35
Solving Differential Equations Using the Exponential

...he climbed a little further...and further...and then just a little further. (Winnie-the-Pooh)

35.1 Introduction

The exponential function plays a fundamental role in modeling and analysis because of its basic properties. In particular it can be used to solve a variety of differential equations analytically as we show in this chapter. We start with generalizations of the initial value problem (31.2) from Chapter *The exponential function*:

$$u'(x) = \lambda u(x) \quad \text{for } x > a, \quad u(a) = u_a, \tag{35.1}$$

where $\lambda \in \mathbb{R}$ is a constant, with solution

$$u(x) = \exp(\lambda(x-a))u_a \quad \text{for } x \geq a. \tag{35.2}$$

Analytic solutions formulas may give very important information and help the intuitive understanding of different aspects of a mathematical model, and should therefore be kept as valuable gems in the scientist and engineer's tool-bag. However, useful analytical formulas are relatively sparse and must be complemented by numerical solutions techniques. In the Chapter *The General Initial Value Problem* we extend the constructive numerical method for solving (35.1) to construct solutions of general initial value problems for systems of differential equations, capable of modeling a very large variety of phenomena. We can thus numerically compute the solution to just about any initial value problem, with more or less computational work, but we are limited to computing one solution for each specific choice of data, and getting qualitative information for a variety of different data may be costly. On the other hand, an analytical solution formula, when available, may contain this qualitative information for direct information.

An analytical solution formula for a differential equation may thus be viewed as a (smart and beautiful) short-cut to the solution, like evaluating an integral of a function by just evaluating two values of a corresponding primitive function. On the other hand, numerical solution of a differential equation is like a walk along a winding mountain road from point A to point B, without any short-cuts, similar to computing an integral by numerical quadrature. It is useful to be able to use both approaches.

35.2 Generalization to $u'(x) = \lambda(x)u(x) + f(x)$

The first problem we consider is a model in which the rate of change of a quantity $u(x)$ is proportional to the quantity with a variable factor of proportionality $\lambda(x)$, and moreover in which there is an external "forcing" function $f(x)$. The problem reads:

$$u'(x) = \lambda(x)u(x) + f(x) \quad \text{for } x > a, \quad u(a) = u_a, \tag{35.3}$$

where $\lambda(x)$ and $f(x)$ are given functions of x, and u_a is a given initial value. We first describe a couple physical situations being modeled by (35.3).

Example 35.1. Consider for time $t > 0$ the population $u(t)$ of rabbits in West Virginia with inital value $u(0) = u_0$ given, which we assume has time dependent known birth rate $\beta(t)$ and death rate $\delta(t)$. In general, we would expect that rabbits will migrate quite freely back and forth across the state border and that the rates of the migration would vary with the season, i.e. with time t. We let $f_i(t)$ and $f_o(t)$ denote the rate of migration into and out of the state respectively at time t, which we assume to be known (realistic?). Then the population $u(t)$ will satisfy

$$\dot{u}(t) = \lambda(t)u(t) + f(t), \quad \text{for } t > a, \quad u(a) = u_a, \tag{35.4}$$

with $\lambda(t) = \beta(t) - \delta(t)$ and $f(t) = f_i(t) - f_o(t)$, which is of the form (35.3). Recall that $\dot{u} = \frac{du}{dt}$.

Example 35.2. We model the amount of solute such as salt in a solvent such as water in a tank in which there is both inflow and outflow, see Fig. 35.1. We let $u(t)$ denote the amount of solute in the tank at time t and suppose that we know the initial amount u_0 at $t = 0$. We suppose that a mixture of solute/solvent, of concentration C_i in say grams per liter, flows into the

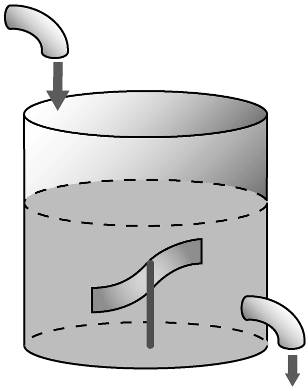

Fig. 35.1. An illustration of a chemical mixing tank

tank at a rate σ_i liters per second. We assume there is also outflow at a rate of σ_o liters per second, and we assume that the mixture in the tank is well mixed with a uniform concentration $C(t)$ at any time t.

