
Contents

1	Roots of Polynomials	1
1.1	Inequalities for roots	1
1.1.1	The Fundamental Theorem of Algebra	1
1.1.2	Cauchy's theorem	2
1.1.3	Laguerre's theorem	5
1.1.4	Apolar polynomials	7
1.1.5	The Routh-Hurwitz problem	11
1.2	The roots of a given polynomial and of its derivative	12
1.2.1	The Gauss-Lucas theorem	12
1.2.2	The roots of the derivative and the focal points of an ellipse	14
1.2.3	Localization of the roots of the derivative	15
1.2.4	The Sendov-Ilieff conjecture	18
1.2.5	Polynomials whose roots coincide with the roots of their derivatives	20
1.3	The resultant and the discriminant	20
1.3.1	The resultant	20
1.3.2	The discriminant	23
1.3.3	Computing certain resultants and discriminants	25
1.4	Separation of roots	27
1.4.1	The Fourier-Budan theorem	27
1.4.2	Sturm's Theorem	30
1.4.3	Sylvester's theorem	31
1.4.4	Separation of complex roots	35
1.5	Lagrange's series and estimates of the roots of a given polynomial	37
1.5.1	The Lagrange-Bürmann series	37
1.5.2	Lagrange's series and estimation of roots	40
1.6	Problems to Chapter 1	41
1.7	Solutions of selected problems	42

2	Irreducible Polynomials	47
2.1	Main properties of irreducible polynomials	47
2.1.1	Factorization of polynomials into irreducible factors	47
2.1.2	Eisenstein's criterion	50
2.1.3	Irreducibility modulo p	51
2.2	Irreducibility criteria	52
2.2.1	Dumas's criterion	52
2.2.2	Polynomials with a dominant coefficient	56
2.2.3	Irreducibility of polynomials attaining small values	58
2.3	Irreducibility of trinomials and founomials	59
2.3.1	Irreducibility of polynomials of the form $x^n \pm x^m \pm x^p \pm 1$	59
2.3.2	Irreducibility of certain trinomials	63
2.4	Hilbert's irreducibility theorem	65
2.5	Algorithms for factorization into irreducible factors	68
2.5.1	Berlekamp's algorithm	68
2.5.2	Factorization with the help of Hensel's lemma	71
2.6	Problems to Chapter 2	73
2.7	Solutions of selected problems	74
3	Polynomials of a Particular Form	77
3.1	Symmetric polynomials	77
3.1.1	Examples of symmetric polynomials	77
3.1.2	Main theorem on symmetric polynomials	79
3.1.3	Muirhead's inequalities	81
3.1.4	The Schur functions	83
3.2	Integer-valued polynomials	85
3.2.1	A basis in the space of integer-valued polynomials	85
3.2.2	Integer-valued polynomials in several variables	87
3.2.3	The q -analogue of integer-valued polynomials	88
3.3	The cyclotomic polynomials	89
3.3.1	Main properties of the cyclotomic polynomials	89
3.3.2	The Möbius inversion formula	90
3.3.3	Irreducibility of cyclotomic polynomials	91
3.3.4	The expression for Φ_{mn} in terms of Φ_n	93
3.3.5	The discriminant of a cyclotomic polynomial	94
3.3.6	The resultant of a pair of cyclotomic polynomials	95
3.3.7	Coefficients of the cyclotomic polynomials	96
3.3.8	Wedderburn's theorem	97
3.3.9	Polynomials irreducible modulo p	99
3.4	Chebyshev polynomials	100
3.4.1	Definition and main properties of Chebyshev polynomials	100
3.4.2	Orthogonal polynomials	105
3.4.3	Inequalities for Chebyshev polynomials	107
3.4.4	Generating functions	109

