (surfaces of prescribed mean curvature) are treated as examples.

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