To get a differential equation for $u(t)$, we compute the change $u(t+\Delta t) - u(t)$ during the interval $[t, t+\Delta t]$. The amount of solute that flows into the tank during that time interval is $\sigma_i C_i \Delta t$, while the amount of solute that flows out of the tank during that time equals $\sigma_o C(t)\Delta t$, and thus

$$u(t+\Delta t) - u(t) \approx \sigma_i C_i \Delta t - \sigma_o C(t)\Delta t, \tag{35.5}$$

where the approximation improves when we decrease Δt. Now the concentration at time t will be $C(t) = u(t)/V(t)$ where $V(t)$ is the volume of fluid in the tank at time t. Substituting this into (35.5) and dividing by Δt gives

$$\frac{u(t+\Delta t) - u(t)}{\Delta t} \approx \sigma_i C_i - \sigma_o \frac{u(t)}{V(t)}$$

and taking the limit $\Delta t \to 0$ assuming $u(t)$ is differentiable gives the following differential equation for u,

$$\dot{u}(t) = -\frac{\sigma_o}{V(t)} u(t) + \sigma_i C_i.$$

The volume $V(t)$ is determined simply by the flow rates of fluid in and out of the tank. If there is initially V_0 liters in the tank then at time t, $V(t) = V_0 + (\sigma_i - \sigma_o)t$ because the flow rates are assumed to be constant. This gives again a model of the form (35.3):

$$\dot{u}(t) = -\frac{\sigma_o}{V_0 + (\sigma_i - \sigma_o)t} u(t) + \sigma_i C_i \quad \text{for } t > 0, \quad u(0) = u_0. \tag{35.6}$$

The Method of Integrating Factor

We now return to derive an analytical solution formula for (35.3), using the method of *integrating factor*. To work out the solution formula, we begin with the special case

$$u'(x) = \lambda(x)u(x) \quad \text{for } x > a, \quad u(a) = u_a, \tag{35.7}$$

where $\lambda(x)$ is a given function of x. We let $\Lambda(x)$ be a primitive function of $\lambda(x)$ such that $\Lambda(a) = 0$, assuming that $\lambda(x)$ is Lipschitz continuous on $[a, \infty)$. We now multiply the equation $0 = u'(x) - \lambda(x)u(x)$ by $\exp(-\Lambda(x))$, and we get

$$0 = u'(x)\exp(-\Lambda(x)) - u(x)\exp(-(x))\lambda(x) = \frac{d}{dx}(u(x)\exp(-\Lambda(x))),$$

where we refer to $\exp(-\Lambda(x))$ as an integrating factor because it brought the given equation to the form $\frac{d}{dx}$ of something, namely $u(x)\exp(-\Lambda(x))$, equal to zero. We conclude that $u(x)\exp(-\Lambda(x))$ is constant and is therefore equal to u_a since $u(a)\exp(-\Lambda(a)) = u(a) = u_a$. In other words, the solution to (35.7) is given by the formula

$$u(x) = \exp(\Lambda(x))u_a = e^{\Lambda(x)}u_a \quad \text{for } x \geq a. \tag{35.8}$$

We can check by differentiation that this function satisfies (35.7), and thus by uniqueness is the solution. To sum up, we have derived a solution formula for (35.7) in terms of the exponential function and a primitive function $\Lambda(x)$ of the coefficient $\lambda(x)$.

Example 35.3. If $\lambda(x) = \frac{r}{x}$ and $a = 1$ then $\Lambda(x) = r\log(x) = \log(x^r)$, and the solution of

$$u'(x) = \frac{r}{x}u(x) \quad \text{for } x \neq 1, \quad u(1) = 1, \tag{35.9}$$

is according to (35.8) given by $u(x) = \exp(r\log(x)) = x^r$. We may define x^r for

Duhamel's Principle

We now continue with the general problem to (35.3). We multiply by $e^{-\Lambda(x)}$, where again $\Lambda(x)$ is the primitive function of $\lambda(x)$ satisfying $\Lambda(a) = 0$, and get

$$\frac{d}{dx}\left(u(x)e^{-\Lambda(x)}\right) = f(x)e^{-\Lambda(x)}.$$

Integrating both sides, we see that the solution $u(x)$ satisfying $u(a) = u_a$ can be expressed as

$$u(x) = e^{\Lambda(x)}u_a + e^{\Lambda(x)}\int_a^x e^{-\Lambda(y)}f(y)\,dy. \tag{35.10}$$

This formula for the solution $u(x)$ of (35.3), expressing $u(x)$ in terms of the given data u_a and the primitive function $\Lambda(x)$ of $\lambda(x)$ satisfying $\Lambda(a) = 0$, is referred to as *Duhamel's principle* or the *variation of constants formula.*