3.5	Bernoulli polynomials	112
3.5.1	Definition of Bernoulli polynomials	112
3.5.2	Theorems of complement, addition of arguments and multiplication	115
3.5.3	Euler's formula	116
3.5.4	The Faulhaber-Jacobi theorem	117
3.5.5	Arithmetic properties of Bernoulli numbers and Bernoulli polynomials	120
3.6	Problems to Chapter 3	125
3.6.1	Symmetric polynomials	125
3.6.2	Integer-valued polynomials	126
3.6.3	Chebyshev polynomials	126
3.7	Solution of selected problems	127
4	Certain Properties of Polynomials	133
4.1	Polynomials with prescribed values	133
4.1.1	Lagrange's interpolation polynomial	133
4.1.2	Hermite's interpolation polynomial	136
4.1.3	The polynomial with prescribed values at the zeros of its derivative	137
4.2	The height of a polynomial and other norms	139
4.2.1	Gauss's lemma	139
4.2.2	Polynomials in one variable	141
4.2.3	The maximum of the absolute value and S. Bernstein's inequality	145
4.2.4	Polynomials in several variables	148
4.2.5	An inequality for a pair of relatively prime polynomials	151
4.2.6	Mignotte's inequality	152
4.3	Equations for polynomials	154
4.3.1	Diophantine equations for polynomials	154
4.3.2	Functional equations for polynomials	162
4.4	Transformations of polynomials	166
4.4.1	Tschirnhaus's transformation	166
4.4.2	5th degree equation in Bring's form	168
4.4.3	Representation of polynomials as sums of powers of linear functions	169
4.5	Algebraic numbers	173
4.5.1	Definition and main properties of algebraic numbers	173
4.5.2	Kronecker's theorem	175
4.5.3	Liouville's theorem	176
4.6	Problems to Chapter 4	179

5	Galois Theory	181
5.1	Lagrange's theorem and the Galois resolvent	181
5.1.1	Lagrange's theorem	181
5.1.2	The Galois resolvent	185
5.1.3	Theorem on a primitive element	189
5.2	Basic Galois theory	191
5.2.1	The Galois correspondence	191
5.2.2	A polynomial with the Galois group S_5	195
5.2.3	Simple radical extensions	196
5.2.4	The cyclic extensions	197
5.3	How to solve equations by radicals	199
5.3.1	Solvable groups	199
5.3.2	Equations with solvable Galois group	200
5.3.3	Equations solvable by radicals	201
5.3.4	Abelian equations	204
5.3.5	The Abel-Galois criterion for solvability of equations of prime degree	208
5.4	Calculation of the Galois groups	212
5.4.1	The discriminant and the Galois group	212
5.4.2	Resolvent polynomials	213
5.4.3	The Galois group modulo p	216
6	Ideals in Polynomial Rings	219
6.1	Hilbert's basis theorem and Hilbert's theorem on zeros	219
6.1.1	Hilbert's basis theorem	219
6.1.2	Hilbert's theorem on zeros	221
6.1.3	Hilbert's polynomial	224
6.1.4	The homogeneous Hilbert's Nullstellensatz for p -fields	231
6.2	Gröbner bases	233
6.2.1	Polynomials in one variable	234
6.2.2	Division of polynomials in several variables	235
6.2.3	Definition of Gröbner bases	235
6.2.4	Buchberger's algorithm	237
6.2.5	A reduced Gröbner basis	239
7	Hilbert's Seventeenth Problem	243
7.1	The sums of squares: introduction	243
7.1.1	Several examples	243
7.1.2	Artin-Cassels-Pfister theorem	248
7.1.3	The inequality between the arithmetic and geometric means	251
7.1.4	Hilbert's theorem on non-negative polynomials $p_4(x, y)$	253
7.2	Artin's theory	259
7.2.1	Real fields	259
7.2.2	Sylvester's theorem for real closed fields	263

7.2.3	Hilbert's seventeenth problem	266
7.3	Pfister's theory	270
7.3.1	The multiplicative quadratic forms	270
7.3.2	C_i -fields	273
7.3.3	Pfister's theorem on the sums of squares of rational functions	274
8	Appendix	279
8.1	The Lenstra-Lenstra-Lovász algorithm	279
8.1.1	The general description of the algorithm	279
8.1.2	A reduced basis of the lattice	280
8.1.3	The lattices and factorization of polynomials	283
	References	289
	Index	297



<http://www.springer.com/978-3-642-03979-9>

Polynomials

Prasolov, V.V.

2004, XIII, 301 p. 8 illus., Softcover

ISBN: 978-3-642-03979-9