We can check the validity of (35.10) by directly computing the derivative of $u(x)$:

$$\begin{aligned} u'(x) &= \lambda e^{\Lambda(x)} u_a + f(x) + \int_0^x \lambda(x) e^{\Lambda(x)-\Lambda(y)} f(y)\, dy \\ &= \lambda(x) \left(e^{\Lambda(x)} u_a + \int_0^x e^{\Lambda(x)-\Lambda(y)} f(y)\, dy \right) + f(x). \end{aligned}$$

Example 35.4. If $\lambda(x) = \lambda$ is constant, $f(x) = x$, $a = 0$ and $u_0 = 0$, the solution of (35.3) is given by

$$\begin{aligned} u(x) &= \int_0^x e^{\lambda(x-y)} y\, dy = e^{\lambda x} \int_0^x y e^{-\lambda y}\, dy \\ &= e^{\lambda x} \left(\left[-\frac{y}{\lambda} e^{-\lambda y} \right]_{y=0}^{y=x} + \int_0^x \frac{1}{\lambda} e^{-\lambda y}\, dy \right) = -\frac{x}{\lambda} + \frac{1}{\lambda^2} \left(e^{\lambda x} - 1 \right). \end{aligned}$$

Example 35.5. In the model of the rabbit population (35.4), consider a situation with an initial population of 100, the death rate is greater than the birth rate by a constant factor 4, so $\lambda(t) = \beta(t) - \delta(t) = -4$, and there is a increasing migration into the state, so $f(t) = f_i(t) - f_o(t) = t$. Then (35.10) gives

$$\begin{aligned} u(t) &= e^{-4t} 100 + e^{-4t} \int_0^t e^{4s} s\, ds \\ &= e^{-4t} 100 + e^{-4t} \left(\frac{1}{4} s e^{4s} |_0^t - \frac{1}{4} \int_0^t e^{4s}\, ds \right) \\ &= e^{-4t} 100 + e^{-4t} \left(\frac{1}{4} t e^{4t} - \frac{1}{16} e^{4t} + \frac{1}{16} \right) \\ &= 100.0625 e^{-4t} + \frac{t}{4} - \frac{1}{16}. \end{aligned}$$

Without the migration into the state, the population would decrease exponentially, but in this situation the population decreases only for a short time before beginning to increase at a linear rate.

Example 35.6. Consider a mixing tank in which the input flow at a rate of $\sigma_i = 3$ liters/sec has a concentration of $C_i = 1$ grams/liter, and the outflow is at a rate of $\sigma_o = 2$ liters/sec, the initial volume is $V_0 = 100$ liters with no solute dissolved, so $u_0 = 0$. The equation is

$$\dot{u}(t) = -\frac{2}{100+t} u(t) + 3.$$

We find $\Lambda(t) = 2\ln(100+t)$ and so

$$\begin{aligned}
u(t) &= 0 + e^{2\ln(100+t)} \int_0^t e^{-2\ln(100+s)} 3\, ds \\
&= (100+t)^2 \int_0^t (100+s)^{-2} 3\, ds \\
&= (100+t)^2 \left(\frac{-3}{100+t} + \frac{3}{100} \right) \\
&= \frac{3}{100} t(100+t).
\end{aligned}$$

As expected from the conditions, the concentration increases steadily until the tank is full.

35.3 The Differential Equation $u''(x) - u(x) = 0$

Consider the second order initial value problem

$$u''(x) - u(x) = 0 \quad \text{for } x > 0,\ u(0) = u_0,\ u'(0) = u_1, \tag{35.11}$$

with two initial conditions. We can write the differential equation $u''(x) - u(x) = 0$ formally as

$$(D+1)(D-1)u = 0,$$

where $D = \frac{d}{dx}$, since $(D+1)(D-1)u = D^2u - Du + Du - u = D^2u - u$. Setting $w = (D-1)u$, we thus have $(D+1)w = 0$, which gives $w(x) = ae^{-x}$ with $a = u_1 - u_0$, since $w(0) = u'(0) - u(0)$. Thus, $(D-1)u = (u_1 - u_0)e^{-x}$, so that by Duhamel's principle

$$\begin{aligned}
u(x) &= e^x u_0 + \int_0^x e^{x-y}(u_1 - u_0)e^{-y}\, dy \\
&= \frac{1}{2}(u_0 + u_1)e^x + \frac{1}{2}(u_0 - u_1)e^{-x}.
\end{aligned}$$

We conclude that the solution $u(x)$ of $u''(x) - u(x) = 0$ is a linear combination of e^x and e^{-x} with coefficients determined by the initial conditions. The technique of "factoring" the differential equation $(D^2 - 1)u = 0$ into $(D+1)(D-1)u = 0$, is very powerful and we now proceed to follow up this idea.

35.4 The Differential Equation $\sum_{k=0}^{n} a_k D^k u(x) = 0$

In this section, we look for solutions of the *linear differential equation with constant coefficients*:

$$\sum_{k=0}^{n} a_k D^k u(x) = 0 \quad \text{for } x \in I, \tag{35.12}$$

where the coefficients a_k are given real numbers, and I is a given interval. Corresponding to the *differential operator* $\sum_{k=0}^{n} a_k D^k$, we define the polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ in x of degree n with the same coefficients a_k as the differential equation. This is called the *characteristic polynomial* of the differential equation. We can now express the differential operator formally as

$$p(D)u(x) = \sum_{k=0}^{n} a_k D^k u(x).$$

For example, if $p(x) = x^2 - 1$ then $p(D)u = D^2 u - u$.

The technique for finding solutions is based on the observation that the exponential function $\exp(\lambda x)$ has the following property:

$$p(D)\exp(\lambda x) = p(\lambda)\exp(\lambda x), \tag{35.13}$$

which follows from repeated use of the Chain rule. This translates the differential operator $p(D)$ acting on $\exp(\lambda x)$ into the simple operation of multiplication by $p(\lambda)$. Ingenious, right?

We now seek solutions of the differential equation $p(D)u(x) = 0$ on an interval I of the form $u(x) = \exp(\lambda x)$. This leads to the equation

$$p(D)\exp(\lambda x) = p(\lambda)\exp(\lambda x) = 0, \quad \text{for } x \in I,$$

that is, λ should be a root of the polynomial equation

$$p(\lambda) = 0. \tag{35.14}$$

This algebraic equation is called the *characteristic equation* of the differential equation $p(D)u = 0$. To find the solutions of a differential equation $p(D)u = 0$ on the interval I, we are thus led to search for the roots $\lambda_1, \ldots \lambda_n$, of the algebraic equation $p(\lambda) = 0$ with corresponding solutions $\exp(\lambda_1 x), \ldots, \exp(\lambda_n x)$. Any linear combination

$$u(x) = \alpha_1 \exp(\lambda_1 x) + \ldots + \alpha_n \exp(\lambda_n x), \tag{35.15}$$

with α_i real (or complex) constants, will then be a solution of the differential equation $p(D)u = 0$ on I. If there are n distinct roots $\lambda_1, \ldots, \lambda_n$, then the *general solution* of $p(D)u = 0$ has this form. The constants α_i will be determined from initial or boundary conditions in a specific situation.

If the equation $p(\lambda) = 0$ has a multiple roots λ_i of multiplicity r_i, then the situation is more complicated. It can be shown that the solution is a sum of terms of the form $q(x)\exp(\lambda_i x)$, where $q(x)$ is a polynomial of degree at most $r_i - 1$. For example, if $p(D) = (D-1)^2$, then the general solution of $p(D)u = 0$ has the form $u(x) = (\alpha_0 + \alpha_1 x)\exp(x)$. In the Chapter *N-body systems* below we study the the constant coefficient linear second order equation $a_0 + a_1 Du + a_2 D^2 u = 0$ in detail, with interesting results!

The translation from a differential equation $p(D)u = 0$ to an algebraic equation $p(\lambda) = 0$ is very powerful, but requires the coefficients a_k of $p(D)$ to be independent of x and is thus not very general. The whole branch of *Fourier analysis* is based on the formula (35.13).

Example 35.7. The characteristic equation for $p(D) = D^2 - 1$ is $\lambda^2 - 1 = 0$ with roots $\lambda_1 = 1, \lambda_2 = -1$, and the corresponding general solution is given by $\alpha_1 \exp(x) + \alpha_2 \exp(-x)$. We already met this example just above.

Example 35.8. The characteristic equation for $p(D) = D^2 + 1$ is $\lambda^2 + 1 = 0$ with roots $\lambda_1 = i, \lambda_2 = -i$, and the corresponding general solution is given by

$$\alpha_1 \exp(ix) + \alpha_2 \exp(-ix).$$

with the α_i complex constants. Taking the real part, we get solutions of the form

$$\beta_1 \cos(x) + \beta_2 \sin(x)$$

with the β_i real constants.

35.5 The Differential Equation $\sum_{k=0}^{n} a_k D^k u(x) = f(x)$

Consider now the nonhomogeneous differential equation

$$p(D)u(x) = \sum_{k=0}^{n} a_k D^k u(x) = f(x), \tag{35.16}$$

with given constant coefficients a_k, and a given right hand side $f(x)$. Suppose $u_p(x)$ is any solution of this equation, which we refer to as a *particular solution*. Then any other solution $u(x)$ of $p(D)u(x) = f(x)$ can be written

$$u(x) = u_p(x) + v(x)$$

where $v(x)$ is a solution of the corresponding homogeneous differential equation $p(D)v = 0$. This follows from linearity and uniqueness since $p(D)(u - u_p) = f - f = 0$.

Example 35.9. Consider the equation $(D^2 - 1)u = f(x)$ with $f(x) = x^2$. A particular solution is given by $u_p(x) = -x^2 - 2$, and thus the general solution is given by

$$u(x) = -x^2 - 2 + \alpha_1 \exp(x) + \alpha_2 \exp(-x).$$

35.6 Euler's Differential Equation

In this section, we consider Euler's equation

$$a_0 u(x) + a_1 x u'(x) + a_2 x^2 u''(x) = 0, \tag{35.17}$$

which has variable coefficients $a_i x^i$ of a very particular form. Following a grand mathematical tradition, we guess, or make an *Ansatz* on the form of the solution, and assume that $u(x) = x^m$ for some m to be determined. Substituting into the differential equation, we get

$$a_0 x^m + a_1 x (x^m)' + a_2 x^2 (x^m)'' = (a_0 + (a_1 - 1)m + a_2 m^2) x^m,$$

and we are thus led to the auxiliary algebraic equation

$$a_0 + (a_1 - 1)m + a_2 m^2 = 0$$

in m. Letting the roots of this equation be m_1 and m_2, assuming the roots are real, any linear combination

$$\alpha_1 x^{m_1} + \alpha_2 x^{m_2}$$

Fig. 35.2. Leonard Euler: "…I soon found an opportunity to be introduced to a famous professor Johann Bernoulli…True, he was very busy and so refused flatly to give me private lessons; but he gave me much more valuable advice to start reading more difficult mathematical books on my own and to study them as diligently as I could; if I came across some obstacle or difficulty, I was given permission to visit him freely every Sunday afternoon and he kindly explained to me everything I could not understand…"

is a solution of (35.17). In fact the general solution of (35.17) has this form if m_1 and m_2 are distinct and real.

Example 35.10. The auxiliary equation for the differential equation $x^2u'' - \frac{3}{2}xu' - 2u = 0$ is $m^2 - \frac{7}{2}m - 2 = 0$ with roots $m_1 = -\frac{1}{2}$ and $m_2 = 4$ and thus the general solution takes the form

$$u(x) = \alpha_1 \frac{1}{\sqrt{x}} + \alpha_2 x^4.$$

Leonard Euler (1707-83) is the mathematical genius of the 18th century, with an incredible production of more than 800 scientific articles half of them written after he became completely blind in 1766, see Fig. 35.2.

Chapter 35 Problems

35.1. Solve the initial value problem (35.7) with $\lambda(x) = x^r$, where $r \in \mathbb{R}$, and $a = 0$.

35.2. Solve the following initial value problems: a) $u'(x) = 8xu(x), u(0) = 1$, $x > 0$, b) $\frac{(15x+1)u(x)}{u'(x)} = 3x$, $u(1) = \mathrm{e}$, $x > 1$, c) $u'(x) + \frac{x}{(1-x)(1+x)}u = 0$, $u(0) = 1$, $x > 0$.

35.3. Make sure that you got the correct answer in the previous problem, part c). Will your solution hold for $x > 1$ as well as $x < 1$?

35.4. Solve the following initial value problems: a) $xu'(x) + u(x) = x$, $u(1) = \frac{3}{2}$, $x > 1$, b) $u'(x) + 2xu = x$, $u(0) = 1$, $x > 0$, c) $u'(x) = \frac{x+u}{2}$, $u(0) = 0$, $x > 0$.

35.5. Describe the behavior of the population of rabbits in West Virginia in which the birth rate exceeds the death rate by 5, the initial population is 10000 rabbits, and (a) there is a net migration out of the state at a rate of $5t$ (b) there is a net migration out of the state at a rate of $\exp(6t)$.

35.6. Describe the concentration in a mixing tank with an initial volume of 50 liters in which 20 grams of solute are dissolved, there is an inflow of 6 liters/sec with a concentration of 10 grams/liter and an outflow of 7 liters/sec.

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